Differentiability of the Value Function in Continuous–Time Economic Models*

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Abstract. In this paper we provide some sufficient conditions for the differentiability of the value function in a class of infinite-horizon continuous-time models of convex optimization arising in economics. We dispense with an interiority condition which is quite restrictive in constrained optimization and it is usually hard to check in applications. The differentiability of the value function is used to prove Bellman's equation as well as the existence and continuity of the optimal feedback policy. We also establish uniqueness of the vector of dual variables under some conditions that rule out existence of asset pricing bubbles.

Keywords. Constrained optimization, value function, differentiability, envelope theorem, duality theory.

1 Introduction

In this paper we study the differentiability of the value function for a class of concave infinite–horizon continuous–time problems of wide application in economics. We extend the envelope theorem of Benveniste and Scheinkman [5] to optimization problems with constraints. We dispense with an interiority condition for the state and control variables

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that is usually quite restrictive in economic applications. This interiority condition may rule out periods of zero consumption, irreversibility of investment, bounded capacity, binding monetary constraints, and various financial market restrictions such as short-sale constraints and collateral requirements. Indeed, in his well-known introduction of control theory to economic growth, Arrow [2] formulated an economic problem with inequality constraints to account for feasibility, irreversibility, market clearing, and non-negative restrictions. There are usually no primitive assumptions that may prevent these constraints from being saturated, and hence one cannot generally invoke the envelope theorem of Benveniste and Scheinkman [5].

In continuous-time models, the differentiability of the value function allows for a simple formulation of Bellman's equation and the maximum principle. Hence, from the differentiability of the value function we obtain that the feedback control or policy is a continuous function. For finite-horizon problems, it is known [cf. Goebel [10]] that if the value function is differentiable then the path of dual variables or supporting prices is unique. We shall extend this uniqueness result for the infinite–dimensional case. Several papers deal with existence of dual variables that belong to the superdifferential of the value function [e.g., Araujo and Scheinkman [1], and Aubin and Clarke [4], and Benveniste and Scheinkman [6]]. Our focus here is on the uniqueness of these dual variables. As discussed later, the problem can be quite complex for infinite–horizon economies with constraints: Uniqueness may be lost in the presence of asset pricing bubbles. Even though we lack a systematic analysis of pricing bubbles, from the study of certain market economies [e.g. see Santos and Woodford [16]] it is known that some free–disposal conditions may preclude the existence of bubbles. We impose a monotonicity restriction so that any path of dual variables must satisfy a standard transversality condition. As shown later, in the absence of this restriction there could be a continuum of initial dual variables that support a given optimal solution.

The starting point of our analysis is our earlier paper [11] on the differentiability of the value function in discrete–time optimization. The continuous–time formulation, however, is technically more involved and requires to make use of infinite–dimensional optimization. But this formulation offers more structure because the dynamical system that generates optimal trajectories is a now a flow: An optimal orbit is conformed by a continuous arc rather than by a countable number of points. This technical difference will be manifested in various stronger results and examples. Theorem 3.2 below shows that differentiability of the value function at the initial point x_0 implies differentiability of the function along the whole optimal trajectory, whereas this result is not guaranteed in the discrete–time formulation. Also, in the scalar case the value function is always differentiable at non–stationary points in the continuous–time case, but this is not so in discrete–time.

In Section 2 we lay out the continuous-time optimization problem. Section 3 contains our main results on the differentiability of the value function. In Section 4 we apply these results to derive Bellman's equation and the uniqueness of the dual variables. Some examples follow in Section 5. A more technical review of our findings will be offered in Section 6. Various mathematical definitions can be found in the Appendix, as well as additional proofs.

2 The dynamic optimization problem

We consider an infinite–horizon optimization problem which is approximated by a sequence of finite–horizon objectives. For finite horizons – rather than for the original optimization problem – we shall make use of a Banach space framework which will be analytically convenient for differentiability. The proof of differentiability of the value function will follow from a limit argument over finite horizons.

2.1 Mathematical setting

Let $t \geq 0$ be the initial date of the optimization problem. Let $I_t = [t, T]$, with $T = \infty$ or $T < \infty$. Let $\beta(s,t) = \exp\left(-\int_t^s \delta(r)\,dr\right)$ be a discount factor over the time interval $[t,s]$, $0 \leq t \leq s$. Function $\delta \geq 0$ is bounded with $\int_t^\infty \delta(r)\,dr = \infty$. Hence, $\beta(\infty, t) = 0$ for all t, and $\beta(t,t) = 1$. Assume that for each $r \in I_t$, there exists a constant $\rho > 0$ such that $\int_r^\infty \beta(s,t)\,ds \leq \rho\beta(r,t)$ for all $r > t$. If δ is a constant discount rate, then $\rho = (1/\delta)$ as $\int_r^\infty e^{-\delta(s-t)}\,ds = (1/\delta)e^{-\delta(r-t)}$.

Let μ_t be the measure on I_t with density $d\mu_t(s) = \beta(s,t)\,ds$. Then, $\mu_t(I_t) < \infty$ for all t. Let $L_n^1(I_t;\mu_t)$ be the set of equivalence classes of (Lebesgue)–measurable functions x_t in $\mathbb{R}^n$ which are μ_t integrable. That is, $\int_{I_t} |x_t(s)|\,d\mu_t(s) < \infty$, where $|x_t(s)|$ is a given norm for $x_t(s)$. It follows that $L_n^1(I_t;\mu_t)$ is a Banach space with norm

$$\|x_t\|_{1,\mu_t} = \int_{I_t} |x_t(s)|\beta(s,t)\,ds.$$

Let $\mu_t^\top$ be the measure on I_t with density $d\mu_t^\top(s) = \beta(s,t)^{-1}\,ds = ds/\beta(s,t)$. The space $L_n^\infty(I_t;\mu_t^\top)$ consists of measurable functions p_t on I_t such that $|p_t(s)|\beta(s,t)^{-1}$ is bounded, except possibly on a set of measure zero. It is also a Banach space with the norm

$$\|p_t\|_{\infty,\mu_t^\top} = \operatorname*{ess\,sup}_{s\in I_t} |p_t(s)|\beta(s,t)^{-1} = \inf_{\substack{y(s)=p_t(s)\\ \text{Lebesgue–a.e.}}} \sup_{s\in I_t} |y(s)|\beta(s,t)^{-1}.$$

These two spaces conform a dual pair under the bilinear form

$$\langle x_t, p_t \rangle = \int_{I_t} x_t(s) p_t(s)\, ds, \qquad x_t \in L^1_n(I_t; \mu_t), \quad p_t \in L^\infty_n(I_t; \mu_t^\top).$$

In what follows, $\dot{x}_t(s)$ is the time derivative of function x_t at time s.

2.2 Continuous–time optimization

The continuous–time optimization problem can now be posed as follows. Given an initial state x_0 and the initial date $t \geq 0$, find a path $\{(x_t^*, \dot{x}_t^*)\}$ that solves the maximization program

$$\begin{aligned} &V(t, x_0) = \max \int_t^T \ell(x_t(s), \dot{x}_t(s)) \beta(s,t)\, ds \\ &\text{subject to } (x_t(s), \dot{x}_t(s)) \in \Omega \quad \text{for all } s \in [t, \infty] \quad \text{and} \quad x_t(t) = x_0. \end{aligned} \tag{1}$$

(A1) *$X \subseteq \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^{2n}$ are convex sets with nonempty interior. For each $x \in X$ the set $\Omega_x = \{u : (x,u) \in \Omega\}$ is non-empty.*

(A2) *Function $\ell : \Omega \longrightarrow \mathbb{R}$ is concave, continuous, and differentiable of class C^1 at every point $(x,u) \in \Omega$.*

(A3) *An optimal solution $\{(x_t^*, \dot{x}_t^*)\}$ does exist, and $(x_t^*, \dot{x}_t^*) \in L^1_n(I_t; \mu_t) \times L^1_n(I_t; \mu_t)$ for all $I_t = [t, \widehat{T}]$ with $\widehat{T} < \infty$.*

Existence of an optimal solution is guaranteed under various standard assumptions [cf. Dmitruk and Kuz´kina [7]]. We then have that the value function $V(t, \cdot)$ in (1) is well defined on X. Note that we only require $(x_t^*, \dot{x}_t^*) \in L^1_n(I_t; \mu_t) \times L^1_n(I_t; \mu_t)$ for all $I_t = [t, \widehat{T}]$, with finite $\widehat{T} < \infty$. This latter condition usually follows from mild convexity assumptions [cf., Fleming and Rishel [9]].

2.3 Some regularity conditions for differentiability of the value function

The following conditions will allow us to dispense with the interiority assumption of Benveniste and Scheinkman [5]. First, if x_t reaches the boundary of X then the value function V may not be differentiable. By backward induction, this lack of differentiability may extend to other points in the optimal path. We therefore assume

(IS) *An optimal path* $x_t^*(s) \in \operatorname{int} X$ *for every* $s \in I_t$.

Rincón–Zapatero and Santos [11] provide some examples of non–differentiability when the assumption of interiority of the state variables (IS) is not satisfied, but as shown below for continuous–time one–dimensional optimization this interiority requirement is generally not needed.

(LI) Ω *can be defined by a finite set of inequalities*

$$g^i(x,u) \geq 0 \quad \text{for} \quad i = 1, \dots, m,$$

where the functions g^i *are* C^1 *in a neighborhood of* Ω. *Let* $g_\sigma = \{g^i : g^i(x,u) = 0\}$. *Then, matrix* $D_2 g_{\sigma(t)}(x_t^*(s), \dot{x}_t^*(s))$ *has full rank over the optimal path* $\{x_t^*(s), \dot{x}_t^*(s)\}$ *for almost all* $s \in [t,T]$.

The notation is as follows: $D_1 g$ and $D_2 g$ are the Jacobian matrices of $(g^1, \dots, g^m)$ with respect to x and $u = \dot{x}$, respectively. As is well-known, linear independence (LI) implies that matrix $D_2 g_{\sigma(s)}^\top$ has a generalized right-inverse $D_2 g_{\sigma(s)}^+$, and guarantees uniqueness of the Kuhn–Tucker multipliers in static differentiable programs. It is important to note that (LI) requires that at least one control variable appears in every saturated constraint; for if not, one of the rows in matrix $D_2 g_\sigma$ is made up of zeros, violating the rank condition. Therefore (LI) rules out pure state constraints. As in (IS), general results on the differentiability of the value function cannot be expected in the presence of pure state constraints.

We also postulate a free disposal assumption to insure non-existence of asset pricing bubbles (NB) for decentralized economies. Let the $n \times n$–matrix $G(\sigma; x, u) = -(D_1 g_\sigma^\top D_2 g_\sigma^+)(x,u)$, where g_σ comes from (LI) and indicates the constraints that are active at (x,u). Assumption (LI) guarantees that function g_σ is measurable. To shorten the notation we will write $G_t^*(s) = G_t(\sigma(s); x_t^*(s), \dot{x}_t^*(s)) = -(D_1 g_{\sigma(s)}^\top D_2 g_{\sigma(s)}^+)(x_t^*(s), \dot{x}_t^*(s))$. Note that if no constraint is saturated at time s, then $G(0; \cdot, \cdot)$ is the null matrix.

(NB) *(i)* $X = \mathbb{R}_+^n$. *(ii) For all* (x,u) *function* $\ell(x,u)$ *is increasing in* x *and decreasing in* u. *(iii) Over the optimal solution* $\{(x_t^*(s), \dot{x}_t^*(s))\}$, *for each* $s \geq t$ *we have* $D_1 \ell(x_t^*(s), \dot{x}_t^*(s)) + G_t^*(s) D_2 \ell(x_t^*(s), \dot{x}_t^*(s)) \geq 0$ *and* $G_t^*(s) \geq 0$. *Moreover, there is a constant* $\alpha > 0$ *such that* $x_{tj}^*(s) \geq \alpha$ *for each coordinate* $j = 1, \dots, n$. *And (iv) for every time* $\widehat{T} > 0$ *there are* $T \geq \widehat{T}$ *and a vector* $\dot{x}$ *with all negative coordinates such that* $g(x_t^*(s), \dot{x}_t^*(T) + \dot{x}) \geq 0$ *and a constant* $0 < \lambda < 1$ *with* $g(\lambda x_t^*(s), \lambda \dot{x}_t^*(s)) \geq 0$ *for all* $s \geq T$.

The existence of asset pricing bubbles in economies with constraints is a rather complex topic that has not been systematically explored. Conditions (NB)(i)–(NB)(iii) are taken from Santos and Woodford [16]. We impose (NB)(iv) because we are using more general constraints. In the absence of this latter condition, it is not feasible to burst out an asset pricing bubble by optimization behavior. Condition (NB)(iv) will also emerge in our discussion of dual variables in Section 4.

For the sake of comparison, we include the interiority assumption postulated by Benveniste and Scheinkman [5].

(IN) *There exist an open and convex set $U \subset X$ and an open neighborhood $B \subset \mathbb{R}^{2n}_+$ and a time $h > 0$, such that $(x^*_t(s), \dot{x^*}_t(s)) + B \subset \Omega$ for all $x_0 \in U$ and almost all $s \in [t, t+h]$.*

That is, there exists an ε-neighborhood of the optimal path $\{(x^*_t(s), \dot{x^*}_t(s))\}$ that belongs to Ω at some initial phase.

3 Results

3.1 Mathematical preliminaries

We start with the following property for concave optimization problems [cf. Aubin [3], Proposition 4.3]. Here, E and F are Banach spaces, and $\partial v(x)$ is the superdifferential of a concave function v.

Proposition 3.1 *Let f be a proper concave function from $E \times F$ to $\mathbb{R} \cup \{-\infty\}$. Consider the function $v : E \longrightarrow \mathbb{R} \cup \{-\infty\}$ defined by*

$$v(x) = \sup_{u \in F} f(x, u).$$

If $\overline{u} \in F$ satisfies $v(x) = f(x, \overline{u})$, then the following conditions are equivalent

$$q \in \partial v(x)$$
$$(q, 0) \in \partial f(x, \overline{u}).$$

Remark 3.1 It follows that $(q, 0) \in \partial f(x, \overline{u})$ if and only if $\overline{u} \in \arg\max f(x, u)$ since function f is concave. Indeed, the condition $q \in \partial v(x)$ is independent of the maximizer chosen.

We now transform a problem with constraints into one of unrestricted maximization by incorporating the indicator function of the feasible set Ω into the integrand of problem (1). Let

$$\mathscr{L}(x,u) = \ell(x,u) - I_\Omega(x,u),$$

where $I_\Omega(x,u) = 0$ if $(x,u) \in \Omega$ and $+\infty$ otherwise.

Assumptions (A1)–(A3) imply that $\mathscr{L}$ is a proper, upper semicontinuous and concave function. Then, problem (1) can now be stated as

$$\begin{aligned} V(t,x_0) = \max & \int_t^T \mathscr{L}(x_t(s), \dot{x}_t(s))\beta(s,t)\, ds \\ \text{subject to} \quad & x(t) = x_0. \end{aligned} \tag{2}$$

Our first step is to compute the superdifferential of the integrand in (2) for $T < \infty$, and then provide a characterization of the superdifferential of the value function. Let $J_t : I_t \times [L^1_n(I_t;\mu_t)]^2 \longrightarrow \mathbb{R} \cup \{-\infty\}$ be given by

$$J_{t,T}(x_t,u_t) = \begin{cases} \displaystyle\int_t^T \mathscr{L}(x_t(s),u_t(s))\beta(s,t)\, ds & \text{if } \mathscr{L}(x_t(s),u_t(s)) \in L^1_n(I_t;\mu_t), \\ -\infty & \text{otherwise.} \end{cases} \tag{3}$$

Lemma 3.1 *Function $J_{t,T}$ is proper, upper semicontinuous, and concave. Moreover,*

$$\partial J_{t,T}(x_t,u_t) = \Big\{(p_t,q_t) \in \big[L^\infty_n(I_t;\mu_t^\top)\big]^2 \,:\, -(p_t(s),q_t(s)) \in \beta(s,t)\partial\mathscr{L}(x_t(s),u_t(s)) \;\, a.e.\Big\}.$$

Proof. By (A1)–(A3) it is clear that function $J_{t,T}$ is proper, upper semicontinuous, and concave. The superdifferential of function $J_{t,T}$ follows from the characterization of the subdifferential of functionals defined by means of integrals provided in [12, 15] and the established duality pairing; see the Appendix for further details. □

Lemma 3.2 *Let $x_0 \in \operatorname{int}(X)$. Then, $q_0 \in \partial V(t,x_0)$ if and only if there exists $(p_t,q_t) \in L^\infty_n(I_t;\mu_t^\top) \times L^\infty_n(I_t;\mu_t^\top)$ and $\xi_{t,T} \in \partial V(T, x_t^*(T))$ such that*

$$\begin{aligned} q_0 &= -\int_t^T p_t(s)\, ds + \beta(T,t)\xi_{t,T} \\ q_t(s) &= -\int_s^T p_t(r)\, dr + \beta(T,t)\xi_{t,T} \\ -(p_t(s),q_t(s)) &\in \beta(s,t)\partial\mathscr{L}(x_t^*(s),\dot{x}_t^*(s)) \quad a.e.\ t \le s \le T. \end{aligned}$$

An immediate consequence of this lemma is the envelope theorem of Benveniste and Scheinkman [5], where the above indicator function $I_\Omega(x,u) = 0$ in an ε–tube of the optimal path.

Theorem 3.1 (Benveniste and Scheinkman [5]) *Suppose that (A1)–(A3) and (IN) are satisfied. Then, the value function is differentiable at x_0 and the derivative*

$$DV(t, x_0) = -D_2\ell(x_0, \dot{x}_t^*(t)).$$

Proof.

By condition (IN) we get $\mathscr{L}((x_t^*(s), \dot{x}_t^*(s)) = \ell(x_t^*(s), \dot{x}_t^*(s))$ for $s \in [t, t+h]$. Then, by Lemma 3.2 the path $q_t(s)$ is absolutely continuous with $q_t(s) = -D_2\ell(x_t^*(s), \dot{x}_t^*(s))$ a.e. $s \in [t, t+h]$. Hence,

$$q_0 = q_t(t) = \lim_{s \to t^+} \frac{1}{s-t} \int_t^s -D_2\ell(x_t^*(r), \dot{x}_t^*(r))\, dr,$$

is unique. It follows that $\partial V(t, x_0)$ is singled–valued. Consequently, $V(t, \cdot)$ is differentiable at x_0. Moreover, by (A2) we obtain $q_t(t) = -D_2\ell(x_t^*(t), \dot{x}_t^*(t))$. □

3.2 Differentiability of the value function in constrained optimization

As before, $G_t^*(s) = G_t(\sigma(s); x_t^*(s), \dot{x}_t^*(s)) = -(D_1 g_{\sigma(s)}^\top D_2 g_{\sigma(s)}^+)(x_t^*(s), \dot{x}_t^*(s))$. It should be understood that assumptions (A1)–(A3), (IS), (LI) and (NB) will be in force for all our main results in this section.

Proposition 3.2 *Let $x_0 \in \operatorname{int}(X)$. Let $T < \infty$. Then, $q_0 \in \partial V(t, x_0)$ if and only if there exists $q_t \in L_n^\infty(I_t; \mu_t^\top)$, $-(\overline{p}_t(s), \overline{q}_t(s)) \in \beta(s,t)\partial\ell(x_t^*(s), \dot{x}_t^*(s))$ a.e., and $\xi_{t,T} \in \partial V(T, x_t^*(T))$ such that q_t is the unique absolutely continuous solution in $L_n^\infty(I_t; \mu_t^\top)$ of the linear differential system*

$$\dot{q}_t(s) = \overline{p}_t(s) + G_t^*(s)(\overline{q}_t(s) - q_t(s)), \tag{4}$$

with initial condition

$$q_0 = q_t(t) = -\int_{I_t} \overline{p}_t(s) + G_t^*(s)(\overline{q}_t(s) - q_t(s))\, ds + \beta(T, t)\xi_{t,T}.$$

Proof. By well–known properties of convex analysis

$$\partial\mathscr{L}(x, u) = \partial\ell(x, u) - \partial I_\Omega(x, u) = \partial\ell(x, u) - N_\Omega(x, u), \tag{5}$$

where I_Ω is the indicator function and N_Ω is the normal cone of the convex set Ω [Rockafellar [13]]. Now, by concavity the normal cone to Ω at (x, u) is given by

$$-N_\Omega(x, u) = \left\{ \sum_{i \in \sigma(x,u)} \lambda^i (D_1 g^i(x, u), D_2 g^i(x, u)) + (z, 0) \,:\, \lambda^i \geq 0, \quad z \in N_X(x) \right\},$$

where $i = 1, 2, \ldots, \sigma$ refers to those constraints which are saturated at (x, u), and $N_X(x)$ the normal cone to X at $x \in X$. Note that $N_X(x_t^*(s)) = \{0\}$ because $x_t^*(s)$ is an interior point of X as asserted in (IS).

By Lemma 3.2, we have that $q_0 \in \partial V(t, x_0)$ if and only if there exists $(p_t, q_t) \in [L_n^\infty(I_t; \mu_t^\top)]^2$ such that

$$q_0 = -\int_{I_t} p_t(s)\, ds + \beta(T, t)\xi_{t,T} \tag{6}$$

$$q_t(s) = -\int_{I_s} p_t(r)\, dr + \beta(T, t)\xi_{t,T} \tag{7}$$

$$-(p_t(s), q_t(s)) \in \beta(s, t)\partial \mathscr{L}(x_t^*(s), \dot{x}_t^*(s)) \quad \text{a.e.} \tag{8}$$

By (5) and (8), $p_t = \overline{p}_t + \widehat{p}_t$ and $q_t = \overline{q}_t + \widehat{q}_t$, where $-(\overline{p}_t, \overline{q}_t) \in \beta(s, t)\partial\ell(x_t^*, \dot{x}_t^*)$ a.e., and $-(\widehat{p}_t, \widehat{q}_t) \in \beta(s, t)N_\Omega(x_t^*, \dot{x}_t^*)$ a.e. Thus, combining these equalities with the characterization of the normal cone $N_\Omega(x_t^*, \dot{x}_t^*)$, we obtain

$$\widehat{p}_t(s) = \beta(s, t) \sum_{i \in \sigma(s)} \lambda_t^i(s) D_1 g^i(x_t^*(s), \dot{x}_t^*(s)),$$

$$\widehat{q}_t(s) = \beta(s, t) \sum_{i \in \sigma(s)} \lambda_t^i(s) D_2 g^i(x_t^*(s)), \dot{x}_t^*(s))$$

a.e., for some $\lambda_t^i(s) \geq 0$, $i = 1, \ldots, m$. By (IS), we can then substitute out

$$\lambda_t(s) = \beta(s, t)^{-1} D_2 g_{\sigma(s)}^+(x_t^*(s), \dot{x}_t^*(s))\, \widehat{q}_t(s),$$

so that

$$\widehat{p}_t(s) = -G_t^*(s)\widehat{q}_t(s) = G_t^*(s)(\overline{q}_t(s) - q_t(s)).$$

Plugging $\widehat{p}_t(s)$ into (7) we obtain

$$q_t(s) = -\int_{I_s} \Big(\overline{p}_t(r) + G_t^*(r)(\overline{q}_t(r) - q_t(r))\Big)\, dr + \beta(T, t)\xi_{t,T}. \tag{9}$$

It follows that q_t is absolutely continuous since $\dot{q}_t(s)$ exists a.e. and

$$\dot{q}_t(s) = \overline{p}_t(s) + G_t^*(s)(\overline{q}_t(r) - q_t(r))$$

at points of differentiability. □

Remark 3.2 From Proposition 3.2 we observe that there is a diffeomorphism between the superdifferentials $\partial V(t, x_0)$ and $\partial V(T, x_t^*(T))$. That is, there exists only one function, q_t, joining q_0 with $\beta(T, t)\xi_{t,T}$. This is because ℓ is smooth and the saturated constraints satisfy (LI). Hence, there are unique points $(\widehat{p}_t, \widehat{q}_t)$ in the normal cone to the feasible set at the optimal solution. The flow mapping linking points $q_0 \in \partial V(t, x_0)$ with points $\xi_{t,T} \in \partial V(T, x_t^*(T))$ is illustrated in Figure 1.

FIGURE 1

Consider the linear homogeneous system $\dot{z}(s) = z(s)G_t^*(s)$ and the associated fundamental matrix $\Phi_t(s)$ with $\Phi_t(t) = I_n$, where I_n is the identity matrix. Note that this is the unique matrix satisfying $\dot{\Phi}_t(s) = \Phi_t(s)G_t^*(s)$ for every $s \geq t$ a.e. Moreover, the inverse $\Phi_t^{-1}(s)$ exists and satisfies $\dot{\Phi}_t^{-1}(s) = -G_t^*(s)\Phi_t^{-1}(s)$ a.e.

Theorem 3.2 *Suppose* $x_0 \in \text{int}\,(X)$. *Let*

$$\lim_{T\to\infty} \beta(t,T)\Phi_t(T)q_t(T) = 0. \tag{10}$$

Then, $V(t,\cdot)$ *is differentiable at* x_0 *and* $V(s,\cdot)$ *is also differentiable along the optimal trajectory* $x_t^*(s)$, *for every* $s \geq t$. *Moreover, the derivative*

$$DV(t,x_0) = \int_t^\infty \Phi_t(s)\big(D_1\ell(x_t^*(s),\dot{x}_t^*(s)) + G_t^*(s)D_2\ell(x_t^*(s),\dot{x}_t^*(s))\big)\beta(s,t)\,ds. \tag{11}$$

Proof. Let q_0, q_0' in $\partial V(t,x_0)$. Let $q_t(s;q_0)$ be a solution of (4) with initial condition $q_t(t) = q_0$. Then, $q_t(s;q_0)$ is unique by Proposition 3.2 and Remark 3.2. It is known from the theory of linear ODEs that

$$q_t(s;q_0) = \Phi_t^{-1}(s)q_0 + \Phi_t^{-1}(s)\int_t^s \Phi_t(r)(\overline{p}_t(r) + G^*(r)\overline{q}_t(r))\,dr. \tag{12}$$

Hence, for some $\xi_{t,T}, \xi'_{t,T} \in \partial V(T, x_t^*(T))$ for which $q_{t,T} = \beta(t,T)\xi_{t,T}$, $q'_{t,T} = \beta(t,T)\xi'_{t,T}$

$$|q_0 - q_0'| \leq |\beta(T,t)\Phi_t(T)\xi_{t,T} - \beta(T,t)\Phi_t(T)\xi'_{t,T}| \to 0, \quad \text{as } T\to\infty.$$

Convergence of this last term is due to the above condition (10). Thus, $q_0 = q_0'$, and $\partial V(t,x_0)$ is a singleton. Therefore, $V(t,\cdot)$ is differentiable at x_0.

We next show that $V(s,\cdot)$ is differentiable at $x_t^*(s)$, $s > t$. Note that every element in $\partial V(s,x_t^*(s))$ is the image of $q_t(s;q_0)\beta^{-1}(s,t)$, that is

$$\bigcup_{q_0\in\partial V(t,x_0)} \{q_t(s;q_0)\} = \beta(s,t)\partial V(s,x_t^*(s))$$

for every $s \geq t$. By uniqueness of solutions to linear ODEs, $q_t(s;q_0)$ is unique in view of the uniqueness of q_0. Since $\partial V(t,x_0)$ is singled valued, $\partial V(s,x_t^*(s))$ is also single-valued. Consequently, $V(s,\cdot)$ is differentiable at $x_t^*(s)$.

The expression for the derivative (11) obtains from (12) for $s = T$, $q_t(T) = \beta(T,t)\xi_{t,T}$ and using (10). More precisely, solving for q_0 and taking limits as $T\to\infty$ we get

$$q_0 = DV(t,x_0) = -\int_t^\infty \Phi_t(s)\big(\overline{p}_t(s) + G^*(s)\overline{q}_t(s)\big)\,ds.$$

Then, recall that $\overline{p}_t(s) = -\beta(s,t)D_1\ell(x_t^*(s), \dot{x}_t^*(s))$ and $\overline{q}_t(s) = -\beta(s,t)D_2\ell(x_t^*(s), \dot{x}_t^*(s))$. □

It remains to establish the non-bubble condition

Proposition 3.3 *Suppose $x_0 \in \operatorname{int}(X)$. Then*

$$\lim_{T\to\infty} \beta(t,T)\Phi_t(T)q_t(T) = 0. \tag{13}$$

See the Appendix.

3.3 The scalar case

In the one dimensional case with a constant discount factor we have that differentiability is attained without assumption (IS). In higher dimensions the argument does not work, since an absolute continuous curve has zero Lebesgue measure.

Corollary 3.1 *Let $n = 1$ and suppose that the discount rate δ is constant. Consider that $x_0 \in \operatorname{int}(X)$ is such that the optimal path $x(s)$ from x_0 satisfies $\dot{x}_t^*(s) \neq 0$ on some interval $t \leq s \leq T$. Then, V is differentiable at x_0.*

Proof. We argue by contradiction. If V is not differentiable at x_0, then by Proposition 3.2 we get that V is not differentiable at $x(s)$ for any $s \geq t$ either. Hence, V is not differentiable in a set of positive Lebesgue measure, by assumption. This leads to a contradiction with the concavity of V, since a real concave function has at most countably many points of non–differentiability. □

Actually, since the optimal trajectory x_t^* is absolutely continuous, it must be that the set $\{x(s) : t \leq s \leq T\}$ is a singleton if and only if $\dot{x}_t^*$ is zero over the interval $[t, T]$. Therefore, in the one dimensional case with a constant discount rate, the value function is differentiable at all interior points of the state space, with the possible exception of stationary points. We study now the differentiability of the value function at stationary points for a general state space $X \subset \mathbb{R}^n$.

3.4 Differentiability at stationary points

Here we dispense with Assumption (NB)(iv). By an optimal stationary point we mean a constant optimal solution $x^* = x_t^*(s)$ for almost all s, so that $\dot{x}_t^*(s) = 0$ for all s.

Corollary 3.2 *Assume that the discount rate δ is constant. Let $x^* \in \operatorname{int}(X)$ be an optimal stationary point. Suppose that all coordinates of vector $D_1\ell(x^*, 0)+G(x^*, 0)D_2\ell(x^*, 0)$ are positive. Then, V is differentiable at x^*.*

Proof. Since the discount rate δ is constant, the value function is time–independent. Using equation (12) in Theorem 3.2 and the identity $q(T) = \beta(T,0)\xi_{t,T}$ we know that $q_0 \in \partial V(x_0)$ if and only if for any T there exists $\xi_T \in \partial V(x^*(T))$ such that

$$q_0 = \int_0^T \beta(s,0)\Phi(s)(D_1\ell(x^*(s),\dot{x}^*(s)) + G^*(s)D_2\ell(x^*(s),\dot{x}^*(s)))\,ds + \beta(T,0)\Phi(T)\xi_T.$$

As x^* is a stationary point this equality reads

$$q_0 = \int_0^T e^{(G(x^*,0)-\delta I_n)s}(D_1\ell(x^*,0) + G(x^*,0)D_2\ell(x^*,0))\,ds + e^{(G(x^*,0)-\delta I_n)T}\xi_T. \tag{14}$$

Note that now the fundamental matrix is $\Phi(s) = e^{G(x^*,0)s}$; moreover, both q_0, ξ_T belong to $\partial V(x^*)$ for any T, and by assumption, each component of vector $D_1\ell(x^*,0) + G(x^*,0)D_2\ell(x^*,0)$ is strictly positive. Hence, $e^{(G(x^*,0)-\delta I_n)T}$ tends to the null matrix as $T \to \infty$. Therefore, V is differentiable at x^* because q_0 is univocally defined as

$$q_0 = \left(\int_0^\infty e^{(G(x^*,0)-\delta I_n)s}\,ds\right)(D_1\ell(x^*,0) + G(x^*,0)D_2\ell(x^*,0)).$$

□

3.5 Some counterexamples

3.5.1 Necessity of assumption (IS)

We will show the necessity of (IS) in a simple specification of the optimal growth model that will be studied in detail in Section 5.1. We assume a constant discount $\delta > 0$, $X = [0,\infty)$, a linear utility $U(c) = c$, and a linear production function $f(k) = \alpha k$ for some $\alpha > 0$ and k in $[0,1]$. For $k \geq 1$, we assume that f is smooth, concave and $\lim_{k\to\infty} f'(k) = 0$. According to Dmitruk and Kuz´kina ([7], Th. 1), the problem admits a solution for any discount factor $\delta > 0$, and every trajectory is bounded.

For $0 < k_0 < 1$, the admissible trajectory is given by $\dot{k}(s) = \alpha k(s)$ if $\alpha \geq \delta$, and $\dot{k}(s) = 0$, otherwise. Let T such that $k(T) = k_0 e^{\alpha T} = 1$. By Lemma 7.1 in the Appendix

$$\begin{aligned} V(k_0) &= \sup_{0\leq \dot{k}_t \leq f(k_t)} \left\{ \int_0^T e^{-\delta s}(f(k(s)) - \dot{k}(s))\,ds + e^{-\delta T}V(k(T)) \right\} \\ &\geq e^{-\delta T}V(k(T)) = e^{-\delta T}V(1) = V(1)k_0^{\delta/\alpha}. \end{aligned}$$

The value function is continuous on X, with $V(k) > 0$ for any $k > 0$ and $V(0) = 0$. Hence, the above inequality determines that $\partial V(0) = \emptyset$ if $\alpha > \delta$.

3.5.2 Necessity of assumption (LI)

Even in the scalar case, condition (LI) cannot be weakened. Consider the following problem

$$V(x_0) = -\max \int_0^\infty -e^{-\delta t} x(t)\, dt, \qquad \delta > 0,$$

subject to the constraints: $\dot{x} \geq -2x$ and $\dot{x} \geq -\frac{1}{2}x$. This set of feasible choices Ω is depicted in Figure 2. At point $x_0 = 0$ both constraints are saturated, thus (LI) does not hold since the problem is one–dimensional. It is clear that in this simple problem optimality requires $x(t)$ to be as small as possible. In the region where $x > 0$ the smallest admissible derivative, $\dot{x} = -\frac{1}{2}x$. Hence, for $x_0 > 0$ the optimal path is $x(t) = x_0 e^{-t/2}$. It follows that $x(t) > 0$ for every t, since the stationary point $x_0 = 0$ is never reached in finite time. In the region where $x < 0$ any derivative is positive, thus x increases. The smallest derivative is $\dot{x} = -2x$. Hence, for $x_0 < 0$ the optimal path is $x(t) = x_0 e^{-2t} < 0$ for every t. Obviously, $x_0 = 0$ is an optimal stationary point.

Therefore, the value function

$$V(x_0) = \begin{cases} \dfrac{x_0}{2+\delta}, & \text{if } x_0 < 0; \\ \dfrac{x_0}{\frac{1}{2}+\delta}, & \text{if } x_0 \geq 0. \end{cases}$$

This function is not differentiable at $x_0 = 0$.

FIGURE 2

3.5.3 Necessity of assumption (NB)

The existence of bubbles complicates our method of proof. Indeed, our strategy of proof is to show that the vector of dual variables q_0 is unique; bubbles may generate multiple explosive paths for different initial conditions q_0. These explosive paths may occur in the absence of assumption (NB).

Let $x_0 \in \mathbb{R}_+$, and

$$\begin{aligned} &V(t, x_0) = \max \int_t^\infty U(c_t(s)) e^{-\delta t}\, ds \\ &\text{s.t. } c_t(s) = \delta x_t(s) - \dot{x}_t(s) \geq 0,\ x_t(s) + \dot{x}_t(s) \geq 1,\ s \in [t, \infty],\ x_t(t) = x_0. \end{aligned} \tag{15}$$

This can be viewed as a Lucas–tree model of asset pricing where $\delta x_t(s)$ is the dividend payment, and consumption $c_t(s) = \delta x_t(s) - \dot{x}_t(s) \geq 0$. There is also the borrowing restriction $\dot{x}_t(s) \geq 1 - x_t(s)$. That is, the agents need to keep a minimum amount of saving when $x_t(s) \leq 1$. Let $\ell(x_t(s), \dot{x}_t(s) = U(\delta x_t(s) - \dot{x}_t(s))$.

For $x_t(t) \geq 1$, the optimal solution is $x_t(s) = x_t(t)$ and $\dot{x}_t(s) = 0$ for all $s \geq t$. And for $x_t(t) \leq 1$, the optimal solution must follow the law of motion $\dot{x}_t(s) = 1 - x_t(s)$. Assume that $x_t(t) < 1$ is sufficiently close to 1 so that consumption is positive: $c_t(s) = \delta x_t(s) - \dot{x}_t(s) > 0$. In this case the value function is differentiable. We can compute the derivative using the equation:

$$q_0 = \int_0^T e^{(-1-\delta)s}(D_1\ell(x_t(s), \dot{x}_t(s)) - D_2\ell(x_t(s), \dot{x}_t(s)))\, ds + e^{(-1-\delta)T} q_T. \tag{16}$$

Although the value function is differentiable our method of proof fails. The problem is that this equation has multiple solutions (q_0, q_T), which cannot be ruled out by our proof of the main theorem since a feasible path requires $\dot{x}_t(s) = 1 - x_t(s) > 0$. Hence, even if q_T is above the fundamental value, it is not optimal or feasible to reduce asset holdings x_T. Therefore, the optimal solution is associated with a continuum of dual variables that cannot be ruled out by optimization behavior.

4 Duality theory and Bellman's equation

We first show uniqueness of dual arcs satisfying a transversality condition. This uniqueness result easily follows from the differentiability of the value function and some properties of partial superdifferentials of saddle functions discussed in the Appendix. We also derive Bellman's equation and show the continuity of the optimal feedback control or policy function. Of course, if the policy function is continuous then the optimal solution $x_t^*(s)$ is a C^1 function of s.

We begin with the Hamiltonian of the optimization problem, which is defined as

$$H(x, q) = \sup_u \{\mathscr{L}(x, u) + qu\}. \tag{17}$$

Combining Lemma 3.2 with Proposition 7.2 in the Appendix, an optimal solution $u = x_t^*$ must satisfy the Hamiltonian inclusions

$$\begin{aligned} -\dot{q}_t(s) &\in \beta(s,t)\partial_x H(x_t^*(s), q_t(s)), \\ \dot{x}_t^*(s) &\in \beta(s,t)\partial_q H(x_t^*(s), q_t(s)), \end{aligned} \tag{18}$$

for almost all $s \in [t, T]$. Here, $\partial_x H$ denotes the superdifferential of the concave function $x \mapsto H(x, q)$ for a fixed q, and $\partial_q H$ denotes the subdifferential of the convex function $q \mapsto H(x, q)$ for a fixed x. For any pair (x_t^*, q_t) satisfying the Hamiltonian inclusions with $x_t^*(t) = x_0$, we say that q_t is the dual variable. It has the interpretation of a shadow price.

Theorem 4.1 *Let the pair* (x_t^*, q_t) *satisfy the Hamiltonian inclusions* (18) *with* $x_t^*(t) = x_0$. *Assume that the following transversality condition holds*[1]

$$\lim_{T\to\infty} q_t(T)x_t^*(T) = 0. \tag{19}$$

Then, the path of dual variables $q_t(s)$ *is unique.*

Bellman's equation is a fundamental tool in solving dynamic programming problems. As is well known, Bellman's equation holds if the value function is smooth; moreover, the optimal policy correspondence is obtained as the *arg max* of this equation. Therefore, the differentiability of the value function is helpful for the existence and numerical solution of Bellman's equation. Let us rewrite (17) as

$$H(x,q) = \sup_{u\in\Omega_x} \{\ell(x,u) + qu\}.$$

Assuming a constant discount rate: $\delta(s) = \delta$ for every s, we get Bellman's equation as

$$-\delta V(x) + H(x, DV(x)) = 0 \quad \text{for all } x \in \operatorname{int} X.$$

That is,

$$-\delta V(x) + H(x, DV(x)) = -\delta V(x) + \sup_{u\in\Omega_x} \{\ell(x,u) + DV(x)u\} = 0 \quad \text{for all } x \in \operatorname{int} X.$$

Let us define the optimal policy correspondence $u \in h(x) = \partial_q H(x, DV(x))$ that is, the set of admissible values of $u \in \Omega_x$ that solves $\max_{u\in\Omega_x}\{\ell(x,u) + qu\}$.

Proposition 4.1 *Assume that the multivalued mapping* $x \rightrightarrows \Omega_x$ *is continuous and that* Ω_x *is a compact set for every* $x \in X$. *Assume also that* ℓ *is strictly concave with respect to* u. *Then, the optimal* $\dot{x}_t^*$ *is given by a continuous function* $\dot{x}_t^* = h(x_t)$ *in* $\operatorname{int}(X)$, *where* $h(x) = \partial_q H(x, DV(x))$.

Proof. Since V is differentiable on $\operatorname{int}(X)$, function $(x,u) \mapsto \ell(x,u) + DV(x)u$ is continuous. Hence, by Berge's Theorem, h is upper hemicontinuous. Moreover, by the strict concavity of ℓ in u, the maximizer $h(x)$ is unique, and thus h is a continuous function. Finally, the expression $h(x) = \partial_q H(x, DV(x))$ follows from the first-order conditions. □

[1] It is well known [cf., [5]] that assumption (NB) implies (19).

5 Examples

5.1 The one–sector growth model with irreversible investment

Consider the following version of the neoclassical growth model:

$$\begin{aligned}
&\max_{c_t(s),i_t(s)} \int_t^\infty \beta(s,t)U(c_t(s))\,ds \qquad \text{subject to}\\
&\dot{k}_t(s) = i_t(s) - \gamma k_t(s),\\
&c_t(s) + i_t(s) = f(k_t(s)),\\
&k_t(s) \geq 0, \quad c_t(s) \geq 0, \quad i_t(s) \geq 0, \quad k_t(t) = k_0.
\end{aligned}$$

The notation is as follows: $k_t(s)$ is capital at time s, $c_t(s)$ is consumption, and $i_t(s)$ is investment. The utility function, $U : \mathbb{R}_+ \longrightarrow \mathbb{R}$, is increasing, concave, differentiable over $[0,\infty)$ with $U'(0^+) < \infty$ or $U'(0^+) = \infty$. The production function, $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, is bounded, increasing, concave, and differentiable in $\mathbb{R}_+$, with $f'(0^+) = \infty$.

As is well understood, the problem can be mapped into the variables $(k_t, \dot{k}_t)$ corresponding to our original framework:

$$\begin{aligned}
&\max_{k_t(s),\dot{k}_t(s)} \int_t^\infty \beta(s,t)U(f(k_t(s)) - \gamma k_t(s) - \dot{k}_t(s))\,ds \qquad \text{subject to}\\
&-\gamma k_t(s) \leq \dot{k}_t(s) \leq f(k_t(s)) - \gamma k_t(s), \quad k_t(s) \geq 0.
\end{aligned}$$

Then, the instantaneous utility function is $\ell(k,u) = U(f(k) - \gamma k - u)$ with derivatives

$$D_1\ell(x,u) = U'(c)(f'(k) - \gamma), \quad D_2\ell(k,u) = -U'(c).$$

The constraints are $g^1(k,u) = u + \gamma k$, $g^2(k,u) = f(x) - \gamma k - u$. The feasible set is depicted in Figure 3.

FIGURE 3

It follows that

$$\begin{aligned}
G(\{1\};k,u) &= -D_1g^1(k,u)D_2g^1(k,u) = -\gamma,\\
G(\{2\};k,u) &= -D_1g^2(k,u)D_2g^2(k,u) = f'(k) - \gamma,\\
G(\{1,2\},k,u) &= 0.
\end{aligned}$$

Therefore,

$$G_t^*(s) = \begin{cases} -\gamma, & \text{if } \sigma_t(s) = 1;\\ f'(k_t^*(s)) - \gamma, & \text{if } \sigma_t(s) = 2, \end{cases}$$

and $\Phi_t(s) = e^{\int_t^s G_t^*(r)\,dr}$.

We are now ready to check that all our regularity conditions are satisfied. First, assumption (IS) holds since $f'(0^+) = \infty$. From our above computations, it is readily seen that assumption (LI) holds because both constraints $g^1(k,u)$ and $g^2(k,u)$ cannot be saturated at the same time. Regarding (NB), one can also check that this condition holds. Indeed, (NB) is trivially satisfied when there are periods in which all constraints stop binding. Condition $f'(0^+) = \infty$ precludes $g^1(k,u) = 0$ for all T, and optimality precludes $g^2(k,u) = 0$ for all T since consumption c must be positive for some time periods. Therefore, for every $\widehat{T}$ there is a time interval $[T', T]$ in which none of the constraints is saturated and $T > T' > \widehat{T}$. In this case, $G_t^*(T)$ would be the null matrix.

We have then proved the following

Proposition 5.1 *In the one–sector growth model with irreversible investment the value function is differentiable at interior points. Moreover, the derivative*

$$DV(t,k_0) = \int_t^\infty e^{\int_t^s (G_t^*(r)-\delta(r))\,dr} U'(c_t^*(s))(f'(k_t^*(s)) - \gamma - G_t^*(s))\,ds.$$

Note that the envelope theorem of Benveniste and Scheinkman [5] cannot be invoked for cases in which some constraint could be binding. The irreversibility assumption may bind if capital is high enough, and zero consumption may be obtained if capital is low enough. We could also discuss the case $f'(o^+) < \infty$ in which the irreversibility constraint may bind all the time. Note that if the discount rate $\delta > 0$ is constant then differentiability of the value function follows from our above results for the scalar case.

5.2 A monetary economy

Consider the following cash-in-advance model

$$\begin{aligned}
&\max_{(c_t(s), m_t(s), k_t(s), \dot{k}_t(s))} \int_t^\infty \beta(s,t) U(c_t(s))\,ds \qquad \text{subject to} \\
&\dot{k}_t(s) + \dot{m}_t(s) = f(k_t(s)) - \gamma k_t(s) - c_t(s) + x_t(s) - \pi_t(s) m_t(s), \\
&m_t(s) \geq c_t(s) + \dot{k}_t(s) + \gamma k_t(s), \\
&k_t(s) \geq 0, \quad c_t(s) \geq 0.
\end{aligned}$$

Here, c_t is consumption, m_t is a stock of real monetary holdings, k_t is capital, x_t is the value of government transfers rebated to the consumer as a consequence of the inflation tax, and π_t is the rate of inflation. Both U and f satisfy the same properties as in the previous example. For simplicity, the cash–in–advance constraint $m_t \geq c_t + \dot{k}_t(s) + \gamma k_t(s)$ applies to the purchase of the consumption good and gross investment.

Let us rewrite this problem in terms of the state variables (k, m) so that the instantaneous objective:

$$\ell((k,m),(\dot{k},\dot{m})) = U(f(k) - \gamma k + x - \pi m - \dot{k} - \dot{m}),$$

and the constraints:

$$\begin{aligned}
&g^1((k,m),(\dot{k},\dot{m})) = f(k) - \gamma k + x - \pi m - \dot{k} - \dot{m} \geq 0, && \text{(non-negative consumption);}\\
&g^2((k,m),(\dot{k},\dot{m})) = \gamma k + \dot{k} \geq 0, && \text{(irreversible investment);}\\
&g^3((k,m),(\dot{k},\dot{m})) = m + \dot{m} - f(k) - x + \pi m \geq 0, && \text{(cash–in–advance).}
\end{aligned}$$

We are therefore confronted with a two–dimensional problem. As in the growth model, the pure state constraint $k \geq 0$ is not binding, as $f'(0^+) = \infty$. Thus, optimal trajectories (k_t^*, m_t^*) lie in the interior of the state space $X = \mathbb{R}_+^2$, and (IS) is satisfied. In order to check (LI) we consider the Jacobian matrices $D_2(g^1, g^2)$, $D_2(g^1, g^3)$, $D_2(g^2, g^3)$ and $D_2(g^1, g^2, g^3)$ and verify the full–rank assumption. Of course, if only one constraint is saturated, then (IS) follows trivially. Matrices

$$D_2(g^1,g^2) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad D_2(g^1,g^3) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad D_2(g^2,g^3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

have all maximal rank. The three constraints (g^1, g^2, g^3) can only be binding for zero money holdings, $m = 0$. This case has been ruled out. Therefore, (LI) is always satisfied.

In order to check the asymptotic condition (13), from our arguments in the previous example we know that there are periods in which constraints g^1 (zero consumption) and g^2 (irreversible investment) will not be saturated. Hence, let us focus on the simple case in which only g^3 (cash-in-advance) is binding for all $s \geq t$. Then, $G_t(\{3\}; ((m), (\dot{m}))) = -D_1(g^3)^\top D_2(g^3)^{-\top} = -(1+\pi)$. Therefore, $\Phi_t(T)e^{-\delta T} = e^{\int_t^T (G_t^*(r) - \delta)\, dr} = e^{\int_t^T (-1-\pi-\delta)\, dr}$. Of course, this expression goes to zero, and hence (13) will always hold whenever the set of optimal solutions (k, m) remains in a compact set separated from the boundary of $\mathbb{R}_+^2$.

There are a few points worth mentioning here. First, in our theoretical results we resort to a standard free–disposal assumption to guarantee (13), which implies the nonexistence of asset pricing bubbles for a related economy. However, in applications there could be other restrictions[2] that may also guarantee (13). Second, our asymptotic condition (13) should not be confused with transversality condition (19). The transversality

[2]For instance if g^2 is always binding then our monotonicity assumption (NB)(iv) is not satisfied, but again here we have that $\Phi_t(T)e^{-\delta T} = e^{(-1-\pi-\delta)T}$ converges to zero.

condition is about asymptotic values (i.e., price times quantity), whereas (13) is about asymptotic shadow prices for constraints that are always binding. For instance, in the literature of the optimum quantity of money, it is well known that (19) implies $\pi > -\delta$. For our asymptotic condition (13) the requirement is simply $\pi > -1 - \delta$. Further, (13) is vacuously satisfied for time intervals in which none of the constraints is saturated.

6 Concluding remarks

This paper contains several results on the differentiability of the value function for a class of infinite–horizon continuous–time optimization problems with saturated constraints. One main goal of our exercise is to dispense with the interiority condition of Benveniste and Scheinkman [5]. We additionally show that the path of dual variables is unique, and derive a version of Bellman's equation for constrained optimization so that the feedback control or policy function is a continuous mapping.

As illustrated in our examples above, there are many economic models with saturated constraints that violate the interiority condition of Benveniste and Scheinkman [5]. To circumvent this interiority condition, we postulate three additional assumptions which seem indispensable. First, we require the path of state variables to lie in the interior of the domain; for if not, the value function may have kinks or the subgradient may be undefined. Second, we require a linear independence assumption on the saturated constraints. And third, we require non–existence of asset pricing bubbles for an associated economy. These latter explosive paths can be ruled out by some well known free–disposal assumptions or some contractivity conditions embedded in the saturated constrains.

The analysis presents several differences with respect to the discrete–time case considered in our previous paper [11]. In discrete–time, Bellman's equation is guaranteed under general assumptions. (Indeed, this equation holds for bounded, non–continuous objective functions.) In continuous–time, we need certain smoothness conditions for Bellman's equation to be satisfied. Moreover, iterations must proceed over time intervals rather than over simple dates as every time t has measure zero. Hence, the continuous–time problem requires the use of infinite–dimensional optimization. We then transform a problem with constraints into one of unconstrained optimization, and build the analysis over finite–horizon optimization problems in a Banach–space setting.

Notwithstanding, the continuous-time formulation is more structured since optimal trajectories are conformed by continuous arcs rather than by a sequence of countable points. This is reflected in stronger results and sharper examples. For instance, in the one–dimensional case the value function if differentiable under general conditions. Also, as illustrated in several examples above the assumptions are usually easier to check in the continuous–time formulation because a switch from a binding constraint to another becomes easier to track down in continuous time.

7 Appendix

For a given Banach space E and its dual $E^\top$, let $\langle \cdot, \cdot \rangle$ be the associated bilinear form over $E \times E^\top$: For fixed $x \in E$ mapping $\langle x, \cdot \rangle$ defines a continuous linear functional on $E^\top$ and for fixed $p \in E^\top$ mapping $\langle \cdot, p \rangle$ defines a continuous linear functional on E.

For a bounded linear mapping $A : E \longrightarrow F$ between Banach spaces E and F, with dual spaces $E^\top$ and $F^\top$, respectively, the adjoint is the unique linear mapping $A^\top : F^\top \longrightarrow E^\top$ satisfying

$$\langle x, A^\top p \rangle = \langle Ax, p \rangle, \qquad \forall x \in E, \quad \forall p \in F^\top.$$

Let us now recall some basic definitions from convex analysis. Assume that $f : F \longrightarrow \mathbb{R} \cup \{\infty\}$ is an upper semicontinuous, concave function. Then, the effective domain of f is $D(f) = \{x \in F : f(x) < \infty\}$. Function f is called proper if $D(f) \neq \emptyset$. The set

$$\partial f(x) = \{p \in F^\top : \langle x - x', p \rangle \leq f(x) - f(x') \quad \forall x' \in F\}$$

is the superdifferential of function f at x. An element $p \in \partial f(x)$ is called a supergradient of f at x. The domain of ∂f is $D(\partial f) = \{x \in F : \partial f(x) \neq \emptyset\}$. The superdifferential of f is always well defined at interior points of the domain of f, that is $\operatorname{int} D(f) \subseteq D(\partial f)$.

Let $A : E \longrightarrow F$ be a continuous linear operator. Assume that there is $\widetilde{x} \in E$ such that $A(\widetilde{x}) \in \operatorname{int} D(f)$. Then, the following equality holds *for every* $x \in E$, see [8], Prop. 5.7:

$$\partial(f \circ A)(x) = (A^\top \circ \partial f)(A(x)), \quad \forall x \in E. \tag{20}$$

Let us then introduce the families of linear mappings $A_t : \mathbb{R}^n \times L^1_n(I_t; \mu_t) \longrightarrow [L^1_n(I_t; \mu_t)]^2$:

$$A_t(x_0, u_t) = (x_t, u_t), \quad \text{where} \quad x_t(s) = x_0 + \int_t^s u_t(r)\, dr \tag{21}$$

and $B_t : \mathbb{R}^n \times L^1_n(I_t; \mu_t) \longrightarrow \mathbb{R}^n$:

$$B_t(x_0, u_t) = x_0 + \int_t^T u_t(r)\, dr. \tag{22}$$

Proposition 7.1 *1. Operator A_t in (21) is linear and continuous. Its adjoint*

$$A_t^\top : \left[L_n^\infty(I_t;\mu_t^\top)\right]^2 \longrightarrow \mathbb{R}^n \times L_n^\infty(I_t;\mu_t^\top)$$

is defined as

$$A_t^\top(p_t,q_t) = \Big(\int_{I_t} p_t(s)\,ds, \int_{I_s} p_t(r)\,dr + q_t\Big).$$

2. Operator B_t given in (22) is linear and continuous. Its adjoint

$$B_t^\top : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times L_n^\infty(I_t;\mu_t^\top)$$

is defined as

$$B_t^\top(y_0) = (y_0, y_0).$$

Proof. 1. Obviously A_t is linear. Let us show that it is well defined and continuous. We have

$$\int_{I_t} |x_t(s)|\beta(s,t)\,ds \le |x_0|\mu_t(I_t) + \int_{I_t} \beta(s,t)\int_t^s |u_t(r)|\,dr.$$

By an application of Fubini's theorem to the second term in the right-hand side we get

$$\begin{aligned}\int_{I_t} \beta(s,t)\int_t^s |u_t(r)|\,drds &= \int_{I_t} |u_t(r)|\int_{I_r} \beta(s,t)\,ds\,dr\\ &\le \rho\int_{I_t} |u_t(r)|\beta(r,t)\,dr < \infty,\end{aligned}$$

since $u \in L_n^1(I_t;\mu_t)$, and by assumption $\int_r^\infty \beta(s,t)\,ds \le \rho\beta(r,t)$. It is easy to prove from these inequalities that the mapping is continuous.

To find the adjoint $A_t^\top$, consider $(x_0,u) \in \mathbb{R}^n \times L_n^1(I_t;\mu_t)$ and $(p_t,q_t) \in [L_n^\infty(I_t;\mu_t^\top)]^2$. Then, using the duality pairings

$$\begin{aligned}\langle A_t(x_0,u),(p_t,q_t)\rangle &= \langle x_0 + \int_t^s u(r)\,dr, p_t\rangle + \langle u, q_t\rangle\\ &= x_0\int_{I_t} p_t(s)\,ds + \int_{I_t}\Big(\int_t^s u(r)\,dr\Big)p_t(s)\,ds + \langle u,q_t\rangle.\end{aligned}$$

Changing the order of integration in the second summand and applying Fubini's Theorem, we find

$$\begin{aligned}\langle A_t(x_0,u),(p_t,q_t)\rangle &= x_0\int_{I_t} p_t(s)\,ds + \int_t^T u(s)\int_{I_s} p_t(r)\,dr\,ds + \langle u,q_t\rangle\\ &= \langle x_0, \int_{I_t} p_t(s)\,ds\rangle + \langle u, \int_{I_s} p_t(r)\,dr\rangle + \langle u,q_t\rangle\\ &= \langle (x_0,u), A_t^\top(p_t,q_t)\rangle.\end{aligned}$$

The result for $A_t^\top$ is thus established.

2. Linearity and continuity of B_t is proved similarly. Moreover, by related computations we get

$$\langle B_t(x_0,u),y_0\rangle = \langle x_0,y_0\rangle + \langle \int_t^T u(s)\,ds, y_0\rangle = \langle (x_0,u), B_t^\top(y_0)\rangle.$$

Hence, $B_t^\top(y_0) = (y_0,y_0)$. □

We now write the model in recursive form. This formulation is made possible by the semigroup property of the discount factor $\beta(T,s)\beta(s,t) = \beta(T,t)$ for every $t \le s \le T$, and the intertemporal separability of the objective and constraints.

Lemma 7.1 (Dynamic Pogramming Principle) *For every $t \le T < \infty$, the value function*

$$V(t,x_0) = \sup\left\{\int_t^T \beta(s,t)\ell(x(s),\dot{x}(s))\,ds + \beta(T,t)V(T,x(T))\right\}$$

s. t. $(x_t(s),\dot{x}_t(s)) \in \Omega$ for all $s \in [t,T]$ and $x_t(t) = x_0$. Moreover, the optimal solution is given by the optimal pair $(x_t^*(s), \dot{x^*}_t(s))$ to problem (1) over $[t,T]$.

Now, for $T > t$ let $\mathcal{J}_{t,T} : \mathbb{R}^n \times L^1_n(I_t;\mu_t) \longrightarrow \mathbb{R}\cup\{-\infty\}$ be defined as in (3). That is,

$$\mathcal{J}_{t,T}(x_0,u) = J_{t,T}(A_t(x_0,u)) + \beta(T,t)V(T,B_t(x_0,u)).$$

It follows from Lemma (7.1) that the value function

$$V(t,x_0) = \sup_{u\in L^1_n(I_t;\mu_t)} \mathcal{J}_{t,T}(x_0,u). \tag{23}$$

By assumptions (A1)–(A3), mapping $V(t,\cdot)$ is well defined and concave over $\operatorname{int}(X)$ at each t, and $\partial V(T,x_t^*(T))$ is not empty for every T.

The following lemma characterizes the superdifferential of $\mathcal{J}_{t,T}$. In the sequel, $p_t(I_s)$ will denote $\int_s^T p_t(r)\,dr$.

Lemma 7.2 *Assume that J_t well-defined in a neighborhood of $A_t(x_0,u) \in \operatorname{int} X$, and V is well defined in a neighborhood of $x_t(T)$. Moreover, the solution $A_t(x_0,u)$ always belongs to the interior of X. Then,*

$$\partial\mathcal{J}_{t,T}(x_0,u) = \Big\{\big(-p_t(I_t)+\beta(s,t)\xi_{t,T}, -p_t(I_s) - q_t + \beta(s,t)\xi_{t,T}\big) : -(p_t(s),q_t(s)) \in \beta(s,t)\partial(\mathscr{L}\circ A_t)(x_0,u),\ \xi_{t,T} \in \partial V(T,x(T))\ a.e.\Big\}.$$

Proof. By the concavity of these functions, we must have

$$\partial \mathcal{J}_{t,T} = \partial(J_{t,T} \circ A_t) + \beta(T,t)\partial(V(T,\cdot) \circ B_t).$$

Also, by (20)

$$\partial(J_{t,T} \circ A_t) = A_t^\top \circ \partial J_{t,T} \circ A_t$$

and

$$\partial(V(T,\cdot) \circ B_t) = B_t^\top \circ \partial V(T,\cdot) \circ B_t.$$

Combining Lemmas 3.1 and 7.1, an element of $A_t^\top(\partial J_{t,T}(A_t(x_0,u)))$ must be of the form $(-p_t(I_t), -p_t(I_s) - q_t)$, with $-(p_t(s), q_t(s)) \in \beta(s,t)\partial\mathscr{L}\big(A_t(x_0,u)\big)$, as well as a typical element of the set $\beta(T,t)B_t^\top(\partial V(T, B_t(x_0,u)))$ must be of the form $\beta(T,t)(\xi_{t,T}, \xi_{t,T})$ with $\xi_{t,T} \in \partial V(T, x(T))$. □

Proof of Lemma 3.2.

Note that at the optimal solution $A_t(x_0, \dot{x}_t^*)$ all the conditions of Lemma 7.2 are satisfied. By Proposition 3.1 we then have $q_0 \in \partial V(t,x_0)$ if and only if $(q_0, 0) \in \partial\mathcal{J}_{t,T}(x_0, \dot{x}_t^*)$. Now, the proof follows as a straightforward consequence of the above characterizations of the subdifferential of $\mathcal{J}_{t,T}$ at $(x_0, \dot{x}_t^*)$.

More precisely, by Lemma 7.2 we must have

$$q_t(s) = -\int_s^T p_t(r)\,dr + \beta(T,s)\xi_{t,T}$$

with

$$-(p_t(r), q_t(r)) \in \beta(r,t)\partial\mathscr{L}(x_t^*(r), \dot{x}_t^*(r)) \quad \text{a.e. } t \leq r \leq T.$$

□

Proof of Proposition 3.3. Under the stated non–negativity conditions it is easy to see that at every point x_0 the superdifferential $\partial V(t,x_0)$ must be composed of non–negative numbers. Then this optimization problem can be reconverted into an asset pricing model with real assets along the lines of [16]; see especially their footnote 10. This asset pricing model considers a matrix of returns – which in this case it is given by the vector $\big(D_1\ell(x_t^*(s), \dot{x}_t^*(s)) + G_t^*(s)D_2\ell(x_t^*(s), \dot{x}_t^*(s))\big)$ – and a non–negative matrix of transformation of securities – which in this case it is given by matrix $\Phi_t(s)$. As it is clear from Theorem 3.1 we only need to focuss on boundary solutions at $t = 0$, which can be identified with long–lived assets. Then, for every optimal path $(x_t^*, \dot{x}_t^*)$

we can generate a sequence of asset prices $q_t(s) \in \partial V(s, x_t(s))$ so that the asset pricing equation $q_t(s) = -\int_s^T p_t(r)\, dr + \beta(T,t)\xi_{t,T}$ holds for $\xi_{t,T} \in \partial V(T, x_t(T))$. By the proof of Theorem 3.2 this equation can be rewritten as $q_t(s) = \int_t^T \Phi_t(s)\big(D_1\ell(x_t^*(s), \dot{x}_t^*(s)) + G_t^*(s)D_2\ell(x_t^*(s), \dot{x}_t^*(s))\big)\beta(s,t)ds + \Phi_t(T)\beta(T,t)\xi_{t,T}$. We can also introduce a single consumption good at each date with relative price equal to unity, and assume that the marginal utility of consumption at the optimal point is equal to one. Asset holdings can be defined in a rather arbitrary way, as the agent can be endowed with new securities at each date so as to replicate the optimal path $(x_t^*, \dot{x}_t^*)$. Hence, under the stated assumptions it follows from [16] that the bubble term $B_0 = 0$. □

The next proposition can be found in Aubin (1993, Problem 22).

Proposition 7.2 *Let H be a proper, concave, upper semicontinuous function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R} \cup \{-\infty\}$. Let*

$$H(x,q) = \sup_{u \in \mathbb{R}^m} \{f(x,u) + qu\}.$$

Then, $x \mapsto H(x,q)$ is a concave mapping for a fixed q, and $q \mapsto H(x,q)$ is a convex mapping for a fixed x. Moreover, the following conditions are equivalent

$$\begin{gathered} -(p,q) \in \partial f(x,u) \\ -p \in \partial_x H(x,q) \quad \text{and} \quad u \in \partial_q H(x,q) \quad . \end{gathered}$$

Proof of Theorem 4.1. Suppose that the pair (x_t^*, q_t) satisfies the Hamiltonian inclusions (18) with $x_t^*(t) = x_0$. It is well known that this condition along with (NB) and (19) constitute a sufficient criterium for optimality of $(x_t^*, \dot{x}_t^*)$ for problem (1). For instance, the proof given in Benveniste and Scheinkman [5] can be easily adapted to our framework; we do not repeat the details here. Let us then assume that $(x_t^*, \dot{x}_t^*)$ is an optimal path with two associated paths of dual variables q_t and q_t' satisfying both the Hamiltonian inclusions (18) and the transversality condition (19). For x_0 fixed, let

$$\begin{aligned} V_T(t, x_0) &= \max \int_t^T \mathscr{L}(x_t(s), \dot{x}_t(s))\beta(s,t)\, ds + q_t(T)x_t(T) \\ &\text{subject to} \quad x(t) = x_0, \end{aligned} \tag{24}$$

and

$$\begin{aligned} V_T'(t, x_0) &= \max \int_t^T \mathscr{L}(x_t(s), \dot{x}_t(s))\beta(s,t)\, ds + q_t'(T)x_t(T) \\ &\text{subject to} \quad x(t) = x_0. \end{aligned} \tag{25}$$

Note that the added linear parts $q_t(T)x_t(T)$ and $q'_t(T)x_t(T)$ are chosen so that $(x^*_t, \dot{x}^*_t)$ with $x^*_t(t) = x_0$ is the optimal solution for both optimization problems. We can readily see that functions $V_T(t, x_0)$ and $V'_T(t, x_0)$ are concave; moreover, by the same arguments as in Lemma 3.2 these functions are of class C^1 in x. By the transversality condition (19), the sequences of functions $\{V_T(t, x_0)\}_{T\geq 0}$ and $\{V'_T(t, x_0)\}_{T\geq 0}$ converge pointwise to function $V(t, x_0)$ as $T \to \infty$. Hence, the sequences of derivative functions $\{DV_T(t, x_0)\}_{T\geq 0}$ and $\{DV'_T(t, x_0)\}_{T\geq 0}$ converge uniformly to function $DV(t, x_0)$ on every compact set $K \subset \text{int}(X)$ [see [13], Theorem 25.7]. By Remark 3.2 the convergence of these derivatives to a unique common value $DV(t, x_0)$ implies that $q_t(T) = q'_t(T)$. Therefore, we get uniqueness of the path of dual variables q_t. $\square$

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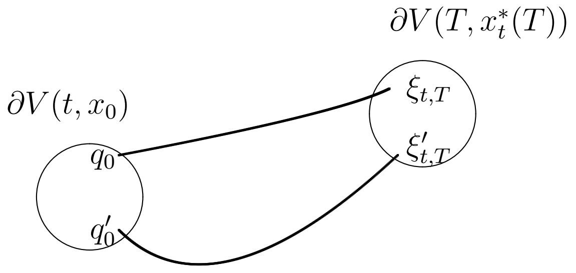

Figure 1: The flow mapping between $\partial V(t, x_0)$ and $\partial V(T, x_t^*(T))$.

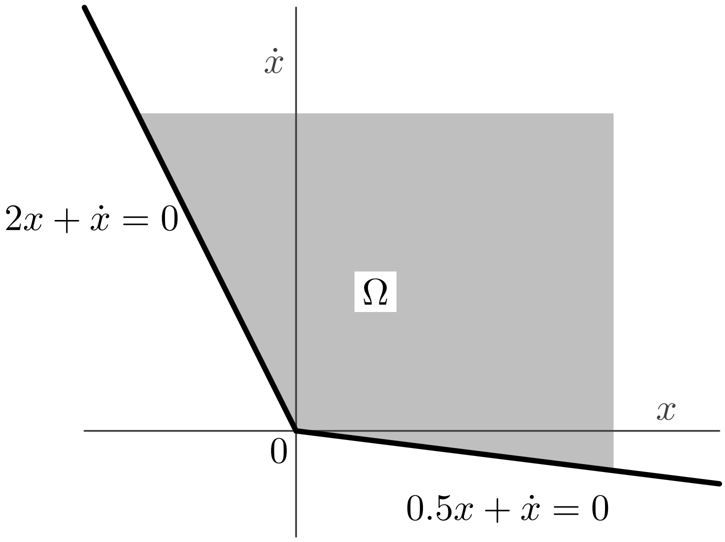

Figure 2: A feasible set where (LI) does not hold

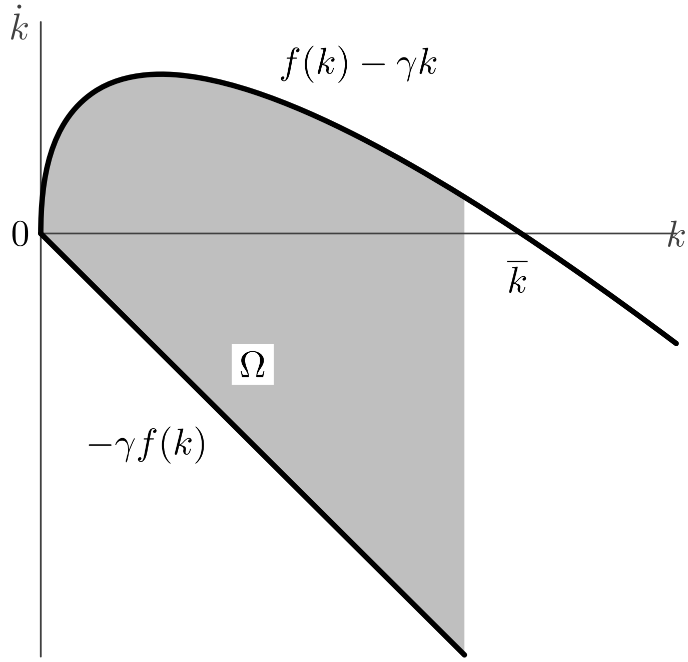

Figure 3: Feasible set Ω in the optimal growth model