Partially-honest Nash implementation: Characterization results

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Abstract

This paper studies implementation problems in the wake of a recent new trend of implementation theory which incorporates a non-consequentialist flavor of the evidence from experimental and behavioral economics into the issues. Specifically, following the seminal works by Matsushima (2008) and Dutta and Sen (2009), the paper considers *implementation problems with partially honest agents*, which presume that there exists at least one individual in the society who concerns herself with not only outcomes but also honest behavior at least in a limited manner. Given this setting, the paper provides a general characterization of Nash implementation with partially-honest individuals. It also provides the necessary and sufficient condition for Nash implementation with partially-honest individuals by mechanisms with some types of strategy-space reductions. As a consequence, it shows that, in contrast to the case of the standard framework, the equivalence between Nash implementation and Nash implementation with strategy space reduction no longer holds.

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1 Introduction

The theory of (Nash) implementation aims to reach *goals* in situations in which the planner does not have all the relevant necessary information, but needs to elicit it from the agents.[1] To this end, she designs a mechanism or game form in which agents will act strategically in accordance with the solution concept of Nash equilibrium. When the (Nash) equilibrium outcomes of the mechanism coincide with the goals set by the planner, these goals are implementable. Seminal paper on implementation is Maskin (1999) who proves that a social choice correspondence (SCC) - which summarizes the planner's goals - is (Maskin) *monotonic* if it is implementable; when there are at least three agents, an SCC is implementable if it is monotonic and satisfies an auxiliary condition called *no-veto power*. Moore and Repullo (1990), Dutta and Sen (1991), Danilov (1992), Sjöström (1991) and Yamato (1992) refined Maskin's characterization result by providing necessary and sufficient conditions for an SCC to be implementable.[2]

A fundamental tenet of implementation theory is the consequentialism axiom. Its core idea is that the ranking of outcomes of agents should be independent of the process that generates these outcomes. An immediate implication of this axiom for implementation theory is that agents should be indifferent between a lie and a truthful statement if they result in the same material payoffs. This axiom, however, is inconsistent with the well documented behavior that agents may display concern for *procedures*; that is, they may care about *how* outcomes are generated and, therefore, their ranking of outcomes may be structurally dependent on the outcome-generating process (Camerer, 2003; Sen, 1997). Remarkably, a considerable amount of experimental data suggests that agents may display preferences for truth-telling; that is, an agent lies *only* when she prefers the outcome obtained from false-telling over the outcome obtained from truth-telling (Gneezy, 2005; Hurkens and Kartik, 2009). This paper aims at narrowing the gap between these two strands. It follows the non-consequentialist approach by accommodating concerns for truthful revelation of agents; but like mainstream theory, it keeps the idea that even these agents respond primarily to material incentives.[3] The paper refers to agents having preferences for truth-telling as

[1] Henceforth, by implementation we mean Nash implementation.

[2] For excellent introductions to the theory of implementation, see, for instance, Jackson (2001) and Maskin and Sjöström (2002).

[3] In its turn, the impressive body of evidence accumulated by psychologists over the past

being *partially-honest*.

Its general thrust goes as follows. Assume, as an example, that the message conveyed by each agent to the planner involves the announcement of a preference profile (i.e., agents' preferences over outcomes). A message is truthful if it involves the announcement of the true preference profile. A partially-honest agent is an agent who strictly prefers to announce a truthful message rather than a lie when the former (given a message announced by other agents) produces an outcome which is at least as good as the one that would be achieved if the agent lied (keeping constant the other agents' messages). Suppose that agent h is a partially-honest agent, who believes that the other agents will send the message m_{-h}, and let m_h be the truthful message of agent h and m'_h be not truthful. Moreover, let both the message profile (m_h, m_{-h}) and the message profile (m'_h, m_{-h}) result in the same outcome x. Then, unlike an agent who is concerned solely with outcomes, the partially-honest agent h strictly prefers (m_h, m_{-h}) to (m'_h, m_{-h}). Put differently, the agent at issue has preferences over message profiles in which she cares about two dimensions in lexicographic order: primarily to her outcome, secondarily to her truth-telling behavior.

Seminal works on the role of honesty in implementation theory are Matsushima (2008) and Dutta and Sen (2009), which show that the assumption that the planner is aware of the existence of partially-honest agents but ignores their identities drastically improves the scope of implementation. Yet, the significant impact of the presence of partially-honest agents upon implementation theory has not been fully appreciated - as described below. In the line with these works, this paper also investigates implementation problems with partially-honest agents, where an SCC is *partially-honestly implementable* if there is a mechanism whose equilibrium outcomes are determined with each profile of preferences over message profiles, and coincide with the optimal outcomes set by this SCC.

Given this setting, the paper provides, in section 3.1, a minimal set of necessary conditions for partially-honest implementation, though the above

two decades has caused scholars to study the implications of weakening other fundamental assumptions in a variety of ways, and has already turned in a number of alternatives back to the standard implementation model (for instance, Eliaz, 2002; Renou and Schlag, 2009; Bergemann et al., 2010; Cabrales and Serrano, 2010). Noteworthy, the first paper on 'behavioral implementation theory' goes back to 1986, in which Hurwicz solves the implementation problem without positing the completeness and the transitivity of agents' preferences (Hurwicz, 1986).

seminal works solely study sufficienct conditions. Due to this result in the paper, it is possible to examine which of the SCCs cannot be partially-honestly implemented. For instance, as shown in section 4, the (strong) $Pareto\ SCC$ defined in abstract social choice environments is not partially-honestly implementable. Furthermore, under mild and reasonable domain restrictions of preferences and mechanisms, the paper shows that a slight strengthening of these conditions is necessary and sufficient for partially-honest implementation in more than two person societies. The set of conditions is much weaker than the necessary and sufficient condition given by Moore and Repullo (1990) for the standard Nash implementation, and in particular it contains no variant of the Maskin monotonicity-like condition. For instance, in rationing problems when agents have single-plateaued preferences, it can be shown from this characterization that the $Pareto\ SCC$ is partially-honestly implementable, though this SCC violates the Moore and Repullo (1990) condition, and also satisfies neither monotonicity nor no-veto power.

Note that the aforementioned theorem of this paper applies a canonical mechanism to show the sufficiency part. This type of mechanism requests agents to announce a feasible social outcome, an agent index, and moreover a profile of agents' preferences on outcomes, which is not an attractive feature, given that an important role of the mechanism is to economize on communication. Facing this issue, the paper pays attention to informational decentralization of mechanisms by considering mechanisms with strategy space reductions. While sub-section 3.2 assumes s-mechanism (Saijo, 1988) in which the message conveyed by each agent to the planner involves the announcement of only her own and her neighbor's preferences - in addition to an outcome and an agent index, sub-section 3.3 assumes self-relevant mechanisms (Tatamitani, 2001) in which each agent announces - *inter alia* - only her own preference. Then, the paper identifies a minimal set of necessary conditions for partially-honest implementation by s-mechanisms (*resp.*, self-relevant mechanisms); moreover, it shows that a slight strengthening of these conditions fully identifies the class of partially-honest implementable SCCs by s-mechanisms (*resp.*, self-relevant mechanisms). Notably, these conditions respectively contain the weaker variants of (Maskin) monotonicity-type conditions, each of which respectively restricts the class of partially-honestly implementable SCCs by s-mechanisms and by self-relevant mechanisms. These findings have at least two immediate consequences. First, there is a trade-off between what the planner can achieve when there are partially-honest agents in the society and the strengthening of informational decentralization in mechanisms. Second, this conflict breaks down the equivalence between implementation and implementation by s-mechanism which holds in the standard framework (Lombardi and Yoshihara, 2010).

Finally, the paper turns to study partially-honest implementation problems in two-agent societies. This issue has recently been analyzed by Dutta and Sen (2009) on the assumption that agents' preferences are linear orders. Their contribution is that, even in the more problematic case of two agents, the stringent condition of monotonicity is no longer required. The paper extends their analysis to the domain of weak orders in view of its potential applications to bargaining and negotiating. The paper identifies the class of partially-honest implementable SCCs, not only in the case that the planner knows that exactly one agent is partially-honest, but also in the more subtle case that she only knows that there are partially-honest agents.

The paper is organized as follows. Section 2 describes the formal environment. Section 3 reports the analysis for the many-person case, whereas Section 4 briefly discusses its implications. Section 5 reports the analysis for the two-agent case and its implications.

2 The implementation problem

The set of outcomes is denoted by X and the set of agents is $N = \{1, ..., n\}$. Unless otherwise specified, we assume that the cardinality of X is $\#X \geq 2$, while the cardinality of N is $n \geq 3$. Let $\mathcal{R}(X)$ be the set of all possible weak orders on X.[4] Let $\mathcal{R}_\ell \subseteq \mathcal{R}(X)$ be the (non-empty) set of all admissible weak orders for agent $\ell \in N$.[5] Let $\mathcal{R}^n \subseteq \mathcal{R}_1 \times ... \times \mathcal{R}_n$ be the set of all admissible profiles of weak orders (or states). A generic element of $\mathcal{R}^n$ is denoted by R, where its ℓth component is $R_\ell \in \mathcal{R}_\ell$, $\ell \in N$.[6] The symmetric and asymmetric factors of any $R_\ell \in \mathcal{R}_\ell$ are, in turn, denoted P_ℓ and I_ℓ, respectively.[7] For any $R \in \mathcal{R}^n$ and any $\ell \in N$, let $R_{-\ell}$ be the list of elements of R for all agents except ℓ, i.e., $R_{-\ell} \equiv (R_1, ..., R_{\ell-1}, R_{\ell+1}, ..., R_n)$. Given a list $R_{-\ell}$ and

[4] A weak order is a complete and transitive binary relation. A relation R on X is *complete* if, for all $x, x' \in X$, $(x, x') \in R$ or $(x', x) \in R$; *transitive* if, for all $x, x', x'' \in X$, if $(x, x') \in R$ and $(x', x'') \in R$, then $(x, x'') \in R$.

[5] The weak set inclusion is denoted by $\subseteq$, while the strict set inclusion is denoted by $\subsetneq$.

[6] $(x, y) \in R_\ell$ stands for "x is at least as good as y".

[7] $(x, y) \in P_\ell$ if and only if $(x, y) \in R_\ell$ and $(y, x) \notin R_\ell$ and P_ℓ stands for "strictly better than". On the other hand, $(x, y) \in I_\ell$ if and only if $(x, y) \in R_\ell$ and $(y, x) \in R_\ell$ and I_ℓ stands for "indifferent to".

$R_\ell \in \mathcal{R}_\ell$, we denote by $(R_{-\ell}, R_\ell)$ the preference profile consisting of these R_ℓ and $R_{-\ell}$. For any preference profile $R \in \mathcal{R}^n$ and any $\varnothing \neq S \subseteq N$, let R_{-S} be the list of elements of R for all agents in $N \backslash S$. Given a list R_{-S} and a list $R_S \in \times_{\ell \in S} \mathcal{R}_\ell$, we denote by (R_{-S}, R_S) the preference profile consisting of these R_S and R_{-S}. Let $\mathcal{P}^n \subseteq \mathcal{R}^n$ be the set of all admissible profiles of linear orders.[8] Let $L(R_\ell, x)$ denote agent i's lower contour set at $(R_\ell, x) \in \mathcal{R}_\ell \times X$, that is, $L(R_\ell, x) \equiv \{y \in X | (x, y) \in R_\ell\}$. For any $R_\ell \in \mathcal{R}_\ell$ and $Y \subseteq X$, let $\max_{R_\ell} Y$ be the set of optimal outcomes in Y according to R_ℓ, that is, $\max_{R_\ell} Y \equiv \{x \in Y | (x, y) \in R_\ell \text{ for all } y \in Y\}$. For any $R_\ell \in \mathcal{R}_\ell$, $\partial L(R_\ell, x) = \{x\}$ means $\{x\} = \max_{R_\ell} L(R_\ell, x)$.

A *social choice correspondence* (SCC) F on $\mathcal{R}^n$ is a correspondence $F : \mathcal{R}^n \twoheadrightarrow X$ with $\varnothing \neq F(R) \subseteq X$ for all $R \in \mathcal{R}^n$. Denote the class of such correspondences by $\mathcal{F}$. An SCC F on $\mathcal{R}^n$ is (Maskin) *monotonic* if, for all $R, R' \in \mathcal{R}^n$ with $x \in F(R)$, $x \in F(R')$ if $L(R_\ell, x) \subseteq L(R'_\ell, x)$ for all $\ell \in N$. An SCC F on $\mathcal{R}^n$ satisfies i) *no-veto power* if, for all $R \in \mathcal{R}^n$, $x \in F(R)$ if $x \in \max_{R_\ell} X$ for at least $n-1$ agents; ii) *unanimity* if, for all $R \in \mathcal{R}^n$, $x \in F(R)$ if $x \in \max_{R_\ell} X$ for all $\ell \in N$.

A *mechanism* is a pair $\gamma \equiv (M, g)$, where $M \equiv M_1 \times ... \times M_n$, with each M_i being a (non-empty) set, and $g : M \rightarrow X$. It consists of a message space M, where M_ℓ is the message space for agent $\ell \in N$, and an outcome function g. Denote the admissible class of mechanisms by Γ. Let $m_\ell \in M_\ell$ denote a generic message (or strategy) for agent ℓ. A message profile is denoted $m \equiv (m_1, ..., m_n) \in M$. For any $m \in M$ and $\ell \in N$, let $m_{-\ell} \equiv (m_1, ..., m_{\ell-1}, m_{\ell+1}, ..., m_n)$. Let $M_{-\ell} \equiv \times_{i \in N \backslash \{\ell\}} M_i$. Given an $m_{-\ell} \in M_{-\ell}$ and an $m_\ell \in M_\ell$, denote by $(m_\ell, m_{-\ell})$ the message profile consisting of these m_ℓ and $m_{-\ell}$. For any $m \in M$ and $\varnothing \neq S \subseteq N$, let $m_{-S} \equiv (m_\ell)_{\ell \in N \backslash S}$. Let $M_{-S} \equiv \times_{\ell \in N \backslash S} M_\ell$. Given $m_{-S} \in M_{-S}$ and $m_S \in M_S$, denote by (m_S, m_{-S}) the message profile consisting of these m_S and m_{-S}.

A mechanism γ induces a class of (*non-cooperative*) *games* $\{(\gamma, R) | R \in \mathcal{R}^n\}$. Given a game (γ, R), we say that $m^* \in M$ is a (pure strategy) *Nash equilibrium* at R if and only if, for all $\ell \in N$, $\left(m^*, \left(m_\ell, m^*_{-\ell}\right)\right) \in R_\ell$ for all $m_\ell \in M_\ell$. Given a game (γ, R), let $NE(\gamma, R)$ denote the set of Nash equilibria message profiles of (γ, R), whereas $NA(\gamma, R)$ represents the corresponding set of Nash equilibrium outcomes.

A mechanism γ *implements* F in *Nash equilibria*, or simply *implements*

[8] A linear order is a complete, transitive, and antisymmetric binary relation. A binary relation R on X is *antisymmetric* if, for all $x, x' \in X$, $x = x'$ if $(x, x') \in R$ and $(x', x) \in R$.

F, if and only if $F(R) = NA(\gamma, R)$ for all $R \in \mathcal{R}^n$. If such mechanism exists, then F is *(Nash)-implementable*.

Given a mechanism γ, for each $\ell \in N$, let us define truth-telling correspondence $T_\ell^\gamma : \mathcal{R}^n \times \mathcal{F} \twoheadrightarrow M_\ell$ such that for each $(R, F) \in \mathcal{R}^n \times \mathcal{F}$, $\varnothing \neq T_\ell^\gamma(R, F) \subseteq M_\ell$. An interpretation of the set $T_\ell^\gamma(R, F)$ is that, given the mechanism γ and the current state (R, F), agent ℓ behaves truthfully at the message profile $m \in M$ if and only if $m_\ell \in T_\ell^\gamma(R, F)$. In other words, $T_\ell^\gamma(R, F)$ is the set of *truthful message* of ℓ under the mechanism γ, when the current social state is $R \in \mathcal{R}^n$ and the social goal is given by F. Note that the type of elements of M_ℓ constituting $T_\ell^\gamma(R, F)$ depends on the type of mechanism γ that one may consider. For example, if the message conveyed by each agent to the planner involves the announcement of a preference profile, a feasible social outcome and an agent integer index, and sending the truthful preference profile constitutes the relevant truthful message for each $(R, F) \in \mathcal{R}^n \times \mathcal{F}$, then M_ℓ may be defined by $M_\ell \equiv M_\ell^1 \times M_\ell^2$, where there is a bijection $\sigma_\ell : \mathcal{R}^n \to M_\ell^1$ such that $T_\ell^\gamma(R, F) = \{\sigma_\ell(R)\} \times M_\ell^2$ for each $(R, F) \in \mathcal{R}^n \times \mathcal{F}$.

For any $\ell \in N$ and $R \in \mathcal{R}^n$, let $\succsim_\ell^R$ be agent ℓ's (weak) order over M under the state R. The asymmetric factor of $\succsim_\ell^R$ is denoted $\succ_\ell^R$, while the symmetric part is denoted $\sim_\ell^R$. For any $R \in \mathcal{R}^n$, let $\succsim^R$ denote the profile of (weak) orders over M under the state R, that is, $\succsim^R \equiv \left(\succsim_\ell^R\right)_{\ell \in N}$.

Definition 1. Given a mechanism γ, an agent $h \in N$ is a *partially-honest agent* if, for any $R \in \mathcal{R}^n$, and any $m \equiv (m_h, m_{-h}), m' \equiv (m_h', m_{-h}) \in M$, the following properties hold:

(i) if $m_h \in T_h^\gamma(R, F)$, $m_h' \notin T_h^\gamma(R, F)$, and $(g(m), g(m')) \in R_h$, then $(m, m') \in \succ_h^R$;
(ii) otherwise, $(m, m') \in \succsim_h^R$ if and only if $(g(m), g(m')) \in R_h$.

If agent $\ell \in N$ is not a partially-honest agent, then for each game (γ, R), for all $m, m' \in M$: $(m, m') \in \succsim_\ell^R$ if and only if $(g(m), g(m')) \in R_\ell$.

Unless otherwise specified, the following informational assumption holds throughout the paper.

Assumption 1. *There are* partially-honest agents *in N. The planner is aware of it but ignores the identity of these agents.*

Let $\mathcal{H} \subseteq \{H \subseteq N \mid H \neq \varnothing\}$ be the class of subsets in N. Note that $\mathcal{H}$ is considered as the potential class of partially-honest agents' groups. That is, if $H \in \mathcal{H}$, this H is a *potential group of partially-honest agents* in N. By

Assumption 1, the planner knows that $\mathcal{H}$ is non-empty, and perhaps, she may know what subsets of N belong to $\mathcal{H}$, but she never knows which element of $\mathcal{H}$ is the true set of partially-honest agents in the society.

A mechanism γ induces a class of (*non-cooperative*) *games with partially-honest agents* $\left\{\left(\gamma, \succcurlyeq^{R}\right) | R \in \mathcal{R}^{n}\right\}$. Given a game $\left(\gamma, \succcurlyeq^{R}\right)$, we say that $m^{*} \in M$ is a (pure strategy) *Nash equilibrium with partially-honest agents* at R if and only if, for all $\ell \in N$, $\left(m^{*}, \left(m_{\ell}, m^{*}_{-\ell}\right)\right) \in \succcurlyeq^{R}_{\ell}$ for all $m_{\ell} \in M_{\ell}$. Given a game $\left(\gamma, \succcurlyeq^{R}\right)$, let $NE\left(\gamma, \succcurlyeq^{R}\right)$ denote the set of Nash equilibria message profiles of $\left(\gamma, \succcurlyeq^{R}\right)$, whereas $NA\left(\gamma, \succcurlyeq^{R}\right)$ represents the corresponding set of Nash equilibrium outcomes. Then:

Definition 2. An SCC F on $\mathcal{R}^{n}$ is *partially-honest (Nash) implementable* if there exists a mechanism $\gamma = (M, g) \in \Gamma$ such that $F\left(R\right) = NA\left(\gamma, \succcurlyeq^{R}\right)$ for all $R \in \mathcal{R}^{n}$.

To conclude, let us introduce two mild conditions imposed on the models of this paper. One is a condition on the domain of agents' preferences, while the other is a condition on the domain of mechanisms admissible in the society. The first condition basically requires that the class of available profiles of preferences is sufficiently rich. Examples of preference domains satisfying such a condition would be the set of all profiles of weak orders, linear orders, and single peaked preferences on X. Moreover, it is vacuously satisfied in the classical economic environments. Hence, our models are applicable to those environments. The condition can be stated as follows.

Rich Domain (RD): For any $i \in N$, any $R \in \mathcal{R}^{n}$, and any $x \in X$, if $R'_{i} \in \mathcal{R}_{i}\left(X\right)$ is such that $L\left(R'_{i}, x\right) = L\left(R_{i}, x\right)$ with $\partial L\left(R'_{i}, x\right) = \{x\}$, then $\left(R'_{i}, R_{-i}\right) \in \mathcal{R}^{n}$ holds.

Next, our informational assumption is that the planner knows that there are partially-honest agents but ignores their identities. The partially-honest agent is an agent who prefers to be truthful if a lie is not beneficial for her. Given this structure, the existence of truthful messages is presumed, since, otherwise, the issue reduces to the standard implementation problem. Moreover, the admissible class of mechanisms should be constituted by those which involve a *simple scheme* to punish such a partially-honest agent if she takes a false message. As such one, let us consider a type of mechanism in which, if an outcome x is F-optimal at the state R and the outcome function g selects x as the resulting outcome of the messages announced by agents, a partially-honest agent can find a truthful message which results in the same outcome x - keeping constant the messages of all other agents. In such a mechanism, any false statement by a partially-honest agent can be punished independently of the detailed information about the real state of the society. This condition on the class of admissible mechanisms Γ can be stated as follows.

Simple Punishment (SP): For any $R, R' \in \mathcal{R}^n$, any $x \in F(R)$, any $i \in N$, and any $m \in M$ such that $g(m) = x$, there is $m'_i \in T_i^{\gamma}(R', F)$ such that $g(m'_i, m_{-i}) = g(m)$.

A mechanism γ is a *mechanism with simple punishment* if it satisfies **SP**. Denote the class of mechanisms with **SP** by Γ_{SP}.

Before closing this section, it may be worth noting that the simple punishment property is satisfied by all classical mechanisms in the literature of Nash implementation (see, for instance, Repullo, 1987; Moore and Repullo, 1990; Saijo, 1988; Dutta and Sen, 1991; Tatamitani, 2001).

3 Characterization results

In this section, we analyze partially-honest implementation of SCCs in the many-person case.

Sub-section 3.1 basically imposes no restriction on the types of admissible mechanisms except for $\Gamma = \Gamma_{SP}$. Under this setting, we begin by showing a minimal set of necessary conditions for partially-honest implementation with no restriction on Γ. Then, given $\Gamma = \Gamma_{SP}$, we prove that a slight strengthening of this minimal set of necessary conditions fully characterizes partially-honest implementation when the message conveyed by each agent involves the announcement of a preference profile, an outcome and an agent integer index - *canonical mechanism*.

Canonical mechanisms are not so attractive in most economic settings, where an important feature of the mechanism is to economize on communication. We then pay attention to informational decentralization in mechanisms. While sub-section 3.2 assumes that the message conveyed by each agent to the planner involves the announcement of only her own and her neighbor's preferences - in addition to a feasible social outcome and an integer - *s-mechanism*, sub-section 3.3 assumes that each agent announces - inter alia - only her own preferences, *self-relevant mechanisms*. We identify a minimal set of necessary conditions for partially-honest implementation by s-mechanisms (*resp.*, self-relevant mechanisms); finally, given $\Gamma = \Gamma_{SP}$, we report that a slight strengthening of these necessary conditions for s-mechanisms (*resp.*, self-relevant mechanisms) fully characterizes partially-honest implementation by s-mechanisms (*resp.*, self-relevant mechanisms).

The sets of conditions that are necessary and sufficient for partially-honest implementation are more complex than those obtained by Moore and Repullo (1990), Tatamitani (2001), and Lombardi and Yoshihara (2010), but they are remarkably weaker and do provide additional insights; we refer the reader to Section 4 for more details.

3.1 Partially-honest implementation: A general characterization

In implementation theory, it is *Maskin's Theorem* (Maskin, 1999) which shows that an SCC F is implementable if it satisfies monotonicity and no-veto power in the many-person case; conversely, any implementable SCC is monotonic. Since Maskin's Theorem, there have been impressive advances in the implementation theory. Specifically, in societies with at least three agents, Moore and Repullo (1990) established that an SCC F is implementable if and only if it satisfies Condition μ defined below.

Condition μ (for short, μ): There is a set $Y \subseteq X$ and, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R,x))_{\ell \in N}$ such that $x \in C_\ell(R,x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$, the following (i)-(iii) are satisfied:
(i) if $C_\ell(R,x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for all $i \in N$, if $y \in C_i(R,x) \subseteq L(R^*_i, y)$ and $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \setminus \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in \max_{R^*_\ell} Y$ for all $\ell \in N$, then $y \in F(R^*)$.[9]

Condition μ(i) is equivalent to monotonicity, while Conditions μ(ii)-μ(iii) are weaker versions of no-veto power.

In this sub-section, we begin by taking an arbitrary SCC that can be partially-honest implemented, and showing that it must satisfy Condition μ^* below. We then prove that a slight strengthening of Condition μ^* - Condition

[9] We refer to the condition that requires only one of the conditions (i)–(iii) in Condition μ as Conditions μ(i)–μ(iii) each. Note that Condition μ implies Conditions μ(i)–μ(iii), but the converse is not true. We use similar conventions below.

μ^{**}- fully identifies the class of partially-honest implementable SCCs under mild conditions on agents' preferences and mechanisms.

Condition μ^* (for short, μ^*): There is a set $Y \subseteq X$ and, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R,x))_{\ell \in N}$ such that $x \in C_\ell(R,x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$, the following (i)-(iii) are satisfied:
(i) if $C_\ell(R,x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$ and $x \notin F(R^*)$, then $(x,x') \in I^*_h$ for some $x' \in C_h(R,x)$ and some $h \in H \subseteq \mathcal{H}$;
(ii) for all $i \in N$, if $y \in C_i(R,x) \subseteq L(R^*_i, y)$, $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \setminus \{i\}$, and $y \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that:
(a) if $H = \{i\}$, then $(y,y') \in I^*_i$ for some $y' \in C_i(R,x) \setminus \{y\}$;
(b) if $i \in H$ and $\#H > 1$, then $R^* \neq R$ or $(y,y') \in I^*_i$ for some $y' \in C_i(R,x) \setminus \{y\}$;
(iii) if $y \in \max_{R^*_\ell} Y$ for all $\ell \in N$ and $y \notin F(R^*)$, then there is an $\ell \in N$ such that $(y,y') \in I^*_\ell$ for some $y' \in Y \setminus \{y\}$.

We are now ready to present our first main result, which shows that Condition μ^* is a minimal set of necessary conditions for the partially-honest implementation.

Theorem 1. *Let Assumption 1 hold. If an* SCC F on $\mathcal{R}^n$ *is* partially-honest implementable, *then it satisfies* Condition μ^*.

Proof. Let Assumption 1 hold. Let $h \in N$ be a partially-honest agent. Let $\gamma \equiv (M,g)$ be a mechanism which partially-honest implements F. Let $Y \equiv g(M)$. Take any $R \in \mathcal{R}^n$ and $x \in F(R)$. Then, there is a strategy $m \in NE(\gamma, \succcurlyeq^R)$ such that $g(m) = x$. Then, $\{x\} \subseteq g(M_\ell, m_{-\ell}) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$. This is true even for h. In fact, if $m_h \notin T^\gamma_h(R,F)$, $m \in NE(\gamma, \succcurlyeq^R)$ implies that $(g(m), g(m'_h, m_{-h})) \in P_h$ for all $m'_h \in T^\gamma_h(R,F)$. Let $C_\ell(R,x) \equiv g(M_\ell, m_{-\ell})$ for all $\ell \in N$. We show that F satisfies Conditions μ^*(i)-μ^*(iii). Take any $R^* \in \mathcal{R}^n$.

Suppose that $C_\ell(R,x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$ and $x \notin F(R^*) = NA(\gamma, \succcurlyeq^{R^*})$. Since $g(M_\ell, m_{-\ell}) \subseteq L(R^*_\ell, x)$ and $m \notin NE(\gamma, \succcurlyeq^{R^*})$ it follows that there is an $H \in \mathcal{H}$ such that, for some $h \in H$, $m_h \notin T^\gamma_h(R^*, F)$ and $(g(m'_h, m_{-h}), g(m)) \in R^*_h$ for some $m'_h \in T^\gamma_h(R^*, F)$. Moreover, as $C_h(R,x) \equiv g(M_h, m_{-h}) \subseteq L(R^*_h, x)$, $(g(m'_h, m_{-h}), g(m)) \in I^*_h$. Thus, F satisfies Condition μ^*(i).

Let $i \in N$ and suppose that $y \in C_i(R,x) \subseteq L(R^*_i, y)$ and $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \setminus \{i\}$. As $y \in C_i(R,x) = g(M_i, m_{-i})$, it follows that there is an

$m'_i \in M_i$ such that $g(m'_i, m_{-i}) = y$. Thus, $g(M_i, m_{-i}) \subseteq L(R^*_i, y)$. Let $\hat{m} \equiv (m'_i, m_{-i})$. Let $y \notin F(R^*) = NA(\gamma, \succsim^{R^*})$. Then, as $g(M_\ell, \hat{m}_{-\ell}) \subseteq L(R^*_\ell, y)$ and $\hat{m} \notin NE(\gamma, \succsim^{R^*})$, it follows from the same reasoning as the case of μ^*(i) that there is an $H \in \mathcal{H}$ such that, for some $h \in H$, $\hat{m}_h \notin T^\gamma_h(R^*, F)$ and $(g(m^*_h, \hat{m}_{-h}), g(\hat{m})) \in I^*_h$ for some $m^*_h \in T^\gamma_h(R^*, F)$.

Let $H = \{i\}$, and assume, to the contrary, that $\{y\} = \max_{R^*_i} C_i(R, x)$, so that $g(m^*_i, \hat{m}_{-i}) = g(\hat{m}) = y$ for all $m^*_i \in T^\gamma_i(R^*, F)$. Since there cannot be any further deviation by $g(M) = Y$, we have that $y \in NA(\gamma, \succsim^{R^*})$, a contradiction. Thus, F satisfies μ^*(ii.a).

Let $\#H > 1$ and $i \in H$. Suppose $R^* = R$. Then, $C_i(R, x) \subseteq L(R_i, x)$ and $C_i(R, x) \subseteq L(R^*_i, y)$ imply that $(x, y) \in I^*_i$. From $x \in F(R)$, $y \notin F(R^*)$, and $R^* = R$, it follows that $x \neq y$. Moreover, suppose $\{y\} = \max_{R^*_i} C_i(R, x)$. This immediately implies that $R^* \neq R$ as $(y, x) \in P^*_i$ and $(x, y) \in R_i$. Therefore, F satisfies μ^*(ii.b).

Let $y \in \max_{R^*_\ell} Y$ for all $\ell \in N$. Suppose $y \notin F(R^*) = NA(\gamma, \succsim^{R^*})$. Since $y \in Y = g(M)$, there is an $\hat{m} \in M$ such that $g(\hat{m}) = y$. Therefore, $y \in \max_{R^*_\ell} g(M)$ for all $\ell \in N$. Assume, to the contrary, that $\{y\} = \max_{R^*_\ell} Y$ for all $\ell \in N$. As $y \notin NA(\gamma, \succsim^{R^*})$, it follows that $\hat{m} \notin NE(\gamma, \succsim^{R^*})$. Then, there is an $H \in \mathcal{H}$ such that, for some $h \in H$, $\hat{m}_h \notin T^\gamma_h(R^*, F)$ and $(g(m^*_h, \hat{m}_{-h}), g(\hat{m})) \in I^*_h$ for some $m^*_h \in T^\gamma_h(R^*, F)$. Let H' be the set of all such $h \in H$. Take any $h_1 \in H'$. Then, there is an $m^1_{h_1} \in T^\gamma_{h_1}(R^*, F)$ such that $(g(m^1_{h_1}, \hat{m}_{-h_1}), y) \in I^*_{h_1}$. As $\{y\} = \max_{R^*_{h_1}} g(M)$, $g(m^1_{h_1}, \hat{m}_{-h_1}) = y$. Let $m^1 \equiv (m^1_{h_1}, \hat{m}_{-h_1})$. Since $y \notin F(R^*) = NA(\gamma, \succsim^{R^*})$ and $m^1 \notin NE(\gamma, \succsim^{R^*})$, there should exist an $h_2 \in H' \backslash \{h_1\}$ and an $m^2_{h_2} \in T^\gamma_{h_2}(R^*, F)$ such that $(g(m^2_{h_2}, m^1_{-h_2}), y) \in I^*_{h_2}$. As $\{y\} = \max_{R^*_{h_2}} g(M)$, $g(m^2_{h_2}, m^1_{-h_2}) = y$. Let $m^2 \equiv (m^2_{h_2}, m^1_{-h_2})$. Again, as $m^2 \notin NE(\gamma, \succsim^{R^*})$, there should exist an $h_3 \in H' \backslash \{h_1, h_2\}$ and an $m^3_{h_3} \in T^\gamma_{h_3}(R^*, F)$ such that $(g(m^3_{h_3}, m^2_{-h_3}), y) \in I^*_{h_3}$. As $\#H' = s \leq n$, the above reasoning will stop after at most s iterations. Let m^s be the strategy profile corresponding to the iteration s and $g(m^s) = y$. Then, $y \in NA(\gamma, \succsim^{R^*})$, a contradiction. Therefore, F satisfies μ^*(iii). ■

We also introduce another new condition, Condition μ^{**}, which lies strictly between Condition μ^* and Condition μ. It can be stated as follows.

Condition μ^{**} (for short, μ^{**}): There is a set $Y \subseteq X$ and, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R, x))_{\ell \in N}$ such that $x \in C_\ell(R, x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$, the following (i)-(iv) are satisfied:

(i) if $C_\ell(R,x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$ and $x \notin F(R^*)$, then $(x,x') \in I^*_h$ for some $x' \in C_h(R,x)$ and some $h \in H \subseteq \mathcal{H}$.
(ii) for all $i \in N$, if $y \in C_i(R,x) \subseteq L(R^*_i, y)$, $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \backslash \{i\}$, and $y \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that:
(a) if $H = \{i\}$, then $(y,y') \in I^*_i$ for some $y' \in C_i(R,x) \setminus \{y\}$;
(b) if $i \in H$ and $\#H > 1$, then $R^* \neq R$ or $(y,y') \in I^*_i$ for some $y' \in C_i(R,x) \setminus \{y\}$;
(c) if $i \notin H$, then $R \neq R^*$;
(iii) if $y \in \max_{R^*_\ell} Y$ for all $\ell \in N$, then $y \in F(R^*)$.
(iv) for all $i \in N$, if $L(R^*_i, x) = L(R_i, x)$, $x \in \max_{R^*_\ell} Y$ for all $\ell \in N \backslash \{i\}$, $R^*_{-i} = R_{-i}$, and $x \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that $H \neq \{i\}$.

Our second main result is given by applying Condition μ^{**} as follows.

Theorem 2. *Let Assumption 1 and $\Gamma = \Gamma_{SP}$ hold, and suppose that $\mathcal{R}^n$ satisfies* **RD**. *An* SCC F *on* $\mathcal{R}^n$ *is* partially-honest implementable *if and only if it satisfies* Condition μ^{**}.

Proof. Let Assumption 1 hold and let $\mathcal{R}^n$ satisfy **RD**. Let $h \in N$ denote a partially-honest agent.

1. The necessity of Condition μ^{**}.

Let F on $\mathcal{R}^n$ be an SCC which is partially-honest implementable by a mechanism $\gamma \equiv (M,g) \in \Gamma_{SP}$. Let $Y \equiv g(M)$. Take any $R \in \mathcal{R}^n$ and any $x \in F(R)$. The, there is an $m(R,x) \in NE(\gamma, \succcurlyeq^R) \subseteq M$ such that $g(m(R,x)) = x$. Moreover, $m_h(R,x) \in T^\gamma_h(R,F)$ for every partially-honest agent $h \in H$. For, assume, to the contrary, that $m_h(R,x) \notin T^\gamma_h(R,F)$ for some $h \in H$. As $\gamma \in \Gamma_{SP}$, we have that agent h can change $m_h(R,x)$ to an $m_h \in T^\gamma_h(R,F)$ and obtain $g(m(R,x)) = g(m_h, m_{-h}(R,x)) = x$, which contradicts from $m(R,x) \in NE(\gamma, \succcurlyeq^R)$. For all $\ell \in N$, let $C_\ell(R,x) \equiv g(M_\ell, m_{-\ell}(R,x))$. Then, $C_\ell(R,x) \equiv g(M_\ell, m_{-\ell}(R,x)) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$. Take any $R^* \in \mathcal{R}^n$.

By similar argument used in Theorem 1 it follows that F satisfies Condition μ^*. Thus, we only show that F satisfies μ^{**}(ii.c)-μ^{**}(iv).

Let $i \in N$; suppose that $y \in C_i(R,x) \subseteq L(R^*_i, y)$, $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \backslash \{i\}$, and $y \notin F(R^*)$. Since $y \in C_i(R,x) \equiv g(M_i, m_{-i}(R,x))$, it follows that $g(m_i, m_{-i}(R,x)) = y$ for some $m_i \in M_i$. Assume, to the contrary, that $R = R^*$ and $i \notin H$ for all $H \in \mathcal{H}$. Since $m_h(R,x) \in T^\gamma_h(R^*, F)$ for all $h \in H$ and there cannot be any profitable deviation, we have that $(m_i, m_{-i}(R,x)) \in NE(\gamma, \succcurlyeq^{R^*})$, a contradiction.

Suppose that $y \in \max_{R_\ell^*} Y$ for all $\ell \in N$. Then there is an $\bar{m} \in M$ such that $g(\bar{m}) = y$. Consider $\bar{R} \equiv (\bar{R}_\ell)_{\ell \in N} \in \mathcal{R}^n$ such that $L(\bar{R}_\ell, y) = L(R_\ell^*, y)$ with $\partial L(\bar{R}_\ell, y) = \{y\}$ for all $\ell \in N$. As $\mathcal{R}^n$ satisfies **RD**, such a profile is admissible. Then, since F satisfies Condition μ^*(iii) by Theorem 1, it follows that $y \in F(\bar{R})$. Suppose that there is an $\varnothing \neq S \subseteq N$ such that $\bar{m}_\ell \notin T_\ell^\gamma(R^*, F)$ for all $\ell \in S$, otherwise $g(\bar{m}) \in F(R^*)$, as sought. Then, by **SP**, for each $\ell \in S$, there is an $\bar{m}'_\ell \in T_\ell^\gamma(R^*, F)$ such that $g(\bar{m}'_\ell, \bar{m}_{-\ell}) = y$. By repeatedly applying **SP** from $\ell_1 \in S$ to $\ell_s \in S$, where $S = \{\ell_1, \ldots, \ell_s\}$, it follows that $g(\bar{m}'_S, \bar{m}_{-S}) = y$. Thus, $(\bar{m}'_S, \bar{m}_{-S}) \in NE(\gamma, \succcurlyeq^{R^*})$ and so $y \in NA(\gamma, \succcurlyeq^{R^*}) = F(R^*)$, as we sought. Therefore, F satisfies Condition μ^{**}(iii).

Suppose that $L(R_i, x) = L(R_i^*, x)$, $x \in \max_{R_\ell^*} Y$ for all $\ell \in N \backslash \{i\}$, $R_{-i} = R_{-i}^*$, and $x \notin F(R^*)$. Since $x \in F(R)$ and $R_{-i}^* = R_{-i}$, $R_i^* \neq R_i$ holds. By $x \notin F(R^*) = NA(\gamma, \succcurlyeq^{R^*})$, $m(R, x) \notin NE(\gamma, \succcurlyeq^{R^*})$ holds. However, as $x \in \max_{R_\ell^*} g(M)$ for all $\ell \in N \backslash \{i\}$ and $g(M_i, m_{-i}(R, x)) \subseteq L(R_i^*, x) = L(R_i, x)$, $m(R, x) \notin NE(\gamma, \succcurlyeq^{R^*})$ implies that there is an $H \in \mathcal{H}$ such that, for some $h \in H$, $m_h(R, x) \notin T_h^\gamma(R^*, F)$ and $(g(m_h, m_{-h}(R, x)), g(m(R, x))) \in I_h^*$ for some $m_h \in T_h^\gamma(R^*, F)$. Assume, to the contrary that, $H = \{i\}$. Then, it follows that the unique deviator is agent i. Since γ satisfies **SP**, there is an $m_i^* \in T_i^\gamma(R^*, F)$ such that $g(m_i^*, m_{-i}(R, x)) = g(m(R, x)) = x$. This implies $(m_i^*, m_{-i}(R, x)) \in NE(\gamma, \succcurlyeq^{R^*})$ so that $x \in NA(\gamma, \succcurlyeq^{R^*}) = F(R^*)$, a contradiction. Therefore, F satisfies Condition μ^{**}(iv).

2. The sufficiency of Condition μ^{**}.

Conversely, suppose that F satisfies Condition μ^{**}. Let $\diamond \in N$ be an arbitrary agent index. Let $\gamma \equiv (M, g)$ be a mechanism having $M_\ell \equiv \mathcal{R}^n \times Y \times N$ with a generic element $m_\ell = (R^\ell, x^\ell, k^\ell)$ for each $\ell \in N$, where R^ℓ is the preference profile announced by agent $\ell \in N$, while x^ℓ and k^ℓ are the outcome and the integer announced by the agent at issue, respectively.

Define the outcome function $g : M \to X$ as follows:

Rule 1: If $m \in M$ is such that for some $(\bar{R}, x) \in \mathcal{R}^n \times Y$ with $x \in F(\bar{R})$, $(R^\ell, x^\ell) = (\bar{R}, x)$ for all ℓ, then $g(m) = x$;
Rule 2: If $m \in M$ is such that there is a unique agent $i \in N$ such that for some $(\bar{R}, x) \in \mathcal{R}^n \times Y$ with $x \in F(\bar{R})$, $(\bar{R}, x) = (R^\ell, x^\ell)$ for all $\ell \neq i$, and $(R^i, x^i) \neq (\bar{R}, x)$ with $R^i \neq \bar{R}$, then

$$g(m) = \begin{cases} x^i & \text{if } x^i \in C_i(\bar{R}, x), \\ x & \text{otherwise;} \end{cases}$$

Rule 3: If $m \in M$ is such that there is a unique agent $i \in N$ such that for some $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y$ with $x \in F\left(\bar{R}\right)$, $\left(\bar{R}, x\right) = \left(R^\ell, x^\ell\right)$ for all $\ell \neq i$, and $(R^i, x^i) \neq \left(\bar{R}, x\right)$ with $R^i = \bar{R}$, then $g(m) = x$;
Rule 4: Otherwise, $g(m) = x^{\ell^*(m)}$ where $\ell^*(m) = \sum_{i \in N} k^i \pmod{n}$.[10]

By the definition of g, it follows that any $\gamma = (M, g)$ satisfies **SP**, that is, $\gamma \in \Gamma_{SP}$. Moreover, for each $\ell \in N$, the truth-telling correspondence $T_\ell^\gamma : \mathcal{R}^n \times \mathcal{F} \twoheadrightarrow M_\ell$ is given by: $T_\ell^\gamma(R, F) = \{R\} \times Y \times N$ for each $(R, F) \in \mathcal{R}^n \times \mathcal{F}$, where Y may change according to F.

Let us show that γ partially-honest implements F. Take any $R \in \mathcal{R}^n$. Since F satisfies μ^{**}, $F(\mathcal{R}^n) \subseteq Y$.

To show that $F(R) \subseteq NA\left(\gamma, \succcurlyeq^R\right)$, let $x \in F(R)$ and suppose that, for all $\ell \in N$, $m_\ell = (R, x, \diamond) \in M_\ell$. Notice that $m_\ell \in T_\ell^\gamma(R, F)$ for all $\ell \in N$. *Rule 1* implies that $g(m) = x$. Suppose that $\ell \in N$ deviates from m_ℓ to $m_\ell^* = \left(R^\ell, x^\ell, \diamond\right) \in M_\ell$. It follows from *Rules 1-3* that $g(M_\ell, m_{-\ell}) \subseteq C_\ell(R, x)$. Since F satisfies μ^{**}, it follows that $g(M_\ell, m_{-\ell}) \subseteq L(R_\ell, x)$. As it holds for any $\ell \in N$, it follows that $m \in NE\left(\gamma, \succcurlyeq^R\right)$ and so $x \in NA\left(\gamma, \succcurlyeq^R\right)$.

Conversely, to show that $NA\left(\gamma, \succcurlyeq^R\right) \subseteq F(R)$, let $m \in NE\left(\gamma, \succcurlyeq^R\right)$. Consider the following cases.

Case 1: m corresponds to *Rule 1*.

Then, for some $\bar{R} \in \mathcal{R}^n$ and $x \in F\left(\bar{R}\right)$, $\left(\bar{R}, x\right) = \left(R^\ell, x^\ell\right)$ for all $\ell \in N$ and $g(m) = x$. Suppose that $R \neq \bar{R}$. Then, $m_\ell \notin T_\ell^\gamma(R, F)$ for all $\ell \in N$. Take any $m'_h \in T_\ell^\gamma(R, F)$ such that $x^h = x$. *Rule 2* implies that $g(m'_h, m_{-h}) = x$ so that $((m'_h, m_{-h}), m) \in \succ_h^R$, a contradiction. Otherwise, $R = \bar{R}$ and so $x \in F(R)$.

Case 2: m corresponds to *Rule 2*.

Then, there exists an $i \in N$ such that $\left(\bar{R}, x\right) = \left(R^\ell, x^\ell\right) \neq (R^i, x^i)$ for all $\ell \in N \backslash \{i\}$, where $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y$ with $x \in F\left(\bar{R}\right)$ and $R^i \neq \bar{R}$. By the definition of g, $g(M_\ell, m_{-\ell}) = Y$ for all $\ell \in N \backslash \{i\}$ and $C_i\left(\bar{R}, x\right) \subseteq g(M_i, m_{-i})$. Thus, $m \in NE\left(\gamma, \succcurlyeq^R\right)$ implies that $Y \subseteq L(R_\ell, g(m))$ for all $\ell \in N \backslash \{i\}$ and $C_i\left(\bar{R}, x\right) \subseteq L(R_i, g(m))$.

Suppose that $m_h \notin T_h^\gamma(R, F)$ for some $h \in H \backslash \{i\}$. Then, agent $h \in H \backslash \{i\}$ can induce *Rule 4* by deviating to a suitable $m'_h \in T_h^\gamma(R, F)$ so as to obtain $g(m'_h, m_{-h}) = g(m)$, which contradicts that $m \in NE\left(\gamma, \succcurlyeq^R\right)$. Therefore, $m_h \in T_h^\gamma(R, F)$ for all $h \in H \backslash \{i\}$.

[10] If the remainder is *zero*, the winner of the game is agent n.

Suppose that $\#H > 1$ and $i \in H$. We show that this case contradicts that $m \in NE\left(\gamma, \succcurlyeq^R\right)$. As $\#H > 1$, we have that $m_h \in T_h^\gamma(R, F)$ for all $h \in H \backslash \{i\}$; moreover, $R = \bar{R}$ and $x \in F(R)$. Since m falls into *Rule 2*, it follows that $R^i \neq R$, so $m_i \notin T_i^\gamma(R, F)$. It follows from $x \in C_i\left(\bar{R}, x\right) \subseteq L(R_i, g(m))$ and $g(m) \in C_i\left(\bar{R}, x\right) \subseteq L(R_i, x)$ that $(x, g(m)) \in I_i$. Agent i can deviate to $m_i' = (R, x, k^i) \in T_i^\gamma(R, F)$ so that she induces *Rule 1* and obtains $g(m_i', m_{-i}) = x$, which contradicts that $m \in NE\left(\gamma, \succcurlyeq^R\right)$. We conclude that $\#H \not> 1$ or $i \notin H$.

Suppose that $\#H \geq 1$ and $i \notin H$. Condition μ^{**}(ii.c) implies that $g(m) \in F(R)$. Otherwise, let $H = \{i\}$. Observe that $R \neq \bar{R}$, otherwise a contradiction that $m \in NE\left(\gamma, \succcurlyeq^R\right)$ can be obtained by the same reasoning used in the case that $i \in H$ and $\#H > 1$. Therefore, let $R \neq \bar{R}$. Notice that $m_i \in T_i^\gamma(R, F)$, otherwise agent i can induce *Rule 2* by deviating to an $m_i' = (R, g(m), k^i) \in T_i^\gamma(R, F)$ and obtain $g(m_i', m_{-i}) = g(m)$, which contradicts that $m \in NE\left(\gamma, \succcurlyeq^R\right)$. Take an $\hat{R}_i \in \mathcal{R}_i(X)$ such that $L\left(\hat{R}_i, y\right) = L(R_i, g(m))$, with $\partial L\left(\hat{R}_i, g(m)\right) = \{g(m)\}$. As $\mathcal{R}^n$ satisfies **RD**, we have that $\hat{R} \equiv \left(\hat{R}_i, R_{-i}\right) \in \mathcal{R}^n$. Then, μ^*(ii.a) implies that $g(m) \in F\left(\hat{R}\right)$. Since F satisfies μ^{**}, there exists a profile $\left(C_\ell\left(\hat{R}, g(m)\right)\right)_{\ell \in N}$ such that $C_\ell\left(\hat{R}, g(m)\right) \subseteq L\left(\hat{R}_\ell, g(m)\right) \cap Y$ for all $\ell \in N$. As $L\left(\hat{R}_i, g(m)\right) = L(R_i, g(m))$, $R_{-i} = \hat{R}_{-i}$, and $H = \{i\}$, Condition μ^{**}(iv) implies that $g(m) \in F(R)$.

Case 3: m corresponds to *Rule 3*.

Then, there exists an $i \in N$ such that $\left(\bar{R}, x\right) = \left(R^\ell, x^\ell\right) \neq (R^i, x^i)$ for any $\ell \in N \backslash \{i\}$, where $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y$, with $x \in F\left(\bar{R}\right)$, $R^i = \bar{R}$ and $g(m) = x$. By the definition of g, $g(M_\ell, m_{-\ell}) = Y$ for all $\ell \in N \backslash \{i\}$ and $C_i\left(\bar{R}, x\right) \subseteq g(M_i, m_{-i})$. Thus, $m \in NE\left(\gamma, \succcurlyeq^R\right)$ implies that $Y \subseteq L(R_\ell, x)$ for all $\ell \in N \backslash \{i\}$ and $C_i\left(\bar{R}, x\right) \subseteq L(R_i, x)$. Suppose that $\bar{R} \neq R$. Then, $m_h \notin T_h^\gamma(R, F)$ for all $h \in H$. Suppose that $h \neq i$. Agent h can induce *Rule 4* by unilaterally deviating to $m_h' = \left(R, x, k^h\right) \in T_h^\gamma(R, F)$. By choosing k^h so as $h = \ell^*(m_{-h}, m_h')$, she obtains $g(m_{-h}, m_h') = x$. Then, $((m_{-h}, m_h'), m) \in \succ_h^R$ which contradicts $m \in NE\left(\gamma, \succcurlyeq^R\right)$. Otherwise, let $h = i$. As agent h can induce *Rule 2* by deviating to $m_h' = (R, x, \diamond) \in T_h^\gamma(R, F)$, we have that $g(m_{-h}, m_h') = x$, which again leads to a contradiction. Therefore, $\bar{R} = R$ and so $x \in F(R)$.

Case 4: m corresponds to *Rule 4*.

From the definition of g and the supposition that $m \in NE\left(\gamma, \succcurlyeq^R\right)$, it follows that $g(m) \in \max_{R_\ell^*} g(M)$ for all $\ell \in N$. Thus, by μ^{**}(iii), $g(m) \in F(R)$. ■

3.2 Partially-honest implementation by s-mechanisms

In this sub-section, we pay attention to informational decentralization and efficiency of admissible mechanisms and consider implementation by mechanisms with a smaller strategy space - *s-mechanisms*: each agent announces, in addition to a feasible social outcome and an integer, her own and her neighbor's preferences (Saijo, 1988). We will find below that the class of SCCs partially-honest implementable by such mechanisms is dwindled down with respect to the class of SCCs identified by Theorem 2.

We define partially-honest implementation by s-mechanisms as follows.

Definition 3. A mechanism $\gamma = (M, g)$ is an s-mechanism if, for any $\ell \in N$, $M_\ell \equiv \mathcal{R}_\ell \times \mathcal{R}_{\ell+1} \times Y \times N$, with $\ell + 1 = 1$ if $\ell = n$, where $Y \subseteq X$.

Note that, if γ is an s-mechanism, then $T_\ell^\gamma(R, F) \equiv \{(R_\ell, R_{\ell+1})\} \times Y \times N$ for any $(R, F) \in \mathcal{R}^n \times \mathcal{F}$.

Definition 4. An SCC F on $\mathcal{R}^n$ is partially-honest implementable by an s-mechanism if there exists an s-mechanism $\gamma \equiv (M, g)$ such that:
(i) for all $R \in \mathcal{R}^n$, $F(R) = NA\left(\gamma, \succcurlyeq^R\right)$; and
(ii) for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, if $m_\ell = \left(R_\ell, R_{\ell+1}, x, k^\ell\right) \in M_\ell$ for all $\ell \in N$, with $\ell + 1 = 1$ if $\ell = n$, then $m \in NE\left(\gamma, \succcurlyeq^R\right)$ and $g(m) = x$.

In Definition 4, it is required not only that all F-optimal outcomes coincide with partially-honest Nash equilibrium outcomes of the game $\left(\gamma, \succcurlyeq^R\right)$ defined by an s-mechanism - for any state $R \in \mathcal{R}^n$ -, but also that such an s-mechanism satisfies *forthrightness*. Forthrightness requires that if the outcome x is F-optimal at the state R and each agent announces truthfully her preference R_ℓ and her neighbor's preference $R_{\ell+1}$ and announces this x, then the message profile should be a partially-honest Nash equilibrium of an s-mechanism and its equilibrium outcome be the announced outcome x.

Forthrightness was originally introduced in economic environments by Dutta, Sen, and Vohra (1995) and Saijo, Tatamitani, and Yamato (1996), and it has desirable implications. A mechanism satisfying forthrightness is simple in the sense that it is easy to compute the outcome of an equilibrium strategy profile. Moreover, if a mechanism fails to satisfy this condition, it is subject to information smuggling; that is, the strategy space can be reduced to an arbitrary smaller dimensional space. Thus, any partially-honest implementable SCC by s-mechanisms would be partially-honest implementable by a 'further strategy space reduction mechanism' like *self-relevant mechanisms* (Tatamitani, 2000), unless forthrightness is required. This indicates that there is no legitimate reason for characterizing partially-honest implementation by s-mechanisms without forthrightness. Hence, to make sense of partially-honest implementation by s-mechanisms, we require this regularity condition in Definition 4.

The issue of what constitutes the necessary and sufficient condition for implementation by s-mechanisms in the standard framework has been recently addressed by Lombardi and Yoshihara (2010), who introduce a new condition - *Condition* M_s -, which is similar to Condition M appearing in Sjöström (1991). This condition can be stated as follows.

Condition M_s (for short, M_s): There is a set $Y \subseteq X$ and, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R_\ell, x))_{\ell \in N}$ such that $x \in C_\ell(R_\ell, x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$, the following (i)-(iii) are satisfied:
(i) if $C_\ell(R_\ell, x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for all $i \in N$, if $y \in C_i(R_i, x) \subseteq L(R^*_i, y)$ and $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \setminus \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in \max_{R^*_\ell} Y$ for all $\ell \in N$, then $y \in F(R^*)$.

In what follows, we state a condition - Condition M^*_s- which is slightly weaker than Condition M_s and show that if an SCC F is partially-honest implementable by any s-mechanism, then it must satisfy it.

Condition M^*_s (for short, M^*_s): There is a set $Y \subseteq X$ and, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R_\ell, x))_{\ell \in N}$ such that $x \in C_\ell(R_\ell, x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$, the following (i)-(iii) are satisfied:
(i) if $C_\ell(R_\ell, x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$ and $x \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that, for some $H' \subseteq H$ and all $h \in H'$, $(R_h, R_{h+1}) \neq \left(R^*_h, R^*_{h+1}\right)$;
(ii) for all $i \in N$, if $y \in C_i(R_i, x) \subseteq L(R^*_i, y)$, $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \setminus \{i\}$, and $y \notin F(R^*)$, then for some $H \in \mathcal{H}$ and some $H' \subseteq H$:
(a) if $H' = \{i\}$, then $(y, y') \in I^*_i$ for some $y' \in C_i(R_i, x) \setminus \{y\}$;
(b) otherwise, $\left(R'_h, R'_{h+1}\right) \neq \left(R^*_h, R^*_{h+1}\right)$ for all $h \in H' \setminus \{i\}$;

(iii) if $y \in \max_{R^*_\ell} Y$ for all $\ell \in N$ and $y \notin F(R^*)$, then there is an $\ell \in N$ such that $(y, y') \in I^*_\ell$ for some $y' \in Y \setminus \{y\}$.

Theorem 3. *Let Assumption 1 hold. If an* SCC F on $\mathcal{R}^n$ *is* partially-honest implementable by s-mechanisms, *then it satisfies* Condition M^*_s.

Proof. Let Assumption 1 hold. Let $h \in N$ be a partially-honest agent; let $\diamond \in N$ be an arbitrary agent index. Let $\gamma \equiv (M, g)$ be an s-mechanism which partially-honest implements F. Let $Y \equiv g(M)$. Take any $R \in \mathcal{R}^n$ and $x \in F(R)$. For all $\ell \in N$, let $C_\ell(R_\ell, x) \equiv g(M_\ell, m_{-\ell}(R, x))$ where $m_{-\ell}(R, x)$ is such that $m_i(R, x) = (R_i, R_{i+1}, x, s^i) \in M_i$ for all $i \in N \setminus \{\ell\}$, with $i+1 = 1$ if $i = n$. By forthrightness, it follows that $m(R, x) = (m_\ell(R, x), m_{-\ell}(R, x))$ is an equilibrium message profile, $m(R, x) \in NE(\gamma, \succcurlyeq^R)$, and $g(m(R, x)) = x$. Then, $C_\ell(R, x) \equiv g(M_\ell, m_{-\ell}(R, x)) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$. We show that F satisfies Conditions M^*_s(i)-M^*_s(iii). As the proof that F meets M^*_s(iii) directly follows by employing the same reasoning used in Theorem 1, we omit it here. Take any $R^* \in \mathcal{R}^n$.

Suppose that $C_\ell(R_\ell, x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$ and $x \notin F(R^*) = NA(\gamma, \succcurlyeq^{R^*})$. Then, for some $\ell \in N$ and some $m'_\ell \in M_\ell$, $(g(m'_\ell, m_{-\ell}(R, x)), g(m(R, x))) \in R^*_\ell$. As $g(M_\ell, m_{-\ell}(R, x)) \subseteq L(R^*_\ell, x)$ we have that $(g(m'_\ell, m_{-\ell}(R, x)), g(m(R, x))) \in I^*_\ell$; and, agent ℓ is a partially-honest agent, $\ell = h$. As $m(R, x) \notin NE(\gamma, \succcurlyeq^{R^*})$ there is an $H \in \mathcal{H}$ such that, for some $H' \subseteq H$ and all $h \in H'$, $m_h(R, x) \notin T^\gamma_h(R^*, F)$; moreover, there is an $m_h \in T^\gamma_h(R^*, F)$ for each $h \in H'$, otherwise a contradiction. Since $T^\gamma_h(R^*, F) = \{(R^*_h, R^*_{h+1})\} \times Y \times N$ and $T^\gamma_h(R, F) = \{(R_h, R_{h+1})\} \times Y \times N$, it follows from $m_h(R, x) \in T^\gamma_h(R, F) \setminus T^\gamma_h(R^*, F)$ that $(R^*_h, R^*_{h+1}) \neq (R_h, R_{h+1})$ for all $h \in H'$. Thus, F satisfies Condition M^*_s(i).

Pick any $i \in N$; suppose that $y \in C_i(R_i, x) \subseteq L(R^*_i, y)$, $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \setminus \{i\}$, and $y \notin F(R^*) = NA(\gamma, \succcurlyeq^{R^*})$. Since $y \in C_i(R_i, x) \equiv g(M_i, m_{-i}(R, x))$, it follows that $g(m_i, m_{-i}(R, x)) = y$ for some $m_i \in M_i$. Let $(m_i, m_{-i}(R, x)) \equiv m$. Moreover, as $y \notin NA(\gamma, \succcurlyeq^{R^*})$, it follows that, for some $\ell \in N$ and some $m'_\ell \in M_\ell$, $(g(m'_\ell, m_{-\ell}), g(m)) \in R^*_\ell$. As $g(M_\ell, m_{-\ell}) \subseteq L(R^*_\ell, y)$, we have that $(g(m'_\ell, m_{-\ell}), g(m)) \in I^*_\ell$; and, agent ℓ is a partially-honest agent, $\ell = h$. Therefore, there is an $H \in \mathcal{H}$ such that, for some $H' \subseteq H$ and all $h \in H'$, $m_h \notin T^\gamma_h(R^*, F)$, that is, $H' = \{h \in H | m_h \notin T^\gamma_h(R^*, F)\} \neq \varnothing$ for some $H' \subseteq H$; furthermore, for all $h \in H'$, there is an $m^*_h \in T^\gamma_h(R^*, F)$ such that $(g(m^*_h, m_{-h}), g(m)) \in I^*_h$.

Let $H' = \{i\}$ and $\{y\} = \max_{R^*_i} C_i(R_i, x)$. It follows that $g(m^*_i, m_{-i}) = y$, which leads to $(m^*_i, m_{-i}) \in NE(\gamma, \succcurlyeq^{R^*})$, a contradiction. Thus, F satisfies M^*_s(ii.a). To examine M^*_s(ii.b), assume, to the contrary, that for all $H' \subseteq H$

with $H' \neq \{i\}$, $(R_h, R_{h+1}) = \left(R^*_h, R^*_{h+1}\right)$ for some $h \in H' \setminus \{i\}$. Then, it follows from the definition of H' that either $H' = \varnothing$ if $i \notin H'$ or $H' = \{i\}$. In either case case, we have a contradiction. ■

We also introduce another new condition, which lies between Condition M^*_s and Condition M_s.

Condition M^{**}_s (for short, M^{**}_s): There is a set $Y \subseteq X$ and, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R_\ell, x))_{\ell \in N}$ such that $x \in C_\ell(R_\ell, x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$; finally, for all $R^* \in \mathcal{R}^n$, the following (i)-(iv) are satisfied:
(i) if $C_\ell(R_\ell, x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$ and $x \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that, for some $H' \subseteq H$ and all $h \in H'$, $(R_h, R_{h+1}) \neq \left(R^*_h, R^*_{h+1}\right)$;
(ii) for all $i \in N$, if $y \in C_i(R_i, x) \subseteq L(R^*_i, y)$, $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \setminus \{i\}$, and $y \notin F(R^*)$, then for some $H \in \mathcal{H}$ and some $H' \subseteq H$:
(a) if $H' = \{i\}$, then $(y, y') \in I^*_i$ for some $y' \in C_i(R_i, x) \setminus \{y\}$;
(b) otherwise, $(R_h, R_{h+1}) \neq \left(R^*_h, R^*_{h+1}\right)$ for all $h \in H' \setminus \{i\}$;
(iii) if $y \in \max_{R^*_\ell} Y$ for all $\ell \in N$, $y \in F(R^*)$;
(iv) for all $i \in N$, if $L(R^*_i, x) = L(R_i, x)$, $x \in \max_{R^*_\ell} Y$ for all $\ell \in N \setminus \{i\}$, $R^*_{-i} = R_{-i}$, and $x \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that $H \neq \{i\}$.

We are ready to show that this new condition is necessary and sufficient for partially-honest implementation by s-mechanisms under the same mild requirements employed in Theorem 2.

Theorem 4. *Let Assumption 1 and* $\Gamma = \Gamma_{SP}$ *hold, and let* $\mathcal{R}^n$ *satisfy* **RD**. *An* SCC F *on* $\mathcal{R}^n$ *is* partially-honest implementable by an s-mechanism *if and only if* F *satisfies* Condition M^{**}_s.

Proof. Let Assumption 1 hold and let $\mathcal{R}^n$ satisfy **RD**. Let $h \in N$ denote a partially-honest agent.

1. The necessity of Condition M^{**}_s.

Suppose that F is partially-honest implementable by an s-mechanism $\gamma = (M, g) \in \Gamma_{SP}$. From Theorem 3, it follows that F satisfies Condition M^*_s. Finally, by using the same reasoning using in Theorem 2, it can readily be obtained that F satisfies Condition M^{**}_s(iii) and Condition M^{**}_s(iv).

2. The sufficiency of Condition M^{**}_s.

Conversely, let $\gamma \equiv (M, g)$ be an s-mechanism. Suppose that F on $\mathcal{R}^n$ satisfies Condition M^{**}_s. Fix any $m \in M$, $R \in \mathcal{R}^n$, and $x \in Y$, and let $m_\ell = \left(R^\ell_\ell, R^\ell_{\ell+1}, x^\ell, k^\ell\right) \in M_\ell$, with $\ell + 1 = 1$ if $\ell = n$, and where the announcement of agent $\ell \in N$ about agent $\ell+1$'s preferences is $R^{\ell}_{\ell+1}$. Let $\diamond \in N$ be an arbitrary agent index. We say that the message profile $m \in M$ is:

(i) *consistent* with R and x if, for all $\ell \in N$, $R^{\ell}_{\ell} = R^{\ell-1}_{\ell} = R_{\ell}$ and $x^{\ell} = x$, where $\ell - 1 = n$ if $\ell = 1$;
(ii) m_{-i} *quasi-consistent* with x and R, where $i \in N$, if for all $\ell \in N$, $x^{\ell} = x$, and for all $\ell \in N \backslash \{i, i+1\}$, $R^{\ell}_{\ell} = R^{\ell-1}_{\ell} = R_{\ell}$, $R^{i-1}_{i} = R_i$, $R^{i+1}_{i+1} = R_{i+1}$, and $[R^{i}_{i} \neq R_i$ or $R^{i}_{i} \neq R_{i+1}]$, where $j-1 = n$ if $j = 1$ for $j \in \{i, \ell\}$;
(iii) m_{-i} *consistent* with x and R, where $i \in N$, if, for all $\ell \in N \backslash \{i\}$, $x^{\ell} = x \neq x^{i}$, and, for all $\ell \in N \backslash \{i, i+1\}$, $R^{\ell}_{\ell} = R^{\ell-1}_{\ell} = R_{\ell}$, $R^{i-1}_{i} = R_i$ and $R^{i+1}_{i+1} = R_{i+1}$, where $j - 1 = n$ if $j = 1$ for $j \in \{i, \ell\}$.

Define the outcome function $g : M \to X$ as follows: For any $m \in M$,

Rule 1: m is consistent with $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y$ such that $x \in F\left(\bar{R}\right)$, then $g(m) = x$.
Rule 2: For some $i \in N$, m_{-i} is quasi-consistent with $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y$ such that $x \in F\left(\bar{R}\right)$, then $g(m) = x$.
Rule 3: For some $i \in N$, m is m_{-i} consistent with $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y$ such that $x \in F\left(\bar{R}\right)$, and $C_i\left(\bar{R}_i, x\right) \neq Y$, then

$$g(m) = \begin{cases} x^i & \text{if} \quad x^i \in C_i\left(\bar{R}_i, x\right) \\ x & \text{otherwise} \end{cases} .$$

Rule 4: Otherwise, $g(m) = x^{\ell^*(m)}$ where $\ell^*(m) \equiv \sum_{i \in N} k^i \pmod{n}$.[11]

By the definition of g, it follows that any $\gamma = (M, g)$ satisfies **SP**, that is, $\gamma \in \Gamma_{SP}$. Moreover, for each $\ell \in N$, the truth-telling correspondence $T^{\gamma}_{\ell} : \mathcal{R}^n \times \mathcal{F} \twoheadrightarrow M_{\ell}$ is given by: $T^{\gamma}_{\ell}(R, F) = \{(R_{\ell}, R_{\ell+1})\} \times Y \times N$ for each $(R, F) \in \mathcal{R}^n \times \mathcal{F}$, where Y may change according to F. We show that γ partially-honest implements F. Take any $R \in \mathcal{R}^n$.

Since F satisfies M^{**}_{s}, it follows that, for all $(R, x) \in \mathcal{R}^n \times X$ with $x \in F(R)$, $x \in Y$. The proof of $F(R) \subseteq NA\left(\gamma, \succcurlyeq^R\right)$ follows the same argument in Lombardi and Yoshihara (2010), so we omit it here. Conversely, to show that $NA\left(\gamma, \succcurlyeq^R\right) \subseteq F(R)$, let $m \in NE\left(\gamma, \succcurlyeq^R\right)$. Consider the following cases.

Case 1: m falls into *Rule 1*.

[11] If the remainder is *zero*, the winner of the game is agent n.

Then, m is consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F\left(\bar{R}\right)$. Thus, $g(m) = x$. By the definition g and $m \in NE\left(\gamma, \succsim^R\right)$, we have that $C_\ell\left(\bar{R}_\ell, x\right) \subseteq L(R_\ell, x)$ for all $\ell \in N$. Suppose that $m_h \notin T_h^\gamma(R, F)$ for some $h \in H$. Suppose that $C_h\left(\bar{R}_h, x\right) = Y$. By changing her strategy m_h into $m'_h \in T_h^\gamma(R, F)$, agent h can trigger the modulo game and choose an integer index k^h so that $\ell = \ell^*(m'_h, m_{-h}) \neq h$. This implies $g(m'_h, m_{-h}) = x$. Hence, $m \notin NE\left(\gamma, \succsim^R\right)$, a contradiction. Otherwise, let $C_h\left(\bar{R}_h, x\right) \neq Y$. By changing her strategy m_h into $m'_h = (R_h, R_{h+1}, x, \diamond) \in T_h^\gamma(R, F)$, the message profile (m'_h, m_{-h}) falls into *Rule 2*, so that $g(m'_h, m_{-h}) = x$. Again, we conclude that $m \notin NE\left(\gamma, \succsim^R\right)$, a contradiction. Therefore, $m_h \in T_h^\gamma(R, F)$ for all $h \in H$. Condition M_s^{**}(i) implies $x \in F(R)$.

Case 2: m falls into *Rule 2*.

Then m is m_{-i} quasi-consistent with $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y$ and $x \in F\left(\bar{R}\right)$. Thus, $g(m) = x$. We proceed according the following sub-cases: 1) $R_i^i \neq \bar{R}_i$ and $R_i^i \neq \bar{R}_{i+1}$ and 2) $R_i^i \neq \bar{R}_i$ and $R_i^i = \bar{R}_{i+1}$.[12]

Sub-case 2.1. $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i \neq \bar{R}_{i+1}$

By the definition g and $m \in NE\left(\gamma, \succsim^R\right)$, we have that $C_i\left(\bar{R}_i, x\right) \subseteq L(R_i, x)$ and $x \in \max_{R_\ell} Y$ for all $\ell \in N \backslash \{i\}$. By the definition of g, we also have that $m_h \in T_h^\gamma(R, F)$ for all $h \in H$, otherwise a contradiction can be obtained. Observe that if agent i is a partially-honest agent, it must be the case that $R_i^{i-1} \neq R_i$ or $R_{i+1}^{i+1} \neq R_{i+1}$. To show this, suppose that $R_i^{i-1} = R_i$ and $R_{i+1}^{i+1} = R_{i+1}$. Then, agent $i \in H$ can change m_i into $m'_i = (R_i, R_{i+1}, x, k^i) \in T_i^\gamma(R, F)$ and induce *Rule 1*. Then, $g(m'_i, m_{-i}) = x$ and so $((m'_i, m_{-i}), m) \in \succ_i^R$, which contradicts that $m \in NE\left(\gamma, \succsim^R\right)$. Therefore, if $m \in NE\left(\gamma, \succsim^R\right)$ falls into *Rule 2* and $i \in H$, it has to be the case that $R_i^{i-1} \neq R_i$ or $R_{i+1}^{i+1} \neq R_{i+1}$. It follows that $i-1 \notin H$ or $i+1 \notin H$ if $i \in H$.

Suppose that $\#H > 1$. Since $m_h \in T_h^\gamma(R, F)$ for each $h \in H \backslash \{i\}$, Condition M_s^{**}(ii.b) implies that $x \in F(R)$. Otherwise, let $\#H = 1$. If $H \subseteq N \backslash \{i\}$, Condition M_s^{**}(ii.b) again implies that $x \in F(R)$. Finally, suppose that $H = \{i\}$. By following the same reasoning used in *Case 2* of the proof of Theorem 2, **RD**, Condition M_s^{**}(ii.a), and Condition M_s^{**}(iv) together imply that $x \in F(R)$.

Sub-case 2.2. $R_i^i \neq \bar{R}_i$ and $R_{i+1}^i = \bar{R}_{i+1}$

Let $R_i^i = R'_i$. We distinguish whether $x \in F\left(\bar{R}'\right)$ where $\bar{R}' \equiv \left(\bar{R}_{-i}, R'_i\right)$ or not. Suppose that $x \notin F\left(\bar{R}'\right)$. Then, since $x \in F\left(\bar{R}\right)$, the same reasoning

[12] The sub-case $R_i^i = \bar{R}_i$ and $R_{i+1}^i \neq \bar{R}_{i+1}$ is not explicitly considered as it can be proved similarly to the *sub-case 2.2* shown below.

used above for *sub-case 2.1* carries over into this sub-case, so that $x \in F(R)$. Otherwise, let $x \in F(\bar{R}')$. Then, there are two potential deviators, $i-1$ and i. Agent $\ell \in N \setminus \{i-1, i\}$ can attain any $y \in Y \setminus \{x\}$ by inducing *Rule 4*, so that $x \in \max_{R_\ell} Y$ as $m \in NE(\gamma, R)$. Consider agent $i-1$. Take any $y \in C_{i-1}(\bar{R}_{i-1}, x) = C_{i-1}(R_{i-1}^{i-2}, x)$. Suppose that $C_{i-1}(\bar{R}_{i-1}, x) \neq Y$. By changing m_{i-1} to $m_{i-1}^* = (R_{i-1}^{i-1}, R_i^{i-1}, y, \diamond) \in M_{i-1}$, agent $i-1$ can obtain $y = g(m_{i-1}^*, m_{-(i-1)})$ via *Rule 3*. In the case that $C_{i-1}(\bar{R}_{i-1}, x) = Y$, by changing m_{i-1} to $m_{i-1}^* = (R_{i-1}^{i-1}, R_i^{i-1}, y, k^{i-1}) \in M_{i-1}$, agent $i-1$ can attain $y = g(m_{i-1}^*, m_{-(i-1)})$ via *Rule 4* with appropriately choosing k^{i-1}. It follows that $C_{i-1}(\bar{R}_{i-1}, x) \subseteq g(M_{i-1}, m_{-(i-1)})$; and $C_{i-1}(\bar{R}_{i-1}, x) \subseteq L(R_{i-1}, x)$ as $m \in NE(\gamma, R)$. Consider agent i. Again, take any $y \in C_i(\bar{R}_i, x) = C_i(R_i^{i-1}, x)$. Suppose that $C_i(\bar{R}_i, x) \neq Y$. By changing m_i to $m_i^* = (R_i^i, R_{i+1}^i, y, \diamond) \in M_i$, agent i can obtain $y = g(m_i^*, m_{-i})$ via *Rule 3*. In the case that $C_i(\bar{R}_i, x) = Y$, by changing m_i to $m_i^* = (R_i^i, R_{i+1}^i, y, k^i) \in M_i$, agent i can attain $y = g(m_i^*, m_{-i})$ via *Rule 4* with appropriately choosing k^i. It follows that $C_i(\bar{R}_i, x) \subseteq g(M_i, m_{-i})$; and $C_i(\bar{R}_i, x) \subseteq L(R_i, x)$ as $m \in NE(\gamma, R)$. By definition of g, we also have that $m_h \in T_h^\gamma(R, F)$ for all $h \in H$. Therefore, $x \in F(R)$ by M_s^{**}(i).

Case 3: m falls into *Rule 3*.

Then, m is m_{-i} consistent with x and $\bar{R} \in \mathcal{R}^n$, where $x \in F(\bar{R})$. Moreover, $C_i(\bar{R}_i, x) \neq Y$. By the definition g and $m \in NE(\gamma, \succcurlyeq^R)$, we have that $g(m) \in C_i(\bar{R}_i, x) \subseteq L(R_i, g(m))$ and $g(m) \in \max_{R_\ell} Y$ for all $\ell \in N \setminus \{i\}$.[13] Moreover, by the definition of g, we also have that $m_h \in T_h^\gamma(R, F)$ for all $h \in H$, otherwise a contradiction can be obtained. Suppose that $\#H > 1$. Condition M_s^{**}(ii.b) implies that $g(m) \in F(R)$. Otherwise, let $\#H = 1$. If $H \subseteq N \setminus \{i\}$, Condition M_s^{**}(ii.b) implies that $g(m) \in F(R)$. Finally, suppose that $H = \{i\}$. By following the same reasoning used in *Case 2* of Theorem 2, it follows from **RD**, Condition M_s^{**}(ii.a) and Condition M_s^{**}(iv) that $g(m) \in F(R)$.

Case 4: m falls into *Rule 4*.

Then, the outcome is determined by the winner of the modulo game. Thus, $Y \subseteq g(M_\ell, m_{-\ell})$ for all $\ell \in N$. As $m \in NE(\gamma, \succcurlyeq^R)$, $g(m) \in \max_{R_\ell} Y$ for all $\ell \in N$. It follows from the definition of g that $m_h \in T_h^\gamma(R, F)$ for all $h \in H$, otherwise a contradiction from $m \in NE(g, \succcurlyeq^R)$. Condition M_s^{**}(iii) implies that $g(m) \in F(R)$. ■

[13] A detailed and exhaustive argument is provided in Lombardi and Yoshihara (2010).

3.3 Partially-honest implementation by self-relevant mechanisms

From the viewpoint of informational decentralization in mechanisms, it is desirable that an agent discloses information related only to her own characteristics (Hurwicz, 1960, 1972). In this sub-section, we then focus on implementation by *self-relevant mechanisms*. A self-relevant mechanism is a mechanism in which each agent reveals her preference only and announces a feasible outcome and an agent index (Tatamitani, 2001).

Our first task here is to find a necessary condition for an SCC to be partially-honest implementable by self-relevant mechanisms. To this end, we introduce a condition, *Condition* λ^*, which is weaker than Tatamitani's *Condition* λ. Moreover, we introduce a slight strengthening of Condition λ^* - *Condition* λ^{**}-, which is necessary and sufficient for partially-honest implementation by self-relevant mechanisms under mild requirements on $\mathcal{R}^n$ and Γ.

Implementation by self-relevant mechanisms in the standard framework is given by Tatamitani (2001) as follows.

Definition 5. A mechanism $\gamma \equiv (M, g)$ is a self-relevant mechanism if, for any $\ell \in N$, $M_\ell \equiv \mathcal{R}_\ell \times Y \times N$.

Note that, if γ is a self-relevant mechanism, then $T_\ell^\gamma (R, F) \equiv \{R_\ell\} \times Y \times N$ for any $(R, F) \in \mathcal{R}^n \times \mathcal{F}$.

Definition 6. An SCC F is partially-honest implementable by a self-relevant mechanism if there exists a self-relevant mechanism $\gamma \equiv (M, g)$ such that:
(i) for all $R \in \mathcal{R}^n$, $F(R) = NA\left(\gamma, \succsim^R\right)$; and
(ii) for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, if $m_\ell = \left(R_\ell, x, k^\ell\right) \in M_\ell$ for all $\ell \in N$, then $m \in NE\left(\gamma, \succsim^R\right)$ and $g(m) = x$.

Tatamitani (2001) shows that Condition λ defined below is necessary and sufficient for implementation by self-relevant mechanisms. Before stating it, we need additional notation. For any $\ell \in N$, any $R_{-\ell} \in \mathcal{R}^{n-1}$, and $x \in X$, let $F_\ell^{-1}(R_{-\ell}, x) \equiv \{R'_\ell \in \mathcal{R}_\ell | x \in F(R'_\ell, R_{-\ell})\}$, and $\Lambda_\ell^F(R_{-\ell}, x) \equiv \cap_{R_\ell \in F_\ell^{-1}(R_{-\ell}, x)} L(R_\ell, x)$. Given $(R, x) \in \mathcal{R}^n \times X$, define $D(R, x) \equiv \left\{\ell \in N | F_\ell^{-1}(R_{-\ell}, x) \neq \varnothing\right\}$. Now, Condition λ can be stated as follows.

Condition λ (for short, λ): There is a set $Y \subseteq X$ and, for all $(R, x) \in \mathcal{R}^n \times X$ with $D(R, x) \neq \varnothing$, there is a profile of sets $(C_\ell(R_{-\ell}, x))_{\ell \in N}$ such

that $x \in C_\ell(R_{-\ell}, x) \subseteq \Lambda_\ell^F(R_{-\ell}, x) \cap Y$ for all $\ell \in D(R, x)$; finally, for all $R^* \in \mathcal{R}^n$, the following (i)-(iv) are satisfied:
(i) if $x \in F(R)$ and $C_\ell(R_{-\ell}, x) \subseteq L(R_\ell^*, x)$ for all $\ell \in N$, then $x \in F(R^*)$;
(ii) for all $i \in D(R, x)$, if $y \in C_i(R_{-i}, x) \subseteq L(R_i^*, y)$ and $y \in \max_{R_\ell^*} Y$ for all $\ell \in N \backslash \{i\}$, then $y \in F(R^*)$;
(iii) if $y \in \max_{R_\ell^*} Y$ for all $\ell \in N$, then $y \in F(R^*)$;
(iv) there exists an outcome $p(R, x) \in X$ such that:
(a) $p(R, x) \in C_\ell(R_{-\ell}, x)$ for all $\ell \in D(R, x)$;
(b) if $C_i(R_{-i}, x) \subseteq L(R_i^*, p(R, x))$ for all $i \in D(R, x)$ and $p(R, x) \in \max_{R_\ell^*} Y$ for all $\ell \in N \backslash D(R, x)$, then $p(R, x) \in F(R^*)$.

Condition λ is markedly stronger than Condition μ. Notable parts of Condition λ are Condition λ(i) and Condition λ(iv). Condition λ(i) is much stronger than (Maskin) monotonicity.[14]

We introduce Condition λ^* below -which is a weakening of Condition λ- and show that it is necessary for partially-honest implementation by self-relevant mechanisms.

Condition λ^* (for short, λ^*): There is a set $Y \subseteq X$ and, for all $(R, x) \in \mathcal{R}^n \times X$ with $D(R, x) \neq \varnothing$, there is a profile of sets $(C_\ell(R_{-\ell}, x))_{\ell \in N}$ such that $x \in C_\ell(R_{-\ell}, x) \subseteq \Lambda_\ell^F(R_{-\ell}, x) \cap Y$ for all $\ell \in D(R, x)$; finally, for all $R^* \in \mathcal{R}^n$, the following (i)-(iv) are satisfied:
(i) if $x \in F(R)$, $C_\ell(R_{-\ell}, x) \subseteq L(R_\ell^*, x)$ for all $\ell \in N$, and $x \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that, for some $H' \subseteq H$ and for all $h \in H'$, $R_h \neq R_h^*$;
(ii) for all $i \in D(R, x)$, if $y \in C_i(R_{-i}, x) \subseteq L(R_i^*, y)$, $y \in \max_{R_\ell^*} Y$ for all $\ell \in N \backslash \{i\}$, and $y \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that for some $H' \subseteq H$:
(a) if $H' = \{i\}$, then $(y, y') \in I_i^*$ for some $y' \in C_i(R_{-i}, x) \setminus \{y\}$;
(b) otherwise, $R_h \neq R_h^*$ for all $h \in H' \setminus \{i\}$;
(iii) if $y \in \max_{R_\ell^*} Y$ for all $\ell \in N$ and $y \notin F(R^*)$, then there is an $\ell \in N$ such that $(y, y') \in I_\ell^*$ for some $y' \in Y \setminus \{y\}$;
(iv) there exists an outcome $p(R, x) \in X$ such that:
(a) $p(R, x) \in C_\ell(R_{-\ell}, x)$ for all $\ell \in D(R, x)$;
(b) if $C_i(R_{-i}, x) \subseteq L(R_i^*, p(R, x))$ for all $i \in D(R, x) \neq \varnothing$, $p(R, x) \in \max_{R_\ell^*} Y$ for all $\ell \in N \backslash D(R, x)$, and $p(R, x) \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that $R_h \neq R_h^*$ for some $h \in H$.

[14] For a detailed analysis on how restrictive Condition λ(i) is, see Tatamitani (2002).

Theorem 5. *Let Assumption 1 hold. If an* SCC F on $\mathcal{R}^n$ *is* partially-honest implementable by self-relevant mechanisms, *then it satisfies* Condition λ^*.

Proof. Let Assumption 1 hold. Let $\gamma \equiv (M, g)$ be a self-relevant mechanism which partially-honest implements F. Let $Y \equiv g(M)$. Take any $(R, x) \in \mathcal{R}^n \times X$ with $D(R, x) \neq \varnothing$. Denote by $h \in N$ a partially-honest agent. For any $i \in D(R, x)$, let $C_i(R_{-i}, x) \equiv g(M_i, m_{-i})$, where m_{-i} is such that $m_\ell = (R_\ell, x, \diamond) \in M_\ell$ for all $\ell \in N \backslash \{i\}$. Therefore, $C_i(R_{-i}, x) \subseteq Y$. Next, we show that $x \in C_i(R_{-i}, x) \subseteq \Lambda_i^F(R_{-i}, x)$. To see this, take any $R_i' \in F_i^{-1}(R_{-i}, x)$ and let $m_i' = (R_i', x, \diamond) \in M_i$. By forthrightness, $(m_i', m_{-i}) \in NE\left(\gamma, \succcurlyeq^{(R_i', R_{-i})}\right)$ and $g(m_i', m_{-i}) = x$. So, $x \in g(M_i, m_{-i}) \equiv C_i(R_{-i}, x) \subseteq L(R_i', x)$. Since it holds for any $R_i' \in F_i^{-1}(R_{-i}, x)$, we have that $x \in C_i(R_{-i}, x) \subseteq \Lambda_i^F(R_{-i}, x)$. Note that the proof that F meets Conditions λ^*(i)-λ^*(iii) can be obtained by following the reasoning used in Theorem 3, so we omit them here. Finally, we show that F satisfies Condition λ^*(iv).

Take any $(R, x) \in \mathcal{R}^n \times X$ with $D(R, x) \neq \varnothing$. Let $m \in M$ be such that $m_\ell = (R_\ell, x, \diamond) \in M_\ell$ for all $\ell \in N$. Let $g(m) \equiv p(R, x)$. Then, $p(R, x) \in g(M_\ell, m_{-\ell}) = C_\ell(R_{-\ell}, x) \subseteq Y$ for all $\ell \in D(R, x)$. Furthermore, suppose that $C_i(R_{-i}, x) \subseteq L(R_i^*, p(R, x))$ for all $i \in D(R, x)$, $p(R, x) \in \max_{R_\ell^*} Y$ for all $\ell \in N \backslash D(R, x)$, and $p(R, x) \notin F(R^*)$. This implies that $m \notin NE\left(\gamma, \succcurlyeq^{R^*}\right)$. Then, there is an $H \in \mathcal{H}$ such that $m_h \notin T_h^\gamma(R^*, F) = \{R_h^*\} \times Y \times N$ for some $h \in H$. Since $m_h = (R_h, x, \diamond)$, by definition, $R_h \neq R_h^*$ holds. Thus, F satisfies Condition λ^*(iv). ■

Condition λ^* and Condition λ are very close to each other; indeed, Condition λ^* incorporates not only a monotonicity-type condition but also a punishment-type condition, λ^*(iv). It follows from Theorem 5 that the class of partially-honest implementable SCCs by self-relevant mechanisms is further dwindled down with respect to the class of partially-honest implementable SCCs by s-mechanisms - Theorem 4.

To fully identifies the class of partially-honest implementable SCCs by self-relevant mechanisms, let us introduce a slight strengthening of Condition λ^* - Condition λ^{**}- which can be stated as follows.

Condition λ^{**} (for short, λ^{**}): There is a set $Y \subseteq X$ and, for all $(R, x) \in \mathcal{R}^n \times X$ with $D(R, x) \neq \varnothing$, there is a profile of sets $(C_\ell(R_{-\ell}, x))_{\ell \in N}$ such that $x \in C_\ell(R_{-\ell}, x) \subseteq \Lambda_\ell^F(R_{-\ell}, x) \cap Y$ for all $\ell \in D(R, x)$; finally, for all $R^* \in \mathcal{R}^n$, the following (i)-(v) are satisfied:

(i) if $x \in F(R)$, $C_\ell(R_{-\ell}, x) \subseteq L(R^*_\ell, x)$ for all $\ell \in N$, and $x \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that, for some $H' \subseteq H$ and for all $h \in H'$, $R_h \neq R^*_h$;
(ii) for all $i \in D(R, x)$, if $y \in C_i(R_{-i}, x) \subseteq L(R^*_i, y)$, $y \in \max_{R^*_\ell} Y$ for all $\ell \in N \backslash \{i\}$, and $y \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that for some $H' \subseteq H$:
(a) if $H' = \{i\}$, then $(y, y') \in I^*_i$ for some $y' \in C_i(R_{-i}, x) \setminus \{y\}$;
(b) otherwise, $R_h \neq R^*_h$ for all $h \in H' \setminus \{i\}$;
(iii) if $y \in \max_{R^*_\ell} Y$ for all $\ell \in N$, then $y \in F(R^*)$;
(iv) there exists an outcome $p(R, x) \in X$ such that:
(a) $p(R, x) \in C_\ell(R_{-\ell}, x)$ for all $\ell \in D(R, x)$; and
(b) if $C_i(R_{-i}, x) \subseteq L(R^*_i, p(R, x))$ for all $i \in D(R, x) \neq \varnothing$, $p(R, x) \in \max_{R^*_\ell} Y$ for all $\ell \in N \backslash D(R, x)$, and $p(R, x) \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that $R_h \neq R^*_h$ for some $h \in H$;
(v) for all $i \in N$, if $x \in F(R)$, $L(R^*_i, x) = L(R_i, x)$ and $x \in \max_{R^*_\ell} Y$ for all $\ell \in N \setminus \{i\}$, $R^*_{-i} = R_{-i}$, and $x \notin F(R^*)$, then there is an $H \in \mathcal{H}$ such that $H \neq \{i\}$.

We are ready to show that this new condition is necessary and sufficient for partially-honest implementation by self-relevant mechanisms under the same mild requirements on $\mathcal{R}^n$ and Γ used in Theorem 2.

Theorem 6. *Let Assumption 1 and $\Gamma = \Gamma_{SP}$ hold, and let $\mathcal{R}^n$ satisfy* **RD**. *An* SCC F on $\mathcal{R}^n$ *is* partially-honest implementable by a self-relevant mechanism *if and only if it satisfies* Condition λ^{**}.

Proof. Let Assumption 1 hold and let $\mathcal{R}^n$ satisfy **RD**. Denote by $h \in N$ a partially-honest agent.

1. The necessity of Condition λ^{**}.

Suppose that F is partially-honest implementable by a self-relevant mechanism $\gamma = (M, g) \in \Gamma_{SP}$. It is clear that F satisfies Condition λ^*. Further, as in the proof of Theorem 2, we can see that F satisfies Condition λ^{**}(iii) and Condition λ^{**}(v). Thus, F satisfies Condition λ^{**}.

2. The sufficiency of Condition λ^{**}.

Conversely, let F on $\mathcal{R}^n$ satisfy Condition λ^{**}, and let $\gamma \equiv (M, g)$ be a self-relevant mechanism. Let $m_\ell = \left(R^\ell, x^\ell, k^\ell\right) \in \mathcal{R}_\ell \times Y \times N$ denote a generic message of agent $\ell \in N$, where R^ℓ_ℓ is the preference profile announced by agent $\ell \in N$, while x^ℓ and k^ℓ are the outcome and the integer announced by the agent at issue, respectively. The proof of the statement can be obtained by using the mechanism devised by Tatamitani (2001). We report it only for completeness.

Define the outcome function $g : M \to X$ as follows. For all $m \in M$ and $\left(\bar{R}, x\right) \in \mathcal{R}^n \times Y$,

Rule 1: If $\left(R_\ell^\ell, x^\ell\right) = \left(\bar{R}_\ell, x\right)$ for all $\ell \in N$ with $x \in F\left(\bar{R}\right)$, then $g\left(m\right) = x$.
Rule 2: If there exists $i \in N$ such that $\left(R_\ell^\ell, x^\ell\right) = \left(\bar{R}_\ell, x\right) \neq \left(R_i^i, x^i\right)$ for all $\ell \in N \backslash \{i\}$ with $x^i \neq x \in F\left(\bar{R}\right)$, $i \in D\left(\left(R_i^i, \bar{R}_{-i}\right), x\right)$, and $C_i\left(\bar{R}_{-i}, x\right) \neq Y$, then

$$g\left(m\right) = \begin{cases} x^i & \text{if} \quad x^i \in C_i\left(\bar{R}_{-i}, x\right) \\ x & \qquad \text{otherwise} \end{cases} .$$

Rule 3: If $\left(R_\ell^\ell, x^\ell\right) = \left(\bar{R}_\ell, x\right)$ for all $\ell \in N$, $x \notin F\left(\bar{R}\right)$, and $D\left(\bar{R}, x\right) \neq \varnothing$, then $g\left(m\right) = p\left(\bar{R}, x\right)$.
Rule 4: Otherwise, $g\left(m\right) = x^{\ell^*(m)}$ where $\ell^*\left(m\right) \equiv \sum_{i \in N} k^i \left(\text{mod } n\right)$.[15]

By the definition of g, it follows that $\gamma = (M, g) \in \Gamma_{SP}$. Moreover, for each $\ell \in N$, the truth-telling correspondence $T_\ell^\gamma : \mathcal{R}^n \times \mathcal{F} \twoheadrightarrow M_\ell$ is given by: $T_\ell^\gamma\left(R, F\right) = \{R_\ell\} \times Y \times N$ for each $(R, F) \in \mathcal{R}^n \times \mathcal{F}$, where Y may change according to F. We show that γ partially-honest implements F. Take any $R \in \mathcal{R}^n$.

Since F satisfies λ^{**}, it follows that, for all $R \in \mathcal{R}^n$ and all $x \in F\left(R\right)$, $x \in Y$. The proof that $F\left(R\right) \subseteq NA\left(\gamma, \succcurlyeq^R\right)$ follows Tatamitani (2001)'s argument, so we omit it here. Conversely, to show that $NA\left(\gamma, \succcurlyeq^R\right) \subseteq F\left(R\right)$, let $m \in NE\left(\gamma, \succcurlyeq^R\right)$. Consider the following cases.

Case 1: m falls into *Rule 1*.

Then, m is such that, for all $\ell \in N$, $m_\ell = \left(\bar{R}_\ell, x, \diamond\right)$ and $x \in F\left(\bar{R}\right)$. By definition of g and the assumption that $m \in NE\left(\gamma, \succcurlyeq^R\right)$, we have that $C_\ell\left(R_{-\ell}, x\right) \subseteq L\left(R_\ell, x\right)$ for all $\ell \in N$. Suppose $m_h \notin T_h^\gamma\left(R, F\right)$ for some $h \in H$. Suppose that $C_h\left(\bar{R}_{-h}, x\right) = Y$. By changing her strategy m_h to $m_h' = \left(R_h, x^h, k^h\right) \in T_h^\gamma\left(R, F\right)$ with $x^h \in Y \backslash \{x\}$, agent h induces *Rule 4* and can obtain $\ell = \ell^*\left(m_{-h}, m_h'\right) \neq h$. Then, $g\left(m_h', m_{-h}\right) = x$. Hence, $\left(\left(m_{-h}, m_h'\right), m\right) \in \succ_h^R$, a contradiction. Otherwise, let $C_h\left(\bar{R}_{-h}, x\right) \neq Y$. By changing her strategy m_h to $m_h' = \left(R_h, x^h, k^k\right) \in T_h^\gamma\left(R, F\right)$ with $x^h \in Y \backslash C_h\left(\bar{R}_{-h}, x\right)$, $\left(m_h', m_{-h}\right)$ falls into *Rule 2* as $h \in D\left(\bar{R}_{-h}, R_h\right)$. Then, $g\left(m_h', m_{-h}\right) = x$. Again, $\left(\left(m_h', m_{-h}\right), m\right) \in \succ_h^R$, a contradiction. We conclude that $m_h \in T_h^\gamma\left(R, F\right)$ for all $h \in H$. Condition λ^*(i) implies $x \in F\left(R\right)$.

[15]If the remainder is *zero*, the winner of the game is agent n.

Case 2: m falls into *Rule 2*.

Then, m is such that $m_\ell = \left(\bar{R}_\ell, x, \diamond\right)$ for any $\ell \in N \backslash \{i\}$ and $m_i = (R_i^i, x^i, \diamond)$, with $x^i \neq x$, $i \in D\left(\left(\bar{R}_{-i}, R_i^i\right), x\right)$, and $C_i\left(\bar{R}_{-i}, x\right) \neq Y$. By the definition of g, we have that $C_i\left(\bar{R}_{-i}, x\right) \subseteq g\left(M_i, m_{-i}\right)$.

Next, we claim that $g\left(M_\ell, m_{-\ell}\right) = Y$ for all $\ell \in N \backslash \{i\}$. We proceed according to whether $\#Y = 2$ and $n = 3$ or not.

Sub-case 2.1. not[$\#Y = 2$ and $n = 3$]

Suppose that $\#Y > 2$. Take any $\ell \in N \backslash \{i\}$. Then, agent ℓ can induce the modulo game by choosing any $y \in Y \backslash \{x, x^i\}$ and changing m_ℓ into $m_\ell^* = \left(\bar{R}_\ell, y, k^\ell\right)$. To attain y, agent ℓ has only to adjust k^ℓ by which $\ell^*\left(m_\ell^*, m_{-\ell}\right) = \ell$. To attain x (resp., x^i), agent ℓ has only to adjust k^ℓ by which $\ell^*\left(m_\ell^*, m_{-\ell}\right) = j$ for $j \in N \backslash \{\ell, i\}$ (*resp.*, $\ell^*\left(m_\ell^*, m_{-\ell}\right) = i$). Therefore, $Y \subseteq g\left(M_\ell, m_{-\ell}\right)$ for any $\ell \in N \backslash \{i\}$. Otherwise, let $\#Y = 2$. Then, $n > 3$. Take any $\ell \in N \backslash \{i\}$. Choosing $x^\ell = x^i$, agent ℓ can make $\#\left\{\ell \in N | x^\ell = x\right\} \geq 2$ and $\#\left\{\ell \in N | x^\ell \neq x\right\} \geq 2$. As the outcome is determined by *Rule 4*, agent ℓ can attain any outcome in Y by appropriately choosing k^ℓ. Therefore, $Y \subseteq g\left(M_\ell, m_{-\ell}\right)$ for any $\ell \in N \backslash \{i\}$.

Sub-case 2.2. $\#Y = 2$ and $n = 3$

Then, $Y = \{x, x^i\}$ and $N = \{i, \ell, \ell'\}$. Moreover, $g(m) = x$. Agent ℓ can change her strategy m_ℓ to $m_\ell^* = \left(\bar{R}_\ell, x^i, k^\ell\right)$. If $\ell' \notin D\left(\bar{R}, x^i\right)$, then the outcome is determined by *Rule 4*. Therefore, agent ℓ can attain $x^i \in Y$ by appropriately choosing the integer index k^ℓ. Otherwise, let $\ell' \in D\left(\bar{R}, x^i\right)$. Suppose that $C_{\ell'}\left(\bar{R}_{-\ell'}, x^i\right) \neq Y$. As $\left(m_\ell^*, m_{-\ell}\right)$ falls into *Rule 2*, it follows that $g\left(m_i^*, m_{-i}\right) = x^i$, as sought. Otherwise, agent ℓ can attain x^i by appropriately choosing the integer index k^ℓ as the outcome is determined by *Rule 4*.

As $m \in NE\left(\gamma, \succcurlyeq^R\right)$, we have that $C_i\left(\bar{R}_{-i}, x\right) \subseteq L\left(R_i, g(m)\right)$ and $g(m) \in \max_{R_\ell} Y$ for all $\ell \in N \backslash \{i\}$. Moreover, by the definition of g, we have that $m_h \in T_h^\gamma(R, F)$, otherwise a contradiction that $m \in NE\left(g, \succcurlyeq^R\right)$ can be obtained (see Annex). Suppose that $\#H > 1$, Condition λ^*(ii.b) implies that $g(m) \in F(R)$. Otherwise, let $\#H = 1$. Suppose that $H \subseteq N \backslash \{i\}$. Again, Condition λ^*(ii.b) implies that $g(m) \in F(R)$. Finally, let $H = \{i\}$. By following the same reasoning used in *Case 2* of Theorem 2, it follows from **RD**, Condition λ^*(ii.a), and Condition λ^*(v) that $g(m) \in F(R)$.

Case 3: m falls into *Rule 3*.

Then, m is such that $m_\ell = \left(R_\ell^\ell, x^\ell, \diamond\right) = \left(\bar{R}_\ell, x, \diamond\right) \in M_\ell$, $x \notin F\left(\bar{R}\right)$, and $D\left(\bar{R}, x\right) \neq \varnothing$. By *Rule 3*, $g(m) = p\left(\bar{R}, x\right)$.

Take any $i \in D\left(\bar{R}, x\right)$. We show that $C_i\left(\bar{R}_{-i}, x\right) \subseteq g\left(M_i, m_{-i}\right)$. As $i \in D\left(\bar{R}, x\right)$, there exists $R_i' \in \mathcal{R}_i$ such that $x \in F\left(R_i', \bar{R}_{-i}\right)$. By changing m_i to $m_i' = \left(R_i', x, \diamond\right)$, agent i can induce *Rule 1* and obtain $g\left(m_i', m_{-i}\right) = x$. Take any $x^i \in C_i\left(\bar{R}_{-i}, x\right) \setminus \{x\}$. Suppose $C_i\left(\bar{R}_{-i}, x\right) \neq Y$. By changing m_i to $m_i' = \left(\bar{R}_i, x^i, k^i\right)$, agent i can induce *Rule 2* and obtain $g\left(m_i', m_{-i}\right) = x^i$. Suppose $C_i\left(\bar{R}_{-i}, x\right) = Y$, then the modulo game is triggered and agent i can attain x^i by choosing k^i appropriately.

Take any $\ell \in N \backslash D\left(\bar{R}, x\right)$. We show that $Y \subseteq g\left(M_\ell, m_{-\ell}\right)$. Then, agent ℓ can induce the modulo game by choosing any $x^\ell \in Y \backslash \{x\}$ and changing m_ℓ into $m_\ell' = \left(\bar{R}_\ell, x^\ell, k^\ell\right)$. To attain x and x^ℓ, agent ℓ has only to adjust k^ℓ by which $\ell^*\left(m_\ell', m_{-\ell}\right) = i$ for $i \in N \backslash \{\ell\}$ and $\ell^*\left(m_\ell^*, m_{-\ell}\right) = \ell$, respectively.

Therefore, we obtained that $C_i\left(\bar{R}_{-i}, x\right) \subseteq g\left(M_i, m_{-i}\right)$ for any $i \in D\left(\bar{R}, x\right)$ and $Y \subseteq g\left(M_\ell, m_{-\ell}\right)$ for any $\ell \in N \backslash D\left(\bar{R}, x\right)$. As $m \in NE\left(\gamma, \succcurlyeq^R\right)$, it follows that $C_i\left(\bar{R}_{-i}, x\right) \subseteq L\left(R_i, p\left(\bar{R}, x\right)\right)$ for any $i \in D\left(\bar{R}, x\right)$ and $p\left(\bar{R}, x\right) \in \max_{R_\ell} Y$ for all $\ell \in N \backslash D\left(\bar{R}, x\right)$. Moreover, by the definition of g, we have that $m_h \in T_h^\gamma\left(R, F\right)$ for any $h \in H$, otherwise a contradiction that $m \in NE\left(g, \succcurlyeq^R\right)$ can be obtained (see Annex). Condition λ^*(iv) implies that $p\left(\bar{R}, x\right) \in F\left(R\right)$.

Case 4: m falls into *Rule 4*.

Then, $g\left(m\right) = x^{\ell^*(m)}$ where $\ell^*\left(m\right) \in N$ is the winner of the modulo game. We show that $Y \subseteq g\left(M_\ell, m_{-\ell}\right)$ for any $\ell \in N$. Take any $i \in N$ and consider the following two sub-cases. Let $\left(R_\ell^\ell\right)_{\ell \in N} \equiv \bar{R}$.

Sub-case 4.1: For all $\ell, \ell' \in N \backslash \{i\}$, $x^\ell = x^{\ell'}$.

Suppose that $x^\ell = x$ for all $\ell \in N$. As m falls into *Rule 4*, it follows that $x \notin F\left(\bar{R}\right)$ and $D\left(\bar{R}, x\right) = \varnothing$, so that $i \notin D\left(\bar{R}, x\right)$. By changing m_i to $m_i' = \left(\bar{R}_i, x^i, k^i\right)$ with $x^i \in Y$, agent i can trigger the modulo game and obtain $g\left(m_i', m_{-i}\right)$ by choosing k^i appropriately. Therefore, $Y \subseteq g\left(M_i, m_{-i}\right)$. On the other hand, suppose that $x^\ell = x$ for all $\ell \in N \backslash \{i\}$ and $x^i \neq x$. Take any $\hat{x}^i \in Y \backslash \{x\}$. Since $g\left(m\right) = x^{\ell^*(m)}$ where $\ell^*\left(m\right) \in N$, it follows that either $i \notin D\left(\bar{R}, x\right)$ or $i \in D\left(\bar{R}, x\right)$ and $C_i\left(\bar{R}_{-i}, x\right) = Y$. Therefore, by deviating from m_i to $m_i' = \left(\bar{R}_i, \hat{x}^i, k^i\right)$, agent i can trigger the modulo game. To attain x and $\hat{x}^i$, agent i has only to adjust k^i so that $\ell^*\left(\left(m_i', m_{-i}\right)\right) \in N \backslash \{i\}$ and $\ell^*\left(\left(m_i', m_{-i}\right)\right) = i$, respectively. Again, we have that $Y \subseteq g\left(M_i, m_{-i}\right)$.

Sub-case 4.2: For some $\ell, \ell' \in N \backslash \{i\}$, $x^\ell \neq x^{\ell'}$.

Suppose that $\#Y = 2$, so that $Y = \left\{x^\ell, x^{\ell'}\right\}$. By changing m_i to $m_i' = \left(\bar{R}_i, x^i, k^i\right)$, agent i induces the modulo game. To attain x^ℓ and $x^{\ell'}$, agent i has only to adjust the integer index k^i so that $\ell^*\left(\left(m_i', m_{-i}\right)\right) = \ell$ and

$\ell^*((m'_i, m_{-i})) = \ell'$, respectively. Otherwise, let $\#Y > 2$. Take any $x^i \in Y \setminus \{x^\ell, x^{\ell'}\}$. By deviating from m_i to $m'_i = (\bar{R}_i, x^i, k^i)$, agent i can trigger the modulo game. To attain x^ℓ, $x^{\ell'}$, and x^i, agent i has only to adjust k^i so that $\ell^*((m'_i, m_{-i})) = \ell$, $\ell^*((m'_i, m_{-i})) = \ell'$, and $\ell^*((m'_i, m_{-i})) = i$, respectively. Again, we have that $Y \subseteq g(M_i, m_{-i})$.

Since $Y \subseteq g(M_i, m_{-i})$ for any $i \in N$ and $m \in NE(\gamma, \succcurlyeq^R)$, we have that $g(m) \in \max_{R_\ell} Y$ for all $\ell \in N$. Moreover, by the definition of g, it is clear that $m_h \in T_h^\gamma(R, F)$ for all $h \in H$. Condition λ^{**}(iii) implies that $g(m) \in F(R)$. ■

4 Implications

In this section, we briefly discuss the implications of the results reported in section 3.

In the many-person case, the only requirement of *no-veto power* is sufficient for partially-honest implementation; that is, the *Dutta-Sen Theorem*. This requirement is not necessary for implementation but only sufficient. Moreover, there are many interesting SCCs which fail to satisfy it. Last but not least, though the no-veto power condition is a weak requirement in many environments, it is by no means universally acceptable (Benoît and Ok, 2006, 2008). Theorem 2 avoids these problems by providing a necessary and sufficient condition -Condition μ^{**}- for the partially-honest implementation. In sub-section 4.1, first, we show that even the weak Condition μ^* imposes non-trivial restrictions on F; second, we show that the class of partially-honest implementable SCCs becomes wider with respect to the Dutta-Sen Theorem by discussing the implementability of an SCC which fails to satisfy Conditions μ(i) and μ(ii), but is partially-honest implementable by virtue of Theorem 2.

Theorem 4 provides a necessary and sufficient condition - Condition M_s^{**}- for the partially-honest implementation by s-mechanisms. Even though Condition M_s^{**} is weaker than Condition M_s, it imposes non-trivial restrictions on F. Moreover, it incorporates a Maskin monotonicity-type condition. In sub-section 4.2, we analyze how the monotonicity-type condition incorporated in Condition M_s^{**} affects the ability to partially-honest implement. The analysis reveals that this condition is restrictive, though it is weaker than (Maskin) monotonicity. This result has at least two immediate consequences. First, there is a trade-off between what the planner can achieve when there are partially-honest agents in the society and the strengthening of informational decentralization in mechanisms. Second, this conflict breaks down the equivalent relationship between implementation and implementation by s-mechanisms holding in the classical implementation framework (Lombardi and Yoshihara, 2010).

4.1 Impossibility and possibility of partially-honest Implementation

Despite Condition μ^* being a very weak requirement, we first show that even this condition imposes non-trivial restrictions on the class of partially-honest implementable SCCs. For example, the *Pareto SCC* is not partially-honest implementable, as we argue next.

An SCC F^{PO} on $\mathcal{R}^n$ is *Pareto* if, for all $R \in \mathcal{R}^n$,

$$F^{PO}(R) = PO(R) \equiv \{x \in X | \nexists y \in X\colon \ (y,x) \in R_i \ \ (\forall i \in N) \ \ \& \ \ (y,x) \in P_i \ \ (\exists i \in N)\}.$$

Proposition 1. *F^{PO} on $\mathcal{R}^n$ is not* partially-honest implementable.

Proof. Assume, to the contrary, that F^{PO} satisfies Condition μ^{**}. Let $N = \{1,2,3\}$ with $\#N = 3$, $X = \{x,y,z\}$ with $\#X = 3$, and $\mathcal{R}^3 = \{R, R^*\}$, where agents' preferences are as follows:

R			R^*		
1	2	3	1	2	3
x	y	z	x	x,y	x,y
y	z	x	y	z	z
z	x	y	z		

where, as usual, $\begin{smallmatrix} x \\ y \end{smallmatrix}$ means that the agent in question strictly prefers x to y, while x, y means that the agent at issue is indifferent between x and y.

Since $y \in PO(R)$, there exists a profile $(C_\ell(R,y))_{\ell \in N}$ such that $y \in C_\ell(R,y) \subseteq L(R_\ell, y) \cap Y$ for all $\ell \in N$. Since $PO(R) = X$, it follows that $Y = X$. Notice that Condition μ^*(ii.a) is vacuously satisfied if $H = \{i\} \subseteq \{2,3\}$. Then, let $H = \{1\}$. Observe $y \in \max_{R^*_\ell} X$ for all $\ell \in \{2,3\}$ and $y \in C_1(R,y) \subseteq L(R_1,y) = L(R^*_1, y)$. Condition μ^*(ii.a) implies that $y \in F^{PO}(R^*) \neq PO(R^*) = \{x\}$, a contradiction. ■

The Dutta-Sen Theorem is silent with respect to the partially-honest implementability of SCCs which violate *no-veto power*. In what follows, we discuss that there is an SCC which fails to satisfy Conditions μ(i) and μ(ii), but is partially-honest implementable by virtue of Theorem 2. Since Condition μ(ii) is weaker than no-veto power, this indicates that the class of partially-honest implementable SCCs is wider with respect to the Dutta-Sen Theorem.

Additional notation is needed. An amount $M \in \mathbb{R}_{++}$ of some infinitely divisible commodity has to be allocated among a set of agents N, with $n \geq 3$. An allocation is a list $x \in \mathbb{R}^n_+$ such that $\sum x_\ell = M$.[16] Let $X \equiv \left\{x \in \mathbb{R}^n_+ | \sum x_\ell = M\right\}$ be the set of feasible allocations. Each agent is equipped with a continuous and single-plateaued preference relation R_ℓ defined on X as follows: there exists a continuous and quasi-concave real-valued function $u_{R_\ell} : [0, M] \to \mathbb{R}$ such that for any $x, x' \in X$, $u_{R_\ell}(x_\ell) \geq u_{R_\ell}(x'_\ell) \Leftrightarrow (x, x') \in R_\ell$. For $\ell \in N$, the preference relation R_ℓ defined on X is called *single-plateaued* when there exist two numbers $\bar{x}_\ell, \underline{x}_\ell \in [0, M]$ such that $\underline{x}_\ell \leq \bar{x}_\ell$ and for all $x_\ell, y_\ell \in [0, M]$: (i) if $x_\ell < y_\ell \leq \underline{x}_\ell$ or $x_\ell > y_\ell \geq \bar{x}_\ell$, then for any $x', y' \in X$ with $x'_\ell = x_\ell$ and $y'_\ell = y_\ell$, $(y', x') \in P_\ell$; (ii) if $x_\ell, y_\ell \in [\underline{x}_\ell, \bar{x}_\ell]$, then for any $x', y' \in X$ with $x'_\ell = x_\ell$ and $y'_\ell = y_\ell$, $(x', y') \in I_\ell$. The interval $p(R_\ell) \equiv [\underline{x}_\ell, \bar{x}_\ell]$ is the *plateau* of R_ℓ, $\underline{x}$ is the left end-point of the plateau of R_ℓ, and $\bar{x}$ is the right end-point. Let $\bar{\mathcal{R}}_\ell$ be the class of all such preference relations for agent ℓ. Note that by definition of $R_\ell \in \bar{\mathcal{R}}_\ell$, it follows that R_ℓ is single-peaked if $\underline{x}_\ell = \bar{x}_\ell$.

Given $x_\ell \in [0, M]$, let $r_\ell(x_\ell)$ be the consumption bundle on the other side of agent ℓ's plateau amounts that she finds indifferent to x_ℓ if such consumption exists, and the end-point of $[0, M]$ on the other side of her plateau amounts otherwise. Given a profile of preferences $R \in \bar{\mathcal{R}}^n$, $p(R) \equiv (p(R_1), ..., p(R_n))$ denotes its associated profile of plateau amounts.

Let F be defined on $\bar{\mathcal{R}}^n$ such that $\varnothing \neq F(R) \subseteq X$ for all $R \in \bar{\mathcal{R}}^n$.

Proposition 2. *Let F^{PO} on $\bar{\mathcal{R}}^n$ be* Pareto. *Then, (i) F^{PO} satisfies neither of* Conditions μ(i) and μ(ii)*; (ii) F^{PO} satisfies* Condition μ^{**}.

Proof. Let F^{PO} on $\bar{\mathcal{R}}^n$ be *Pareto*, that is, $F^{PO}(R) = PO(R)$ for all $R \in \bar{\mathcal{R}}^n$.

We illustrate part (i) by considering the following the three-agent example.[17]

Let $M = 1$, $N \equiv \{1, 2, 3\}$, with $\#N = 3$, and $R, R^* \in \bar{\mathcal{R}}^n$ be such

[16] When its bounds are not explicitly indicated, a summation should be understood to cover all agents.

[17] The *Pareto SCC* is monotonic and satisfies no-veto power when $\bar{\mathcal{R}}^n$ consists only of single-peaked preference profiles.

that $R_1 = R_1^*$, $p(R) = \left(\frac{1}{4}, 1, [0,1]\right)$, and $p(R^*) = \left(\frac{1}{4}, \left[\frac{1}{2}, 1\right], [0,1]\right)$. Let $x = \left(\frac{1}{6}, \frac{5}{6}, 0\right)$ and $y = \left(\frac{1}{5}, \frac{4}{5}, 0\right)$. First note that $x, y \in X$, $x \in PO(R)$, and $L(R_1, x) = L(R_1^*, x) = \left\{z \in X \mid 0 \leq z_1 \leq \frac{1}{6} \text{ or } r_1(x_1) \leq z_1 \leq 1\right\}$, $L(R_2, x) = \left\{z \in X \mid 0 \leq z_2 \leq \frac{5}{6}\right\}$ and $L(R_3, x) = L(R_3^*, x) = X$. Moreover, note that $y \notin L(R_1, x)$ while $y \in L(R_2, x)$; and $L(R_2^*, x) = X$. Suppose that PO satisfies Conditions μ(i) and μ(ii). Note that $x, y \in \max_{R_2^*} X \cap \max_{R_3^*} X$. Note also that, for any $C_1(R, x)$ which is a subset of $L(R_1, x)$ with $x \in C_1(R, x)$, $C_1(R, x) \subseteq L(R_1^*, x)$. Thus, according to Condition μ(ii), $x \in PO(R^*)$ should hold. However, $x \notin PO(R^*)$ as y Pareto dominates it, a contradiction. Also, since $x \in PO(R)$ and $L(R_\ell, x) \subseteq L(R_\ell^*, x)$ for all $\ell \in N$, Condition μ(i) implies that $x \in PO(R^*)$ should hold, thus a contradiction, since $x \notin PO(R^*)$.

To show part (ii), let $(R, x, \ell) \in \bar{\mathcal{R}}^n \times X \times N$ with $x \in F^{PO}(R)$, and let $C_\ell(R, x) \equiv L(R_\ell, x)$. Also, $X = Y$ as F^{PO} satisfies unanimity. We will show that F^{PO} satisfies Condition μ^{**} under these specifications. Pick any arbitrary $(R, R^*, x) \in \bar{\mathcal{R}}^n \times \bar{\mathcal{R}}^n \times X$ with $x \in F^{PO}(R)$. Condition μ^{**}(i) is always vacuously satisfied. Condition μ^{**}(iii) is satisfied as F^{PO} is Pareto.

Take any $(H, i) \in \mathcal{H} \times N$. Suppose that $y \in C_i(R, x) = L(R_i, x) \subseteq L(R_i^*, y)$ and $y \in \max_{R_\ell^*} X$ for all $\ell \in N \setminus \{i\}$. We check for μ^{**}(ii) and μ^{**}(iv).

Let $H = \{i\}$ and $y \notin F^{PO}(R^*)$. We show that $\{y\} \neq \max_{R_i^*} C_i(R, x)$. As $y \notin F(R^*)$, it follows that there is an allocation $z \in X$ such that $(z, y) \in R_j^*$ for all $j \in N$ and $(z, y) \in P_j^*$ for some $j \in N$. As $y \in \max_{R_\ell^*} X$ for all $\ell \in N \setminus \{i\}$, it follows that $(z, y) \in P_i^*$ and $(z, y) \in I_\ell^*$ for all $\ell \in N \setminus \{i\}$; moreover, $z \notin L(R_i^*, y) \supseteq L(R_i, x)$ as $(z, x) \in P_i^*$. Then, y is not a plateau amount for agent i and so $L(R_i^*, y) \neq X$. Let $y' \equiv (y_i, w_{-i}) \neq y$ where $w_{-i} \in \mathbb{R}_+^{n-1}$ such that $\sum_{\ell \in N \setminus \{i\}} w_\ell = \sum_{\ell \in N \setminus \{i\}} y_\ell$. The allocation y' exists and belongs to the set $L(R_i, x)$ as $(x, y) \in R_i$ and $(y, y') \in I_i$. As $y' \in L(R_i, x) \setminus \{y\}$ and $(y, y') \in I_i^*$, we have that $\{y\} \neq \max_{R_i^*} C_i(R, x)$. Hence, F^{PO} satisfies Condition μ^{**}(ii.a).

Let $i \in H$ and $\#H > 1$, $R^* = R$, and $\{y\} = \max_{R_i^*} C_i(R, x)$. It follows that $x = y$ and so $y \in F^{PO}(R^*)$. Therefore, F^{PO} satisfies Condition μ^{**}(ii.b).

Let $i \notin H$ and $R^* = R$. It follows that $(y, x) \in I_i$ and $(y, x) \in I_\ell$ for all $\ell \in N \setminus \{i\}$. Suppose that $y \notin F^{PO}(R)$. Then, there is a $z \in X$ such that $(z, y) \in R_j$ and $(z, y) \in P_j$ for some $j \in N$. By transitivity of R_j for all $j \in N$, it follows that z Pareto dominates x under the state R. Then, $x \notin F^{PO}(R)$, a contradiction. Therefore, F^{PO} satisfies Condition μ^{**}(ii.c).

Let $H = \{i\}$, $x = y$, $R_{-i} = R^*_{-i}$, and $L(R_i, x) = L(R^*_i, x)$. We show that $x \in F^{PO}(R^*)$. Assume, to the contrary, that $x \notin F^{PO}(R^*)$. Then, there is an allocation $z \in X$ such that $(z, x) \in R^*_j$ for all $j \in N$ and $(z, x) \in P^*_j$ for some $j \in N$. As $x \in \max_{R^*_\ell} X$ for all $\ell \in N \setminus \{i\}$, it follows that $(z, x) \in P^*_i$ and $(z, x) \in I^*_\ell$ for all $\ell \in N \setminus \{i\}$; and $(z, x) \in I_\ell$ for all $\ell \in N \setminus \{i\}$ as $R_{-i} = R^*_{-i}$. Moreover, $z \notin L(R^*_i, x) = L(R_i, x)$ as $(z, x) \in P^*_i$. It follows that $x \notin F^{PO}(R)$, a contradiction. Hence, F^{PO} satisfies μ^{**}(iv). ■

4.2 On partially-honest non-implementable SCCs by s-mechanisms

The class of SCCs which are partially-honest implementable by s-mechanisms is smaller than the class of SCCs which are partially-honest implementable. The reason is that Condition μ^{**} does not incorporate any kind of monotonicity property, while Condition M^*_s incorporates a weakening of the (Maskin) monotonicity condition.

In what follows, we show the *top-cycle SCC* is not partially-honest implementable by any s-mechanism, while it is partially-honest implementable. Before proving this result, additional notation is needed. Let $\#X$ be finite. For $x, y \in X$ with $x \neq y$, let $N_R(x, y)$ be the set of agents with $(x, y) \in R_i$ at state $R \in \mathcal{P}^n$, $N_R(x, y) \equiv \{i \in N | (x, y) \in R_i\}$.[18] Given $x, y \in X$ and $R \in \mathcal{P}^n$, we say that x is majority preferred to y at the profile R, denoted $(x, y) \in T_R$, if $\#N_R(x, y) \geq \#N_R(x, y)$. For the sake of simplicity, suppose that n is an odd number so as the majority relation T_R on X is a tournament for any $R \in \mathcal{P}^n$.[19] The set of all top-cycle outcomes at state R is defined as follows:

$$x \in TC(R) \Leftrightarrow \forall y \in X \setminus \{x\}, \text{ there are } x^0, x^1, \ldots, x^m \in X, \text{ with } m \in \mathbb{Z}_{++}, \text{ such that } \left(x^k, x^{k+1}\right) \in T_R \text{ for } k = 0, \ldots, m-1, \text{ with } x^0 = x \;\&\; x^m = y.$$

An SCC F^{TC} on $\mathcal{P}^n$ is the *top-cycle SCC* if, for all $R \in \mathcal{P}^n$, $F^{TC}(R) = TC(R)$.

Proposition 3. *(i) F^{TC} is* partially-honest implementable; *(ii) F^{TC} is not* partially-honest implementable by any s-mechanism.

[18] $\mathcal{P}^n \subseteq \mathcal{R}^n$ is the set of all available profiles of linear orders.

[19] A relation T on X is a tournament if it is complete and asymmetric.

Proof. Observe that Condition μ^{**}(i) is vacuously satisfied by any SCC. Then, to see that F^{TC} is partially-honest implementable, it suffices to observe that F^{TC} satisfies the requirement of no-veto power which, in turn, implies Conditions μ^{**}(ii)-μ^{***}(iv). This completes part (i) of the statement.

To show part (ii), assume, to the contrary, that F^{TC} is partially-honest implementable by an s-mechanism. Then, F^{TC} satisfies Condition M_s^*, and, in particular, Condition M_s^*(i). Let $N = \{1,2,3\}$ with $\#N = 3$, $X = \{x,y,z\}$ with $\#X = 3$, and $\mathcal{R}^3 = \{R, R^*\}$, where agents' preferences are as follows:

R			R^*		
1	2	3	1	2	3
x	y	z	x	y	x
y	z	x	y	z	z
z	x	y	z	x	y

With abuse of notation, we write xT_Ry for $(x,y) \in T_R$. In terms of the tournament relation, we have that $xT_RyT_RzT_Rx$, while $xT_{R^*}a$ for all $a \in \{y,z\}$ and $yT_{R^*}z$. Since $y \in TC(R) = X$, there is a a profile of sets $(C_\ell(R_\ell, y))_{\ell \in N}$ such that $y \in C_\ell(R_\ell, y) \subseteq L(R_\ell, y) \cap X$ for all $\ell \in N$. Since $(R_\ell, R_{\ell+1}) \neq (R_\ell^*, R_{\ell+1}^*)$ for $\ell \in \{2,3\}$, it follows that Condition M_s^*(i) is vacuously satisfied if $H \cap \{2,3\} \neq \varnothing$. The only case that we are left to verify is $H = \{1\}$. As $(R_1, R_2) = (R_1^*, R_2^*)$ and $L(R_\ell, y) = L(R_\ell^*, y)$ for all $\ell \in N$, Condition M_s^*(i) implies that $y \in F(R^*) \neq TC(R^*) = \{x\}$, a contradiction. ■

The *egalitarian-equivalent solution* (Pazner and Schmeidler, 1978), defined in the classical exchange economies, is another well-known example of a non-monotonic SCC. This solution is not partially-honest implementable by any s-mechanism either, though it is partially-honest implementable by virtue of Theorem 2.

We close this section by noting that Tatamitani (2002) shows that in standard one-to-one matching environments - where staying single is feasible - the *stable rule solution* is not Nash implementable by any self-relevant mechanism as it violates Condition λ(i). It follows that this SCC is not partially-honest implementable either.[20] However, as the stable rule solution equals the *weak core SCC* of the associated coalitional game environment and the weak core solution is monotonic and satisfies Condition M_s^{**}, it follows

[20]This solution, however, is implementable by a self-relevant mechanism under a different notion of partial-honesty (Lombardi and Yoshihara, 2011).

that the stable rule solution is partially-honest implementable by virtue of Theorem 4.

5 Two-agent implementation problems

Seminal papers on two-agent Nash implementation are those of Moore and Repullo (1990) and Dutta and Sen (1991) who independently refined Maskin's characterization result (Maskin, 1999) by providing necessary and sufficient conditions for an SCC to be implementable.[21] As Dutta and Sen's Condition β and Moore and Repullo's Condition $\mu 2$ coincide in substance, we state only Condition $\mu 2$.

Condition $\mu 2$ (for short, $\mu 2$): There is a set $Y \subseteq X$ and, for all $R \in \mathcal{R}^n$ and all $x \in F(R)$, there is a profile of sets $(C_\ell(R,x))_{\ell \in N}$ such that $x \in C_\ell(R,x) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$; furthermore, Condition μ holds; finally, for all $R^* \in \mathcal{R}^n$, the following (iv) is satisfied:
(iv) for each $(x', R') \in X \times \mathcal{R}^2$ with $x' \in F(R')$,
(a) there is an $e \equiv e(x', R', x, R) \in C_1(R', x') \cap C_2(R, x)$, with $e(x, R, x, R) = x$;
(b) if $C_1(R', x') \subseteq L(R_1^*, e)$ and $C_2(R, x) \subseteq L(R_2^*, e)$, then $e \in F(R^*)$.

Condition $\mu 2$ is markedly stronger than Condition μ, as it requires a punishment condition (Condition $\mu 2$(iv)). While the first part of Condition $\mu 2$(iv) guarantees the existence of a punishment outcome, the second part requires that if the punishment outcome is an equilibrium outcome, it should be F-optimal. Notable parts of Condition $\mu 2$ are the monotonicity condition (Condition μ(i)) and the punishment condition.

The two-agent implementation problem with partially-honest agents has recently been analyzed by Dutta and Sen (2009) on the assumption that agents' preferences are linear orders. Their contribution is that, even in the more problematic case of the two agents society, the scope of implementation is enlarged as the stringent condition of monotonicity is no longer required. This section extends their analysis to the domain of weak orders in view of its potential applications to bargaining and negotiating.

In the next two sub-sections, we identify the class of partially-honest implementable SCCs, not only in the case that the planner knows that exactly one agent is partially-honest, but also in the more subtle case that she only

[21] See also Busetto and Codognato (2009).

knows that there is at least one partially-honest agent. We present two new conditions which are not only necessary and sufficient conditions for SCCs to be partially-honest implementable but also markedly weaker than Condition $\mu 2$. In particular - and in line with earlier results and Theorem 2-, our characterizations confirm that even in a two-agent society the presence of partially-honest agents drastically improves the scope, though limits still remain. In particular, what still limits implementability is the punishment condition.

Sub-section 5.3 reports briefly the implications of our results.

5.1 Exactly one partially-honest agent

In this sub-section, we make the informational assumption that there is exactly one partially-honest agent; the planner is aware of this fact but ignores her identity. We begin by proving that if an SCC F is partially-honest implementable, then it must satisfy Condition $\mu 2^*$ below. Although such a condition is quite complex, it is in fact very weak.

Condition $\mu 2^*$ (for short, $\mu 2^*$): There is a set $Y \subseteq X$ and, for all $i \in N$, all $R \in \mathcal{R}^2$, and all $x \in F(R)$, there is a set $C_\ell(R, x)$ such that $x \in C_\ell(R, x) \subseteq L(R_\ell, x) \cap Y$; Conditions μ^*(i), μ^*(ii.a), and μ^*(iii) hold; finally, for all $R^* \in \mathcal{R}^2$, the following condition (iv) is satisfied:
(iv) for each $(x', R') \in X \times \mathcal{R}^2$ with $x' \in F(R')$,
(a) there is an $e \equiv e(x', R', x, R) \in C_1(R', x') \cap C_2(R, x)$, with $e(x, R, x, R) = x$;
(b) if $x' \neq x$, $R' \neq R$, $C_1(R', x') \subseteq L(R_1^*, e)$, $C_2(R, x) \subseteq L(R_2^*, e)$, and
(b.1) if $H = \{1\}$ and $\{e\} = \max_{R_1^*} C_1(R', x')$, then $e \in F(R^*)$;
(b.2) if $H = \{2\}$ and $\{e\} = \max_{R_2^*} C_2(R, x)$, then $e \in F(R^*)$.

Theorem 7. *Let Assumption 1 hold and $\mathcal{H} = \{\{1\}, \{2\}\}$. If an* SCC F on $\mathcal{R}^2$ *is* partially-honest implementable, *then it satisfies* Condition $\mu 2^*$.

Proof. Let Assumption 1 hold and let $\mathcal{H} = \{\{1\}, \{2\}\}$. Let $h \in N$ be the unique partially-honest agent. Let $\gamma \equiv (M, g)$ be a mechanism which partially-honest implements F on $\mathcal{R}^2$. The proof that the set Y and the profile $(C_\ell(R, x))_{\ell \in N}$ exist follows from Theorem 1. Moreover, from Theorem 1, it follows that F satisfies Conditions μ^*(i), μ^*(ii.a), and μ^*(iii), so we omit them here. Finally, we show that F meets Condition $\mu 2^*$(iv). Take any $R^* \in \mathcal{R}^2$.

Pick any $(x', R', x, R) \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$ with $x \in F(R)$ and $x' \in F(R')$. Then, there is an equilibrium strategy $m \equiv (m_1, m_2) \in NE(\gamma, \succcurlyeq^R)$ such that $g(m) = x$. Similarly, $m' \equiv (m'_1, m'_2) \in NE(\gamma, \succcurlyeq^{R'})$ and $g(m') = x'$.

Let $e \equiv e(x', R', x, R) = g(m_1, m'_2)$. Then, $e \in C_1(R', x') = g(M_1, m'_2)$ and $e \in C_2(R, x) = g(m_1, M_2)$. Thus, F satisfies $\mu 2^*$(iv.a). Finally, it is also clear that F satisfies Condition $\mu 2^*$(iv.b) as, for instance, in the case $\mu 2^*$(iv.b.1) if $e \notin F(R^*)$, the only deviator is the partially-honest agent 1 but her deviation to an $m'_h \in T_h^\gamma(R^*, F)$ results in the same outcome e as $\{e\} = \max_{R_1^*} C_1(R', x')$. We conclude that F satisfies $\mu 2^*$(iv). ■

Even though Condition $\mu 2^*$ is a weak condition, it imposes non-trivial restrictions on partially-honest implementable SCCs. For instance, the Pareto SCC F^{PO} on $\mathcal{R}^2$ violates Condition $\mu 2^*$(iv.a).

The class of partially-honest implementable SCCs is fully identified by Condition $\mu 2^{**}$ (stated below) under the mild domain restrictions as in Theorem 2.

Condition $\mu 2^{**}$ (for short, $\mu 2^{**}$): There is a set $Y \subseteq X$ and, for all $i \in N$, all $R \in \mathcal{R}^2$, and all $x \in F(R)$, there is a set $C_\ell(R, x)$ such that $x \in C_\ell(R, x) \subseteq L(R_\ell, x) \cap Y$; Condition μ^{**} holds; finally, for all $R^* \in \mathcal{R}^2$, the following condition (v) is satisfied:
(v) for each $(x', R') \in X \times \mathcal{R}^2$ with $x' \in F(R')$,
(a) there is an $e \equiv e(x', R', x, R) \in C_1(R', x') \cap C_2(R, x)$ with $e(x, R, x, R) = x$, such that;
(b) if $R = R' = R^*$, $x' \neq x$, $(e, x') \in I_1^*$, and $(e, x) \in I_2^*$, then $e \in F(R^*)$;
(c) if $x' \neq x$, $R' \neq R$, $C_1(R', x') \subseteq L(R_1^*, e)$, $C_2(R, x) \subseteq L(R_2^*, e)$, and $e \notin F(R^*)$, then;
(c.1) if $R = R^*$, then $H = \{2\}$;
(c.2) if $R' = R^*$, then $H = \{1\}$.[22]

Theorem 8. *Let Assumption 1,* $\Gamma = \Gamma_{SP}$*, and* **RD** *hold, and* $\mathcal{H} = \{\{1\}, \{2\}\}$*. An* SCC F on $\mathcal{R}^2$ *is* partially-honest implementable *if and only if it satisfies* Condition $\mu 2^{**}$.

Proof. Let Assumption 1 and **RD** hold and $\mathcal{H} = \{\{1\}, \{2\}\}$. Let $h \in N$ denote a partially-honest agent.

1. The necessity of Condition $\mu 2^{**}$.

[22] We refer to the condition that requires only one of the conditions (i)–(iv) in Condition μ^* as Conditions $\mu 2^{**}$(i)–$\mu 2^{**}$(iv) each.

Let F on $\mathcal{R}^2$ be an SCC which is partially-honest implementable by a mechanism $\gamma \equiv (M, g) \in \Gamma_{SP}$. Let $Y \equiv g(M)$. Take any $R \in \mathcal{R}^2$ and any $x \in F(R)$. Then, there is an $m(R,x) \in NE(\gamma, \succcurlyeq^R) \subseteq M$ such that $g(m(R,x)) = x$. Moreover, $m_h(R,x) \in T_h^\gamma(R,F)$ for every partially-honest agent $h \in H$. To see this, assume, to the contrary, that $m_h(R,x) \notin T_h^\gamma(R,F)$ for some $h \in H$. As $\gamma \in \Gamma_{SP}$, we have that agent h can change $m_h(R,x)$ to $m_h \in T_h^\gamma(R,F)$ and obtain $g(m(R,x)) = g(m_h, m_{-h}(R,x)) = x$, which contradicts that $m(R,x) \in NE(\gamma, \succcurlyeq^R)$. For all $\ell \in N$, let $C_\ell(R,x) \equiv g(M_\ell, m_i(R,x))$ where $i \in N \backslash \{\ell\}$. Then, $C_\ell(R,x) \equiv g(M_\ell, m_i(R,x)) \subseteq L(R_\ell, x) \cap Y$ for all $\ell \in N$. From Theorem 2, it follows that F satisfies Conditions μ^{**}. Next, we show Condition $\mu 2^{**}$(v).

Pick any $(x', R') \in X \times \mathcal{R}^2$ with $x' \in F(R')$, and take any $R^* \in \mathcal{R}^2$. As $x' \in F(R')$, it follows that there is an $m(R',x') \in NE(\gamma, \succcurlyeq^{R'})$ and $g(m(R',x')) = x'$, where $m_h(R',x') \in T_h(R',F)$ for all $h \in H$. Let $e \equiv e(x', R', x, R) = g(m_1(R,x), m_2(R',x'))$. Then, $e \in C_1(R',x') = g(M_1, m_2(R',x'))$ and $e \in C_2(R,x) = g(m_1(R,x), M_2)$. Thus, F satisfies $\mu 2^{**}$(v.a).

It is also clear that F meets Condition $\mu 2^{**}$(v.b) as $R = R' = R^*$ implies that every agent is truthful and e is optimal at state R^*.

Finally, we check $\mu 2^{**}$(v.c). Let $e \equiv e(x', R', x, R) = g(m_1(R,x), m_2(R',x'))$ with $x \neq x'$ and $R \neq R'$. Let $C_1(R',x') \subseteq L(R_1^*, e)$, $C_2(R,x) \subseteq L(R_2^*, e)$, and $e(x', R', x, R) \notin NA(\gamma, \succcurlyeq^{R^*}) = F(R^*)$. Suppose that $R = R^*$. Assume, to the contrary, that $H = \{1\}$. Then, $m_1(R,x) \in T_1^\gamma(R^*, F)$. Since there cannot be any profitable deviation, we have that $e(x', R', x, R) \in NA(\gamma, \succcurlyeq^{R^*})$, a contradiction. Thus, $H = \{2\}$. Similarly, we obtain $H = \{1\}$ if $R' = R^*$. We conclude that F satisfies Condition $\mu 2^{**}$(v).

2. The sufficiency of Condition $\mu 2^{**}$.

Let $\gamma \equiv (M, g)$ be a mechanism. For each $\ell \in N$, let the message space of agent ℓ be defined as follows

$$M_\ell \equiv \left\{ m_\ell = \left(R^\ell, x^\ell, y^\ell, k^\ell\right) \in \mathcal{R}^2 \times X \times Y \times \mathbb{Z}_+ \mid x^\ell \in F\left(R^\ell\right) \right\}, \quad (1)$$

where $\mathbb{Z}_+$ is the set of nonnegative integers.

Define the outcome function $g : M \to X$ as follows: For all $m \in M$,

Rule 1: If $(R^1, x^1) = (R^2, x^2)$ and $k^1 = k^2 = 0$, then $g(m) = x^1$.
Rule 2: If $k^1 > k^2 = 0$, then

$$g(m) = \begin{cases} y^1 & \text{if } \ y^1 \in C_1(R^2, x^2) \\ e \equiv e(x^2, R^2, x^1, R^1) & \text{otherwise.} \end{cases}$$

Rule 3: If $k^2 > k^1 = 0$, then

$$g(m) = \begin{cases} y^2 & \text{if } \ y^2 \in C_2(R^1, x^1) \\ e \equiv e(x^2, R^2, x^1, R^1) & \text{otherwise.} \end{cases}$$

Rule 4: If $(R^1, x^1) \neq (R^2, x^2)$ and $k^1 = k^2 = 0$, then

$$g(m) = \begin{cases} x^1 & \text{if } \ x^1 = x^2 \\ e \equiv e(x^2, R^2, x^1, R^1) & \text{otherwise.} \end{cases}$$

Rule 5: If $k^1 \geq k^2 > 0$, then, $g(m) = y^1$.
Rule 6: Otherwise, $g(m) = y^2$.

By definition of g, we see that $\gamma \in \Gamma_{SP}$. We show that γ partially-honest implements F. Pick any $R \in \mathcal{R}^2$.

Since F satisfies Condition $\mu 2^{**}$, $F(\mathcal{R}^2) \subseteq Y$. Thus, for any $R \in \mathcal{R}^2$ and any $x \in F(R)$, $x \in Y$.

To show that $F(R) \subseteq NA\left(\gamma, \succcurlyeq^R\right)$, let $x \in F(R)$ and suppose that, for all $\ell \in N$, $m_\ell(R, x) = (R, x, x, 0) \in M_\ell$. *Rule 1* implies that $g(m) = x$. By the definition of g, any deviation of agent $\ell \in N$ leads to an outcome in $C_\ell(R, x)$, so that $g(M_\ell, m_i(R, x)) \subseteq C_\ell(R, x)$, where $i \in N \setminus \{\ell\}$. Since $C_\ell(R, x) \subseteq L(R_\ell, x)$, such deviations are not profitable. It follows that $x \in NA\left(\gamma, \succcurlyeq^R\right)$.

Conversely, to show that $NA\left(\gamma, \succcurlyeq^R\right) \subseteq F(R)$, let $m \in NE\left(\gamma, \succcurlyeq^R\right)$. Consider the following cases.

Case 1: m corresponds to Rule 1.

Suppose that m falls into *Rule 1*. Then, $g(m) = x^1$. By the definition of g, it follows that $m_h \in T_h^\gamma(R, F)$ for $h \in H$. Indeed, assume, to the contrary, that $m_h \notin T_h^\gamma(R, F)$ for $h \in H$. Let $h = 1$. Then, by changing m_h to $m'_h = \left(R, x^h, x^1, k^h\right) \in T_h^\gamma(R, F)$ with $x^h \in F(R)$ and $k^h > 0$, agent $h = 1$ induces *Rule 2* and obtains $x^1 = g(m'_h, m_2) \in C_h(R^2, x^2)$. Therefore, $((m'_h, m_2), m) \in \succ_h^R$ which contradicts that $m \in NE\left(\gamma, \succcurlyeq^R\right)$. Similar reasoning applies if $h = 2$. We conclude that $x^1 = x^2 \in F(R)$.

Case 2: m corresponds to Rule 2.

Then, $g(m_1, M_2) = Y$ and $C_1(R^2, x^2) \subseteq g(M_1, m_2)$. Moreover, since $m \in NE\left(\gamma, \succcurlyeq^R\right)$, it follows that $g(m) \in C_1(R^2, x^2) \subseteq L(R_1, g(m))$ and $Y \subseteq L(R_2, g(m))$. By the definition of g, $m_h \in T_h^\gamma(R, F)$ holds for $h \in H$. Suppose that $H = \{2\}$. Condition $\mu 2^{**}$(ii.c) implies that $g(m) \in F(R)$ as $R^2 = R$. Otherwise, let $H = \{1\}$. Take an $\hat{R}_1 \in \mathcal{R}_1(X)$ such that

$L\left(\hat{R}_1, g(m)\right) = L(R_1, g(m))$ with $\partial L\left(\hat{R}_1, g(m)\right) = \{g(m)\}$. As $\mathcal{R}^2$ satisfies **RD**, we have that $\hat{R} \equiv \left(\hat{R}_1, R_2\right) \in \mathcal{R}^2$. Condition $\mu 2^{**}$(ii.a) implies that $g(m) \in F\left(\hat{R}\right)$. Since F satisfies $\mu 2^{**}$, there is a profile of sets $\left(C_\ell\left(\hat{R}, g(m)\right)\right)_{\ell \in N}$ such that $C_\ell\left(\hat{R}, g(m)\right) \subseteq L\left(\hat{R}_\ell, g(m)\right) \cap Y$ for all $\ell \in N$. Since $L\left(\hat{R}_1, g(m)\right) = L(R_1, g(m))$, $R_2 = \hat{R}_2$, and $H = \{1\}$, Condition $\mu 2^{**}$(iv) implies that $g(m) \in F(R)$.

Case 3: m corresponds to *Rule 3.*

The proof can be obtained by simply readapting the proof of *Case 2*, so we omit it here.

Case 4: m corresponds to *Rule 4.*

Then, $m = (m_1(R^1, x^1), m_2(R^2, x^2))$, $C_1(R^2, x^2) \subseteq g(M_1, m_2(R^2, x^2))$, and $C_2(R^1, x^1) \subseteq g(m_1(R^1, x^1), M_2)$. Then, $g(m) \in \{x^1, e(R^2, x^2, R^1, x^1)\}$. As $m \in NE\left(\gamma, \succcurlyeq^R\right)$, it follows that $C_1(R^2, x^2) \subseteq L(R_1, g(m))$ and $C_2(R^1, x^1) \subseteq L(R_2, g(m))$. Notice that $m_h \in T_h^\gamma(R, F)$ for $h \in H$. Suppose that $x^1 = x^2$, so that $g(m) = x^1$. Then, $x^1 \in F(R)$ as $m_h \in T_h^\gamma(R, F)$ for $h \in H$. Otherwise, let $x^1 \neq x^2$. Then, $g(m) = e(R^2, x^2, R^1, x^1) \in C_1(R^2, x^2) \cap C_2(R^1, x^1)$. Suppose that $R^1 = R^2$. Then, as F satisfies Condition $\mu 2^{**}$, it follows that $(e, x^2) \in I_1$ and $(e, x^1) \in I_2$. Condition $\mu 2^{**}$(v.b) implies that $g(m) \in F(R)$. Finally, let $R^1 \neq R^2$. Suppose that $H = \{1\}$, so that $R^1 = R$. Condition $\mu 2^{**}$(v.c.1) implies that $e(R^2, x^2, R^1, x^1) \in F(R)$ as $H \neq \{2\}$. Otherwise, let $H = \{2\}$, and so $R^2 = R$. Condition $\mu 2^{**}$(v.c.2) implies that $e(R^2, x^2, R^1, x^1) \in F(R)$ as $H \neq \{1\}$.

Cases 5: m corresponds to *Rule 5* or *Rule 6.*

Then, $g(m_1, M_2) = Y$ and $g(M_1, m_2) = Y$. By definition of g and $m \in NE\left(\gamma, \succcurlyeq^R\right)$, we have that $Y \subseteq L(R_1, g(m))$ and $Y \subseteq L(R_2, g(m))$. Moreover, $m_h \in T_h^\gamma(R, F)$ for $h \in H$. Condition $\mu 2^{**}$(iii) implies that $g(m) \in F(R)$. ■

5.2 There are partially-honest agents

In this sub-section, we make the informational assumption that the planner knows that agents are partially-honest but she ignores their identities. This assumption is much weaker than the informational assumption made in the previous sub-section, since the planner also ignores the exact number of partially-honest agents. We begin by proving that if an SCC F is partially-honest implementable, then it must satisfy Condition $\mu 2^{\circ}$ below, which is stronger than Condition $\mu 2^{*}$.

Condition $\mu 2^{\circ}$ (for short, $\mu 2^{\circ}$): There is a set $Y \subseteq X$ and, for all $i \in N$, all $R \in \mathcal{R}^2$, and all $x \in F(R)$, there is a set $C_\ell(R, x)$ such that $x \in C_\ell(R, x) \subseteq L(R_\ell, x) \cap Y$; furthermore, Condition $\mu 2^{*}$ holds; finally, for all $R^* \in \mathcal{R}^2$, the following condition (v) is satisfied:
(v) for all $i \in N$ and $H \in \mathcal{H}$, if $H = N$, $R = R^*$, $y \in C_i(R, x) \subseteq L(R_i^*, y)$ and $y \in \max_{R_\ell^*} Y$ for all $\ell \in N \backslash \{i\}$, then $y \in F(R^*)$ if $x = y$.

Theorem 9. *Let Assumption 1. If an* SCC F on $\mathcal{R}^2$ *is* partially-honest implementable, *then it satisfies* Condition $\mu 2^{\circ}$.

Proof. From Theorem 7, it follows that F satisfies Condition $\mu 2^{*}$. As it is obvious that F satisfies Condition $\mu 2^{\circ}$(v), we omit it here. ■

Condition $\mu 2^{\circ}$ does not suffice to guarantee the partial-honest implementability of SCCs. A sufficient condition can be stated as follows.

Condition $\mu 2^{\circ\circ}$ (for short, $\mu 2^{\circ\circ}$): Condition $\mu 2^{**}$ holds; moreover, for all $R, R^* \in \mathcal{R}^2$, the following condition (vi) is satisfied:
(vi) $x \in F(R)$, for all $i \in N$ and $H \in \mathcal{H}$, if $H = N$, $R = R^*$, $y \in C_i(R, x) \subseteq L(R_i^*, y)$ and $y \in \max_{R_\ell^*} Y$ for all $\ell \in N \backslash \{i\}$, then $y \in F(R^*)$.

The above condition becomes necessary and sufficient for the partially-honest implementation if the admissible class Γ of mechanisms is restricted by the following condition.

Strong Punishment (StP): For any $R, R' \in \mathcal{R}^2$, any $i \in N$, and any $m \equiv (m_i, m_\ell) \in M$ such that $g(m) = x$, there is an $m_i' \in T_i^{\gamma}(R', F)$ such that $g(m_i', m_\ell) = g(m)$.

The above condition has a similar flavor to **SP**. However, with condition **StP**, the planner is required to design a mechanism where if x is an attainable outcome at state R -in the sense that there is a message profile leading to it at this state-, then a partially-honest agent should be able to reach the same outcome x by announcing a truthful message (while keeping constant the messages of all others). Different from **SP**, this **StP** demands the existence of such a message profile for each attainable outcome, regardless of whether it is an F-optimal outcome. In this sense, the above condition can be considered as a strong punishment requirement. Similar to **SP**, the requirement of **StP** is satisfied by all classical mechanisms in the literature of

Nash implementation (see, for instance, Repullo, 1987; Moore and Repullo, 1990; Saijo, 1988; Dutta and Sen, 1991; Tatamitani, 2001).

Denote the class of mechanisms satisfying **StP** by Γ_{StP}.

Theorem 10. *Let Assumption 1,* $\Gamma = \Gamma_{StP}$*, and* **RD** *hold. An* SCC F on $\mathcal{R}^2$ *is* partially-honest implementable *if and only if it satisfies* Condition $\mu 2^{\circ\circ}$.

Proof. Let Assumption 1 and **RD** hold. Let $h \in N$ denote a partially-honest agent.

1. The necessity of Condition $\mu 2^{\circ\circ}$.

Suppose that F is partially-honest implementable by $\gamma \equiv (M, g) \in \Gamma_{StP}$. From Theorem 8, Condition $\mu 2^{**}$ is satisfied. Finally, as it is clear that F satisfies Condition $\mu 2^{\circ\circ}$(vi), we thus omit the proof here.

2. The sufficiency of Condition $\mu 2^{\circ\circ}$.

Let $\gamma \equiv (M, g)$ be a mechanism. For each $\ell \in N$, let the message space of agent ℓ be that defined in (1). Define the outcome function $g : M \to X$ as in Theorem 8. Note that γ satisfies **StP** by the definition of g.

Pick any $R \in \mathcal{R}^2$. The proof that $F(R) \subseteq NA\left(\gamma, \succcurlyeq^R\right)$ follows from Theorem 8. Conversely, to show that $NA\left(\gamma, \succcurlyeq^R\right) \subseteq F(R)$, let $m \in NE\left(\gamma, \succcurlyeq^R\right)$. As in Theorem 8, we have to consider several *Cases*. However, all *Cases* but *Cases 2-3* follow from the same arguments used in Theorem 8, so we omit them here. Next, we consider *Case 2* and *Case 3*.

Case 2: m corresponds to *Rule 2*.

Then, $g(m_1, M_2) = Y$ and $C_1(R^2, x^2) \subseteq g(M_1, m_2)$. Moreover, since $m \in NE\left(\gamma, \succcurlyeq^R\right)$, it follows that $g(m) \in C_1(R^2, x^2) \subseteq L(R_1, g(m))$ and $Y \subseteq L(R_2, g(m))$. By the definition of g, we have that $m_h \in T_h^{\gamma}(R, F)$ for all $h \in H$. Suppose that $\#H = 1$. Then, $g(m) \in F(R)$ by *Case 2* of Theorem 8. Suppose that $\#H = 2$. Then, Condition $\mu 2^{\circ\circ}$(vi) implies that $g(m) \in F(R)$, as sought.

Case 3: m corresponds to *Rule 3*.

The proof can be obtained by simply readapting the proof of *Case 2*, so we omit it here. ■

Before closing this sub-section, it may be worth briefly mentioning that if the planner knows that both agents are partially-honest, the class of partially-honest implementable SCCs becomes larger because neither Condition $\mu 2^{**}$(ii), Condition $\mu 2^{**}$(iv), nor Condition $\mu 2^{**}$(v.c) is required. This result is readily obtained by Theorem 10.

Corollary 3. *Let Assumption 1 and* $\mathcal{H} = \{N\}$*. An* SCC F on $\mathcal{R}^2$ *is* partially-honest implementable by a mechanism in Γ_{StP} *if and only if it satisfies* Condition $\mu 2^{\circ\circ}$ *without* Condition $\mu 2^{**}$(ii), Condition $\mu 2^{**}$(iv) *or* Condition $\mu 2^{**}$(v.c)

5.3 Implications

Condition $\mu 2^{\circ\circ}$ - and so Condition $\mu 2^{**}$ - imposes non-trivial restrictions on F. For example, the *Pareto SCC* is not partially-honest implementable by virtue of Proposition 1, as this SCC violates Condition $\mu 2^{**}$(ii.a); moreover, this SCC fails to meet Condition $\mu 2^{**}$(v.a). Despite this, our results are quite permissive.[23] In the following, we justify this assertion by providing sufficient conditions which allow us to give a quick answer to the question of implementability.

One avenue is to introduce a bad outcome $b \in X$ and make the following assumption.

Assumption 2 (Moore and Repullo, 1990, p. 1093). *There exists a* bad outcome $b \in X$ *such that for all* $R \in \mathcal{R}^2$ *and* $i \in N$*,* $(x, b) \in P_i$ *for all* $x \in F(\mathcal{R}^2) \equiv \{y \in X | y \in F(R') \text{ for some } R' \in \mathcal{R}^2\}$.

There are economic environments in which it is easy to find a bad outcome. Consider an exchange economy in which agents have strict monotonic preferences and the SCC assigns only positive consumption bundles. Under free disposal, one can define the null consumption bundle as the bad outcome.

If there is a bad outcome, we can set $e(x, R, x', R') = b$ for each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$ to satisfy Condition $\mu 2^{\circ\circ}$(v) vacuously. Then, Condition $\mu 2^{\circ\circ}$ without Condition $\mu 2^{**}$(v) is sufficient for an SCC to be partially-honest implementable. Even though these conditions can easily be checked by using the algorithm provided by Sjöström (1991), the condition of *restricted veto power*, when combined with Assumption 2, suffices to ensure Condition $\mu 2^{\circ\circ}$.

Restricted veto power: For all $i \in N$, all $R \in \mathcal{R}^2$, all $x \in X$, and all $x' \in F(\mathcal{R}^2) \equiv \{y \in X | y \in F(R) \text{ for some } R \in \mathcal{R}^2\}$, if $x \in \max_{R_\ell} X$ for all $\ell \in N \backslash \{i\}$ and $(x, x') \in R_i$, then $x \in F(R)$ holds.

[23] For a non-dictatorial and weakly Pareto efficient partially-honestly implementable SCC defined on the domain of linear orders which rebuts the negative conclusion of Hurwicz and Schmeidler (1978) we refer the reader to Dutta and Sen (2009).

Restricted veto power is used by Moore and Repullo (1990, p. 1093) to analyze the two-agent case under Assumption 2. We can now state the following result.

Corollary 4. *Let Assumption 1 and Assumption 2 hold. An* SCC F on $\mathcal{R}^2$ *is* partially-honest implementable *if it satisfies* restricted veto power.

Proof. Let Assumption 1 and Assumption 2 hold. Suppose that F on $\mathcal{R}^2$ satisfies restricted veto power. It suffices to show that Assumption 2 and restricted veto power imply Condition $\mu 2^{\circ\circ}$. Let $Y = X$; and for all $R \in \mathcal{R}^2$ and all $x \in F(R)$, let $C_i(R, x) = L(R_i, x)$ for all $i \in N$. Since Assumption 2 holds, for each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$ with $x \in F(R)$ and $x' \in F(R')$, let $e(x', R', x, R) = b$ if $(x, R) \neq (x', R')$, otherwise $e(x, R, x', R') = x$. Then, Condition $\mu 2^{\circ\circ}$(v) is satisfied. As restrict veto power implies Conditions μ(ii)-μ(iii) which, in turn, imply Conditions $\mu 2^{\circ\circ}$(ii)-$\mu 2^{\circ\circ}$(iv) and Condition $\mu 2^{\circ\circ}$(vi), the statement follows. ■

By Corollary 3, we know that an SCC is partially-honest implementable by Condition $\mu 2^{\circ\circ}$ without Condition $\mu 2^{**}$(ii), Condition $\mu 2^{**}$(iv) or Condition $\mu 2^{**}$(v.c) if the planner knows that both agents are partially-honest. Under this informational assumption, we show that unanimity and a weakening of restricted veto power, when combined with Assumption 2, suffice to guarantee partially-honest implementation.[24] The condition can be stated as follows.

Weak restricted veto power: For all $i \in N$, all $R \in \mathcal{R}^2$, and all $x \in X$, if $x \in \max_{R_\ell} X$ for all $\ell \in N \setminus \{i\}$ and $(x, x') \in R_i$ for all $x' \in F(R)$, then $x \in F(R)$ holds.

The above condition is new and considerably weaker than restricted veto power. We can now state the following result.

Corollary 5. *Let Assumption 1 and Assumption 2 hold, and let $\mathcal{H} = \{N\}$. An* SCC F on $\mathcal{R}^2$ *is* partially-honest implementable *if it satisfies* weak restricted veto power *and* unanimity.

Proof. Let Assumption 1 and Assumption 2 hold. Suppose that F on $\mathcal{R}^2$ satisfies weak restricted veto power and unanimity. As it it is clear that weak restricted veto power implies Condition $\mu 2^{\circ\circ}$(vi) and unanimity implies Condition $\mu 2^{\circ\circ}$(iii), the proof can be obtained by simply readapting the proof of Corollary 4, so we omit it here. ■

[24] For the definition of unanimous SCCs, see section 2.

For instance, suppose that two agents bargain over the division of one unit of a perfectly divisible good. If they do not reach an agreement, they both receive nothing. In this framework, non-monotonic strong individually rational bargaining solutions[25] defined on the class of utility possibility sets - such as the *Nash bargaining solution* - are special examples of SCCs applied to Corollary 4 and Corollary 5, setting the disagreement point $d = (0, 0)$ as a bad outcome.[26]

Another interesting weak domain restriction is Assumption 3 below. Before stating it, we need additional notation. Let $SL(R_i, x)$ denote agent i's strict lower contour set at $(R_i, x) \in \mathcal{R}_i \times X$, that is, $SL(R_i, x) \equiv \{y \in X | (x, y) \in P_i\}$.

Assumption 3 (Busetto and Codognato, 2009)**.** *$\mathcal{R}^2$ is such that for all $R^* \in \mathcal{R}^2$, we have***:**
(i) $\max_{R_i^*} SL(R_i, x) \cap \max_{R_j^*} SL(R_i, x) = \varnothing$ for all $i, j \in N$ with $i \neq j$, all $R \in \mathcal{R}^2$, and all $x \in X$;
(ii) $\max_{R_1^*} SL(R_1', x') \cap \max_{R_2^*} SL(R_2, x) = \varnothing$ for each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$ with $(x, R) \neq (x', R')$.

This domain restriction is very mild and much weaker than *Assumption E* imposed by Moore and Repullo (1990, p. 1095) and *Assumptions 5.1-5.2* imposed by Dutta and Sen (1991, p. 125) whenever X is a subset of a finite-dimensional Euclidean space.[27] For example, this restriction is satisfied in environments with continuous and locally non-satiated preferences or in environments in which the set of outcomes is a space of lotteries over a finite set of outcomes and agents' preferences over lotteries are represented by von Neumann-Morgenstern utility functions. Given Assumption 3, we can define a condition that, when combined with others, suffices to ensure Condition $\mu 2^{\circ\circ}$.

Definition 7. An SCC F on $\mathcal{R}^2$ satisfies the *non-empty lower intersection* if for all $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$ with $x \in F(R)$ and $x' \in F(R')$, we have that $SL(R_1', x') \cap SL(R_2, x) \neq \varnothing$.

[25] A bargaining solution is strong individually rational if it provides agents with agreements which give them utilities higher than those they derive from the disagreement point d.

[26] For the Nash bargaining solution defined on the class of utility possibility sets, see Vartiainen (2007).

[27] The formal arguments are provided in Busetto and Codognato (2009).

This property appears in Moore and Repullo (1990) and Dutta and Sen (1991) and holds in many environments. For example, it holds in an exchange economy for which indifference curves never touch the axes and for which the SCC recommends only interior allocations. We can now state our next results.

Corollary 6. *Let Assumption 1 and Assumption 3 hold. An* SCC F on $\mathcal{R}^2$ *is* partially-honest implementable *if it satisfies* non-empty lower intersection *and* restricted veto power.

Proof. Let Assumption 1 and Assumption 3 hold. Suppose that F on $\mathcal{R}^2$ satisfies non-empty lower intersection, weak restricted veto power and unanimity. We show that F is partially-honest implementable. It suffices to show that Condition $\mu 2^{\circ\circ}$ is implied by our suppositions.

For all $i \in N$, $(x, R) \in X \times \mathcal{R}^2$, and $x \in F(R)$, let $C_i(R, x) = SL(R_i, x) \cup \{x\}$ and $Y = X$. It is easy to verify that $C_i(R, x) \subseteq L(R_i, x) \cap Y$. For all $(x', R', x, R) \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$ with $x \in F(R)$ and $x' \in F(R')$, let $e(x', R', x, R) \in SL(R'_i, x') \cap SL(R_\ell, x)$ if $(x, R) \neq (x', R')$, and otherwise, $e(x', R', x, R) = x$. By definition of $e(x', R', x, R)$ and non-empty lower intersection, it is easy to see that Condition $\mu 2^{\circ\circ}$(v.a) is satisfied, while Condition $\mu 2^{\circ\circ}$(v.b) and Condition $\mu 2^{\circ\circ}$(v.c) are vacuously satisfied as $(x, R) \neq (x', R')$.

Pick any $R, R^* \in \mathcal{R}^2$; let $x \in F(R)$, $y \in C_i(R, x) \subseteq L(R^*_i, y)$, and $y \in \max_{R^*_\ell} Y$ for $i, \ell \in N$ with $i \neq \ell$. It cannot be that $y \in C_i(R, x) \setminus \{x\}$, otherwise $y \in \max_{R^*_i} SL(R_i, x) \cap \max_{R^*_j} SL(R_i, x)$, contradicting Assumption 3(i). Let $x = y$. Condition $\mu 2^{\circ\circ}$(vi) is satisfied trivially. Moreover, as restricted veto power implies that $x \in F(R^*)$, it follows that Condition $\mu 2^{\circ\circ}$(ii) and Condition $\mu 2^{\circ\circ}$(iv) are satisfied. Clearly, restricted veto power implies Condition $\mu 2^{\circ\circ}$(iii). The statement follows by observing that Condition $\mu 2^{\circ\circ}$(i) is satisfied. ■

Corollary 7. *Let Assumption 1 and Assumption 3 hold*; *let* $\mathcal{H} = \{N\}$. *An* SCC F on $\mathcal{R}^2$ *is* partially-honest implementable *if it satisfies* non-empty lower intersection *and* unanimity.

Proof. The proof of this statement directly follows from the proof of Corollary 6, and so it is omitted here. ■

Consider a two-agent exchange economy with $\ell \geq 2$ divisible goods, in which agents have continuous and strictly monotonic preferences, and in which indifference curves never touch the axes (for instance, Cobb-Douglas preferences). Suppose that an SCC F selects only interior allocations of the feasible set. In this setting, restricted veto power, unanimity, and non-empty lower intersection are satisfied by this F. The *egalitarian-equivalent solution* is an example of such an SCC.[28] This implies that it is partially-honest implementable, according to Corollary 6 and Corollary 7.

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6 Annex

*Completion of the proof of Case 2 of Theorem 6***.** In the following, we complete the proof of *Case 2* of Theorem 6 by showing that $m_h \in T_h^\gamma(R,F)$ for all $h \in H$. Assume, to the contrary, that $m_h \notin T_h^\gamma(R,F)$ for some $h \in H$.

Suppose that $h = i$. By changing his strategy m_i to $m_i' = (R_i, x^i, \diamond) \in T_i^\gamma(R,F)$, agent $i = h$ obtains $g(m) = g(m_i', m_{-i})$ via *Rule 2* as $i \in D(\bar{R}_{-i}, R_i)$. It follows that agent $h = i$ has a unilateral profitable deviation as $((m_i', m_{-i}), m) \in \succ_i^R$, which contradicts that $m \in NE(\gamma, \succcurlyeq^R)$.

Otherwise, let $h \neq i$. We proceed according to whether $\#Y = 2$ and $n = 3$ or not.

Consider the case *not*[$\#Y = 2$ and $n = 3$]. Suppose that $\#Y > 2$. Then, agent h can induce the modulo game by choosing any $y \in Y \backslash \{x, x^i\}$ and changing m_h to $m_h' = (R_h, y, k^h) \in T_h^\gamma(R,F)$. As the outcome is determined by *Rule 4*, agent h can obtain $g(m) = g(m_h', m_{-h})$ by appropriately choosing k^h. It follows that $((m_h', m_{-h}), m) \in \succ_h^R$, which contradicts that $m \in NE(\gamma, \succcurlyeq^R)$. Suppose that $\#Y = 2$ but $n > 3$. Then, agent h can make $\#\{\ell \in N | x^\ell = x\} \geq 2$ and $\#\{\ell \in N | x^\ell \neq x\} \geq 2$ by changing m_h to $m_h' = (R_h, x^i, k^h) \in T_h^\gamma(R,F)$. As the outcome is determined by *Rule 4*, agent h can obtain $g(m) = g(m_h', m_{-h})$ by appropriately choosing k^h. Again, $((m_h', m_{-h}), m) \in \succ_h^R$, which contradicts that $m \in NE(\gamma, \succcurlyeq^R)$.

Let us consider the case that $\#Y = 2$ and $n = 3$. Then, $g(m) = x$ as $C_i(\bar{R}_{-i}, x) \neq Y$. Suppose that agent h changes m_h to $m_h' = (R_h, x, k^h) \in T_h^\gamma(R,F)$. Suppose that $i \in D(R_i^i, R_h, \bar{R}_{-\{i,h\}})$. Then, $g(m) = g(m_h', m_{-h})$ whenever *Rule 2* applies to (m_h', m_{-h}). Otherwise, the modulo game is triggered and agent h can obtain $g(m) = g(m_h', m_{-h})$ by choosing k^h appropriately. Hence, if $i \in D(R_i^i, R_h, \bar{R}_{-\{i,h\}})$, agent h has a profitable unilateral deviation, which contradicts that $m \in NE(\gamma, \succcurlyeq^R)$. Thus, let $i \notin D(R_i^i, R_h, \bar{R}_{-\{i,h\}})$. It follows that, for all $R_i' \in \mathcal{R}_i$, $x \notin F(R_i', R_h, \bar{R}_{-\{i,h\}})$. Then, (m_h', m_{-h}) falls into *Rule 4* and agent h can obtain $g(m) = g(m_h', m_{-h})$ by choosing k^h appropriately. Agent h has a unilateral profitable deviation in the latter case too, which contradicts that $m \in NE(\gamma, \succcurlyeq^R)$.

We conclude that $m_h \in T_h^\gamma(R,F)$ for all $h \in H$. ■

*Completion of the proof of Case 3 of Theorem 6***.** In the following, we complete the proof of *Case 3* of Theorem 6 by showing that $m_h \in T_h^\gamma(R,F)$ for all $h \in H$. Assume, to the contrary, that $m_h \notin T_h^\gamma(R,F)$ for some $h \in H$. We proceed according to whether $h \in D(\bar{R}, x)$ or not.

Suppose that $h \notin D(\bar{R}, x)$. Then, for all $R'_h \in \mathcal{R}_h$, $x \notin F(R'_h, \bar{R}_{-h})$. By changing m_h to $m'_h = (R_h, x^h, k^h) \in T_h^\gamma(R, F)$, agent h can trigger the modulo game with choosing x^h appropriately. If $p(\bar{R}, x) = x$, by choosing $x^h \in Y \setminus \{x\}$, agent h can design agent $\ell^*(m'_h, m_{-h}) \in N \setminus \{h\}$ as the winner of the modulo game with choosing k^h appropriately. Otherwise, let $p(\bar{R}, x) \neq x$. Then, agent h by choosing $p(\bar{R}, x) = x^h$ can becomes the winner of the modulo game with choosing k^h appropriately. In either case, we have that $g(m) = g(m'_h, m_{-h})$, which contradicts that $m \in NE(\gamma, \succcurlyeq^R)$.

Let us consider the case that $h \in D(\bar{R}, x)$. Suppose that $C_h(\bar{R}_{-h}, x) = Y$. By changing m_h to $m'_h = (R_h, x^h, k^h) \in T_h^\gamma(R, F)$, agent h can trigger the modulo game. By choosing the outcome announcement and the integer index appropriately, agent h can obtain $g(m) = g(m'_h, m_{-h})$, which contradicts that $m \in NE(\gamma, \succcurlyeq^R)$. Otherwise, let $C_h(\bar{R}_{-h}, x) \neq Y$. As $h \in D(\bar{R}, x)$, it follows that $h \in D((R_h, \bar{R}_{-h}), x)$. By changing m_h to $m'_h = (R_h, x^h, \diamond) \in T_h^\gamma(R, F)$, agent h can make the outcome determined by *Rule 2* with choosing x^h appropriately. Suppose that $p(\bar{R}, x) = x$. By announcing $x^h \in Y \setminus C_h(\bar{R}_{-h}, x)$, agent h can obtain $g(m) = g(m'_h, m_{-h})$, and so $((m'_h, m_{-h}), m_h) \in \succ_h^R$, a contradiction. Finally, let $p(\bar{R}, x) \neq x$. Then, by announcing $x^h = p(\bar{R}, x) \in C_h(\bar{R}, x)$, agent h can again obtain $g(m) = g(m'_h, m_{-h})$, and so $((m'_h, m_{-h}), m_h) \in \succ_h^R$, a contradiction. We conclude that every partially-honest agent is reporting truthfully. ■