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First-price auctions, Dutch auctions, and buy-it-now prices with Allais paradox bidders

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This paper investigates first-price and Dutch auctions when bidders have preferences exhibiting the Allais paradox. We characterize an equilibrium for both auctions, paying particular attention to the dynamic inconsistency problems that can arise with such preferences. We show that the Dutch auction systematically yields a higher revenue than the first-price auction. This stands in sharp contrast to the presumption that these auctions are strategically equivalent, which is indeed valid in the expected utility case. We also show that introducing a "buy-it-now price" to the first-price auction increases seller's expected revenue when bidders have Allais paradox preference, while it does not for expected-utility maximizers.

Keywords. Dutch auctions (descending auctions), Allais paradox, buy-it-now-prices.

JEL classification. D44, D81.

1. Introduction

A Dutch auction is a descending auction in which an auctioneer first announces a very high price and gradually lowers it until it is accepted by one of the bidders. The winning bidder obtains the object and pays the prevailing price at that time. The Dutch auction is considered to be a strategically equivalent to the first-price auction. Since the Dutch auction immediately ends once somebody accepts the price, a bidder cannot make his stopping price contingent on other bidders' behaviors. All he can do is to internally determine the price at which he stops, and the bidder with the highest stopping price wins the auction and pays his stopping price. It is clearly equivalent to the first-price sealed-bid auction, in which everybody writes down a price on paper simultaneously and the bidder who submits the highest price wins and pays his price.

Strategic equivalence is the strongest possible equivalence notion and implies revenue equivalence without the need for particular assumptions. In contrast, revenue equivalence results between the first-price (or Dutch) and the second-price (or English) auctions (such as Myerson 1981 and Riley and Samuelson 1981) fail once we drop any of

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the assumptions such as private values, risk neutrality, and identical and independent distributions of bidders' types. Even the revenue equivalence between the second-price and the English auctions depends on the assumption of private values. In contrast, the equivalence between the first-price auction and the Dutch auction does not depend on the various assumptions listed above.

The equivalence does, however, depend on the assumption that bidders have expected-utility preferences. Indeed, expected-utility preferences guarantee that bidders' incentives do not change as the price goes down in the Dutch auction. Absent this dynamic consistency, the two auctions may result in different outcomes.

To see this, suppose that a bidder (a) prefers winning at price \$400 with probability 40% to winning at price \$500 with probability 50%, but (b) prefers winning at price \$500 for sure to winning at price \$400 with probability 80%. This preference clearly violates the independence axiom and, indeed, is a simplified version of the Allais (1953) paradox.

Let us assume that the probabilities that nobody other than the bidder himself writes down or stops at a price above either \$500 or \$400 are 50% and 40%, respectively. How does he behave in the first-price auction and the Dutch auction, respectively?

In the first-price auction, by (a), he prefers a gamble and bids \$400 to win with probability 40%. On the other hand, in the Dutch auction, he *initially* prefers to take a risk by waiting until \$400. However, *once the price drops to \$500*, by updating his winning probability, his choice becomes either winning at \$500 for sure (by stopping immediately) or winning at \$400 with probability 80% (by waiting until \$400), so he is tempted to stop at \$500 by (b). Therefore, when the bidders' preferences exhibit the Allais paradox, we can expect that the Dutch auction yields a higher revenue to the seller than the first-price auction.

Notice that this is not the case when bidders are expected-utility maximizers. For these bidders, (a) and (b) could never both hold. Thus, their incentives at the beginning of the Dutch auction remain unaltered as the price declines. Hence, the two auction formats are indeed strategically equivalent when bidders have expected-utility preferences.

There are several experimental studies of these two auction formats. Earlier laboratory experimental results consistently reject our theoretical predictions based on Allais paradox bidders. Coppinger et al. (1980) and Cox et al. (1982) report that the first-price auction yields a higher revenue than the Dutch auction in laboratories.

On the other hand, the results of recent experiments are consistent with our theoretical predictions. Lucking-Reiley (1999) conducts a field experiment by actually selling magic cards in an Internet auction and reports that the Dutch auction results in a higher revenue than the first-price auction. Also, Katok and Kwasnica (2008) obtain the same result in a laboratory experiment by making price drops in the Dutch auction slower. These two experiments suggest that the Dutch auction tends to result in higher revenue than the first-price auction when the speed at which the price falls is relatively slow. Katok and Kwasnica (2008) and Carare and Rothkopf (2005) explain this observation by hypothesizing that bidders incur the cost of monitoring the Dutch auction or the opportunity cost of time spent in the auction. We discuss in Section 5 how to distinguish their explanations from the one based on the Allais paradox.

There are several previous theoretical papers on this topic.[1] Karni (1988) first points out that these two auctions are equivalent if and only if bidders are expected-utility maximizers. Weber (1982) shows that the first-price auction yields a higher revenue than the Dutch auction when bidders' preference exhibits the *counter* Allais paradox. Nevertheless, none of them characterizes the equilibrium of the Dutch auction when bidders' preferences exhibit the Allais paradox. This paper is the first paper to characterize the equilibrium of the Dutch auction with Allais paradox preferences.

Another closely related theoretical paper is Bose and Daripa (2009), which shows that the seller can extract an entire surplus if bidders have a special class of *ambiguity* aversion preference (the epsilon-contamination model) on which their clearest result depends. In this paper, we assume bidders have a preference over *risks* that exhibits the Allais paradox, but we do not impose any other parametric restriction to characterize the equilibria of auctions.

A robust principle that emerges from our analysis is that, since bidders with Allais preferences become more risk averse when they expect to win the auction with a higher probability, they are subject to exploitation by institutional arrangements that raise the "ex post" probability of making a winning bid. That is, seller's revenues are enhanced by *making bidders optimistic at the time of their decisions*. From this viewpoint, we demonstrate that adding a buy-it-now-price to the first-price auction, which normally hurts the seller because it causes a distortion at the top, turns out to enhance seller's expected revenue against the Allais paradox bidders.

The organization of the paper is as follows. In Section 2, we discuss conditions on bidders' preferences and derive several properties. Section 3 characterizes an equilibrium in the first-price auction. Section 4 characterizes an equilibrium in the Dutch auction. Section 5 compares the seller's revenues in the two auctions. In Section 6, we apply our framework to a buy-it-now-price attached to the first-price auction and the Dutch auction, respectively. Section 7 concludes the paper.

2. Preference over lotteries

There is a single object to be sold. There are n bidders and each bidder i has a type, denoted by x_i, that represents his monetary value of the object and is only known to bidder i. Each x_i is identically and independently distributed over $[0, 1]$ with a distribution function F that admits a positive and continuous density f. Define $G(x) = (F(x))^{n-1}$ and $g = G'$. The probability that a bidder is the highest type conditional on his type being x is $G(x)$.

For $b \geq 0$ and $q \in [0, 1]$, let (b, q) be a lottery such that a bidder is awarded the object and pays b with probability q, and no transaction occurs with probability $1 - q$. Let L be the set of all such lotteries. Denote type x's preference over lotteries in L by $\succeq_x$. We do not need to consider bidder's preferences over lotteries that are not included in L. Notice that any lottery that results from a pure strategy in the first-price auction or the

[1]Nonexpected utilities are used to study English (ascending) auctions and second-price auctions, see, for example, Karni and Zvi (1986, 1989a, 1989b).

Dutch auction is included in L.[2] Therefore, we can restrict our attention to lotteries in L, which makes the model general and highly tractable. Let $\succ_x$ and $\backsim_x$ be the strict preference and the indifference relationship derived from $\succeq_x$, as usual.

Throughout the paper, we always assume the following conditions, which are standard in most of the literature.

- The operator $\succeq_x$ can be represented by a utility function $u(b, q; x)$, which is continuously differentiable in all arguments including x and is normalized to $u(0, 0; x) = 0$.
- For any x and b, $(x, 1) \backsim_x (b, 0)$.
- If $q > 0$, then $(b, q) \succ_x (b', q)$ for any $b < b'$.
- If $b < [>]\, x$, then $(b, q) \succ_x [\prec_x]\, (b, q')$ for any $q > q'$.

In addition, we impose two conditions on $\succeq_x$ as follows:

Condition 1. If $x > b > b'$ and $(b, q) \succeq_x (b', q')$, then $(b, q) \succ_{x'} (b', q')$ for any $x' > x$.

Condition 2. If $x > b > b'$ and $(b, q) \succeq_x (b', q')$, then $(b, q/\alpha) \succeq_x (b', q'/\alpha)$ for any $\alpha \in [q, 1)$.

Condition 1 requires that if a lower type prefers a safer lottery (i.e., more likely to win but pays more on winning), then a higher type must also prefer the safer one. This condition is virtually universal in the literature. Indeed, it is satisfied if $\succeq_x$ can be represented by a type-homogeneous expected utility function exhibiting risk-neutrality or risk-aversion.

Condition 2 can be interpreted as follows: if a bidder considers that the difference between two winning probabilities q and q' is large enough to cause him to accept a higher price b rather than b', then he regards the difference of the two scaled-up probabilities between q/α and q'/α as also large enough to cause him to accept the higher price. Therefore, the subjective difference between 100% and 80% is perceived as greater than or equal to that between 50% and 40%, for example. It is easy to see that an expected utility satisfies Condition 2, but the example discussed in the Introduction also meets it. Condition 2 basically accommodates the common ratio effect reported by many experiments such as Kahneman and Tversky (1979) and MacCrimmon and Larsson (1979).

For convenience, we name some classes of preferences as follows:

Definition 1. Suppose $\{\succeq_x\}_{x\in[0,1]}$ satisfies Conditions 1 and 2.

(i) It is called an expected-utility (EU) preference if, for every $x \in [0, 1]$, $(b, q) \sim_x (b', q')$ with $q > q'$ implies $(b, q/\alpha) \sim_x (b', q/\alpha)$ for any $\alpha \in [q, \infty)$.

[2] One may wonder if a bidder might be strictly better off by playing a mixed strategy, which yields a lottery that is not included in L. However, unless he has a commitment to implement his randomization, he is tempted to play any pure strategy that is strictly better than all other pure strategies and, indeed, such a strategy exists for almost all types. Therefore, we can always ignore mixed strategies.

(ii) It is called an Allais paradox (AP) preference if it is not an EU preference. Furthermore, it is called a strict Allais paradox (SAP) preference if, for every $x \in [0, 1]$, $(b, q) \sim_x (b', q')$ with $q > q'$ implies $(b, q/\alpha) \succ_x (b', q/\alpha)$ for any $\alpha \in [q, 1)$.

Finally, we define a function ϕ to express bidder's attitude toward risks:

$$\phi(b, q; x) = -\frac{u_q(b, q; x)}{u_b(b, q; x)} \cdot q.$$

This is the reduction of the winning price required to keep a bidder indifferent with respect to a 1% decrease in his winning probability (proportional to the current winning probability). Therefore, if a preference has a higher ϕ than another preference, it exhibits more risk-aversion.

Let us examine important properties of ϕ implied by Conditions 1 and 2.

Lemma 1. *Under Condition 1, $\phi(b, q; x)$ is nondecreasing in x.*

Proof. Suppose for some $x < x'$ and (b, q), $\phi(b, q; x) > \phi(b, q; x')$. Then we can find $m \in (\phi(b, q; x'), \phi(b, q; x))$ and $\varepsilon > 0$ such that

$$(b, q) \succ_x (b - m\varepsilon, (1 - \varepsilon)q) \quad \text{and} \quad (b, q) \prec_{x'} (b - m\varepsilon, (1 - \varepsilon)q),$$

which contradicts Condition 1. □

Lemma 1 is simple. Condition 1 requires that a higher type prefers a safer lottery than a lower type, so the higher type's ϕ must be no less than the lower type's ϕ.

Lemma 2. *Under Condition 2, $\phi(b, q; x)$ is nondecreasing in q. Furthermore, (i) it is constant in q if the preference is EU and (ii) for almost all (b, q), it is strictly increasing in q if the preference is SAP.*

Proof. Suppose $\phi(b, q'; x) < \phi(b, q; x)$ for some $q < q'$. Then we can find some $m \in (\phi(b, q'; x), \phi(b, q; x))$ and $\varepsilon > 0$ such that

$$(b - m\varepsilon, (1 - \varepsilon)q'; x) \succ_x (b, q'; x) \quad \text{and} \quad (b - m\varepsilon, (1 - \varepsilon)q; x) \prec_x (b, q; x).$$

The second preference, coupled with Condition 2, implies

$$(b - m\varepsilon, (1 - \varepsilon)q'; x) \prec_x (b, q'; x),$$

which contradicts the first preference.

If (i) is not true, we can find four lotteries $(b, q) \succ_x (b', q')$ but $(b, q) \prec_x (b', q')$, where $q/q' = q/q'$ in a similar way, so it cannot be an EU preference. If (ii) is not true, we can find $(b, q) \sim_x (b', q')$ and $(b, q/\alpha) \sim_x (b', q'/\alpha)$, which violate a definition of the SAP preference. □

Lemma 2 characterizes an important property of an individual EU preference: his attitude toward risks (captured by ϕ) is independent of the current winning probability, which means that for a fixed price level b, he demands the same amount of price reduction when the winning probability is reduced from 50% to 40% and from 100% to 80%. It is easy to see that if his preference is EU with utility function $q \cdot v(b; x)$, we have

$$\phi(b, q; x) = \frac{v(b; x)}{qv'(b; x)} \cdot q = \frac{v(b; x)}{v'(b; x)},$$

which shows that ϕ is indeed independent of q. (Conversely, if ϕ is independent of q, it is indeed an EU preference. We can construct an EU-form representation of a preference with ϕ.)

Alternatively, AP preference's ϕ can be increasing in q, which means that he is more risk-averse when q is high (so he is more optimistic about his chance of winning). Therefore, he may demand a greater price drop when the winning probability is lowered from 100% to 80%, but demand less when it drops from 50% to 40%.

3. The first-price auction

In the first-price auction with a reserve price $r > 0$, bidders, who participate in the auction simultaneously, submit bids $b \in [r, 1]$, and the bidder who has the highest bid wins the object and pays his bid. If no bidder participates, the seller keeps the object. Assume that ties are broken with equal probabilities.

A strategy in the first-price auction is a bidding function $b : [r, 1] \to [r, 1]$, where $b(x)$ is a bid made by a bidder with type x. We can ignore bidders whose types are below r, as they do not participate in the auction.

We focus on a symmetric equilibrium, where all bidders use the same bidding function b^*. Suppose all other bidders follow b^*. Then by bidding b, a bidder wins the auction with probability

$$W^{b^*}(b) = \left(\int_{b^*(x)<b \text{ or } x<r} dF(x)\right)^{n-1} + \sum_{i=1}^{n-1} \frac{\binom{n-1}{i}}{i+1} \left(\int_{b^*(x)=b} dF(x)\right)^{i} \left(\int_{b^*(x)<b \text{ or } x<r} dF(x)\right)^{n-i-1},$$

so bidding b yields a lottery $(b, W^{b^*}(b))$. We drop the superscript on W when this does not lead to confusion. Naturally, we can define an equilibrium of the first-price auction as follows.

Definition 2. A bidding function $b^* : [r, 1] \to [r, 1]$ is a symmetric equilibrium of the first-price auction with reserve price r if and only if

$$(b^*(x), W^{b^*}(b^*(x))) \succeq_x (b, W^{b^*}(b))$$

for all $x \in [r, 1]$ and $b \in [r, 1]$.

We now characterize b^*. First, it is pretty straightforward that Condition 1 implies that b^* must be strictly increasing and continuous. Therefore, in equilibrium, type x

gets payoff $u(b^*(x'), G(x'); x)$ by making the same bid as type x' does, so this function is maximized at $x' = x$. Taking a first-order condition, we have

$$u_b(b^*(x), G(x); x)b^{*\prime}(x) + u_q(b^*(x), G(x); x)g(x) = 0.$$

Rearranging the above condition, we get a differential equation that characterizes an equilibrium:

$$b^{*\prime}(x) = -\frac{u_q(b^*(x), G(x); x)}{u_b(b^*(x), G(x); x)} \cdot g(x) \equiv \phi(b^*(x), G(x); x) \cdot \frac{g(x)}{G(x)}.$$

This is a unique equilibrium as is shown in Appendix A.

Proposition 1. *Under Condition 1, the first-price auction with reserve price $r > 0$ has a unique equilibrium b^*, which is characterized by the differential equation*

$$b^*(r) = r \quad \textit{and} \quad b^{*\prime}(x) = \phi(b^*(x), G(x); x) \cdot \frac{g(x)}{G(x)}. \tag{1}$$

Equation (1) is easy to interpret. If type x pretends to be a slightly lower type $x - \varepsilon$, the percentage drop of his winning probability is $g(x)\varepsilon/G(x)$. To keep him indifferent, his payment on winning must drop by $\phi \cdot g\varepsilon/G$. Equation (1) guarantees that the actual price drop $b^{*\prime}(x)\varepsilon$ is indeed equal to this amount, so he has no incentive to increase or decrease his bid from $b^*(x)$.

4. The Dutch auction

We now analyze the Dutch auction. We consider a bidder who behaves in a sophisticated way in the sense that (i) he fully understands his incentive may change later and rationally predicts what to do if he does not stop the auction right now, and (ii) he compares what he gets from stopping immediately and waiting now, and makes a decision optimally at every price.[3]

The second requirement causes a problem if the price drops continuously. To evaluate what he gets from waiting, he must know which price he actually stops at. It may not be well defined if prices are continuous.

To avoid this problem, we first analyze a discrete-price model in which the seller lowers the price discretely. Formally, we define a *price grid* as $B = \{b^0, b^1, \dots, b^m\}$ with $r = b^0 < b^1 < \cdots < b^m = 1$ and define the grid size of B as $\max_{i=1,\dots,m}(b^i - b^{i-1})$. In the Dutch auction with price grid B, the auctioneer first announces b^m. The bidders simultaneously choose between stopping and waiting. If only one bidder stops, he wins the object and pays b^m, and the auction ends at this point. If more than one bidder stops, each bidder who stops is chosen as the winner with equal probability. If no bidder stops, then the auctioneer announces b^{m-1} and the same process is continued until b^0 is reached. The auctioneer keeps the object if no bidder stops at price b^0.

[3]This formulation is in the same spirit as the "multi-selves" approach to such problems suggested by Strotz (1955–1956) in his seminal paper on time inconsistent preference. This approach is widely adopted (see, for instance, Goldman 1979, Laibson 1997, and O'Donoghue and Rabin 1999).

With price grid B, the bidder's strategy is a mapping from $B \times [r, 1]$ to $\{s, w\}$, denoted by $\{D_b^B(x)\}_{b\in B, x\in[r,1]}$, where $D_b^B(x) = s$ (w) indicates that if price $b \in B$ is announced, a bidder with type x stops (waits). From a strategy D_b^B, we can derive the actual stopping price of each type: $d^B(x) = \max_{b\in B}\{b \in B \mid D_b^B(x) = s\}$, which we call the bidding function induced by D^B.

Given this, we now define an equilibrium. To do so, imagine a bidder who believes the bidding function of other bidders is $\tilde{d}$. At every price, he understands that he cannot control his future behaviors if he waits now. Therefore, he is playing a game with $|B|$ selves for fixed strategies of other bidders. Let us see how he behaves when price b^i is actually reached. This happens with probability[4]

$$S^{\tilde{d}}(b) = \left(\int_{\tilde{d}(x)<b \text{ or } x<r} dF(x)\right)^{n-1}.$$

By Bayes' rule, he gets $(b^i, W^d(b^i)/S^d(b^i))$ from stopping immediately. Then he correctly predicts what to do if he does not stop now. Suppose the next highest price at which he may stop is $b^{i'}$ if he waits now. Then what he obtains from waiting is $(b^{i'}, W^{\tilde{d}}(b^{i'})/S^{\tilde{d}}(b^i))$. He stops if he prefers the former lottery and waits otherwise. Actually, we can derive his behavior by backward induction starting from price b^0 $(= r)$ at which he should stop as long as the valuation exceeds r.

Let D^B be the strategy derived as above and let d^B be the induced bidding function from it. Then we call d^B a *response* to $\tilde{d}$. If a bidding function is a response to itself, it is an equilibrium. This concept is formalized as follows.

Definition 3. For the Dutch auction with reserve price r and price grid B, a bidding function d^B is a response to another bidding function $\tilde{d}$ if and only if there exists strategy D^B such that

(i) Strategy $D_r^B(x) = s$ if and only if $x \geq r$.

(ii) For any $i = 1, \ldots, m$, if $D_{b^i}^B(x) = s$ $[w]$, then

$$(b^i, W^{\tilde{d}}(b^i)/S^{\tilde{d}}(b^i)) \succeq_x [\preceq_x] (b^{i\prime}, W^{\tilde{d}}(b^{i\prime})/S^{\tilde{d}}(b^i)),$$

where $b^{i\prime} = \max\{b \in B \mid D_b^B(x) = s \text{ and } b < b^i\}$.

(iii) Function d^B is derived from D^B (i.e., $d^B(x) = \max_{b\in B}\{b \in B \mid D_b^B(x) = s\}$).

We say d^{*B} is a symmetric equilibrium of the Dutch auction with price grid B if and only if it is a response to itself.

We now establish the existence of an equilibrium with any price grid. To do so, we utilize the technique developed by Athey (2001), who shows the existence of a pure-strategy equilibrium for an incomplete information game when (i) there is a continuous

[4]We drop the superscript on S when it does not cause confusion.

type space, (ii) there is a finite action space, and (iii) a response to any strategy is nondecreasing, that is, a higher type selects a greater action than a lower type.

Thus, we need to prove that whenever d^B is a response to some other bidding function, it must be nondecreasing. However, Condition 1 alone does not guarantee that d^B is nondecreasing, because the next price at which the bidder stops if he waits at some price b depends on his type in general. Consider the following example.

Example 1. Suppose $B = \{\$1, \$2, \$3\}$ and the initial probabilities of winning by stopping at \$1, \$2, and \$3 are .1, .2, and .4, respectively, and assume that the probability of ties is negligible. The preference of type x includes $(\$1, .5) \succ_x (\$2, 1)$ and $(\$1, .25) \prec_x (\$3, 1)$, while the preference of type x' $(> x)$ includes $(\$2, 1) \succ_{x'} (\$1, .5)$ and $(\$2, .5) \succ_{x'} (\$3, 1)$. $\Diamond$

In this example, type x stops at \$1 and \$3, and waits at \$2, while type x' stops at \$1 and \$2, and waits at \$3. Hence, lower type x actually stops earlier than higher type x'. Notice that the above preference does not violate Condition 1. However, it is excluded by Condition 2. To see this, apply Condition 2 to $(\$1, .5) \succ_x (\$2, 1)$. Then we have $(\$1, .25) \succ_x (\$2, .5)$ so $(\$3, 1) \succ_x (\$2, .5)$. Therefore, $(\$3, 1) \succ_{x'} (\$2, .5)$ by Condition 1. This is a contradiction.

Lemma 3. *Under Conditions 1 and 2, if d^B is a response to another bidding function $\tilde{d}^B$, then d^B must be nondecreasing.*

The formal proof of Lemma 3 is provided in Appendix B. Given this and some extra work in Appendix B, we get the existence result by Athey's (2001) technique.

Proposition 2. *Under Conditions 1 and 2, the Dutch auction with reserve price $\underline{b}$ and price grid B has a symmetric equilibrium d^{*B} and it is nondecreasing.*

Now, we let the grid size converge to zero and investigate the limit of the equilibrium bidding function. Consider a sequence of price grids $\{B_k\}$, where the grid size of B_k converges to zero. Let d^{*B_k} be an equilibrium bidding function with price grid B_k. Since d^{*B_k} is uniformly bounded and nondecreasing, the sequence of d^{*B_k} must have a convergent subsequence. We show that any convergent sequence must converge to the particular function d^*, which is characterized in Proposition 3. Therefore, any sequence converges to the unique d^*.

Proposition 3. *Suppose Conditions 1 and 2 are satisfied. Let d^{*B_n} be a symmetric equilibrium of the Dutch auction with price grid B_n and reserve price r. If the grid size of B_n converges to zero as n goes to infinity, then d^{*B_n} converges to d^*, which is characterized by the differential equation*

$$d^*(r) = r \quad \textit{and} \quad d^{*\prime}(x) = \phi(d^*(x), 1; x) \cdot \frac{g(x)}{G(x)}. \tag{2}$$

Proof. Here we provide only the outline of the proof. Showing that d^* is continuous and strictly increasing is similar to the standard first-price auction case, with several complications because of time inconsistent preferences. Therefore, in the limit, $S(d^*(x)) = W(d^*(x)) = G(x)$. We now show equation (2).

Finding the lower bound of $d^{*\prime}$ is relatively easy. Notice that at any price greater than $d^{*B_k}(x)$, type x prefers waiting until $d^{*B_k}(x)$ to stopping immediately. Therefore, it must also be true in the limit and we obtain

$$u(d^*(x+\varepsilon), 1; x) \le u\left(d^*(x), \frac{G(x)}{G(x+\varepsilon)}; x\right)$$

for any $x \in [r, 1]$ and $\varepsilon > 0$, which implies

$$d^{*\prime}(x) \ge \phi(d^*(x), 1; x) \cdot \frac{g(x)}{G(x)}.$$

Alternatively, finding the upper bound of $d^{*\prime}$ is not straightforward because

$$u\left(d^*(x-\varepsilon), \frac{G(x-\varepsilon)}{G(x)}; x\right) \le u(d^*(x), 1; x) \quad \text{for small } \varepsilon > 0$$

may not be true. This is because the equilibrium condition requires that type x prefer stopping at $d^{*B_k}(x)$ to waiting *only* until his next stopping price, which depends on k. Concretely, fix any $\varepsilon > 0$. Since d^{*B_k} converges to the continuous and strictly increasing function d^*, there exists $\varepsilon' \in (0, \varepsilon)$ such that $d^{*B_k}(x) > d^{*B_k}(x-\varepsilon') > d^{*B_k}(x-\varepsilon)$ when k is large. Since by Lemma 3, type x stops at $d^{*B_k}(x-\varepsilon')$ if it is reached, his next stopping price after $d^{*B_k}(x)$ is greater than $d^*(x-\varepsilon)$. Therefore, we can never compare his payoffs from stopping at $d^{*B_k}(x)$ and from waiting until $d^{*B_k}(x-\varepsilon)$ when k is large. This is why the above inequality may not hold even for small $\varepsilon > 0$.

However, by Lemma 3, for any $\varepsilon > 0$ and k, type x must stop at $d^{*B_k}(x-\varepsilon)$ because lower type $x-\varepsilon$ stops at this price. Therefore, we can find prices $d^{*B_k}(x-\varepsilon) = d^0 < d^1 < \cdots < d^{m_k} < d^{*B_k}(x)$ such that when price d^l $(l \ge 1)$ is reached, he prefers stopping immediately to waiting until d^{l-1}.

The upper bound of $d^{*\prime}(x)$ is reached at the greatest difference between $d^{*B_k}(x)$ and $d^{*B_k}(x-\varepsilon)$. This occurs when a bidder is very risk-averse (has greater ϕ) in this region. Since ϕ is nondecreasing in q, the value of ϕ in this region is at most

$$\bar{\phi}^k = \max_{\tilde{x} \in [x, x-\varepsilon]} \phi(d^{*B_k}(\tilde{x}), 1; x).$$

Therefore,

$$\begin{aligned} d^l - d^{l-1} &\le \bar{\phi}^k \cdot \frac{W(d^l) - W(d^{l-1})}{S(d^l)} \\ &\le \bar{\phi}^k \cdot \frac{W(d^l) - W(d^{l-1})}{S(d^{*B_k}(x-\varepsilon))} \end{aligned}$$

for all $l = 0, \ldots, m_k$. By summing the above inequalities for $l = 0, \ldots, m_k$, we obtain

$$d^{*B_k}(x) - d^{*B_k}(x-\varepsilon) \le \bar{\phi}^k \cdot \frac{W(d^{*B_k}(x)) - W(d^{*B_k}(x-\varepsilon))}{S(d^{*B_k}(x-\varepsilon))}.$$

Taking the limit of both sides with respect to k, we obtain

$$d^{*B_k}(x) - d^{*B_k}(x-\varepsilon) \le \max_{\tilde{x}\in[x-\varepsilon,x]} \phi(d^*(\tilde{x}), 1; x) \cdot \frac{G(x) - G(x-\varepsilon)}{G(x-\varepsilon)},$$

which implies

$$d^{*\prime}(x) \le \phi(d^*(x), 1; x) \cdot \frac{g(x)}{G(x)}.$$ □

Remark. Although we derive an equilibrium of the first-price auction with a continuous price space, it can also be obtained as the limit of equilibria with discrete price grids.

It is easy to see that (2) can be obtained by taking the first-order condition

$$\left.\frac{du(d^*(\hat{x}), G(\hat{x})/G(x); x)}{d\hat{x}}\right|_{\hat{x}=x} = 0. \tag{3}$$

Equation (3) can be interpreted as follows: Imagine that the price drops continuously and every bidder follows a bidding function d^*, which is continuous and strictly increasing. Suppose that the price reaches $d^*(x)$. Then the probability of winning if type x waits until $d^*(\hat{x})$ is $G(\hat{x})/G(x)$, so if the left-hand side of (3) is negative, the bidder has an incentive to wait at $d^*(x)$. Alternatively, if the left-hand side is positive, then by continuity it is true at a slightly higher price level, so the bidder is tempted to stop at some higher price. Therefore, d^* must satisfy the first-order condition as given in (3).

Before ending this section, let us discuss several previous works that study the Dutch auction with nonexpected utilities. Karni (1988) shows that the first-price and the Dutch auctions are equivalent if and only if bidders are expected-utility maximizers. Weber (1982) considers a particular class of *counter* Allais paradox preferences (i.e., Condition 2 is reversed) and shows that the first-price auction dominates the Dutch auctions in terms of the revenue in the equilibria.[5]

Unlike this paper, these works implicitly assume that bidders are *naive* in the sense that they do not realize that their future incentives may be different from their current incentives, so they always (wrongly) believe that they wait and stop according to the current incentives. However, there is no work that characterizes an *equilibrium* of the Dutch auction with the Allais paradox and naive bidders. Indeed, it is extremely difficult.

To illustrate how differently the sophisticated and the naive behave, consider the following example.

[5]In their research report in 2001 (made available to the author), Soo Hong Chew and Naoko Nishimura show that Weber's result can be extended when bidders have heterogeneous preferences.

Example 2. Consider the same setting as in Example 1, but type x's preference is now

$$(\$2, 1) \succ_x (\$1, .5) \quad \text{and} \quad (\$1, .25) \succ_x (\$3, 1) \succ_x (\$2, .5).$$

Suppose the current price is \$3. If he is naive, he waits because $(\$1, .25) \succ_x (\$3, 1)$, wrongly believing that he waits at \$2, although he actually stops at \$2 because of $(\$2, 1) \succ_x (\$1, .5)$. A sophisticated bidder understands this and he stops at \$3 since $(\$3, 1) \succ_x (\$2, .5)$. ◊

This suggests that for a *fixed* strategy of all others, a sophisticated bidder's actual bid is higher than a naive bidder's bid because he has another incentive to stop earlier: to prevent his future selves from behaving against his current intention.[6] Indeed, it can happen in the equilibrium. To see this, consider the equilibrium with sophisticated bidders and type x finds that price $d^*(x)$ is actually reached. At this point, what he gets from waiting until $d^*(\hat{x})$ $(\hat{x} \leq x)$ is $(d^*(\hat{x}), G(\hat{x})/G(x))$. Let us look for the marginal gain/loss of waiting:

$$\begin{aligned}\frac{du(d^*(\hat{x}), G(\hat{x})/G(x); x)}{d\hat{x}} &= u_1 \cdot \left(d^{*\prime}(\hat{x}) + \frac{u_2}{u_1} \cdot \frac{g(\hat{x})}{G(x)}\right) \\ &= u_1 \cdot \left(\phi(d^*(\hat{x}), 1; \hat{x}) - \phi\left(d^*(\hat{x}), \frac{G(\hat{x})}{G(x)}; x\right)\right) \cdot \frac{g(\hat{x})}{G(\hat{x})}.\end{aligned}$$

The sign of this expression is not clear because ϕ is increasing both in x and q. It is negative if the effect of the Allais paradox is large (ϕ is rapidly increasing in q), in which case $u(d^*(\hat{x}), G(\hat{x})/G(x); x)$ may not be maximized at $\hat{x} = x$.[7,8]

Therefore, the equilibrium with naive AP bidders cannot be characterized simply by taking the first-order condition, because a naive bidder does not stop unless doing so generates a higher revenue than waiting until *any* lower price. At this point, all we can say is that the sophisticated bidders stop at a weakly higher price than the naive bidder, but both types stop at (weakly) higher prices than the ex ante optimal stopping price for a *fixed* strategy of other bidders.

5. The revenue comparison

In this section, we compare the revenues of the first-price auction and the Dutch auction (when the price grid size converges to zero) with the same reserve price r. Propositions 1 and 3 show that the equilibrium bidding functions b^* of the first-price auction and d^* of the Dutch auction are characterized by (1) and (2).

By Lemma 2, $\phi(b, q; x) \leq \phi(b, 1; x)$, so we have $b^{*\prime}(x) \leq d^{*\prime}(x)$ whenever $b^*(x) = d^*(x)$. Hence, b^* can never be above d^*. Also, if $\phi(b^*(x), G(x); x) < \phi(b^*(x), 1; x)$, then

[6] A similar phenomenon appears in O'Donoghue and Rabin's (1999) "immediate reward model."

[7] In extreme cases, $\hat{x} = x$ locally minimizes $u(d^*(\hat{x}), G(\hat{x})/G(x); x)$. This happens when $\phi_x(d^*(x), 1; x) < \phi_q(d^*(x), 1; x)$ so the $du/d\hat{x} < 0$ in when $\hat{x}$ is close to x.

[8] The characterization of the Dutch auction equilibrium by Weber and Chew–Nishimura is based on the first-order condition. This is possible because ϕ is decreasing in q with counter Allais preferences, so the above derivative is positive for all $\hat{x} < x$.

$d^*(x)$ cannot be equal to $b^*(x)$ because, if so, $d^*(x-\varepsilon) < b^*(x-\varepsilon)$ for any small $\varepsilon > 0$. Furthermore, if the preference is SAP, then $\phi(b, q; x)$ is strictly increasing in q for almost all (b, q), so unless the preference is degenerate, $b^*(x) < d^*(x)$ for all $x > r$.

Proposition 4. *Under Conditions 1 and 2, the Dutch auction yields no smaller revenue than the first-price auction for any realization of bidders' types. Furthermore, if the preference is SAP, the Dutch auction yields a strictly higher revenue than the first-price auction for almost all realizations of bidders' types unless the preference is degenerate in the sense that* $\phi(b^*(x), G(x); x) = \phi(b^*(x), 1; x)$ *for all* $x \in [r, 1]$.

Proposition 4 shows that the Dutch auction dominates the first-price auction in terms of revenue. If a bidder's preference is risk-averse and SAP, then the Dutch auction also outperforms the English (ascending) auction, which is known to be expected-revenue inferior to the first-price auction with risk-averse bidders (Maskin and Riley 1984). Therefore, the Dutch auction is the best among all of the popular auction mechanisms.

Finally, let us discuss another possible explanation of why the Dutch auction dominates the first-price auction as put forth by Katok and Kwasnica (2008) and Carare and Rothkopf (2005). Their explanations are based on the cost of waiting in the Dutch auction (the opportunity cost of time or the cognitive cost), which is in line with Katok and Kwasnica's (2008) experiments, where the Dutch auction generates a higher revenue when the price goes down slowly, and provides an explanation based on the opportunity cost of time spent in the auction.[9]

To distinguish the opportunity cost model from our model based on the Allais paradox, we can vary the stake of the Dutch auction to see how the difference between the first-price and the Dutch auction changes. As the stake gets larger, the opportunity cost model predicts that the difference vanishes, while our model predicts a persistent difference as long as the Allais effects persist at higher stake levels.[10]

6. Buy-it-now price

In some auctions, a seller sets a *buy-it-now price* (BP). Any bidder, by accepting the BP, can obtain the object at that price immediately, while the object is sold via the first-price auction if nobody accepts the BP. As an example, consider a governmental procurement

[9]Another possible reason why a slow Dutch auction generates a higher revenue than a quicker one is as follows: Bidders have AP preferences but may not have enough time to update their beliefs and reconsider their actions during the quick Dutch auction. If so, a quick Dutch auction is almost the same as a first-price auction, as bidders behave according to their initial plan. (The speeds of price drops in these experiments are about .75 to 2% of initial price per second in Cox et al. (1982), 5% of initial price per 5 minutes in Katok and Kwasnica (2008), and about 5% of the current price per day in Lucking-Reiley (1999).) To seriously study this, one needs to explicitly provide a model that involves the bounded rationality of bidders.

[10]The opportunity cost model is not a good explanation for Lucking-Reiley's (1999) field experiment on the Internet. Although the auctions in this experiment are very slow, the costs of waiting for cheaper prices are not very large, because the prices drop only once a day and interested bidders can receive updates of the Dutch auction automatically via daily e-mails.

auction in which the company that offers the lowest price wins the project and receives that price, which is equivalent to the first-price auction. Occasionally, the government decides the winner prior to the auction when that company accepts some reasonable price. Such a price can be interpreted as a BP. Another example of the use of a BP is in some Internet auctions. A seller may set a BP and wait for someone to accept it. When bidders believe that she holds the first-price auction later if nobody accepts the BP, this is the same situation as the previous example.

However, attaching the BP to the first-price auction seems unwise for the seller because it causes "a distortion at the top." When two bidders with relatively high valuations are willing to accept the BP, the object is awarded to the one who accepts the BP earlier and the highest type may lose, unlike in the first-price auction without a BP. Such a distortion typically reduces the seller's expected revenue.

Nevertheless, when bidders' preferences are AP, attaching a BP to the first-price auction has another effect. Suppose nobody accepts the BP and now the object is sold via a first-price auction. Then a bidder can infer that types of his opponents are not so high, which raises the probabilities of winning. Thus he becomes *more risk-averse at the time he places a bid*, which forces him to make a higher bid than the initially optimal one. We show that this positive effect dominates the negative effect when the BP is set appropriately.

To focus on the effect of the Allais paradox, we model a BP in a way that keeps bidders strongly symmetric, as follows.

1. The seller sets a BP ($\bar{b}$) and a reserve price (r), where $\bar{b} \in (r, b^*(1))$.

2. Without knowing the order of arrivals, n bidders arrive sequentially.

3. The first bidder can either accept or reject the BP. If he accepts, he obtains the object at price $\bar{b}$ and the game ends at this point. If he rejects, the second bidder has the same choice. This process continues until someone accepts the BP or all bidders reject the BP. If nobody accepts the BP, the object is sold by a first-price auction with reserve price r.

Thus, a bidders' strategy is a pair of (A, b), where $A \subset [r, 1]$ is a set of types who accept the BP and $b : [r, 1] \to [r, 1]$ is a bidding function in the (post-BP) first-price auction.

First, we consider equilibrium bidding in the first-price auction after the BP is rejected by all bidders. Notice that conditional on the first-price auction taking place (i.e., all bidders decline the BP), the probability of winning when bidding b is

$$\hat{W}(b) = \frac{(\int_{x \notin A, b(x) < b} dF(x))^{n-1}}{(\int_{x \notin A} dF(x))^{n-1}}. \tag{4}$$

Thus, in the first-price auction, a bidder makes optimal bids by referring to $\hat{W}$ as the winning probability of bidding b.

Next, we consider the decision about the BP. By accepting the BP, a bidder wins the object at $\bar{b}$ for sure. Alternatively, suppose he rejects the BP and bids b in the post-BP auction. Conditional on being offered the BP (i.e., all bidders arriving before him decline the BP), the probability that the post-BP auction takes place (i.e., all subsequent bidders decline the BP) is given by

$$\hat{\pi} = \frac{(\int_{x\notin A} dF(x))^{n-1}}{\frac{1}{n}\cdot\sum_{j=0}^{n-1}(\int_{x\notin A} dF(x))^{j}}. \tag{5}$$

Thus, he obtains the object at price b with probability $\hat{\pi}\hat{W}(b)$.[11] Therefore, we can define a symmetric equilibrium of the BP as follows:

Definition 4. The set $(\hat{A}, \hat{b})$ is a symmetric equilibrium of the BP if and only if

$$u(\hat{b}(x), \hat{W}(\hat{b}(x)); x) \geq u(b', \hat{W}(b'); x) \quad \text{for all } x \text{ and } b'$$

and

$$u(\bar{b}, 1; x) \geq u(\hat{b}(x), \hat{\pi}\hat{W}(b(x)); x) \quad \text{if and only if} \quad x \in \hat{A},$$

where $\hat{W}$ and $\hat{\pi}$ are defined by (4) and (5).

By applying the same argument as in Lemma 3, we can establish the monotonicity of a BP equilibrium. That is, $\hat{b}$ is strictly increasing and the BP is accepted only by types above some cutoff type $\hat{x} \in (r, 1)$, who are indifferent between accepting and rejecting the BP. Thus, in the equilibrium, $\hat{W}(b(x))$ and $\hat{\pi}$ become

$$\hat{W}(b(x)) = \frac{G(x)}{G(\hat{x})} \quad \text{for } x \leq \hat{x}$$

$$\hat{\pi} = \frac{G(\hat{x})}{\frac{1}{n}\sum_{j=0}^{n-1}(F(\hat{x}))^{j}}. \tag{6}$$

Particularly for cutoff type $\hat{x}$, $\hat{W}(b(\hat{x})) = 1$, that is, the cutoff type wins for sure when everybody declines the BP. Hence, when he has the BP choice but declines it, he wins the object with probability $\hat{\pi}$. Therefore, we obtain Proposition 5, which characterizes an equilibrium of the BP.

Proposition 5. *Under Conditions 1 and 2, a symmetric equilibrium $(\hat{A}, \hat{b})$ of the BP exists*[12] *and is characterized by the following conditions:*

(i) Variable $\hat{A} = [\hat{x}, 1]$ for some $\hat{x} \in (r, 1)$. That is, a bidder accepts the BP if and only if his type is no less than $\hat{x}$.

(ii) Variable $\hat{b}$ is given by

$$\hat{b}'(x) = \phi\left(\hat{b}(x), \frac{G(x)}{G(\hat{x})}; x\right)\cdot\frac{g(x)}{G(x)} \quad \textit{for } x \in [r, \hat{x}] \tag{7}$$

with $\hat{b}(r) = r$ and $\hat{b}(x) = \hat{b}(\hat{x})$ for $x > \hat{x}$.

[11] We ignore the possibility of ties in the first-price auction.

[12] Since $\hat{\pi}$ increases as $\hat{x}$ goes up, multiple equilibria may exist. However, all of the following propositions hold for any equilibrium of the BP.

(iii) Type $\hat{x}$ is indifferent between accepting and rejecting the BP. That is,

$$u(\bar{b}, 1; \hat{x}) = u(\hat{b}(\hat{x}), \hat{\pi}; \hat{x}),$$

where $\hat{\pi}$ is given by (6).

The existence of an equilibrium and the cutoff type is shown in Appendix E. Given the latter conclusion, the rest of the proposition is straightforward.

Now we compare the expected revenue to the seller between the BP and the first-price auction without a BP. To illustrate the effect of the Allais paradox, first investigate the case with EU bidders.

Since ϕ is independent of q for EU preferences, (7) is equivalent to the equality that characterizes the equilibrium of the first-price auction (without the BP) in Proposition 1. Therefore, if no one has a type greater than $\hat{x}$, both the first-price auction (without the BP) and the BP result in the same outcome, in which the highest bidder wins and pays his bid b^*. Alternatively, when there is a bidder with a type greater than $\hat{x}$, the first-price auction allocates the object to the highest type, while the BP chooses the winner randomly among bidders whose types are greater than $\hat{x}$.

Thus the allocation is ex post efficient in the first-price auction but not in the BP. Maskin and Riley (1984) show that if the payment on winning depends only on the winner's type, the object should be awarded to the bidder with the highest valuation as long as a virtual value is increasing in an actual type.[13] Therefore, attaching a BP to the first-price auction against EU bidders reduces the seller's expected revenue.

Now, we consider SAP bidders. Contrary to the case of EU bidders, the equilibrium bidding function after the BP is rejected by all bidders is higher than that of the first-price auction without a BP, because bidders' winning probabilities get higher *at the time of making a bid*, so they are more risk-averse. We can confirm this by looking at (7). Now

$$\phi\left(\hat{b}(x), \frac{G(x)}{G(\hat{x})}; x\right) > \phi(\hat{b}(x), G(x); x),$$

so it must be that

$$\hat{b}(x) > b^*(x) \quad \text{for } x \in (r, \hat{x}].$$

Thus, attaching a BP to the first-price auction has another effect on the seller's revenue. When the BP is declined by all bidders, the seller obtains a higher revenue than he would get without the BP.

Suppose a seller sets a BP slightly lower than $b^*(1)$, say $\bar{b} = b^*(1) - \varepsilon$. Then the BP is accepted by types close to 1 and the probability of acceptance is proportional to ε. The expected revenue from these types might increase or decrease, but such a change is proportional to ε. Therefore, it is just a second-order effect. Alternatively, when the

[13] Maskin and Riley (1984) characterize the optimal auction for risk-averse bidders, which involves transfers to losers. When we restrict our attention to mechanisms without payments from (or transfers to) losers, the optimal auction is the first-price auction if $x - (1 - F)/f$ is monotone. This can be derived by restricting $a(\theta) = 0$ in their papers.

BP is set, any type who rejects the BP increases his bid in proportion to ε, compared to the first-price auction without a BP. Since the probability that the BP is declined by all bidders is close to 1, this is a first-order positive effect. Therefore, if ε is small, the gain outweighs a possible loss, so the seller's expected revenue is increased. The formal proof is given in Appendix F.

Proposition 6. *(i) Assume the bidders' preferences are EU and satisfy Condition 1. In addition, assume $x - (1 - F(x))/f(x)$ is increasing in x. Then attaching a BP with $\bar{b} < b^*(1)$ to the first-price auction reduces the seller's expected revenue.*

(ii) Suppose the bidders' preferences are SAP. Then there exists $\varepsilon > 0$ such that if $\bar{b} \in (b^*(1) - \varepsilon, b^*(1))$, attaching a BP to the first-price auction strictly increases the seller's expected revenue.

Before ending the discussion, let us consider an alternative form of the BP, in which each bidder submits his bid right away when he declines the BP.

This alternative format is strategically equivalent to the standard BP when bidders have EU preferences, but is no longer equivalent when they are Allais paradox bidders. In the original form of the BP, if a bidder has a chance to place a bid, he knows that everybody has declined the BP, so *his bid is guaranteed to be considered at the time he places it*. In the alternative form of the BP, he only knows that all bidders who arrive before him reject the BP, so *his bid is considered only if all subsequent bidders reject the BP*. Therefore, *at the time of placing his bid*, his chance of winning is lower in the alternative format, which makes him more risk-taking when he places a bid. Therefore, this alternative format is strictly worse for the seller.

Nevertheless, Proposition 6 still holds with a slight modification of the proof. This is because, even in this alternative format, each bidder knows that all other bidders who arrive before him decline the BP, which makes him more risk-averse compared with the beginning of the game. Therefore, if the BP is appropriately chosen, the expected revenue of a seller is strictly higher than the first-price auction without a BP when bidders' preferences are AP.

7. Conclusion

We show that the Dutch auction yields a higher revenue than the first-price auction, provide a complete characterization of symmetric equilibria with sophisticated bidders, and give an explicit treatment of the dynamic inconsistency issues that Allais preferences cause. These results are consistent with the recent field and laboratory experimental results reported by Lucking-Reiley (1999) and Katok and Kwasnica (2008).

Also, we observe several interesting effects of Allais preferences. Attaching a BP to the first-price auction and choosing the BP appropriately yields a strictly higher expected revenue to the seller, contrary to the case when bidders' preferences are EU.

These results reflect a general principle: when bidders have Allais preferences, the seller should make bidders more optimistic *at the time they make a decision*. Applying this principle to other economic environments is one possible direction for future research.

There are a few recent papers that examine the impact of Allais preferences in strategic models. Nakajima (2005) shows that the hypothesis that bidders' preferences are EU systematically underestimates the seller's expected revenue when the seller raises the reserve price of the first-price auction. Chew and Nishimura (2003) show that even in the private value case, if the object sold is risky, the second-price and the English auctions do not result in the same outcome with AP bidders. Eliaz et al. (2006) use Allais preferences to explain "choice shift" in group decision making, where each member of the group tends to move from one extreme to the other more than when each member makes a decision on his or her own.

The very interesting open question is the structure of optimal auctions when bidders have Allais preferences. Unfortunately, this is also a very difficult question because (among other issues) we cannot rely on the revelation principle anymore. For instance, the equilibrium of the Dutch auction cannot be replicated by any simultaneous-move game.

Maskin and Riley (1984) characterize the optimal auction when the bidders are risk-averse (and expected-utility maximizers). Their mechanism involves transfers to losers, so that the highest possible type is perfectly insured and all other types are partially insured. Given this result, we can immediately conclude that the Dutch auction is not optimal in general. For instance, if bidders are significantly risk-averse and have very weak Allais paradox preferences, then the expected revenues between the first-price auction and the Dutch auction are very close, while Maskin and Riley's mechanism yields a significantly higher expected revenue than the first-price auction. Since their mechanism needs information from all bidders, it cannot be implemented in a descending manner like the Dutch auction. At this point, we have little idea of what the optimal auction looks like.

Finally, we note that this research suggests a range of interesting experiments. Recent experiments suggest that the slow Dutch auction yields a higher revenue than the first-price auction. As we discuss in Section 5, a good test of our theoretical result would be an experiment with a slow Dutch auction where the effect of the opportunity costs of time is carefully mitigated.

We also present several results about setting a buy-it-now price. It should be relatively easy to experimentally test the nonequivalence between buy prices prior to and during the first-price auctions, as the equilibrium bidding functions are clearly different between the two games.

Appendix A: Proof for Proposition 1

First, we prove that b^* is strictly increasing and continuous in x. Suppose $x < x'$ exists such that $b^*(x) > b^*(x')$. Then by type x's incentive, $(b^*(x), W(b^*(x))) \succeq_x (b^*(x'), W(b^*(x')))$, so $(b^*(x), W(b^*(x))) \succ_{x'} (b^*(x'), W(b^*(x')))$ by the Condition 1, so b^* cannot be an equilibrium. Hence, b^* must be nondecreasing. The proofs that b^* is strictly increasing and continuous are the same as those in standard models and are omitted.

Given this, for any $\varepsilon > 0$,

$$u(b^*(x), G(x); x) \geq u(b^*(x+\varepsilon), G(x+\varepsilon); x).$$

Since u is continuously differentiable, and b^* and G are continuous, there exists $\varepsilon' \in [0, \varepsilon]$ such that

$$\begin{aligned} &u_b(b^*(x+\varepsilon'), G(x+\varepsilon); x)(b^*(x+\varepsilon) - b^*(x)) \\ &\qquad + u_q(b^*(x+\varepsilon'), G(x+\varepsilon'); x)(G(x+\varepsilon) - G(x)) \geq 0. \end{aligned}$$

Therefore,

$$\frac{b^*(x+\varepsilon) - b^*(x)}{\varepsilon} \geq -\frac{u_q(b^*(x+\varepsilon'), G(x+\varepsilon'); x)}{u_b(b^*(x+\varepsilon'), G(x+\varepsilon); x)} \cdot \frac{G(x+\varepsilon) - G(x)}{\varepsilon}.$$

We can apply the same arguments using type $x + \varepsilon$'s incentive and obtain the right derivative. Exactly the same step for $\varepsilon < 0$ shows that the left derivative is also equal to $\phi g(x)/G(x)$.

Alternatively, suppose b^* satisfies (1). Then

$$\begin{aligned} &\frac{du(b^*(\hat{x}), G(\hat{x}); x)}{d\hat{x}} \\ &\qquad = u_b(b^*(\hat{x}), G(\hat{x}); x)\left(b^{*\prime}(\hat{x}) - \phi(b^*(\hat{x}), G(\hat{x}); x) \cdot \frac{g(\hat{x})}{G(\hat{x})}\right) \\ &\qquad \gtreqless u_b(b^*(\hat{x}), G(\hat{x}); x)\left(b^{*\prime}(\hat{x}) - \phi(b^*(\hat{x}), G(\hat{x}); \hat{x}) \cdot \frac{g(\hat{x})}{G(\hat{x})}\right) \quad (\text{for } \hat{x} \lesseqgtr x) \\ &\qquad = 0, \end{aligned}$$

where the second inequality comes from Lemma 1, $u_b < 0$, while the last equality is from (1). Therefore, type x maximizes his utility by bidding $b^*(x)$, so b^* is indeed an equilibrium.

Appendix B: Proof for Lemma 3

Suppose d^B is a response to another bidding function and let D^B be a strategy inducing d^B. We show whenever $D^B_{b^i}(x) = s$, that $D^B_{b^i}(x') = s$ for any $x' > x$ by induction. This is true for $i = 0$, so assume that it is true for any $i \leq k-1$. Let $i' = \max_j\{j < k \mid D^B_{b^j}(x) = s\}$ and $i'' = \max_j\{j < k \mid D^B_{b^j}(x') = s\}$. By the induction hypothesis, $i' \leq i''$.

If $i' = i''$, we can apply the same argument for the first-price auction, so the statement is true for $i = k$. Suppose $i' < i''$ and $D^B_{b^k}(x) = s$. Then by type x's incentives at b^k and $b^{i''}$, we have $(b^k, W(b^k)/S(b^k)) \succeq_x (b^{i'}, W(b^{i'})/S(b^{i'}))$ and $(b^{i'}, W(b^{i'})/S(b^{i''})) \succeq_x (b^{i''}, W(b^{i''})/S(b^{i''}))$. By Condition 2, the second relationship implies $(b^{i'}, W(b^{i'})/S(b^k)) \succeq_x (b^{i''}, W(b^{i''})/S(b^k))$ because $W(b^{i'}) \leq W(b^{i''})/S(b^{i''}) \leq S(b^k)$. Thus, we have $(b^k, W(b^k)/S(b^k)) \succeq_x (b^{i''}, W(b^{i''})/S(b^k))$. Therefore, by Condition 1, we have $(b^k, W(b^k)/S(b^k)) \succ_{x'} (b^{i''}, W(b^{i''})/S(b^k))$, so type x' must stop at b^k.

Appendix C: Proof for Proposition 2

Fix price grid $B = \{b_0, \dots, b_m\}$. For any nondecreasing function $d:[r,1] \to B$ (we drop the superscript on d to simplify the notation), define $a(d) = (a_1^d, \dots, a_m^d)$ as $a_i^d = \sup\{x \mid d(x) \le b_{i-1}\}$. Then set $\Gamma \equiv \{a(d) \mid d:[r,1] \to B, \text{nondecreasing}\}$ as a compact subset of R^m. Now define the following correspondence from Γ to itself:

$$\mathrm{BR}(a(d)) = \{a(\tilde{d}) \mid \tilde{d} \text{ is a response to } d \text{ in the Dutch auction with price grid } B\}.$$

This correspondence is well defined because a response to d is always nondecreasing by Lemma 3, and $\mathrm{BR}(a(d))$ cannot be empty because price grid B is a finite set.

Indeed, $\mathrm{BR}(a(d))$ is a single-valued function. To see this, suppose both $\tilde{d}$ and $\hat{d}$ are responses to d but $a(\tilde{d}) \ne a(\hat{d})$. Then there exists j such that $a_j(\tilde{d}) < a_j(\hat{d})$. Therefore, there exists x and x' such that $a_j(\tilde{d}) < x' < x'' < a_j(\hat{d})$. By construction, $\tilde{d}(x) > b_{j-1}$ and $\hat{d}(x) \le b_{j-1}$ for $x = x', x''$. Therefore, the function

$$d^*(x) = \begin{cases} \tilde{d}(x) & \text{for } x \ne x'' \\ \hat{d}(x) & \text{for } x = x'' \end{cases}$$

is a response to d, but $d^*(x'') < d^*(x')$, which contradicts Lemma 3. Hence, $\mathrm{BR}(a(d))$ is a single-valued function.

We now prove that $\mathrm{BR}(a(d))$ is continuous. Take any sequence $\{a(d_n)\}$ converging to $a(d)$. For each n, define $a(\tilde{d}_n) = \mathrm{BR}(a(d_n))$. Suppose sequence $\{a(\tilde{d}_n)\}$ converges to $a(\tilde{d})$. We show that $a(\tilde{d}) = \mathrm{BR}(a(d))$.

Suppose $x \in (a_i(\tilde{d}), a_{i+1}(\tilde{d}))$. Then $x \in (a_i(\tilde{d}_n), a_{i+1}(\tilde{d}_n))$ for sufficiently large n. Therefore, $\tilde{d}(x) = \tilde{d}_n(x) = b_i$. We show that type x's response to d must involve waiting until b_i and stopping at b_i.

For each n, consider $B_n = \{\hat{b} \in B \mid D_b(x) = s$ and D is a best responding strategy to $d_n\}$. By construction, $\max B_n = b_i$. Since price grid B is a finite set, there exists B' such that $B' = B_n$ for infinitely many n. Without loss of generality, assume $B' = B_n$ for all n. (If not, take such a subsequence of $\{a(d_n)\}$.)

Then a strategy that stops if and only if $b \in B'$ is best responding to each d^n. Therefore, for all $b_k \in B$ with $k > 0$,

$$(b_k, W^{d_n}(b_k)/S^{d_n}(b_k)) \succeq_x (\preceq_x)\ (b'_k, W^{d_n}(b'_k)/S^{d_n}(b)) \quad \text{if } b \in B'\ (b \notin B'),$$

where $b'_k = \max\{\hat{b} \in B'$ and $\hat{b} \le b_k\}$. Since $S^{d_n}(b_j) = G(a_{j+1}(d_n))$, S^{d_n} is continuous in $a(d_n)$. It can also be checked that W^{d_n} is also continuous in $a(d_n)$. Therefore, we have

$$(b, W^d(b)/S^d(b)) \succeq_x (\preceq_x)\ (b', W^d(b')/S^d(b)) \quad \text{if } b \in B'\ (b \notin B'),$$

which implies that stopping if and only if $b \in B'$ is also a best responding strategy to d, so type x actually stops at b_i. We can make the same arguments for all x (except when $x = a_i(\tilde{d})$ for some i) and verify that $\tilde{d}(x)$ is indeed a response to d. Therefore, $a(\tilde{d}(x)) = \mathrm{BR}(a(d(x)))$.

Hence, by Brower's fixed point theorem, BR has a fixed point $a(d_B^*)$, and such d_B^* is an equilibrium of the Dutch auction with price grid B.

Appendix D: Proof for Proposition 3

First we show d^* is a strictly increasing function. Suppose not. Then there exist x and x' $(> x)$ such that for any $\varepsilon > 0$, $d^{*B_k}(x') - d^{*B_k}(x) < \varepsilon$ holds for sufficiently large k, and we can find $b' \in B_k$ satisfying $b' \in (d^{*B_k}(x'), d^{*B_k}(x') + 2\varepsilon)$. Then if type x stops at b', compared to waiting until $d^{*B_k}(x)$, he can increase his winning probability by at least $(G(x') - G(x))/2$, while the payment on winning increases by less than ε. Therefore, he is tempted to stop at b', so d^{*B_k} cannot be an equilibrium. Consequently, d^* must be strictly increasing.

Next we prove d^* is a continuous function. Suppose not. Then there exists x^* such that $\lim_{x'\uparrow x^*} d^*(x') < \lim_{x'\downarrow x^*} d^*(x')$. Since d^* is strictly increasing, for any x, all of $W^{d^{*B_k}}(d^{*B_k}(x))$, $W^{d^{*B_k}}(d^{*B_k}(x))$, $S^{d^{*B_k}}(d^{*B_k}(x))$, and $S^{d^{*B_k}}(d^{*B_k}(x))$ converge to $G(x)$.

Suppose $d^*(x^*) > \lim_{x'\uparrow x^*} d^*(x')$. Then for sufficiently large k, the next stopping price of type x^* after $d^{*B_k}(x^*)$ is either $b^{k\prime} = \lim_{x'\uparrow x^*} d^{*B_k}(x')$ or the price just above $b^{k\prime}$ (let this price be $b^{k\prime\prime}$). This is because (i) type x has no incentive to stop strictly between $b^{k\prime\prime}$ and $d^{*B_k}(x^*)$, since if the price drops below $d^{*B_k}(x^*)$, the probability that some other bidder stops the auction before $b^{k\prime}$ is zero, and (ii) type x must stop at $b^{k\prime\prime}$, since $b^{k\prime\prime}$ is accepted by a type slightly lower than x, by Lemma 3.

As k goes to infinity, the difference between the winning probabilities of stopping at $d^{*B_k}(x^*)$ and waiting until $b^{k\prime}$ or $b^{k\prime\prime}$ converges to zero, because d^{*B_k} converges to a strictly increasing function. However, $d^{*B_k}(x^*) - b^{k\prime\prime}$ is bounded away from zero, so type x is tempted to wait at $d^{*B_k}(x^*)$, because his next stopping price is either $b^{k\prime}$ or $b^{k\prime\prime}$. Therefore, $d^*(x^*) = \lim_{x'\uparrow x^*} d^*(x')$.

Suppose now that $d^*(x^*) < \lim_{x'\downarrow x^*} d^*(x')$. Then for each k, define $b^{k\prime} = \min\{d^{*B_k}(x) \mid x > x^*\}$. Then $\lim b^{k\prime} = \lim_{x'\downarrow x^*} d^*(x')$. Again, as k goes to infinity, the difference between the winning probabilities of stopping $b^{k\prime}$ and waiting until $d^{*B_k}(x^*)$ converges to zero, so types stopping at $b^{k\prime}$ are tempted to wait until $d^{*B_k}(x^*)$, because $b^{k\prime} - d^{*B_k}(x^*)$ is bounded away from zero. Hence, d^{*B_k} cannot be an equilibrium, so $d^*(x^*) = \lim_{x'\downarrow x^*} d^*(x')$. Therefore, d^* is continuous.

Now, we show that d^* satisfies the differential equation (2).

Consider any x and x' such that $x > x'$. Then

$$u\left(d^{*B_k}(x'), \frac{W(d^{*B_k}(x'))}{S(d^{*B_k}(x))}; x'\right) \geq u\left(d^{*B_k}(x), \frac{W(d^{*B_k}(x))}{S(d^{*B_k}(x))}; x'\right)$$

for all k; otherwise type x' would be tempted to stop at $d^{*B_k}(x)$. As u is continuous, the inequality is true at the limit, so we obtain

$$u\left(d^*(x'), \frac{G(x')}{G(x)}; x'\right) \geq u(d^*(x), 1; x'),$$

so for any $x > x'$, we have

$$\begin{aligned} \frac{d^*(x) - d^*(x')}{x - x'} &\geq -\frac{u_q(b, q; x)}{u_b(b, q; x)} \cdot \frac{G(x) - G(x')}{G(x)} \cdot \frac{1}{x - x'} \\ &= \phi(b, q; x) \cdot \frac{1}{q} \cdot \frac{1}{G(x)} \cdot \frac{G(x) - G(x')}{x - x'} \end{aligned} \tag{8}$$

for some $b \in [d^*(x), d^*(x')]$ and $q \in [G(x')/G(x), 1]$.

Next we find the upper bound of the above value. Let

$$\tilde{B}^k = \left\{\tilde{b} \in B^k \mid D_x^{*B_k}(\tilde{b}) = s \text{ and } \tilde{b} \in [d^{*B_k}(x'), d^{*B_k}(x)]\right\},$$

which is a set of prices between $d^{B_k}(x')$ and $d^{B_k}(x)$ at which type x stops if they are reached with price grid B^k. Clearly, $d^{*B_k}(x) \in \tilde{B}^k$ and Lemma 3 implies $d^{*B_k}(x') \in \tilde{B}^k$. Let us order the elements of $\tilde{B}^k$ as $d^{*B_k}(x') = \tilde{b}^0 < \tilde{b}^1 < \tilde{b}^2 < \cdots < \tilde{b}^{m_k} = d^{*B_k}(x)$. Then type x's incentive at each price in $\tilde{B}^k$ implies

$$u\left(\tilde{b}^l, \frac{W^l}{S^{l+1}}; x\right) \leq u\left(\tilde{b}^{l+1}, \frac{W^{l+1}}{S^{l+1}}; x\right)$$

for all $l \in \{0, \dots, m_k - 1\}$, where $S^l = S^{d^{*B_k}}(\tilde{b}^l)$ and $W^l = W^{d^{*B_k}}(\tilde{b}^l)$. The above inequality implies that for any $l \in \{0, \dots, m_k - 1\}$, there exist $\hat{b}^l \in [\tilde{b}^l, \tilde{b}^{l+1}]$ and $q^l \in [W^l, W^{l+1}]$, and

$$\begin{aligned}
\tilde{b}^{l+1} - \tilde{b}^l &\leq -\frac{u_q(\hat{b}^l, q^l/S^{l+1}; x)}{u_b(\hat{b}^l, q^l/S^{l+1}; x)} \cdot \frac{W^{l+1} - W^l}{S^{l+1}} \\
&= \phi(\hat{b}^l, q^l; x) \cdot \frac{W^{l+1} - W^l}{q^l} \\
&\leq \phi(\hat{b}^l, 1; x) \cdot \frac{W^{l+1} - W^l}{q^l} \\
&\leq \max_{b \in [\tilde{b}^0, \tilde{b}^{m_k}]} \phi(b, 1; x) \cdot \frac{W^{l+1} - W^l}{W^0},
\end{aligned}$$

where the equality comes from the definition of ϕ, the second inequality follows from Lemma 2, and the last inequality is because $\hat{b}^l \in [\tilde{b}^0, \tilde{b}^{m_k}]$ and $q^l \geq W^l \geq W^0$. Summing both sides of the above inequality for $l = 0, \dots, m_k - 1$ gives

$$d^{*B_k}(x) - d^{*B_k}(x') \leq \max_{b \in [\tilde{b}^0, \tilde{b}^{m_k}]} \phi(b, 1; x) \frac{W^{m_k+1} - W^0}{W^0}.$$

Taking the limits of both sides, we obtain

$$\frac{d^*(x) - d^*(x')}{x - x'} \leq \max_{b \in [d^*(x'), d^*(x)]} \phi(b, 1; x) \cdot \frac{1}{G(x')} \cdot \frac{G(x) - G(x')}{x - x'}. \tag{9}$$

Hence, (8) and (9) imply

$$\lim_{x' \uparrow x} \frac{d^*(x) - d^*(x')}{x - x'} = \phi(d^*(x), 1; x) \cdot \frac{g(x)}{G(x)}.$$

Exactly the same argument shows that the right derivative of d^* coincides with the left derivative, and so we obtain (2).

Appendix E: Proof for Proposition 5

First we show (i). Take any $x \in A$. Then it must be that

$$u(\bar{b}, 1; x) \geq u(\hat{b}(x), \hat{\pi}\hat{W}(\hat{b}(x)); x).$$

Take any $x' > x$. Since type x bids $\hat{b}(x)$ rather than $\hat{b}(x')$ in the auction after the BP is declined, it must be that

$$u(\hat{b}(x), \hat{W}(\hat{b}(x)); x) \geq u(\hat{b}(x'), \hat{W}(\hat{b}(x')); x).$$

Condition 1 implies $\hat{W}(\hat{b}(x)) \leq \hat{W}(\hat{b}(x'))$, so applying the contraposition of Condition 2, we obtain

$$u(\hat{b}(x), \hat{\pi}\hat{W}(\hat{b}(x)); x) \geq u(\hat{b}(x'), \hat{\pi}\hat{W}(\hat{b}(x')); x).$$

Thus we get

$$u(\bar{b}, 1; x) \geq u(\hat{b}(x'), \hat{\pi}\hat{W}(\hat{b}(x')); x).$$

Therefore, by Condition 1, the above inequality holds strictly for type x', so $x' \in A$. Therefore, $A = [\hat{x}, 1]$ for some $\hat{x} \in [r, 1]$.

Given that result, in the auction after the BP bidders know that all of them have types less than $\hat{x}$, it is immediate that $\hat{b}$ is strictly increasing and continuous, so if the post-BP auction takes place, then bid $\hat{b}(x)$ wins with probability $G(x)/G(\hat{x})$. Hence, $\hat{b}$ is given by the differential equation in the proposition. Obviously $\hat{b}(r) = r$ and $\hat{b}(x) = \hat{b}(\hat{x})$ for all $x > \hat{x}$. (If type $x > \hat{x}$ declines the BP, he wins for sure by bidding $\hat{b}(\hat{x})$ once the auction takes place.) Thus (ii) is proven.

Now we show $\hat{x} \in (r, 1)$. Since $\bar{b} > r$, type r strictly prefers to reject the BP, so it must be $\hat{x} > r$. Alternatively, if no type accepts the BP (so $\hat{x} = 1$), then the differential equation in (ii) is exactly the same as the one that characterizes the first-price auction equilibrium, so we have $\hat{b}(1) = b^*(1)$. Since $\bar{b} < b^*(1)$, the highest type $x = 1$ strictly prefers to accept the BP, so it must be $\hat{x} < 1$. Hence we obtain $\hat{x} \in (r, 1)$ and the cutoff type must be indifferent, which proves (iii).

Finally, we establish the existence. Suppose any type above $\tilde{x}$ accepts the BP and any type below $\tilde{x}$ rejects the BP. Let $\hat{\pi}(\tilde{x})$ be the probability that the BP is rejected by all other bidders conditional on a particular bidder choosing to accept or reject the BP. For $x \leq \tilde{x}$, define $\tilde{b}(x; \tilde{x})$ by

$$\frac{\partial \tilde{b}}{\partial x} = \phi\left(x - \tilde{b}(x; \tilde{x}), \frac{G(x)}{G(\tilde{x})}; x\right) \frac{g(x)}{G(x)}$$

with $\tilde{b}(r; \tilde{x}) = r$. Then $\hat{x} \in (r, 1)$ is an equilibrium cutoff type if and only if it is a solution to

$$u(\bar{b}, 1; \hat{x}) = u(\tilde{b}(\hat{x}; \hat{x}), \hat{\pi}(\hat{x}); \hat{x}). \tag{10}$$

Notice that in (10), the right-hand side is strictly greater than the left-hand side when $\hat{x} = r$, because $\bar{b} > r$, and the left-hand side is strictly greater than the right-hand side when $\hat{x} = 1$, because $\hat{\pi}(1) = 1$, and $\tilde{b}(1; 1) = b^*(1)$, and $\bar{b} < b^*(1)$. Therefore, there exists

$\hat{x} \in (r, 1)$ satisfying (10) because of the continuity of all functions in (10). This establishes the existence of an equilibrium and (iii).

Appendix F: Proof for Proposition 6

Let $b_1 = b^*(1)$ and $\bar{b} = b_1 - \varepsilon$. Suppose $\hat{b}_\varepsilon$ is the equilibrium bidding function after the BP is declined and $\hat{x}_\varepsilon$ is the equilibrium cutoff type. Then the difference between the expected revenues of the BP and of the first-price auction is given by

$$\begin{aligned} R(\varepsilon) &\equiv \int_{\hat{x}_\varepsilon}^{1} (b_1 - \varepsilon - b^*(t))\, dG(t) + \int_{r}^{\hat{x}_\varepsilon} (\hat{b}_\varepsilon(t) - b^*(t))\, dG(t) \\ &\geq \int_{b^{*-1}(b_1-\varepsilon)}^{1} (b_1 - \varepsilon - b^*(t))\, dG(t) + \int_{r}^{b^{*-1}(b_1-\varepsilon)} (\hat{b}_\varepsilon(t) - b^*(t))\, dG(t) \\ &\equiv \tilde{R}(\varepsilon). \end{aligned}$$

The inequality holds because type $x \in [\hat{x}_\varepsilon, b^{*-1}(b_1 - \varepsilon)]$ accepts the BP, which is greater than $\hat{b}_\varepsilon(x) = \hat{b}_\varepsilon(\hat{x}_\varepsilon)$. Now we show $\tilde{R}'(0) > 0$. Since $b^{*-1}(b_1) = 1$, $\hat{b}_0(1) = b^*(1)$, we have

$$\tilde{R}'(0) = \int_r^1 \frac{\partial \hat{b}_0(t)}{\partial \varepsilon}\, dG(t).$$

Notice that

$$\frac{\partial \hat{b}_0(x)}{\partial \varepsilon} \geq 0 \quad \text{for any } x \geq r,$$

so $\tilde{R}'(0) \geq 0$. Suppose $\tilde{R}'(0) = 0$. Then it must be that

$$\frac{\partial \hat{b}_0(x)}{\partial \varepsilon} = 0 \quad \text{for any } x \geq r.$$

Since

$$\hat{b}_\varepsilon(x) = \int_r^x \phi\left(\hat{b}_\varepsilon(t), \frac{G(t)}{G(b^{*-1}(b_1 - \varepsilon))}; t\right) \frac{g(t)}{G(t)}\, dt,$$

we obtain

$$\begin{aligned} \frac{d\hat{b}_0(x)}{d\varepsilon} &= \int_r^x \left(-\phi_b \frac{\partial \hat{b}_0(t)}{\partial \varepsilon} + \phi_q \frac{G(t)g(1)}{b^{*\prime}(1)}\right) \frac{g(t)}{G(t)}\, dt \\ &= \int_r^x \phi_q \frac{g(1)g(t)}{b^{*\prime}(1)}\, dt \\ &= 0, \end{aligned}$$

which is impossible because $\phi_q > 0$ as bidders have SAP preferences. Hence it must be that

$$\frac{\partial \hat{b}_0(x)}{\partial \varepsilon} \geq 0 \quad \text{for any } x \geq r,$$

because any type who rejects the BP bids higher in the auction after the BP is rejected than in the first-price auction. Thus, inside the integral must be positive, $\tilde{R}'(0) > 0$. Hence for small $\varepsilon > 0$, $R(\varepsilon) > \tilde{R}'(\varepsilon) > 0$, so the expected revenue when the seller sets the BP is greater than the first-price auction without a BP, when ε is small.

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