Harvesting technology and catch-to-biomass dependence: The case of small pelagic fish*

Pedro Gajardo† Julio Peña-Torres‡ Héctor Ramírez C.§

December 28, 2009

Abstract

This note deals with a harvesting model for a single stock fishery. In the case of small pelagic fish it seems reasonable to consider harvest functions depending nonlinearly on fishing effort and on fish stock. Empirical evidence about these fish species suggests that marginal catch does not necessarily react in a linear way to changes in fishing effort and fish stock levels. This is in contradiction with traditional fishery models where catch-to-input marginal productivities are normally assumed to be constant. While allowing for non linearities in both catch-to-effort and catch-to-stock parameters, this note extends the traditional analysis by focusing on the dependence of the stationary solutions upon the nonlinear catch-to-biomass parameter. Given the emphasis on the case of small pelagic fish, the analysis considers positive but small values for the catch-to-stock parameter.

Keywords: small pelagic fisheries, harvesting functions, Cobb-Douglas production function, optimal control, maximum principle.

1 Introduction

Small pelagic fish stocks, such as anchovy, sardine, herring and jack mackerel represent an important proportion of world's marine fish harvests (about a third of it, with more than 20 million tons per year worldwide, according to FAO statistics). In some fishing nations (e.g., Peru and Chile), fisheries of this type are important resources for their national economies, both in terms of value added production as well as for regional employment.

Small pelagic stocks are characterized by some peculiar features. On the one hand, they tend to face strong and recurrent cycles of fish abundance. On the other, they usually provide for high catch yields per fishing effort unit. Given these characteristics, different pelagic stocks have experienced fishing collapse. Examples in the XXth century are the sardine fishery in Japan during the early

*Supported by Stic AmSud cooperation program (MIFIMA), FONDECYT grants 1070297 and 1080173, ECOS-CONICYT project C07E03, FONDAP in Applied Mathematics and BASAL project.

†Departamento de Matemática, Universidad Técnica Federico Santa María. Avda. España 1680, Valparaíso, Chile (pedro.gajardo@usm.cl)

‡Facultad de Economía y Negocios/ILADES, Universidad Alberto Hurtado. Erasmo Escala 1835, Santiago, Chile (jpena@uahurtado.cl)

§Departamento de Ingeniería Matemática, Centro de Modelamiento Matemático(CNRS UMI 2807), FCFM, Universidad de Chile, Avda. Blanco Encalada 2120, Santiago, Chile (hramirez@dim.uchile.cl).

1940s, the sardine fishery in California a decade later, the herring population in the North Sea at the end of the 1960s and early 1970s, and the early 1970s collapse of the Peruvian anchovy (Peña-Torres [20]).

However, the abovementioned characteristics are normally absent from traditional mathematical fishery models. This literature has typically focused on the case of linear harvest functions (well known examples are Clark [9, 10]; Plourde and Yeung [23]; Dockner et al. [13]). This approach has obvious advantages in terms of mathematical tractability. Thus, the model that usually describes a single species fish stock's evolution in continuous time is

$$\dot{x}(t) = F(x(t)) - u(t)x(t),$$

where $x(t)$ is the fish stock level at time t, $u(t)$ is fishing effort[1] and F is the species biological growth function.

In contrast, and given this paper's focus on the case of small pelagic fish resources, we consider a Cobb-Douglas form for the harvest function[2] :

$$h(t) = u^{\alpha}(t)x^{\beta}(t), \tag{1.1}$$

where α and β are two non negative parameters such that $\alpha + \beta \geq 1$.

The value of parameter $\alpha > 0$ controls for how fishing effort's marginal catch productivity varies as the fishing effort level changes. Thus, $\alpha = 1$ implies constant marginal productivity of additional fishing effort units. On the other hand, the parameter $\beta \geq 0$ measures how sensitive catch yields are to marginal changes in fish stock level. In the case of constant unit cost of fishing effort, the lower the value of $\beta > 0$ the less sensitive the unit harvest cost will be to variations in fish stock level. Hence, the lower the value of β the more likely should be, ceteris paribus, the occurrence of a fishing collapse outcome.

Regarding the function (1.1), empirical studies suggest that for small pelagic fisheries neither α nor β would necessarily be close to unitary values. Indeed, available evidence suggests positive values but lower than the unit for the case of β (e.g., Opsomer and Conrad [18]; Bjorndal and Conrad [5]). For this type of fish stocks, some authors have even suggested that, for certain ranges of fish abundance, total independence may eventually prevail between catch yields and fish stock levels (e.g., MacCall [17]; Clark [8]; Csirke [11]; Bjorndal [3, 4])[3]. Regarding the value of α, available evidence for several small pelagic fisheries suggests positive values that are either very close to or greater than the unit (e.g. Bjorndal [2, 4]; Bjorndal and Conrad [5]; Opsoner and Conrad [18]; Peña-Torres and Basch [21]; Peña-Torres, Vergara and Basch [22]). In order to maintain mathematical tractability, in this paper we will limit the analysis to studying cases with $\alpha \leq 1$.

[1] The use of a single input variable presupposes that other inputs (e.g., labour, capital) are used in fixed proportions, so input use intensity can be measured up by a single variable.

[2] This is a widely used functional form in economics. See Heathfield and Wibe [15, Ch. 4] about its properties when applied to modeling production functions. However, its use at fishery models has been very uncommon (rare exceptions are Leonard and Van Long [16], Hannesson [14, Page 53] and Peña-Torres [20]). See also Dasgupta and Heal [12] for a classical description of its use at optimal economic growth models for economies with exhaustible natural resource.

[3] Marine biologists (e.g. Csirke [11]) have stated that in small pelagic fisheries mean harvest yields (per unit of fishing effort) are not a good predictor of changes in fish abundance. The hypothesis is that when abundance falls, small pelagic fish stocks tend to reduce the range of their feeding and breeding areas, with concurrent decreases in the number of schools, despite that schools' average size may remain constant. That is, the stock reduces the range of its spatial distribution while simultaneously increasing its density. This behaviour could result in a relation of independence between harvest yields and fish stock abundance.

Given the evidence and conjectures quoted, this paper focuses on harvesting settings in which $\alpha+\beta=1$ and β is initially positive but it may eventually tend to zero. The analysis concentrates on the nature of the resulting stationary equilibria. The emphasis will be on studying the sensitivity of the corresponding equilibria with respect to changes in the value of parameters β, when β has relatively small positive values. In particular, given $\alpha+\beta=1$, we study the effects of changes in the proportion (α/β) when $\beta\to 0$ and $\alpha\to 1$.

For simplicity, we consider the optimal control problem of a social planner who maximizes the total discounted value of the intertemporal flows of the fish stock economic rents, when the harvest function of each agent is given by (1.1). We analyze the steady states of the associated dynamical system describing the asymptotic behavior of these states when β in (1.1) tends to zero.

The outline of this article is the following: In Section 2 we characterize and solve the social planner's problem when $\beta\in]0,1[$ and $\alpha+\beta=1$. Section 3 analyzes the unique steady state equilibrium's behavior as a function of parameter β. Finally, Section 4 discusses the asymptotic behavior of the unique stationary equilibrium when $\beta\to 0$. This section constitutes the main core of our work. The proofs are relegated to the appendix.

2 The social planner's problem

Consider N symmetric fishing units (say vessels) harvesting simultaneously a single-species fish stock. The number of fishing units is exogenous. All these vessels are under the social planner's control.

Given an admissible fishing effort policy $u(\cdot)$, the resulting catch $h(\cdot)$ is given by the Cobb-Douglas function (1.1). Thus, the evolution of the fish stock level $x_u(\cdot)$, starting from an initial condition $x_0>0$ is given by the solution of the following ordinary differential equation:

$$\begin{cases} \dot{x}(t)=F(x(t))-Nu^{\alpha}(t)x^{\beta}(t) & t>0 \\ x(0)=x_0 \end{cases} \tag{2.1}$$

where the biological growth function F is assumed strictly concave and twice continuously differentiable. We shall assume that there exists $K>0$ called *saturation constant* such that $F(0)=F(K)=0$ and $F(x)>0$ for all $x\in]0,K[$.

Example 2.1. Some examples of biological growth function F are the following:

- Logistic function: $F(x)=ax(1-x/K)$;
- Gompertz function: $F(x)=\begin{cases} ax\ln(K/x) & \text{if } x>0 \\ 0 & \text{if } x=0 \end{cases}$

where $a>0$ is a given parameter.

In what follows we assume that $x_0\in]0,K[$, and therefore the trajectory $x(t)$ remains in this interval for all $t>0$ and for any applied fishing effort.

The social planner's problem consists of choosing each vessel's fishing effort $u(t)\geq 0$ in order to maximize the total discounted value of the intertemporal flow of the natural resource's rents given by

$$J(u,x_u):=N\int_0^{+\infty} e^{-rt}(pu^{\alpha}(t)x_u^{\beta}(t)-cu(t))dt \tag{2.2}$$

where $r > 0$ is the (time invariant) discount rate, c is the (constant) cost per unit of fishing effort, and p is the (constant) unit price of harvesting.

Remark 2.2. Notice that, since the quantity $Nu^\alpha(t)x^\beta(t)$ represents the instantaneous total harvesting, we necessarily have, for any time t, the following two inequalities

$$u^\alpha(t)x^\beta(t) \leq Nu^\alpha(t)x^\beta(t) \leq x(t),$$

where the first inequality is a simple consequence of $N \geq 1$. Consequently, $u^\alpha(t) \leq x^\alpha(t)$ and then $u^\beta(t) \leq x^\beta(t)$. So, the instantaneous profit associated with the optimization criteria (2.2) satisfies

$$N(pu^\alpha(t)x^\beta(t) - cu(t)) = Nu^\alpha(t)(px^\beta(t) - cu^\beta(t)) \geq Nu^{\alpha+\beta}(t)(p - c).$$

Therefore, in order to ensure the positivity of the instantaneous profits above, we will implicitly assume from now on that $p > c$.

Hence, for a given initial condition x_0, the infinite horizon control problem is established as follows:

$$(P_{SP}) \quad V(x_0) := \max_{u\in\mathcal{U}}\{J(u, x_u) \ : \ x_u \ \text{ solves (2.1)}\} \tag{2.3}$$

where $J(u, x_u)$ is the criteria given in (2.2) and the admissible control set $\mathcal{U}$ is defined by

$$\mathcal{U} = \{u : [0, +\infty[\longrightarrow [0, \bar{U}[\ : \ u \text{ piecewise continuous}\}.$$

Here $\bar{U} \in]0, +\infty]$ is the maximal fishing effort allowed to each firm, being able to be $+\infty$.

In what follows we focus on the problem (P_{SP}) when the marginal catch productivity is strictly decreasing with respect to the stock level, that is when $\beta \in]0, 1[$. Moreover, for the sake of simplicity, we shall assume $\alpha + \beta = 1$ with $\beta > 0$. The latter allows us to work with a strictly concave Hamiltonian.

The following proposition establishes, for a fixed $\beta \in]0, 1[$, the Pontryagin's maximum principle for the optimal control problem (P_{SP}) .

Proposition 2.3. *Let $u : [0, +\infty[\longrightarrow [0, +\infty[$ be an optimal solution of the infinite horizon problem (P_{SP}) and $x : [0, +\infty[\longrightarrow]0, K[$ the associated fish stock level*[4]*. Then, there exists a function λ differentiable almost everywhere, such that*

(i)

$$\dot{\lambda}(t) = r\lambda(t) - \beta Npvu^\alpha x^{\beta-1} - \lambda(t)(F'(\bar{x}) - \beta Nu^\alpha x^{\beta-1}) \qquad a.e. \ t > 0; \tag{2.4}$$

(ii) the Hamiltonian defined by

$$H(\lambda, x, u) = N(pu^\alpha x^\beta - cu) + \lambda(F(x) - Nu^\alpha x^\beta) \tag{2.5}$$

is maximized in $u(t)$ for every t, that is

$$H(\lambda(t), x(t), u(t)) = \max_{u\geq 0} H(\lambda(t), x(t), u), \tag{2.6}$$

where λ is the current valued shadow (unit) price of x.

[4]We omit the sub-index u for the trajectory $x(\cdot)$ associated to u.

Proof. It is a direct application of the Pontryagin's maximum principle. See for instance [24, 1, 25]. □

Given an optimal policy u and the associated trajectory x, the above result allows to obtain the expression of u in terms of the shadow price λ. Indeed, since the Hamiltonian H is maximized in $u(t)$, the Fermat's rule $\frac{\partial H}{\partial u} = 0$ gives

$$u(x(t), \lambda(t)) = \begin{cases} 0 & \text{if } \lambda(t) \geq p \\ \left(\frac{\alpha(p-\lambda(t))}{c}\right)^{\frac{1}{\beta}} x(t) & \text{if } \lambda(t) < p. \end{cases} \tag{2.7}$$

As expected, the above expression shows that if the social planner assigns a higher value than the harvest price p to keep (*invest on*) an additional unit of the fish stock at sea, then the optimal policy consists in stopping fishing effort completely and immediately.

In what follows, let us consider an optimal policy $u(\cdot)$, the associated trajectory $x(\cdot)$, and the shadow price $\lambda(\cdot)$ given by Proposition 2.3 under the assumption $\alpha + \beta = 1$ (for a fixed β).

From (2.1), (2.4), and (2.7) we obtain a new system for the state x and adjoint state λ given by

$$\begin{cases} \dot{x}(t) = \varphi_1(x(t), \lambda(t)) \\ \dot{\lambda}(t) = \varphi_2(x(t), \lambda(t)); \\ x(0) = x_0 \end{cases} \tag{2.8}$$

where

$$\varphi_1(x, \lambda) := \begin{cases} F(x) & \text{if } \lambda \geq p \\ F(x) - N\phi^\alpha(\lambda)x & \text{if } \lambda < p, \end{cases}$$

$$\varphi_2(x, \lambda) := \begin{cases} \lambda(r - F'(x)) & \text{if } \lambda \geq p \\ \lambda(r - F'(x)) - \beta N\phi^\alpha(\lambda)(p - \lambda) & \text{if } \lambda < p, \end{cases}$$

and

$$\phi(\lambda) := \left(\frac{\alpha(p - \lambda)}{c}\right)^{\frac{1}{\beta}}.$$

Notice that the functions φ_1 and φ_2 are continuously differentiable. This implies the existence and uniqueness of (x, λ) solution of system (2.8).

3 Properties of the stationary equilibrium

The proposition below ensures the existence and uniqueness of a steady state of the system (2.8).

Proposition 3.1. *If $r < F'(0) < N((1-\beta)p/c)^{\frac{1-\beta}{\beta}}$, then the system (2.8) has only one steady state $(x^*(\beta), \lambda^*(\beta))$ satisfying the relation:*

$$\lambda^*(\beta) = p - \left(\frac{c}{1-\beta}\right)\left(\frac{F(x^*(\beta))}{Nx^*(\beta)}\right)^{\frac{\beta}{1-\beta}}. \tag{3.1}$$

Furthermore, the unique steady state is in $]x_r, K[\times]\bar{\lambda}_\beta, p[$, *where*

$$F'(x_r) = r \quad \textit{and} \quad \bar{\lambda}_\beta = p - \frac{c}{(1-\beta)}\left(\frac{F(x_r)}{Nx_r}\right)^{\frac{\beta}{1-\beta}}.$$

Proof. See Appendix A.1. □

Remark 3.2. Notice that the term $N((1-\beta)p/c)^{\frac{1-\beta}{\beta}}$ converges to $+\infty$ when $\beta \to 0$. Then, for small values of β, the hypothesis $F'(0) < N((1-\beta)p/c)^{\frac{1-\beta}{\beta}}$ (assumed in Proposition 3.1 above) always holds true.

Remark 3.3. The stationary solution $\lambda^*(\beta)$ that is relevant to our analysis is necessarily positive, as the latter is the only solution of economic interest. Notice that the positivity of $\lambda^*(\beta)$ will hold for $\beta > 0$ small enough.

Naturally, we need to impose condition $F'(0) > r$ to ensure that the stationary solution $x^*(\beta)$ will be strictly positive. Otherwise, it would be optimal to fully deplete the resource x and thereby being able to invest the obtained harvesting profits at the market return $r > 0$.

Proposition 3.1 also states that the optimal stationary state $x^*(\beta)$ will be strictly above the value x_r. The logic for this is as follows. First of all, the stationary economic optimum implies that no additional gains can be obtained from exploiting x at a different level. Therefore, at the stationary equilibrium, the return obtainable from marginal changes in the level of investment on x must fully coincide with the opportunity cost of doing such an investment. In our problem (PSP), such opportunity cost is given by the parameter $r > 0$.

Now, Euler-Lagrange equations applied to our problem (P_{SP}) leads to:

$$F'(x^*(\beta)) + \frac{Nc\beta}{1-\beta}\frac{\left(\frac{F(x^*(\beta))}{Nx^*(\beta)}\right)^{\frac{1}{1-\beta}}}{\lambda^*(\beta)} = r. \tag{3.2}$$

which is a variant (for $0 < \beta < 1$) of the well-known equation describing the stationary optimal solution (x^*, λ^*) and which is known in the economic literature as the "*fundamental equation of renewable resource exploitation*" (e.g., see Bjorndal and Munro [6], and Hannesson [14, Eq. (2.11)]):

$$F'(x^*) + \frac{C'(u^*)\frac{\partial h}{\partial x}(x^*, u^*)}{p\frac{\partial h}{\partial u}(x^*, u^*) - C'(u^*)} = r, \tag{3.3}$$

where $C(u) = cu$ is the cost of u units of fishing effort.

The left hand-side of equation (3.2) describes the return obtained from investing on x at the stationary level x^*. As (3.2) shows, the return from keeping an additional unit of x at sea comes from two sources. On the one hand, the biological return of keeping an additional unit of x at sea, which is given by $F'(x^*(\beta))$. On the other, the profits resulting from the incremental harvest, given that $\frac{\partial(u^\alpha x^\beta)}{\partial x} > 0$, which holds for $\beta > 0$. This second source of return will thus increase the profitability of investing on x, adding itself to the gain directly consisting of the marginal biological return $F'(x^*)$. Therefore, the intertemporal equilibrium (that is, the optimal investment) condition will be such that $F'(x^*) < F'(x_r) = r$ and, by the strict concavity of function $F(\cdot)$, then $x^* > x_r$.

Lemma 3.4. *The unique steady state of the system* (2.8) *is a saddle point. That is, the Jacobian matrix of the function* $\varphi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda))$ *evaluated in the steady state* $(x^*(\beta), \lambda^*(\beta))$ *has two real eigenvalues with opposite signs.*

Proof. See Appendix A.2. □

The property established above is well known and arises in different economic problems. For instance, in the case of harvesting fishery models, it can be found in [7, 16, 19] when the function F is a logistic map and $\alpha = \beta = 1/2$.

4 Analysis about the asymptotic behavior when $\beta \to 0$

In this part, we study the behavior of the unique steady state given by Proposition 3.1 with respect to variations of parameter β. In particular, we focus on the case of $\beta \to 0$.

For the parameter $\beta \in (0,1)$, we shall denote by $x^*(\beta)$ and $\lambda^*(\beta)$ the corresponding (unique) steady states. The pair $(x^*(\beta), \lambda^*(\beta))$, solves the following system

$$\begin{cases} 0 = \Phi_1(x,\lambda,\beta) := F(x) - N\phi^{1-\beta}(\lambda)x \\ \\ 0 = \Phi_2(x,\lambda,\beta) := -\lambda(F'(x) - r) + \beta N\phi^{1-\beta}(\lambda)(\lambda - p). \end{cases} \tag{4.1}$$

Consequently, we write $u(\beta)$ the associated steady control given by

$$u(\beta) = \left(\frac{\alpha(p - \lambda^*(\beta))}{c}\right)^{\frac{1}{\beta}} x^*(\beta). \tag{4.2}$$

In the above equality, the assumption $\alpha + \beta = 1$ was considered.

Thus, the total equilibrium harvesting is

$$h(\beta) = Nu(\beta)^{\alpha} x^*(\beta)^{\beta} = F(x^*(\beta)). \tag{4.3}$$

Proposition 4.1. *There exist two continuously differentiable functions* $x :]0,1[\longrightarrow (x_r, +\infty)$ *and* $\lambda : (0,1) \longrightarrow (0,p)$ *such that*

$$\begin{aligned} 0 &= \Phi_1(x(\beta), \lambda(\beta), \beta) \\ 0 &= \Phi_2(x(\beta), \lambda(\beta), \beta). \end{aligned} \tag{4.4}$$

Proof. See Appendix A.3. □

Remark 4.2. Note that, since the steady state of system (2.8) is unique, Proposition 4.1 above implies that $(x(\beta), \lambda(\beta)) = (x^*(\beta), \lambda^*(\beta))$, for all $\beta \in (0,1)$. So, from now on, we can use notation $(x(\beta), \lambda(\beta))$ for both meanings without any possibility of confusion.

The above result establishes that there is a continuous dependence of the steady states with respect to parameter β. Furthermore, the next proposition shows that the steady states converge when β goes to zero.

Proposition 4.3. *The steady states converge as follows, when the parameter* β *goes to zero:*

1. $\lim_{\beta \to 0} x(\beta) = x_r$ *;*

2. $\lim_{\beta \to 0} \lambda(\beta) = p - c$ *,*

where x_r *is such that* $F'(x_r) = r$*. Moreover, the limit of the optimal effort at the equilibrium is*

$$\lim_{\beta \to 0} u(\beta) = \frac{F(x_r)}{N}.$$

Proof. See Appendix A.4. □

Remark 4.4. It is well known that the solution of the Pontryaguin system (2.8) associated with problem (P_{SP}) when $\alpha = 1$ and $\beta = 0$ is a turnpike solution approaching as fast as possible to the values $x = x_r$ and $\lambda = p - c$ (see for example [10]). So, Proposition 4.3 establishes that the limit behavior of the steady states solutions $(x(\beta), \lambda(\beta))$, when $\beta \to 0$, is coherent with this limit result.

When $\beta \to 0$ we see from condition (3.2) that there will tend to remain a unique source of return from keeping an additional unit of x at sea, that is the biological growth rate $F'(x^*(\beta))$. This is so because the current period profits tend to be independent of x and therefore the Hamiltonian (or value) function at (4.4), which has to be maximized by choosing the optimal control u^*, varies with changes in x only by the differential effect $F'(x)$. As a result of this, the optimal stationary equilibrium x^* tends to the value x_r.

Proposition 4.5. *The limits of derivatives of steady states with respect to* β *, when* $\beta \to 0$*, are:*

$$\lim_{\beta \to 0} \frac{dx}{d\beta} = -\frac{cF(x_r)}{F''(x_r)(p-c)x_r} > 0; \tag{4.5}$$

$$\lim_{\beta \to 0} \frac{d\lambda}{d\beta} = -c\left[\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1\right]. \tag{4.6}$$

Therefore, for β *small enough, one has that:*

1. $\frac{dx}{d\beta} > 0$*, that is,* $x^*(\beta)$ *decreases when* β *decreases.*

2. (a) if $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1 > 0$ *then* $\frac{d\lambda}{d\beta} < 0$*, that is,* $\lambda(\beta)$ *increases when* β *decreases.*

   (b) if $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1 < 0$ *then* $\frac{d\lambda}{d\beta} > 0$*, that is,* $\lambda(\beta)$ *decreases when* β *decreases.*

Proof. See Appendix A.5. □

The result in Part 1 at Proposition 4.5 is directly related to the economic intuition already analyzed regarding the results at Proposition 3.1: a greater value of $\beta > 0$ increases the profit effect from one of the two sources of positive marginal returns which are obtained from keeping an additional unit of x at sea. Thus, at the steady state equilibrium a greater value of β will imply a higher stationary value for x.

Regarding the result in Part 2 at Proposition 4.5, its economic interpretation is as follows: Parts (a) and (b) define different parametric value ranges which imply, at the stationary equilibrium, opposite signs for the differential effect $\frac{d\lambda}{d\beta}$. The result (2.a) implies that $\frac{d\lambda}{d\beta} < 0$ if the parametric value[5] $F(x_r)/(Nx_r)$ is above a minimum bound. Thus, for a given function $F(x)$ and a given value of N, the result (2.a) will hold as long as the market rate of interest r is above a lower bound; and correspondingly, for a given value of r, the number N of fishing units is below an upper bound. Therefore, the result (2.a) will hold for "relatively high values" of r and/or "relatively low values" of N. Whereas the differential result at (2.b) will hold for the opposite parametric value ranges; that is, the result (2.b) will hold, , for a given value of N, as long as the value of parameter $r > 0$ is "relatively small" (i.e., below an upper bound) or, for a given value of r, the value of parameter N is "relatively large" (i.e., above a lower bound).

It is interesting to notice that the condition that "β is small enough" in Proposition 4.5 implies the validity of one of the two conditions needed for ensuring the existence and uniqueness of a positive steady-state solution in this model. The second condition needed is $r < F'(0)$. The latter defines an upper bound on the feasible values of r such that the stationary solution $x^* > 0$, i.e. a solution of economic interest. Given these two conditions, the results in Part 2 of Proposition 4.5 then define parametric value ranges for N, given a biological function $F(\cdot)$, which determines whether the result (2.a) or (2.b) holds. Indeed, and considering $r \in (0, F'(0))$ and a function $F(\cdot)$ satisfying the properties defined at Section 2, the result (2.b) will always hold as long as $N/e > F'(0)$ (which imposes a lower bound on N, as already explained). Whereas the result (2.a) will always hold as long as $N/e < F(x')/x'$ (i.e., an upper bound on N), where x' is the element such that $F(x') = \max\{F(x) : 0 < x < K\}$ and $K > 0$ is the saturation constant defined at Section 2. Finally, for given values of N such that $F(x')/x' < N/e < F'(0)$, the result (2.b) will hold for "relatively small" values of r (close to zero), whereas the result (2.a) will hold for "relatively high" values of r (close to $F'(0)$).

Moreover, notice that a higher value of $r > 0$ implies a greater opportunity cost of (investing on) keeping an additional unit of x at sea, which in turn implies, *ceteris paribus*, a lower demand valuation (of the optimizing social planner) for keeping an additional unit of x at sea. Thus, for higher values of r the stationary solution for x should be at a lower level and at a higher level for the stationary fishing effort u (keeping constant all other factors). Recall also that, in this model, the social planner's (marginal) valuation of incremental units of investment on x corresponds to the value of the co-state variable $\lambda \geq 0$. A similar effect on the social planner's valuation of marginal investment on x, again keeping constant all other factors, will be associated with relatively small values of N. In contrast, the result (2.b) will hold for (N, r) values such that the social planner's valuation of marginal investments on x is relatively high, either because of a large N or due to a low value of r.

Therefore, and according with the results at Proposition 4.5, $\frac{d\lambda}{d\beta} < 0$ will hold for relatively low stationary values of λ and x, and vice versa for the case when $\frac{d\lambda}{d\beta} > 0$.

In the next propositions we explore other differential effects which underlie the resulting signs for $\frac{d\lambda}{d\beta}$ at Proposition 4.5. Firstly, for $\beta \in (0, 1)$ small enough, it can be proved that, at the stationary equilibrium, $\frac{dh}{d\beta} > 0$, as we see in the next proposition:

Proposition 4.6. *For $\beta \in (0, 1)$ small enough, the equilibrium harvesting $h(\beta)$ decreases when β decreases.*

Proof. See Appendix A.6. □

[5] This term can be interpreted as the (per fishing unit) biological rate of return from marginal investments on x when $x = x_r$.

Secondly, we also know by the result in Part 1 at Proposition 4.5 that, at the stationary equilibrium, $\frac{dx}{d\beta} > 0$ for $\beta \in (0,1)$ small enough. Thirdly, and again for $\beta \in (0,1)$ small enough, it can be proved that, at the stationary equilibrium, $\frac{d}{d\beta}\left(\frac{\partial h}{\partial x}\right) > 0$, as we show in the next proposition

Proposition 4.7. *For $\beta \in (0,1)$ small enough, the marginal equilibrium harvesting $\frac{\partial h}{\partial x}$ decreases when β decreases.*

Proof. See Appendix A.7. □

Therefore, for $\beta \in (0,1)$ small enough, we know that increases in beta will increase the steady state levels of x and h. So, how can it then be that, for $\beta \in (0,1)$ small enough, the sign of $\frac{d\lambda}{d\beta}$ changes as a function of a critical value for the parametric condition $F(x_r)/(Nx_r) > 0$? The answer must lie in the marginal effects of changes in β upon the stationary value of the fishing effort u. The following Proposition and Corollary provide the answer.

Proposition 4.8. *The marginal harvesting productivity of u, at equilibrium, has the same monotonicity properties that the shadow price when β varies. Indeed,*

$$\frac{d}{d\beta}\left(\frac{\partial h}{\partial u}(x(\beta),\lambda(\beta))\right) = \left(\frac{Nc}{(p-\lambda(\beta))^2}\right)\frac{d\lambda(\beta)}{d\beta}. \tag{4.7}$$

Proof. See Appendix A.8. □

Corollary 4.9. *For $\beta \in (0,1)$ small enough, one has that:*

1. *if* $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1 > 0$ *then* $\frac{d}{d\beta}\left(\frac{\partial h}{\partial u}\right) < 0$.
2. *if* $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1 < 0$ *then* $\frac{d}{d\beta}\left(\frac{\partial h}{\partial u}\right) > 0$.

Proof. This is a direct consequence of Propositions 4.5 and 4.8.

□

Naturally, we are interested in the monotonicity properties described at Parts 1 ands 2 of Collorary 4.9 for the cases when $p - c > 0$, which is a necessary condition for obtaining stationary solutions of economic interests, i.e. where x^*, λ^* and u^* are all strictly positive. In this setting, and again for $\beta \in (0,1)$ small enough, the parametric condition which guarantees the validity of the result (2.a) at Proposition 4.5 also implies that the marginal productivity of fishing effort u decreases as β increases, all other factors remaining constant; whereas the opposite effect on the marginal productivity of u occurs for the parametric values such that the result (2.b) at Proposition 4.5 is valid. Thus, relatively low stationary values of λ, which are compatible with relatively high values of r and/or relatively low values of N, will coincide with the result at Part 1 of Corollary 4.9; and vice versa for the result at Part 2 of this Corollary.

The next Proposition 4.10 and its corresponding Corollary 4.11 help to describe the marginal effect on the stationary solution u^* as the value of β changes, for the case when $\beta \in (0,1)$ is small enough.

Proposition 4.10. *The limit of $\frac{du}{d\beta}$ when β goes to zero is:*

$$\lim_{\beta\to 0}\frac{du}{d\beta} = \frac{F(x_r)}{N}\left(\ln\left(\frac{F(x_r)}{Nx_r}\right) - \frac{rc}{F''(x_r)(p-c)x_r}\right).$$

Proof. See Appendix A.9. □

Corollary 4.11. *For $\beta \in (0,1)$ small enough, if $\frac{F(x_r)}{Nx_r} > 1$, then $\frac{du}{d\beta} > 0$ and $\frac{d\lambda}{d\beta} < 0$.*

Proof. The sign of $\frac{du}{d\beta}$ is a direct consequence of Proposition 4.10. Indeed, the strict concavity of F implies that $\frac{F(x_r)}{N}\left(\ln\left(\frac{F(x_r)}{Nx_r}\right) - \frac{rc}{F''(x_r)(p-c)x_r}\right) > 0$ whenever $\frac{F(x_r)}{Nx_r} > 1$. The sign of $\frac{d\lambda}{d\beta}$ is trivially obtained from Proposition 4.5 because $\frac{F(x_r)}{Nx_r} > 1$ implies that $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1 > 0$. □

To obtain economic intuitions from these two last results, let us consider some specific parametric configurations. Firstly, let us consider $r \in (0, F'(0))$ and also impose boundaries on the value of N such that $F(x')/x' < N/e < F'(0)$, where x' is the element such that $F(x') = \max\{F(x) : 0 < x < K\}$ and $K > 0$ is the saturation constant. Thus, we know that result (2.b) at Proposition 4.5 will prevail for r close to zero. Under these conditions, the result at Proposition 4.10 necessarily implies $\frac{du}{d\beta} < 0$. Therefore, for (N, r) values such that the social planner assigns a "relatively high" value $\lambda^* > 0$ to keeping an additional unit of x at sea, if $\beta > 0$ (but small enough) increases then the stationary solutions λ^* and x^* will also both increase, whereas the stationary fishing effort u^* will decline. Notice that a very small value of r implies a greater present value of the stream of future profits resulting from harvesting x, given $p - c > 0$, which in turn implies a greater incentive for investing on additional sustainable (or stationary) units of x. Secondly, if we now maintain the parametric configuration (N, r) which guarantees the validity of result (2.b) at Proposition 4.5, and additionally suppose that the fishing business under analysis is quite profitable, let say implying that $(p - c)$ is very large, then the result at Proposition 4.10 will again imply $\frac{du}{d\beta} < 0$. Because in this case the fishing business is very good and also the present value of the stream of future profits is sufficiently large, given the low value of r and the large fishing fleet (given the relatively large but bounded value of N), in this case the planner's optimal reaction to a greater value of $\beta > 0$ will again be to invest on additional units of stationary x^* at sea while reducing the value of each fishing unit's stationary fishing effort u^*. Thirdly, consider now any given values of N and r such that all three stationary solutions (x^*, λ^*, u^*) are strictly positive. Suppose now that $p - c > 0$ but small enough. In this case, the result at Proposition 4.10 necessarily implies $\frac{du}{d\beta} > 0$, given the strict concavity of function $F(\cdot)$. Therefore, when β increases, and so does the stationary solution $x^* > 0$, and there are profits to be made from harvesting x, but the profit per unit of fishing effort is quite small, the social planner's optimal fishing policy will consist of increasing the stationary (per unit- or boat-) level of fishing effort. Fourthly, now for a given value of $(p - c)$, and again considering (N, r) parametric configurations such that all three stationary solutions (x^*, λ^*, u^*) are strictly positive, if $r > 0$ gets close enough to the value of $F'(0)$, and so does the value x_r gets close to zero, the result at Proposition 4.10 necessarily implies that $\frac{du}{d\beta} > 0$. In this case, the relatively high value of r implies, *ceteris paribus*, that the social planner will assign a relatively low value λ^* at keeping additional (stationary) units of x at sea, because of the smaller present value of the future stream of harvesting profits. Hence, the optimal fishing policy will be to increase the (per-unit) fishing effort from all available fishing units. In a consistent way with this last intuition, Corollary 4.11 simply states that the result $\frac{du}{d\beta} > 0$ will always prevail for sufficiently high values of r, given a value of N, and/or for sufficiently low values of N, given a value of r, in all these cases the parametric configuration being such that $F(x_r)/Nx_r > 1$.

References

[1] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.

[2] T. Bjorndal. Production economics and optimal stock size in a North Atlantic fishery. *Scandinavian Journal of Economics*, 89:145–164, 1987.

[3] T. Bjorndal. The optimal management of North Sea Herring. *J. of Environmental Economics and Management*, 15(1):9–29, 1988.

[4] T. Bjorndal. Production in a schooling fishery: The case of the North Sea Herring fishery. *Land Economics*, 65:49–56, 1989.

[5] T. Bjorndal and J. Conrad. The dynamics of an open access fishery. *Canadian Journal of Economics*, 20:74–85, 1987.

[6] T. Bjorndal and G. R. Munro. *The Economics of Fisheries Management: A Survey*, pages 153–188. Cheltenham, UK: Elgar, 1998/1999. In The International Yearbook of Environmental and Resource Economics (Eds. T. Tietenberg and H. Folmer).

[7] C. Chiang. *Dynamic optimization*. Economic Series. McGraw-Hill, Inc., 1992.

[8] C. Clark. *Concentration Profiles and the Production and Management of Marine Fisheries*. Physica-Verlag, Würzburg, Germany, 1982. in Economic Theory of Natural Resources (Eds. W. Eichhorn, R. Henn, K. Neumann, and R.W. Shepard).

[9] C. W. Clark. Restricted access to common-property fishery resources: A game theoretic analysis. *in Liu, P. (ed.), Dynamic Optimization and Mathematical Economics*, pages 117–132, 1980.

[10] C. W. Clark. *Mathematical bioeconomics*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1990.

[11] J. Csirke. *Small shoaling pelagic fish stocks*, chapter 11. John Wiley & Son, New York, USA, 1988. In Fish Population Dynamics. The implications for Managenment (Ed. J. A. Gulland).

[12] P. Dasgupta and G.M. Heal. *Economic theory and exhaustible resources*. CUP, London, 1979.

[13] E. Dockner, G. Feichtinger, and A. Mehlmann. Noncooperative solutions for a differential game model of fishery. *J. Econom. Dynam. Control*, 13(1):1–20, 1989.

[14] R. Hannesson. *Bioeconomic Analysis of Fisheries*. Fishing New Books. 1993.

[15] D.F. Heathfield and S. Wibe. *An introduction to cost and production functions*. MacMillan, London, 1987.

[16] D. Léonard and N.V. Long. *Optimal control theory and static optimization in economics*. Cambridge University Press, 1992.

[17] A.D. MacCall. Density dependence of catchability coefficient in the California Pacific Sardine sardinops sagax caerulea. *Calif. Coop. Oceanic Fish. Invest. Rep*, 18:136–148, 1976.

[18] J. D. Opsomer and J. Conrad. An open acess analysis of the Northern Anchovy fishery. *J. of Environmental Economics and Management*, 27:21–37, 1994.

[19] J. Peña Torres. Overfishing with dynamic profit optimizing firms. *Documento de trabajo - Departamento de Economía - Universidad de Chile*, 55(136 08/95/50/1), 1995.

[20] J. Peña Torres. Sustainability versus fishing collapse: A review of causes and welfare prescriptions. *Estudios de Economía*, 23(1):83–112, 1996.

[21] J. Peña Torres and M. Basch. Harvesting in a pelagic fishery: The case of northern Chile. *Ann. Oper. Res.*, 94(1-4):295–320, 2000. (Special Issue on Natural Resources).

[22] J Peña-Torres, S. Vergara, and M. Basch. El dilema de la escala productiva frente a ciclos de abundancia: La pesca industrial en Chile. *El Trimestre Económico*, (283):575–612, 2004.

[23] C. Plourde and D. Yeung. A note on Clemhout and Wan's dynamic games of common property resources. *J. Optim. Theory Appl.*, 55(2):327–331, 1987.

[24] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The mathematical theory of optimal processes*. Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt. Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962.

[25] R. Vinter. *Optimal control*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 2000.

A Proofs

A.1 Proof of Proposition 3.1

For the sake of simplicity, we denote the steady states of (2.8) by $x^* = x^*(\beta)$ and $\lambda^* = \lambda^*(\beta)$.

In order to prove the existence of (x^*, λ^*), let us define the following function

$$g(x) = \frac{c}{\alpha}\left(\frac{F(x)}{Nx}\right)^{\frac{\beta}{\alpha}}\left(\beta\frac{F(x)}{x} + r - F'(x)\right) + F'(x)p.$$

Note that

$$g(x) \to \frac{c}{\alpha}\left(\frac{F'(0)}{N}\right)^{\frac{\beta}{\alpha}}\big(\beta F'(0) + r - F'(0)\big) + F'(0)p, \quad \text{when } x \to 0.$$

This last limit is strictly greater than rp. On the other hand, $g(K) = F'(K)p < 0 < rp$. Therefore, there exists $x^* \in [0, K]$ such that $g(x^*) = rp$. Then, defining

$$\lambda^* = p - \frac{c}{\alpha}\left(\frac{F(x^*)}{Nx^*}\right)^{\frac{\beta}{\alpha}},$$

it follows that (x^*, λ^*) is a steady state of (2.8) and relation (3.1) is satisfied.

We now proceed to prove that $(x^*, \lambda^*) \in]x_r, +\infty[\times]\lambda_\beta, p[$. Note that, since F es strictly concave, $x = x_r$ is the only point satisfying $F'(x) = r$. Then, since $F(x_r) \neq 0$ (otherwise $F(x) < 0$ for all $x < x_r$ close enough to x_r which contradicts the positivity of F on $[0, K]$), it necessarily holds that $\lambda < p$. Also, cases when $x^* = 0$ or $\lambda^* = 0$ are trivially discarded. So, (x^*, λ^*) is a steady state if and only if

$$\frac{F(x^*)}{Nx^*} = \phi^\alpha(\lambda^*) \tag{A.1}$$

$$F'(x^*) - r = \Phi(\lambda^*), \tag{A.2}$$

where the auxiliar function Φ is defined as follows

$$\Phi(\lambda) = \frac{\beta N \phi^\alpha(\lambda)(\lambda - p)}{\lambda}. \tag{A.3}$$

Notice that Φ has a minimum over $]-\infty, 0[$ at $\lambda_m = -\beta p/(1-\beta)$ (Indeed, $\Phi'(\lambda) < 0$ when $\lambda \in]-\infty, \lambda_m[$ and $\Phi'(\lambda) > 0$ when $\lambda \in]\lambda_m, 0[$). So, the hypotheses on $F'(0)$ and the strictly concavity of F imply that

$$F'(x^*) - r \leq F'(0) - r < F'(0) < N((1-\beta)p/c)^{\frac{1-\beta}{\beta}} < N(p/c)^{\frac{1-\beta}{\beta}} = \Phi(\lambda_m),$$

which together with (A.2) discards the case when $\lambda^* < 0$. Consequently, x^* should be necessarily strictly greater than x_r, because otherwise left and right terms in equality (A.2) have opposite signs.

We finally note that $\lambda > \bar{\lambda}_\beta$ follows from the relations:

$$\phi^\alpha(\bar{\lambda}_\beta) = \frac{F(\bar{x})}{N\bar{x}} > \frac{F(x^*)}{Nx^*} = \phi^\alpha(\lambda^*)$$

where the inequality is due to the monotonicity of function $x \to F(x)/x$, which is a consequence of the strict concavity of F.

We finish this proof by showing the uniqueness of the steady state (x^*, λ^*). Consider two steady states (x_1, λ_1) y (x_2, λ_2). Since functions $x \to F(x)/x$ and $\lambda \to \phi^\alpha(\lambda)$ are decreasing, we have the following equivalences:

$$x_1 \leq x_2 \Leftrightarrow \frac{F(x_2)}{Nx_2} \leq \frac{F(x_1)}{Nx_1} \Leftrightarrow \phi^\alpha(\lambda_2) \leq \phi^\alpha(\lambda_1) \Leftrightarrow \lambda_1 \leq \lambda_2.$$

On the other hand, it is easy to verify that function Φ is increasing on $]\bar{\lambda}_\beta, p[$. Consequently, we also have the equivalences:

$$\lambda_1 \leq \lambda_2 \Leftrightarrow \Phi(\lambda_1) \leq \Phi(\lambda_2) \Leftrightarrow F'(x_1) \leq F'(x_2) \Leftrightarrow x_2 \leq x_1.$$

We thus conclude that $x_1 = x_2$ and $\lambda_1 = \lambda_2$.

A.2 Proof of Proposition 3.4

For a given $\beta \in (0, 1)$, the Jacobian matrix of the right-hand-side function of (2.8), at the steady state $(x^*(\beta), \lambda^*(\beta))$, is given by

$$J(\beta) := J(x^*(\beta), \lambda^*(\beta)) = \begin{bmatrix} F'(x^*(\beta)) - \frac{F(x^*(\beta))}{x^*(\beta)} & \frac{\alpha F(x^*(\beta))}{\beta(p - \lambda^*(\beta))} \\ -\lambda^*(\beta) F''(x^*(\beta)) & r - F'(x^*(\beta)) + \frac{F(x^*(\beta))}{x^*(\beta)} \end{bmatrix}. \tag{A.4}$$

The determinant of $J(\beta)$ is then computed as follows

$$\det(J(\beta)) = A(\beta) - \frac{B(\beta)}{\beta}, \tag{A.5}$$

where

$$A(\beta) := \left(F'(x^*(\beta)) - \frac{F(x^*(\beta))}{x^*(\beta)}\right)\left(r - \left(F'(x^*(\beta)) - \frac{F(x^*(\beta))}{x^*(\beta)}\right)\right),, \tag{A.6}$$

$$B(\beta) := -\lambda^*(\beta)F''(x(^*\beta))\frac{\alpha F(x^*(\beta))}{(p-\lambda^*(\beta))}. \tag{A.7}$$

So, since F is strictly concave and $\lambda(\beta) > 0$, it follows that $A(\beta) < 0$ and $B(\beta) > 0$. We thus obtain that $\det(J(\beta)) < 0$. On the other hand, the trace of $J(\beta)$ is constant and equal to r, which is a positive value. These two conditions hold on $J(\beta)$ if and only if its two eigenvalues are real numbers with opposite signs, that is, the steady state $(x^*(\beta), \lambda^*(\beta))$ is a saddle point of system (2.8).

A.3 Proof of Proposition 4.1

We have seen in Section A.2 (see equation (A.5)) that $\det J(\beta) < 0$, for all $\beta \in (0, 1)$, where $J(\beta) := J(x^*(\beta), \lambda^*(\beta))$ is the Jacobian matrix of the RHS of (2.8) at the steady state $(x^*(\beta), \lambda^*(\beta))$. Hence, the implicit function theorem implies the existence of two continuously differentiable mappings of β, simply denoted here by $x :]0, 1[\longrightarrow]\bar{x}, +\infty[$ and $\lambda :]0, 1[\longrightarrow]0, p[$ (the range sets of these functions are obtained in Lemma 3.1), satisfying (4.4).

A.4 Proof of Proposition 4.3

Proposition 4.1 implies that $x(\beta)$ and $\lambda(\beta)$ remain in the compact set $C = [x_r, K] \times [0, p]$. Hence, in order to prove the convergences of $x(\beta)$ and $\lambda(\beta)$, we only need to prove that any converging subsequence has x_r and $p - c$, respectively, as their limit points. Consider then any sequence β_k converging to zero, when $k \to +\infty$, such that $x(\beta_k) \to \tilde{x}$ and $\lambda(\beta_k) \to \tilde{\lambda}$ for some $\tilde{x}$ and $\tilde{\lambda}$ in C. Since $\bar{\lambda}_\beta \to p - c$ when $\beta \to 0$ (see Proposition 4.1), we can ensure that $\tilde{\lambda} \geq p - c > 0$. Moreover, first equation in (4.1) gives us the following relation:

$$\lambda(\beta) = p - \left(\frac{c}{1-\beta}\right)\left(\frac{F(x(\beta))}{Nx(\beta)}\right)^{\frac{\beta}{1-\beta}},$$

which implies that $\tilde{\lambda} = p - c$ provided that $F(\tilde{x}) \neq 0$.

Let us prove this claim. We argue by contradiction. Suppose that $F(\tilde{x}) = 0$, then we obtain from (4.1) that $\tilde{\lambda}(F'(\tilde{x}) - r) = 0$, and consequently $F'(\tilde{x}) = r$. This holds only if $\tilde{x} = x_r$ (because, since F es strictly concave, $x = x_r$ is the only point satisfying $F'(x) = r$). However, this contradicts the fact that $F(x_r) \neq 0$ (otherwise $F(x) < 0$ for all $x < x_r$ close enough to x_r which contradicts the positivity of F on $[0, K]$). Hence $\tilde{\lambda} = p - c$. The second equation in (4.1) allows us to conclude that $\tilde{x} = x_r$. The desired convergences of $x(\beta)$ and $\lambda(\beta)$ are thus established.

Finally, the convergence of $u(\beta)$ follows directly from equality (4.3).

A.5 Proof of Proposition 4.5

From the implicit function theorem, the derivative of $x(\beta)$ can be computed as follows

$$\frac{dx(\beta)}{d\beta} = \frac{N}{\beta \det J(\beta)} \left[\frac{-F(x(\beta))}{Nx(\beta)} \left(\frac{1}{1-\beta} \ln \left(\frac{F(x(\beta))}{Nx(\beta)} \right) + 1 \right) \left\{ rx(\beta) - x(\beta)F'(x(\beta)) + \beta F(x(\beta)) \right\} \right.$$
$$\left. - (1-\beta) \frac{F(x(\beta))^2}{Nx(\beta)} \right],$$

where $J(\beta) := J(x(\beta), \lambda(\beta))$ is the Jacobian matrix of the RHS of (2.8) at the steady state $(x(\beta), \lambda(\beta))$. On the other hand, equation (A.5) establishes that $\det J(\beta) = A(\beta) - B(\beta)/\beta < 0$, for all $\beta \in (0,1)$, where $A(\beta)$ and $B(\beta)$ were described in (A.6) and (A.7), respectively. By noting that

$$\lim_{\beta \to 0} A(\beta) = \left(r - \frac{F(x_r)}{x_r} \right) \frac{F(x_r)}{x_r}$$

$$\lim_{\beta \to 0} B(\beta) = -(p-c)F''(x_r)\frac{F(x_r)}{c},$$

we conclude $\beta \det J(\beta) \to (p-c)F''(x_r)\frac{F(x_r)}{c}$, when $\beta \to 0$. This limit value is negative because the strict concativity of F. Therefore,

$$\lim_{\beta \to 0} \frac{dx(\beta)}{d\beta} = -\frac{cF(x_r)}{F''(x_r)(p-c)x_r} > 0.$$

Analogously, from the implicit function theorem, the derivative of $\lambda(\beta)$ can be computed as follows

$$\frac{d\lambda(\beta)}{d\beta} = \frac{N}{\beta \det J(\beta)} \left[\frac{-F(x(\beta))}{Nx(\beta)} \left(\frac{1}{1-\beta} \ln \left(\frac{F(x(\beta))}{Nx(\beta)} \right) + 1 \right) \{ \lambda(\beta)F''(x(\beta))x(\beta) \right.$$
$$\left. + \beta \left(F'(x(\beta)) - \frac{F(x(\beta))}{x(\beta)} \right) (p - \lambda(\beta)) \} + \beta \left(F'(x(\beta)) - \frac{F(x(\beta))}{x(\beta)} \right) (p - \lambda(\beta)) \frac{F(x(\beta))}{Nx(\beta)} \right].$$

Hence

$$\lim_{\beta \to 0} \frac{d\lambda(\beta)}{d\beta} = -c \left[\ln \left(\frac{F(x_r)}{Nx_r} \right) + 1 \right].$$

We have thus concluded (4.5) and (4.6).

Finally, from (4.5) and (4.6), we deduce that $\frac{dx(\beta)}{d\beta} > 0$ and that $\frac{d\lambda(\beta)}{d\beta}$ has the opposite sign of $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1$ when β is small enough. The theorem follows.

A.6 Proof of Proposition 4.6

Propositions 4.3 and 4.5 imply that $x(\beta)$ decreases to x_r when $\beta \to 0$. This in particular implies that $F'(x(\beta)) > 0$ and $\frac{dx(\beta)}{d\beta} > 0$ when β is small enough. Hence, we obtain from (4.3) that

$$\frac{\partial h(x(\beta), u(\beta))}{\partial \beta} = F'(x(\beta))\frac{dx(\beta)}{d\beta} > 0, \quad \text{for all } \beta \text{ small enough.}$$

A.7 Proof of Proposition 4.7

The partial derivative of h with respect to u is given by

$$\frac{\partial h}{\partial x}(x,u) = N\beta \left(\frac{u}{x}\right)^{1-\beta}.$$

Therefore, at the equilibrium $(x(\beta), u(\beta))$, it holds that

$$\frac{\partial}{\partial \beta}\left(\frac{\partial h}{\partial x}(x(\beta),u(\beta))\right) = N\left(\frac{u(\beta)}{x(\beta)}\right)^{1-\beta}\left(1-\beta\ln\left(\frac{u(\beta)}{x(\beta)}\right)\right).$$

However, it follows from (4.2) that $\beta \ln\left(\frac{u(\beta)}{x(\beta)}\right) = \frac{\beta}{\alpha}\ln\left(\frac{F(x(\beta))}{Nx(\beta)}\right)$, which tends to 0 when $\beta \to 0$. We thus conclude that $\frac{\partial}{\partial\beta}\left(\frac{\partial h}{\partial x}(x(\beta),u(\beta))\right)$ is positive when β is small enough.

A.8 Proof of Proposition 4.8

The partial derivative of h with respect to u is given by

$$\frac{\partial h}{\partial u}(x,u) = \alpha N \left(\frac{x}{u}\right)^{\beta}.$$

Then, at the equilibrium $(x(\beta), u(\beta))$, we obtain from (4.2) the expression

$$\frac{\partial h}{\partial u}(x(\beta),u(\beta)) = \frac{Nc}{(p-\lambda(\beta))},$$

and relation (4.7) is obtained by deriving the above equality with respect to β.

A.9 Proof of Proposition 4.10

Define the function $\zeta(\beta,x,u) = F(x) - Nu^{(1-\beta)}x^{\beta}$. From (4.3) one has that

$$\zeta(\beta, x(\beta), u(\beta)) = 0 \qquad \forall\ \beta \in (0,1).$$

Deriving this equality with respect to β we obtain

$$\partial_\beta\zeta + \partial_x\zeta\ \frac{dx}{d\beta} + \partial_u\zeta\ \frac{du}{d\beta} = 0. \tag{A.8}$$

It is straightforward to check that the partial derivatives of the function ζ are given by:

$$\begin{aligned}
\partial_\beta\zeta &= \frac{Nu}{\alpha}\left(\frac{x}{u}\right)^{\beta}\ln\left(\frac{F(x)}{Nx}\right)\\
\partial_x\zeta &= F'(x) - \beta\frac{F(x)}{x}\\
\partial_u\zeta &= -N\alpha\left(\frac{x}{u}\right)^{\beta}.
\end{aligned}$$

So, Proposition 4.3 implies that $\partial_\beta\zeta \to F(x_r)\ln\left(\frac{F(x_r)}{Nx_r}\right)$, $\partial_x\zeta \to r$, $\partial_u\zeta \to -N$ when $\beta \to 0$. These limits, expression (4.5) for the limit of $\frac{dx}{d\beta}$ when $\beta \to 0$, and (A.8) give us the desired result.