Insurance Contracts Designed by Competitive Pooling

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Abstract

We build a model of competitive pooling and show how insurance contracts emerge in equilibrium, designed by the invisible hand of perfect competition. When pools are exclusive, we obtain a unique separating equilibrium. When pools are not exclusive but seniority is recognized, we obtain a different unique equilibrium: the pivotal primary-secondary equilibrium. Here reliable and unreliable households take out a common primary insurance up to its maximum limit, and then unreliable households take out further secondary insurance.

Keywords: Competitive pooling, insurance contracts, adverse selection, signalling, seniority, equilibrium refinement

JEL Classification: D4, D5, D41, D52, D81, D82

1 Introduction

Traditional general equilibrium theory treated insurance as a special case of securities with contingent payoffs. A household with a low endowment in some state could "insure" himself by buying a security which delivered when he most needed the money.

What is missing from this traditional approach is adverse selection. In practice, an insurance company issues a generic contract, to pay in case of "accident." Different clients sign the same insurance contract, but they purchase thereby different securities, because their "accident" states are different. The shareholder in the insurance company in effect holds a pool of liabilities, and suffers from adverse selection because households with more probable accident states may be more than proportionately represented in the pool.

This has been recognized by the burgeoning field of contract theory, in which a classic problem is how insurance companies will design contracts to protect themselves from adverse selection. Rothschild and Stiglitz [3] wrote a pioneering article in this field, in which they described a separating equilibrium that (sometimes) arises when oligopolistic, risk-neutral insurance firms have the power to impose exclusive contracts prohibiting their clients from taking insurance elsewhere. They found that only two contracts would actually be offered, with reliable agents choosing one and unreliable agents choosing the other. Rothschild and Stiglitz noted that there are robust regions in which no equilibrium, as they defined it, exists. Having departed from standard models of perfect competition, in order to study contract design, this phenomenon might have been inevitable.

In this paper we develop a general theory of competitive pooling, taking up the question of insurance from the point at which traditional general equilibrium theory stopped. A primitive example is the land pool in which households contribute part of their land to a common pool, whose collective output is then distributed in proportion to the acres put in. It is a short leap from there to the pooling of promises, like mortgages and insurance, that play such an important role in modern financial markets. Our treatment is firmly in the tradition of perfect competition, but with one significant twist. When there is no trade in a pool, potential investors do not have the requisite information to anticipate what deliveries will be forthcoming from the pool, even if they observe the analogous pool in other economies. By a simple device, reminiscent of the trembling hand in game theory, we fix their anticipations at the most optimistic level consistent with cautious rationality.

The upshot is that equilibrium with competitive pooling always exists.. Though there are no insurance companies deliberately designing contracts, in equilibrium the choices that emerge for households resemble the complex contracts found in the insurance design literature. When we restrict to exclusive pools, we always get a unique equilibrium, similar to the separating equilibrium found by Rothschild and Stiglitz, but with two differences. First, in our equilibrium every quantity limit is marketed, so that a household faces a complete schedule of price for every level of insurance he might take out. Households choose only two of these, because they prefer the two to all the others, not because insurance companies deliberately choose only two to offer. Second, our equilibrium always exists, even when theirs does not, as one would expect from a perfectly competitive theory.

It is not realistic, however, to suppose that insurance companies can prohibit their clients from taking out any more insurance. Therefore we consider a new model in which households can contribute to at most one pool $j \in \mathcal{J} \equiv \{1, ..., J\}$, as before, but thereafter are free to contribute more to a pool $J+1$. We prove that when the rates in pool $J+1$ are contingent on how much a household contributes to his primary pool $j \in \mathcal{J}$, there is a unique equilibrium which we call the pivotal primary-secondary equilibrium.

In this pivotal equilibrium, reliable and unreliable households take out primary insurance up to a prescribed maximum. The rate they pay reflects the "proportional representation" in the pool. The unreliable households take out further insurance on the secondary pool $J+1$, receiving a much worse rate because by doing so they reveal themselves to be bigger risks. The maximum quantity of primary insurance is set at the level at which reliable agents are just indifferent to taking out the first penny of secondary insurance. We call the contract pivotal because if its primary quantity limit were set any lower, reliable households would mix with unreliable households in the secondary pool.

Our primary-secondary insurance contract conforms much better with practice than the exclusive contracts posited in Rothschild–Sliglitz. But our contracts represent a rudimentary form of seniority. In the insurance and reinsurance markets of today, clients can take out many insurance policies, provided that each dollar of insurance is clearly ordered. We describe such a model in our final section, and prove that our pivotal primary-secondary contract survives as an equilibrium. We do not, however, examine the question of uniqueness. (In fact, we suspect equilibrium is not unique.)

2 The Land Cooperative

Imagine households $h \in \mathcal{H} = \{1, ..., H\}$, each of whom owns one acre of land on which there remains no work to be done. The land of household h yields a risky output of $e_s^h \in \mathbb{R}_+$ bushels of wheat depending on the state of nature $s \in \mathcal{S} = \{1, ..., S\}$. We assume $(e_1^h, ..., e_S^h) = e^h \neq 0$.

Every household h is risk-averse and his ex ante utility of consumption is given by a continuous, strictly monotonic and strictly concave function

$$u^h : \mathbb{R}_+^{\mathcal{S}} \to \mathbb{R}.$$

If the output risks are not perfectly correlated, the households might wish to diversify by pooling their land in a cooperative. Each household would then receive

$$K_s = \frac{1}{H} \sum_{h \in \mathcal{H}} e_s^h$$

bushels in state $s \in \mathcal{S}$, in lieu of his contribution.

Such a cooperative could well turn out to be fruitful for all its members. If, for example, individual risks are not just uncorrelated but also identically distributed, and if households have the same von Neumann–Morgenstern utilities, then we would indeed obtain $u^h(K) > u^h(e^h)$ for all $h \in \mathcal{H}$, where $K = (K_1, ..., K_s)$. But if some households begin with relatively high expected e^h, they could easily stand to lose by contributing *all* their land to the cooperative pool. To make membership attractive, it might be needful for the cooperative to amend its rules and solicit *voluntary* contributions. Then at least there would be the guarantee that no household loses from joining.

3 The Voluntary Cooperative

Let each household h contribute a share $\varphi^h \in [0, 1]$ of his land. The pool then holds $\sum_{h \in \mathcal{H}} \varphi^h$ acres of land and produces $K_s(\varphi) = (1/\sum_{i \in \mathcal{H}} \varphi^i) \sum_{h \in \mathcal{H}} \varphi^h e_s^h$, $s \in \mathcal{S}$, per acre, where $\varphi = (\varphi^1, ..., \varphi^H)$. (If $\sum_{i \in \mathcal{H}} \varphi^i = 0$, we define $K_s(\varphi)$ to be arbitrary.) Household h receives

$$\varphi^h K_s(\varphi)$$

bushels in each state $s \in \mathcal{S}$ from his contribution of φ^h acres, and thus finally consumes

$$x_s^h = e_s^h + \varphi^h (K_s(\varphi) - e_s^h)$$

bushels in each state $s \in \mathcal{S}$.

The rules of the cooperative define a noncooperative game. At any (pure-strategy Nash) equilibrium, no agent is worse off than he was before the cooperative was formed, since he can always choose $\varphi^h = 0$. The incentive for household h to contribute $\varphi^h > 0$ arises because in some states s, he has relatively low e_s^h.

Both the quantity $\sum_h \varphi^h$ of land in the pool and its average quality $K(\varphi)$ are endogenous. In equilibrium there may be considerable *adverse selection.* Households with relatively high e_s^h may contribute relatively less land.

4 The Perfectly Competitive Cooperative

In the game, households must anticipate that their contributions alter the pool quality $K(\varphi)$. When the number of households is very large, this quality effect becomes almost negligible. By ignoring it, any one household can concentrate on the far simpler problem of determining how much of the "net trade" $(K - e^h)$ he wishes to acquire.

We now postulate a world in which it is perfectly rational for each household to take K as given, independent of his action. This simplifies the analysis of equilibrium, without compromising the economic phenomena of adverse selection and signalling.

Let us imagine a continuum of households $t \in (0, H]$, where all $t \in (h-1, h]$ are of type h and are identical: $e^t = e^h$, $u^t = u^h$. Given a measurable choice of actions $\varphi : (0, H] \to [0,1]$ (which we also write $\varphi \in [0,1]^{(0,H]}$), the pool holds $\bar{\varphi} \equiv \int_0^H \varphi^t dt$ acres and produces $K_s(\varphi) \equiv \frac{1}{\bar{\varphi}} \int_0^H \varphi^t e_s^t$ per acre, if $\bar{\varphi} > 0$. It is clear that no single household in the continuum $(0, H]$ can affect $K_s(\varphi)$ by changing his actions. From his point of view, the trading opportunities are specified by the fixed vector $K = K(\varphi)$. Household $t \in (h-1, h]$ consumes

$$x_s^t = \chi_s^t(\varphi^t, K) \equiv e_s^h + \varphi^t (K_s - e_s^h)$$

bushels in each state $s \in \mathcal{S}$. His budget set is given by

$$\Sigma^t(K) = \{(\theta, y) \in [0,1] \times \mathbb{R}_+^S : y = \chi^t(\theta, K)\}.$$

We will say that $(K, \varphi, x) \in \mathbb{R}_+^S \times [0,1]^{(0,H]} \times \mathbb{R}_+^{S \times (0,H]}$ is an *equilibrium* for the economy $((u^h, e^h)_{h \in H})$ iff φ and x are measurable and

(1) $K = \frac{1}{\bar{\varphi}} \int_0^H \varphi^t e^t dt$ if $\bar{\varphi} > 0$

(2) $(\varphi^t, x^t) \in \arg\max_{(\theta,y) \in \Sigma^t(K)} u^t(y)$.

Notice that we are silent on how K should be formed when $\bar{\varphi} = 0$. By taking $K = 0$ (or sufficiently small, provided the marginal utilities of u^h are bounded), we can always sustain an *inactive* equilibrium (K, φ, x) in which $\varphi^t = 0$ almost everywhere. With only one cooperative this is not a serious matter, since we lose little by confining our attention to equilibria (K, φ, x) which are *active*, in the sense that $\bar{\varphi} > 0$. But when we consider multiple cooperatives, we will always find that many of them are effectively inactive in equilibrium, and then the choice of their K becomes a crucial issue, which we shall discuss at length. For the moment observe that, by its presence, K "opens" the inactive cooperative's doors for business: every household t knows that he will receive θK in exchange for θe^t; if the cooperative pool is inactive in equilibrium, it is in spite of this trading opportunity, and all households are choosing voluntarily not to go there.

We shall say that equilibrium (K, φ, x) is *type-symmetric* if $\varphi^t = \varphi^h$ (and so $x^t = x^h$) for all $t \in (h-1, h]$. In this case we often denote (φ, x) by $((\varphi^1, ..., \varphi^H), (x^1, ..., x^H))$.

5 Pooling Equilibrium and Adverse Selection

To make our analysis concrete, we shall return frequently to the following canonical example and its straightforward generalization, which we shall call the microeconomic version of the insurance problem.

Let there be $H = 6$ household types, and $S = 3$ states of nature. Suppose households have the same utility

$$u^t(x_1, x_2, x_3) = \sum_{s=1}^{3} \frac{1}{3} u(x_s), \text{ for all } t \in (0, 6],$$

where $u' > 0$, $u'' < 0$, and $\lim_{x \to 0} u'(x) = \infty$. The endowments of the households are given by

$$e^1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}; \; e^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \; e^3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$
$$e^4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \; e^5 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \; e^6 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since households $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are symmetric across states, we suspect that there must be a *symmetric* equilibrium in which $\varphi^t = \varphi^{t+1} = \varphi^{t+2}$, for all $t \in (0, 1]$ and all $t \in (3, 4]$. In these equilibria we perforce get $K_1 = K_2 = K_3 = \kappa$. The analysis collapses to a 2-dimensional picture. Every household begins with an endowment of 1 in his "good" state(s) and 0 in his "bad" state(s). His final consumption x_s will only depend on whether s is good or bad for him. The first three household types (whom we shall call reliable and label R because two-thirds of the states are good for them) have utility of consumption

$$u^R(x_G, x_B) = \tfrac{2}{3} u(x_G) + \tfrac{1}{3} u(x_B).$$

Similarly the unreliable household types $h \in \{4, 5, 6\}$ have utility

$$u^U(x_G, x_B) = \tfrac{1}{3} u(x_G) + \tfrac{2}{3} u(x_B).$$

If the pool quality is $K = (\kappa, \kappa, \kappa)$, then by contributing θ, the household on net gives up $\theta(1-\kappa)$ in his good state, and receives $\theta\kappa$ in his bad state. Thus his consumption must lie on the "κ-price line" joining (1,0) to $(0, \kappa/(1-\kappa))$. We can describe the same situation from a different point of view by observing that after giving up θ bushels in his good state, the agent receives $\theta\kappa$ in both states. His final consumption must therefore also lie on the "θ-quantity line" starting at $(1-\theta, 0)$ and moving northeast at 45°.

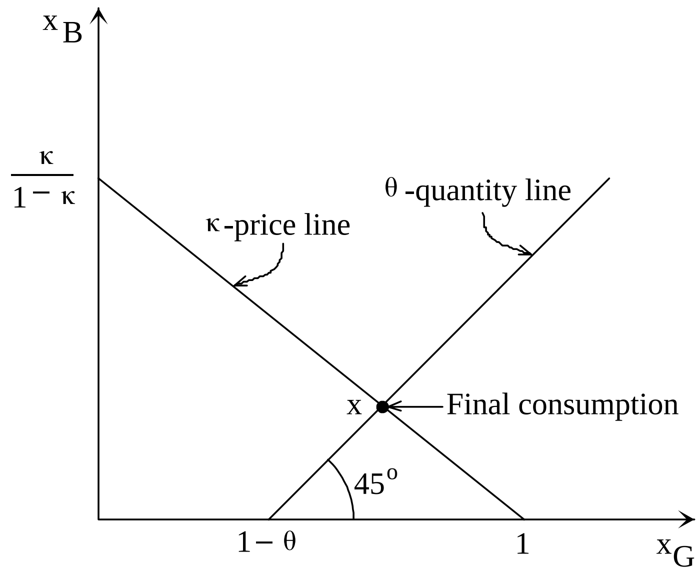

Figure 1

From strict concavity of the utility functions, it is clear that each agent t has a unique optimal choice x^t on the κ-price line. This choice can be implemented by a unique θ^t. Thus it is evident that any symmetric equilibrium must be type symmetric. We call it supersymmetric to mean symmetric and type symmetric, and we denote it $(\kappa, \varphi^U, \varphi^R)$.

In the obligatory cooperative, $\varphi^t \equiv 1$ for every household, so

$$\kappa = \frac{\frac{1}{3}\varphi^U + \frac{2}{3}\varphi^R}{\varphi^U + \varphi^R} = \frac{\frac{1}{3} + \frac{2}{3}}{1+1} = \frac{1}{2}.$$

In the voluntary cooperative, reliable households are likely to curtail their contributions because they recognize that their land delivers more on average than the pool, which is "debased" by the unreliable agents. When $\varphi^R < \varphi^U$ the pool quality is worse than the population average of 1/2, and we say that the pool displays adverse selection.

We can see pictorially why there is a tendency for adverse selection. If at some $\kappa > 0$ the reliable agents voluntarily contribute $0 < \varphi^R < 1$, consuming $x^R = e^R + \varphi^R(K - e^R)$, then their indifference curve I^R through x^R must be tangent to the κ-price line. But the unreliable indifference curve I^U through x^R is flatter, and so the unreliable must be choosing $\varphi^U > \varphi^R$. We make this precise below.

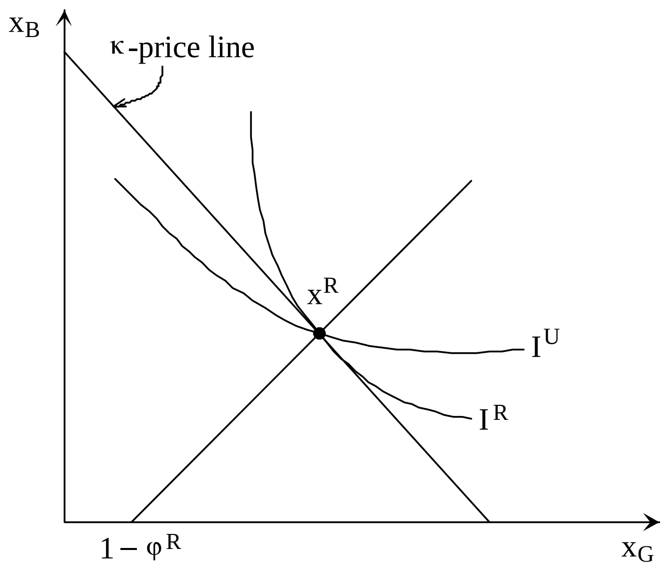

Figure 2

Theorem 1 *There exists at least one active, supersymmetric equilibrium with* $K_1 = K_2 = K_3 = \kappa$. *And every active, symmetric equilibrium has* $0 < \varphi^R < \varphi^U = 1$, *and* $1/3 < \kappa < 1/2$, *i.e., pooling always creates adverse selection.*

Proof Every household acts to

$$\max_{0 \le \theta \le 1} [(1-\pi)u(\theta\kappa) + \pi u(1-\theta(1-\kappa))] \equiv \max_{0 \le \theta \le 1} f_\pi(\theta, \kappa)$$

where $\pi^R = 2/3$ and $\pi^U = 1/3$. Clearly f_π is strictly concave in θ, and so achieves a unique maximum at $\theta_\pi(\kappa)$, a continuous function of κ. Notice that

$$\frac{\partial f_\pi(\theta, \kappa)}{\partial \theta} = (1-\pi)u'(\theta\kappa)\kappa - \pi u'(1-\theta(1-\kappa))(1-\kappa).$$

When $\kappa > 0$, $\partial f_\pi(\theta,\kappa)/\partial\theta|_{\theta=0} = \infty$, so we must have $\theta_\pi(\kappa) > 0$ whenever $\kappa > 0$.

Clearly $\pi > \kappa$ if and only if $\pi(1-\kappa) > (1-\pi)\kappa$. Hence when $\kappa > 0$, the optimal $\theta = \theta_\pi(\kappa)$ must satisfy $u'(\theta\kappa) > u'(1-\theta(1-\kappa))$. This happens precisely when $\theta\kappa < 1-\theta(1-\kappa)$, that is when $\theta < 1$. Conversely, if $\pi < \kappa$, we must have $\theta_\pi(\kappa) = 1$.

Consider now the map $\psi : [1/3, 1/2] \to [1/3, 2/3]$ given by

$$\psi(\kappa) = \frac{\frac{1}{3}\theta_{1/3}(\kappa) + \frac{2}{3}\theta_{2/3}(\kappa)}{\theta_{1/3}(\kappa) + \theta_{2/3}(\kappa)}.$$

Since $\theta_{1/3}(\kappa) > 0$ and $\theta_{2/3}(\kappa) > 0$ for all $\kappa \in [1/3, 2/3]$, the function ψ is continuous and indeed lies in the range $[1/3, 2/3]$. Hence it must have a fixed point $\kappa^* = \psi(\kappa^*)$.

Observe that since $\theta_{1/3}(\kappa^*) > 0$ and $\theta_{2/3}(\kappa^*) > 0$, $1/3 < \kappa^* < 2/3$. Hence $\theta_{1/3}(\kappa^*) = 1$ and $\theta_{2/3}(\kappa^*) < 1$. Therefore $\kappa^* < 1/2$.

Clearly κ^*, $\theta_{1/3}(\kappa^*)$, and $\theta_{2/3}(\kappa^*)$ generate a supersymmetric equilibrium. But conversely, any active symmetric equilibrium being supersymmetric, generates κ, $\theta_{1/3}(\kappa)$, $\theta_{2/3}(\kappa)$ which constitute a fixed point of ψ. ■

In the special case where $u(x) = \log(x)$, we obtain a unique active supersymmetric equilibrium which can be easily calculated. If $\theta_\pi < 1$,

$$\frac{\partial f_\pi(\theta, \kappa)}{\partial \theta} = (1-\pi)\frac{1}{\theta_\pi} - \frac{\pi}{1-\theta_\pi(1-\kappa)}(1-\kappa) = 0.$$

From this we deduce that $\theta_\pi = (1-\pi)/(1-\kappa)$ if $\varphi_\pi < 1$, or else $\varphi_\pi = 1$. Substituting $\pi = 2/3$ for the reliable agents, and recalling from the theorem that $\varphi^U \equiv \theta_{1/3} = 1$, we get $\varphi^R \equiv \theta_{2/3} = 1/3(1-\kappa)$, and

$$\kappa = \frac{\frac{1}{3}1 + \frac{2}{3} \cdot \frac{1}{3(1-\kappa)}}{1 + \frac{1}{3(1-\kappa)}}.$$

Rearranging terms shows that there is a unique solution $K = (5-\sqrt{5})/6 \approx 0.46$.

Then $\varphi^U = 1$, $x_G^U = x_B^U = \kappa \approx .46$, and $u^U \approx \frac{1}{3}\log(.098)$. Also $\varphi^R \approx .62$, $x_G^R = \frac{2}{3}$, $x_B^R \approx .28$, and $u^R \approx \frac{1}{3}\log(.126)$.

This equilibrium is pictured in Figure 3.

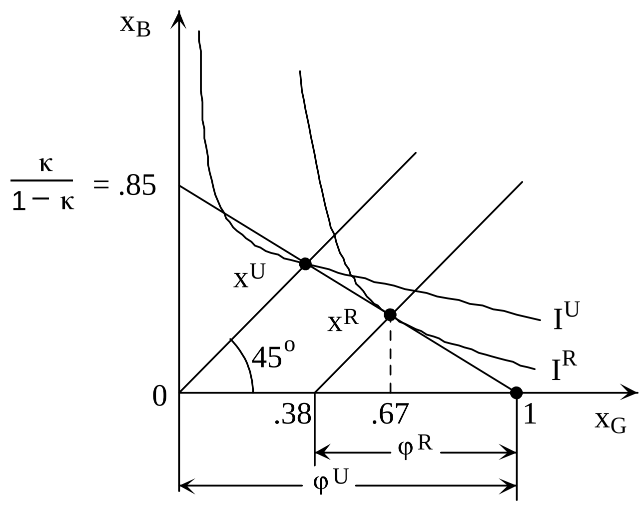

Figure 3

Had the pool restricted to quantities $\theta \in [0, Q]$ with $Q < 1$, the argument in Theorem 1 would only give the conclusion $\varphi^R \leq \varphi^U = Q$, allowing the possibility that adverse selection is suppressed and $\varphi^R = \varphi^U$.

6 Insurance

The classical insurance problem can be embedded in our model of cooperatives, and turns out to be a straightforward generalization of our canonical example.

6.1 The Insurance Problem

As in Rothschild–Stiglitz, we consider a continuum of two types of households: "reliable" (R) and "unreliable" (U), with population measures λ_R and λ_U respectively. Each household knows his own type, but not that of the others. Each household has wealth (for simplicity, 1 dollar) in his "good" (no-accident) state, but nothing in his "bad" (accident) state for which he seeks insurance. Accidents occur independently across households. The unreliable households are more accident-prone than the reliable. Thus if π^h denotes the probability of a good state for type h, we have $\pi^R > \pi^U$.

The utility for x units of money is $u(x)$, invariant of the state as well as household-type. As is standard, we assume that u is strictly concave ($u'' < 0$) and strictly monotonic, and $u'(x) \to \infty$ as $x \to 0$. The consumption of (x_G, x_B) across the two states yields expected utility

$$\pi^h u(x_G) + (1 - \pi^h) u(x_B)$$

to a household of type $h = R, U$. For ease of presentation we take π^h to be a rational number m/n.

6.2 A Microeconomic Representation of Insurance

We recast the Rothschild–Stiglitz story into our framework, building a microfoundation for the insurance problem in the process. The key step is to represent probability distributions of accidents by states of the world which make explicit who has an accident there. This makes it clear that "identical" insurance policies for two households of the same type do not pay off identically, since the households will have accidents in different states, even if their probabilities are the same.

Within our framework of finite states and household types, we cannot maintain both the hypotheses that accidents occur independently, and that the same proportion of each type has an accident in every state. We drop the independence hypothesis, which actually plays no role in the theory anyway.

Since probabilities are rational, let $\pi^R = r/n$ and let $\pi^U = u/n$. To convert the insurance problem into our framework, take $S = n$, and suppose there are $\binom{n}{r} = n!/[r!(n-r)!]$ subtypes of reliable households, each with population measure $\sigma\binom{n}{u}\lambda_R$, where σ is a positive scalar. Similarly, suppose there are $\binom{n}{u}$ subtypes of unreliable households, each with population measure $\sigma\binom{n}{r}\lambda_U$.

Each subtype τ is identified with the set $\mathcal{S}_\tau \subset \mathcal{S}$ of its bad states (r in number if reliable, u in number if unreliable). All households t of subtype τ have endowments equal to 1 is $s \in \mathcal{S} \backslash \mathcal{S}_\tau$, and equal to 0 if $s \in \mathcal{S}_\tau$.

The reader can verify that each household has the right probability of accident (r/n if reliable, u/n if unreliable), and that in *every* state the appropriate fraction of reliable and unreliable households have accidents.

An equilibrium of the insurance problem corresponds to a supersymmetric equilibrium of its representation, to which we therefore confine attention.

Recalling our numerical example of the previous section, note that it corresponds to the insurance problem with $\pi^R = 2/3$, $\pi^U = 1/3$, $\lambda_R = \lambda_U = 1$. Hence, in the micro-economic representation provided by our example, $S = 3$. There are $\binom{3}{1} = 3$ reliable subtypes, each of measure $\binom{3}{2}\sigma = 3 = 1$ (setting $\sigma = 1/3$), and $\binom{3}{2} = 3$ unreliable subtypes, each of measure $\binom{3}{1}\sigma = 1$.

6.3 Restriction to Symmetry

When we come to study the insurance problem, we should realize that every triple $(t, t+1, t+2)$ of households in the microeconomic representation (where $t \in (0, 1]$ or $t \in (3, 4]$) corresponds to one household in the insurance problem. Therefore we shall always restrict attention to the symmetric situations where such triplets choose the same $\varphi^t = \varphi^{t+1} = \varphi^{t+2}$.

Observe that symmetry is much weaker than supersymmetry, which requires in addition that *all* triplets with $t \in (0, 1]$ behave identically, and similarly for *all* triplets with $t \in (3, 4]$. Supersymmetry will be imposed later only on equilibrium actions, while the weaker notion of symmetry will be assumed even when we consider deviations from equilibrium. (Of course, since we have a continuum of households, deviations by three agents give the same effect as deviations by each of the three separately.)

7 The Pooling of Promises

The cooperative required land for membership. But the land was only instrumental. It indicated how much wheat its contributor promised to deliver to the pool, and at the same time it provided a benchmark for measuring his share in the pool. One might have imagined that land put into the cooperative was painted blue, and land held back was painted red. At a glance the villagers could survey the aggregate blue land held by the cooperative.

The modern world has more sophisticated and less cumbersome methods of accounting, which enables the delinking of promises and shares from the land. This leads us to think of a cooperative in which promises are pooled.

Such a cooperative must specify the nature of one unit of promise for each potential contributor. Abstractly we can represent this simply as a vector whose components e_s^h depend on h and s. It should be evident that the mathematics of pooling promises is identical to the land cooperative.

In the modern world one sees many examples of pools of promises, e.g., insurance pools, mortgage pools, credit card pools, and betting pools. Often entry into a pool is signified by a virtual promise which, like our land, is identical across agents. It is understood, however, that different households will actually deliver differently. The mechanisms by which these different deliveries come about involve options and default (and give rise to moral hazard). But as long as actual deliveries are foreseen, the analysis we do in this paper will remain relevant in the study of equilibrium. We have focused attention on the default option in [1].

The modern pools we alluded to take the further step of decoupling contributions to the pool from ownership of the pool. We have discussed this development in [1].

8 Quantity Limits

We imposed the restriction $\varphi^t \leq 1$ because every household was assumed to have one acre of land, and we imagined that the contributed land would have to be turned over and identified as the property of the cooperative. But if we reinterpret contributions as promises, then there is no reason why φ^t could not exceed 1. The household is typically able to deliver even when φ^t is a little larger than 1, out of his receipts from the pool (assuming that the cooperative makes it feasible to *net* deliveries from the pool against promises to the pool). And often the household would like to do so. For example, in Figure 3, we can see that the unreliable household would prefer $\varphi^U > 1$.

Returning to our example, but now *without* the quantity constraint $\varphi^t \leq 1$, household t's budget is expanded to

$$\Sigma^t(K) = \{(\theta, y) \in \mathbb{R}_+ \times \mathbb{R}_+^S : y = \chi^t(\theta, K)\}.$$

With log utilities, it can easily be checked that there is a unique active supersymmetric equilibrium in which $\varphi^U = 6/5$, $\varphi^R = 3/5$, $\kappa = 4/9$, $x_G^U = 1/3$, $x_B^U = 8/15$, $u^U \approx \frac{1}{3}\log(.095)$, and $x_G^R = 2/3$, $x_B^R = 4/15$, $u^R \approx \frac{1}{3}\log(.12)$.

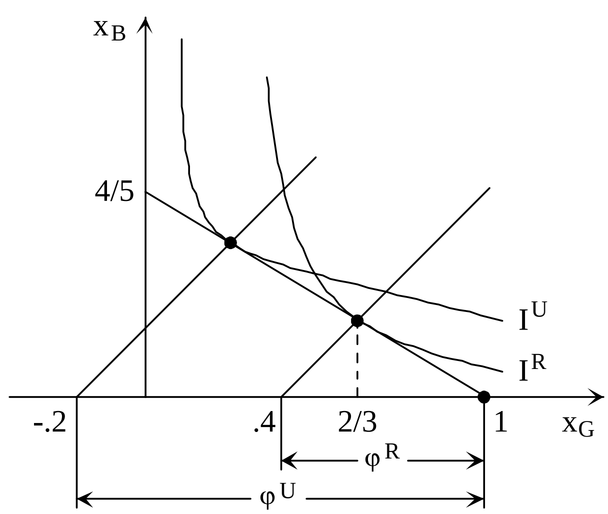

Figure 4

Notice that although each unreliable household t wants to trade $\varphi^t = 6/5 > 1$, the upshot of all the unreliable households doing so is to reduce the quality of the pool from $\kappa \approx .46$ to $\kappa \approx .44$ and to lower *every* household's utility, including their own.

Thus the cooperative can help everybody by imposing a quantity restriction, $\varphi^t \leq 1$.

Needless to say, there is no reason why the quantity restriction $\varphi^t \leq 1$ should be optimal. It is possible that setting a bound $\varphi^t \leq Q < 1$ will make everybody still better off (though not in this example). Reducing the limit Q further will help the reliable households and hurt the unreliable households. Reducing Q even further will hurt both households.

How will the cooperative set its quantity limit Q?

9 Competing Cooperatives

Different quantity limits may, as we saw, impinge on households differently. But if a cooperative cannot discriminate between households, it can set only one quantity limit. This gives an opportunity for a new cooperative to form, with a different quantity limit, to lure away dissatisfied members. How will this competition turn out?

Let us imagine a collection of cooperatives $j \in \mathcal{J} = \{1, ..., J\}$, all entailing the same promises e_s^t, but different quantity restrictions $\varphi_j^t \leq Q_j$.

Now household t chooses a vector $\theta = (\theta_1, ..., \theta_J) \in \mathbb{R}_+^J$, where θ_j denotes the number of promises (or the quantity of land) contributed to pool j. Suppose that for each pool j, the households anticipate deliveries K_s^j in state s, per unit contributed. Denote $K^j = (K_1^j, ..., K_S^j)$ and $K = (K^1, ..., K^J)$. Household t then consumes

$$\chi^t(\theta, K) \equiv e^t + \sum_{j \in \mathcal{J}} \theta_j (K^j - e^t).$$

His budget set is

$$\Sigma^t(K) = \{(\theta, y) \in \mathbb{R}_+^{\mathcal{J}} \times \mathbb{R}_+^{\mathcal{S}} : \theta_j \leq Q_j \text{ for all } j \in \mathcal{J},\, y = \chi^t(\theta, K)\}.$$

When $\sum_{j \in \mathcal{J}} \theta_j \leq 1$, the interpretation of the budget set is evident. Household t is simply dividing his land among different cooperatives, taking into account his anticipations of the various pool qualities. In that case the constraint $e^t + \sum_{j \in \mathcal{J}} \theta_j (K^j - e^t) \geq 0$ is redundant. But we allow for $\sum_j \theta_j > 1$, provided that household t can cover each of his promises out of receipts from the same pool, or from other pools. (Note that the netting of promises and deliveries is now across all pools, so we may view it as *supernetting*.)

9.1 Equilibrium

Abbreviate "almost all t in $(0, H]$" by "a.a.t," and the integral $\int_0^H f(t)dt$ by $\bar{f}$. The vector $(K, \varphi, x) \in \mathbb{R}_+^{\mathcal{J} \times \mathcal{S}} \times \mathbb{R}_+^{(0,H] \times \mathcal{J}} \times \mathbb{R}_+^{(0,H] \times \mathcal{S}}$ is said to be an *equilibrium* of the economy $((u^h, e^h)_{h \in H}, (Q_j)_{j \in \mathcal{J}})$ if φ and x are measurable, and

(1) $K_s^j = \frac{1}{\bar{\varphi}_j} \int_0^H \varphi_j^t e_s^t dt$ if $\bar{\varphi}_j > 0$, $\forall j \in \mathcal{J}$

(2) $(\varphi^t, x^t) \in \arg\max_{(\theta, y) \in \Sigma^t(K)} u^t(y)$ for a.a.t.

10 Cooperatives without Managers, Contracts without Designers

In our framework the cooperative j makes no decisions. It simply stands open for business. Its quantity limit Q_j is its defining characteristic, rather than a strategic choice made by its manager. And its K^j is determined by the forces of perfect competition in equilibrium.

The current orthodox view is that insurance is impossible without strategic intermediaries, actively designing contracts. This view was most elegantly expressed by Rothschild and Stiglitz (1976), who described an economy with perfectly competitive consumers and oligopolistic, risk-neutral insurance companies. These companies designed and marketed insurance contracts (Q, K) specifying the quantity Q of insurance available and its price K.

From the point of view of the household in our model, there is a potentially complex menu of nonlinear contracts. But this sophistication is owing entirely to "the market," not to any manager–designer. We will see that only a few (Q_j, K^j) have $\bar{\varphi}_j > 0$ among all potential $j \in \mathcal{J}$. The set of active contracts, that are played out at equilibrium, is thus sharply determinate. And it is designed entirely by the "invisible hand" of perfect competition.

10.1 Perfect Competition and Entry

In the Rothschild–Stiglitz model of insurance, equilibrium was required to be immune to entry by new insurance companies who might offer contracts $(\tilde{Q}, \tilde{K})$ that would turn a profit by luring households away from their old contracts. One might well ask whether our equilibrium is immune to entry. The answer is that whatever *new* $\tilde{Q}$ could be imagined is already present and embodied by one of the pools j, and it's associated quantity limit $Q_j = \tilde{Q}$. Its price K^j is set by the market. If the price K^j is set sensibly, and if at that price no household wants to join pool j, then we conclude that entrants using contracts $(Q_j, \tilde{K})$ cannot upset the equilibrium, for any $\tilde{K}$.

For this conclusion to be apt, we must be careful to make sure that the associated K^j are appropriate. High $\tilde{K}$ encourage entry, and thus have the potential to upset equilibrium. Of course $\tilde{K}$ cannot be arbitrarily high. It must be set at a level at which pool j can "reasonably expect" its receipts to cover its commitment to deliver $\tilde{K}$. If K^j is the highest $\tilde{K}$ consistent with this commitment, then consistent contracts of type $(Q_j, \tilde{K})$ must have $\tilde{K} \leq K^j$. But since (Q_j, K^j) rendered pool j inactive, so will $(Q_j, \tilde{K})$.

11 Equilibrium Refinement

With only one cooperative, we were content to confine our attention to equilibria in which the pool was active. With many cooperatives, the analogue would be to assume that all pools are active. But, as we have said, in the typical case every equilibrium effectively renders most pools inactive. Thus we have no choice but to confront how anticipations K^j will be formed when pool j is inactive, since it is those anticipations themselves that are responsible for the inactivity.

Our definition permits any pool j to be inactive, i.e., to have $\bar{\varphi}_j = 0$. Many potential pools in the real world are also inactive. One possible explanation is that people anticipate unduly pessimistic deliveries from them and are thus discouraged from joining them. There is nothing so far in our definition to prevent this from happening. When pool j is active, there is a "reality check" on K^j, since (by (1)) K^j must conform to actual deliveries. But for inactive pools j, there are no real deliveries to compare K^j to. If K^j were set suitably low, then no household t would be willing to contribute to pool j, for he would get very little per unit but incur a relatively large obligation to deliver e^t. Indeed, given an arbitrary subset of pools, we can always obtain equilibria which render them inactive by choosing their K^j to be low enough.

We believe that unreasonable pessimism does prevent many real world markets from opening, and provides an important role for government intervention. But it is interesting to study equilibrium in which anticipations are always reasonably optimistic. It is of central importance for us to understand which markets are open and which are not, and we do not want our answer to depend on the agents' whimsical pessimism.

Anticipated deliveries from inactive pools are analogous to beliefs in game theory "off the equilibrium path." Selten [4] dealt with the game theory problem by forcing every agent to tremble and play all his strategies with probability at least $\varepsilon > 0$, and then letting $\varepsilon \to 0$. We shall also invoke a tremble, but in quite a different spirit. Our tremble will be "on the market" and not on households' (players') strategies. Indeed, no household could tremble the way we want: we introduce an external player who delivers more per unit than any of the real households.[1] This extraordinary delivery is what banishes whimsical pessimism.

Consider an external e-agent who contributes $\varepsilon(n) = (\varepsilon_j(n))_{j\in J} \geq 0$ to every pool, and delivers an exogenously fixed vector $e = (e, ..., e)$ per unit contributed. We require that $e > \max_{h\in H} e^h_s$ for all $s \in \mathcal{S}$. Any e satisfying this requirement will be called an *optimistic external delivery*. The vector e indicates the boosting of household anticipations brought about by the external e-agent. We assume that $\varepsilon(n) \to 0$ as $n \to \infty$, so one might interpret this agent as a government which guarantees delivery on the first infinitesimal promises.

Formally, we say that an equilibrium $E = (K, \varphi, x) \in \mathbb{R}^{SJ}_+ \times \mathbb{R}^{(0,H]}_+$ is a *refined equilibrium* if there is a sequence $E_e(n) = (K(n), \varphi(n), x(n), \varepsilon(n)) \in \mathbb{R}^{SJ}_+ \times \mathbb{R}^{(0,H]J}_+ \times \mathbb{R}^{(0,H]\mathcal{S}}_+ \times \mathbb{R}^{\mathcal{J}}_+$ such that e is optimistic, $\varphi(n)$ and $x(n)$ are measurable for all $n = 1, 2, ...$ and

(1) $K(n) \to K$ and $\varphi^t(n) \to \varphi^t$, $x^t(n) \to x^t$ for a.a.t

(2) $(\varphi^t(n), x^t(n)) \in \arg\max_{(\theta,y)\in\Sigma^t(K(n))} u^t(y)$ for a.a.t and all n

[1] Were we to invoke a tremble on strategies, e.g., forcing each household t to contribute $\varepsilon(n) > 0$ to every pool, this would *not* meet our needs.

(3) $\varepsilon_j(n) > 0$ if $\bar{\varphi}_j(n) = 0$, for all $j \in \mathcal{J}$ and all n, and $\varepsilon(n) \to 0$

(4) For all n and for all $j \in \mathcal{J}^* = \{j \in J : \bar{\varphi}_j = 0\}$,

$$K_s^j(n) = \frac{1}{\varepsilon_j(n) + \bar{\varphi}_j(n)} \left[\varepsilon_j(n) e + \int_0^H \varphi_j^t(n) e_s^t dt \right].$$

The external e-agent may boost the delivery rate $K_s^j(n)$ above the level achieved by the real households in $E_e(n)$. As $n \to \infty$ this boost disappears for pools that are active in the limit. But for inactive pools, his presence prevents the limiting anticipations from sinking too low, and steers them away from undue pessimism. In fact, at first glance, one might think that given his extraordinary deliveries, no pool will be inactive in equilibrium. We shall see, however, that quite the opposite is true: many pools will be effectively inactive.

One can prove that a refined equilibrium always exists by explicitly adding an e-external agent to the market (who contributes $\varepsilon_j(n) = 1/n$ on every pool and delivers $(1/n)e$), showing that a $(1/n) - e$-equilibrium exists, and finally letting $n \to \infty$ and taking limits. Computing such an equilibrium would be hard, because it would involve computing a different equilibrium for each n. Our definition captures this spirit, but makes the computation much easier by dropping $\varepsilon_j(n) > 0$ unless $\bar{\varphi}_j(n) = 0$ (where the external boost $\varepsilon_j(n) > 0$ is really needed) and also dropping the condition that $K^j(n) =$ actual deliveries for active $j \in \mathcal{J} \backslash \mathcal{J}^*$, since we know where these $K^j(n)$ must converge anyway.

11.1 Elimination of Trivial Equilibria

Our refinement eliminates trivial equilibria in which K is set absurdly low. Let $\kappa^h \equiv \frac{1}{S} \sum_{s \in \mathcal{S}} e_s^h$ be the average delivery of household type h. Since $e^h \neq 0$, $\kappa^h > 0$ for all $h \in \mathcal{H}$, and $\kappa \equiv \min_{h \in \mathcal{H}} \kappa^h > 0$. It is easy to see that there cannot be any refined equilibrium $(K^1, ..., K^J, \varphi)$ in which for some pool j, $K_s^j < \kappa$ for all $s \in S$.

In the case of $H = 2$, and $J = 1$, the pool is active in a refined equilibrium if and only if there are gains to trade, i.e., there is some $0 < \alpha < 1$ and $0 < \beta < 1$ with $u^1(\alpha e^1 + \beta e^2) > u(e^1)$ and $u^2((1-\alpha)e^1 + (1-\beta)e^2) > u^2(e^2)$.

11.2 Hierarchical Refinement

Suppose that we have a hierarchy on $\mathcal{J}$ given by a partial order $<$ on $\mathcal{J}$. We specialize our notion of refinement to respect this order by strengthening (3) to

(3*) $\varepsilon_j(n) > 0$ if $\bar{\varphi}_j(n) = 0$, for all $j \in \mathcal{J}$ and all n, and $\varepsilon(n) \to 0$; moreover, $\varepsilon_j(n) \leq \frac{1}{n} \varepsilon_i(n)$ whenever $i < j$.

This notion will play a decisive role when we introduce seniority ranking of the pools. (Read "$i < j$" as "i is senior to j.") Our hierarchical refinement requires that the boosting provided by the external agent is an order of magnitude higher on a senior pool compared to any of its juniors.

12 Insurance with Competitive Pooling

Let us return to our canonical example of insurance, with reliable household types $h \in \{1, 2, 3\}$ and unreliable types $h \in \{4, 5, 6\}$. Fix the economy $((u^h, e^h)_{h \in \mathcal{H}}, (Q_j)_{j \in \mathcal{J}})$ with multiple pools $j \in \mathcal{J} = \{1, ..., J\}$ whose quantity limits form a fine grid $Q_1 < Q_2 < \cdots < Q_J$, i.e., $\max_j \{Q_{j+1} - Q_j\}$, Q, and $1/Q_J$ are all small, enabling us to approximate continuous quantities.

Theorem 2 shows that the economy with competitive pools has a continuum of equilibria, all of which entail a piecewise linear schedule in which the price rises with the quantity of insurance taken out. (This rise, in contrast to volume discounts, is the result of adverse selection.)

In Section 13 we show that with exclusivity the whole continuum of equilibria disappears, and a unique (separating) equilibrium takes its place.

In Section 14 we introduce seniority instead of exclusivity, and we find that the separating equilibrium disappears and a unique (pivotal) equilibrium emerges from out of the continuum.

Theorem 2 *There is a continuum of distinct (in consumption) refined symmetric equilibria* (K, φ, x), $K^j = (\kappa^j, ..., \kappa^j)$. *In each such equilibrium, delivery rates* κ^j *take on at most three values:* $1/3$ *or* $\bar{\kappa}$ *or* $1/2$, *where* $1/3 < \bar{\kappa} < 1/2$. *No household contributes to a pool before exhausting the quantity limits of all pools with higher delivery rates. The reliable households actively contribute only to pools* j *with* $\kappa^j \in \{1/2, \bar{\kappa}\}$. *The unreliable households contribute at least as much as the reliable households on every* κ*-level, and are active on at most one level below the reliable. All inactive pools have the same delivery rate* $= \min_{j \in \mathcal{J}} \kappa^j$.

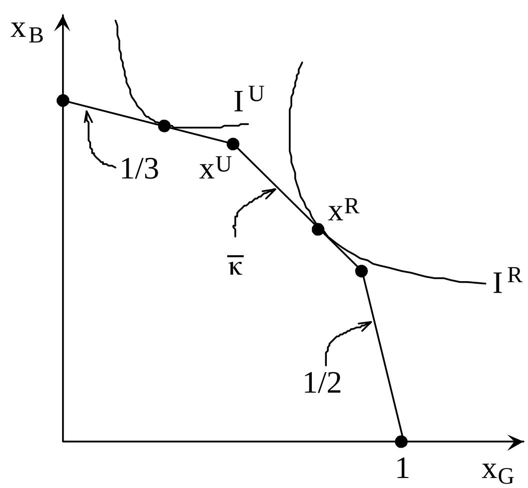

Figure 5

We prove Theorem 2 over the next two subsections.

12.1 The Continuum of Equilibria

We can easily characterize the continuum of equilibria of Theorem 1.

12.1.1 Primary-Secondary Equilibria

Let us first focus on the case where κ takes on at most two values. In all these equilibria both types take out equal amounts φ^* of a primary contract (i.e., each contributes φ^* over a set of primary pools with the same delivery rate), followed by different amounts $0 \leq \tilde{\varphi}^R < \tilde{\varphi}^U$ of a secondary contract. This set of equilibria can be parametrized by φ^*.

At the maximum level of φ^*, we have the equilibrium depicted in Figure 6. (For simplicity we will assume throughout that all points of tangency are exactly achievable via the quantity grid $\{Q_1, ..., Q_J\}$. Otherwise, obvious though tedious modifications need to be made to our statements.)

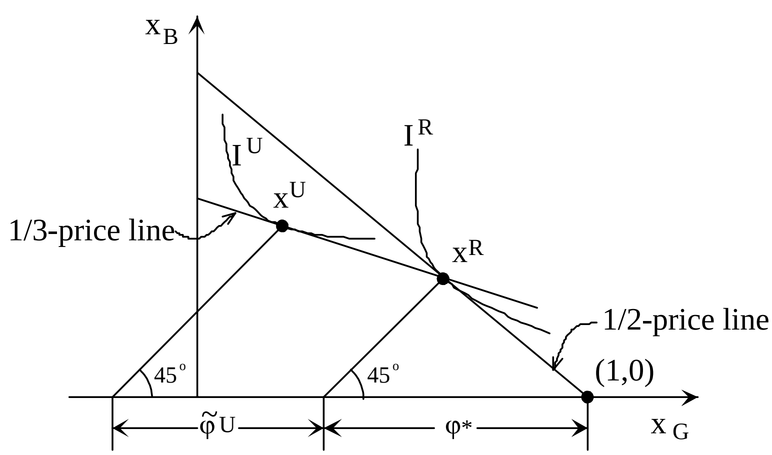

Figure 6

As φ^* is lowered we obtain

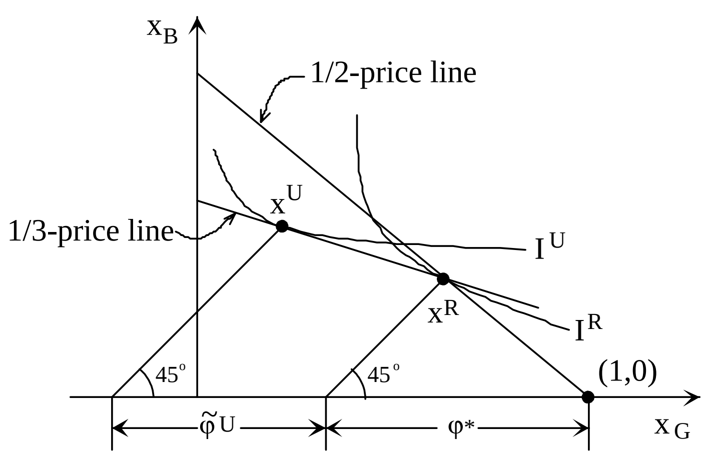

Figure 7

In the middle is the equilibrium of Figure 8, which will assume a prominent role later. We call it *pivotal* because the reliable households are ready to switch to being active on the secondary contract, were its rate improved by a penny. This equilibrium also constitutes a critical point between two regimes: one where both types take out the secondary contract, the other where only the unreliable do so.

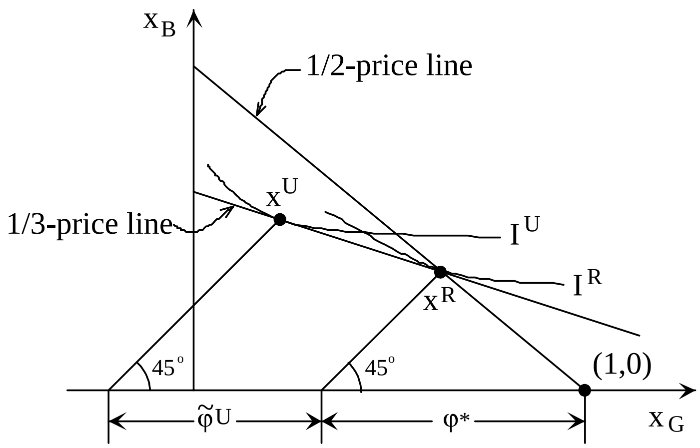

Figure 8: The Pivotal Equilibrium

As φ^* is lowered still further, the reliable households also take out the secondary contract (whose delivery rate $\tilde{\kappa}$ as a result rises above 1/3):

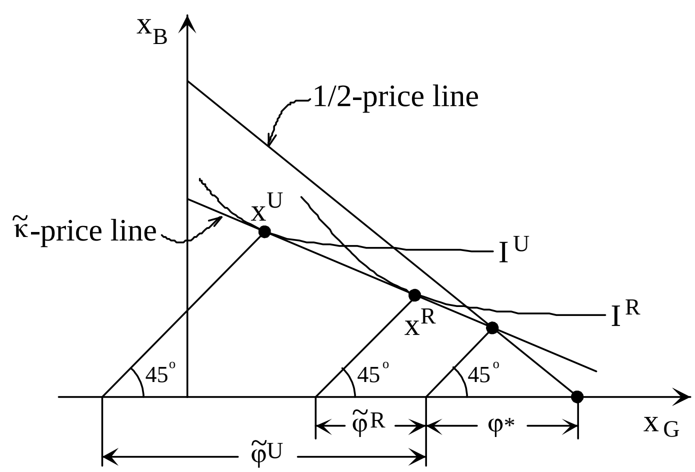

Figure 9

At the bottom end of this continuum, $\varphi^* = 0$ and we get the pure pooling equilibrium of Figure 4.

To show that all these equilibria are refined, we simply put $\kappa^j = \tilde{\kappa}$ for every inactive pool j, where $\tilde{\kappa}$ is the delivery rate of the active secondary contract. In the refining sequence we let households spread themselves over all these pools in the same proportion they are spread on the secondary contract. In fact this even shows that there is an equilibrium of each type in which all pools are active (and thus automatically refined).

12.1.2 Primary-Secondary-Tertiary Equilibrium

These are as in Figure 5. We leave the details to the imagination of the reader.

We have pictured the continuum of equilibria. Their geometry can easily be made algebraically precise, as we do for the pivotal equilibrium in Section 14.3.

12.2 No Other Equilibrium

We now present the proof that there can be no other equilibrium.

Proof Let (κ, φ, x) be an arbitrary refined symmetric equilibrium, and let

$$\mathcal{J}^R = \{j \in \mathcal{J} : \int_0^3 \varphi_j > 0\}$$
$$\mathcal{J}^U = \{j \in \mathcal{J} : \int_3^6 \varphi_j > 0\}$$

be the sets of pools on which R and U households are active. (Since $u'(x) \to \infty$ as $x \to 0$, and $\kappa_j \geq 1/3$ for all $j \in \mathcal{J}$ by refinement, it is evident that $\mathcal{J}^R \neq \phi$ and $\mathcal{J}^U \neq \phi$.) Arrange $\{\kappa^j : j \in \mathcal{J}^R \cup \mathcal{J}^U\}$ in descending order: $\kappa^{j_1} > \kappa^{j_2} > \cdots > \kappa^{j_L}$. Any household t (which may be of R or of U type) could obtain consumption anywhere on the piecewise linear path π given in Figure 10. Since $\min\{\kappa^j : j \in \mathcal{J}^R\} > 1/3 = \kappa^i$ for all $i \in \mathcal{J}^U \backslash \mathcal{J}^R$, the unreliable households are acting alone on at most the last link κ^{j_L}. It follows, from the strict concavity and monotonicity of the utilities, that each type has (and is choosing) a unique optimal consumption on π. By the argument made in the proof to Theorem 1, the unreliable need more insurance, so their consumption x^U must lie on π to the northwest of the reliable consumption x^R, i.e., starting from (1,0) and proceeding to x^U on π, the unreliable keep constant company with the reliable till the latter drop off at x^R (at which point, the unreliable proceed further). On links j_ℓ where the two types act together till the end, κ_{j_ℓ} must be $1/2$. Therefore the part of π on which R households act has at most two distinct κ^j. If there are two links, $\kappa^{j_1} = 1/2$ and $1/3 < \kappa^{j_2} < 1/2$. The theorem now follows. ■

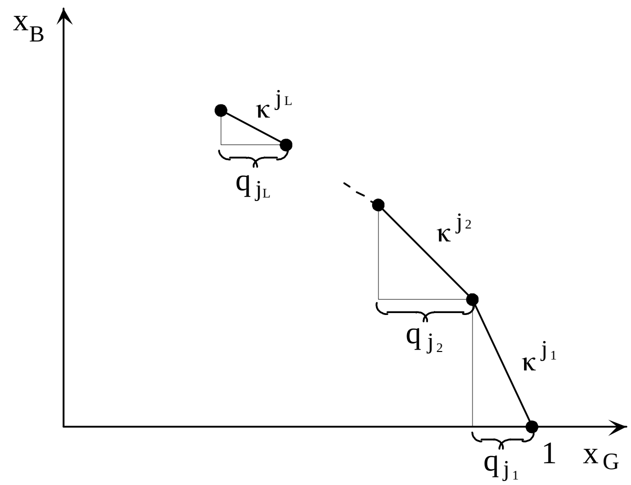

Figure 10

13 Exclusivity and Rothschild–Stiglitz's Separating Equilibrium

Pools can compete more effectively if they are allowed to be *exclusive*, i.e., to prohibit any household from contributing elsewhere if he contributes there. The contribution $\theta \in \mathbb{R}^J_+$ of any household must then satisfy

$$\theta \in \mathbb{R}^J_+(\text{exclusive}) \equiv \{\theta \in \mathbb{R}^J_+ : \theta_i > 0 \Rightarrow \theta_j = 0 \ \forall j \neq i\}.$$

Thus we define the budget set of t to be

$$\Sigma^t_{ex}(K) = \{(\theta, y) \in \Sigma^t(K) : \theta \in \mathbb{R}^J_+(\text{exclusive})\}.$$

Substituting Σ^t_{ex} for Σ^t, we then define refined equilibrium exactly as before.

The exclusivity constraint destroys the entire continuum of equilibria that we obtained in Section 11, when simultaneous access was permitted across all pools. Indeed many of those equilibria had households contribute to more than one pool, and so are ruled out a priori. The only feasible candidates are the pure "pooling" equilibria.

We showed in [2] that, in the presence of the exclusivity constraint, pooling equilibria cannot survive the test of our refinement. Let us informally see why.

Suppose we have a pooling equilibrium (drawn with both types up against the quantity restriction Q_j):

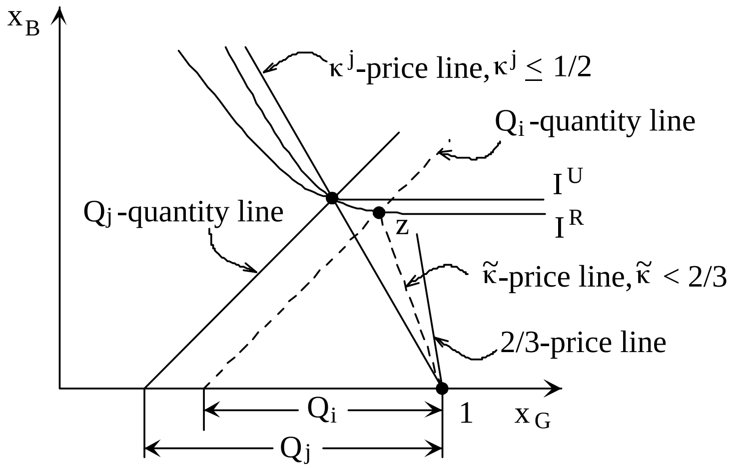

Figure 11

The crucial point is that if U and R act on the same pool, then by the argument of Theorem 1, $\varphi^U \geq \varphi^R$ and $\kappa^j \leq 1/2$, and the R-indifference curve will cut the U-indifference curve from northwest to southeast. How will κ^i be set for an *inactive* pool i with Q_i slightly less than Q_j? Consider the point z formed by the intersection of the Q_i-quantity line and I^R. Then z lies below I^U. Let $\tilde{\kappa}$ be the price line determined by z. By continuity, $\tilde{\kappa} \approx \kappa^j \leq 1/2 < 2/3$. If $\kappa^i > \tilde{\kappa}$, then reliable types will rush to join the pool, and i will not be inactive. On the other hand, if $\kappa^i \leq \tilde{\kappa}$, then no unreliable type will be close to wanting to join pool i. By the refinement $\kappa^i \geq 2/3$, a contradiction.

With exclusivity, a new kind of equilibrium emerges. By the exclusivity hypothesis, each household can contribute to at most one pool. By type-symmetry, all the reliable households will choose one pool i , and the unreliable households will choose another pool j, as indicated in Figure 12.

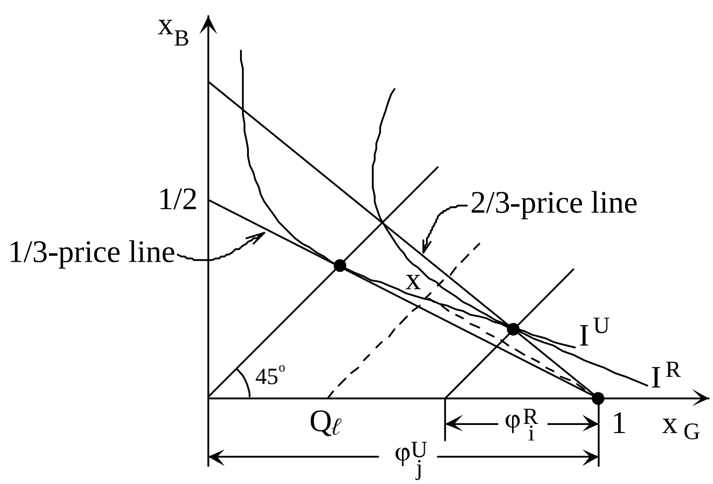

Figure 12

Let us briefly see why equilibrium of this sort always exists. (For rigorous details, see [2].) Take any quantity constraint Q_ℓ with $Q_\ell > \varphi_i^R =$ the quantity taken out by the reliable households. The Q_ℓ quantity line will intersect the I^U indifference curve first, at a point $x = x(Q_\ell)$ that lies below the I^R indifference curve. The price line $\kappa^\ell \equiv x_B/Q_\ell$ would make unreliable households just indifferent to switching pools, but leave the reliable households completely uninterested. In the refining sequence, it is therefore easy to combine a dwindling measure of unreliable households with the dwindling external agent to produce deliveries $\kappa^\ell(n) = \kappa^\ell$. Letting $\kappa^i(n) = 2/3$ and $\kappa^j(n) = 1/3$ throughout, all the households are optimizing.

If on the other hand $Q_\ell < \varphi_i^R$, the Q_ℓ-quantity line will intersect the I^R indifference curve first, at a point y, below the I^U indifference curve. Let $\kappa^\ell = y_B/Q_\ell$. Then $\kappa^\ell > 2/3$, and at that price reliable households are just indifferent to switching, while unreliable households are uninterested. Our refinement sequence will allow for a mixture of the reliable, and the external agent, who collectively will deliver at the rate κ^ℓ.

13.1 Comparison to Rothschild–Stiglitz

Our "separating" equilibrium produces the same consumption and the same active contracts as the Rothschild–Stiglitz separating equilibrium, when the latter exists. However in the Rothschild–Stiglitz equilibrium, households have just two quantity choices. In our equilibrium, every quantity choice Q_j comes with a "price" κ^j. Households choose from an entire schedule (where higher Q_j come with lower κ^j, i.e., higher premiums), though they end up picking only two.

Our equilibrium always exists, whereas the Rothschild–Stiglitz equilibrium robustly fails to exist.

14 Primary and Secondary Pools

It is unrealistic to suppose that an insurance company can prohibit its clients from taking out *any* further insurance from other carriers. Many people (including one of the co-authors of this paper) take out multiple life insurance policies, many professional athletes and musicians have multiple accident insurance contracts, and many agencies have multiple disaster insurance contracts.

We consider the simplest variant of our exclusivity model, in which primary pools $1, ..., J$ remain mutually exclusive, but none of them can prohibit households from contributing to a secondary pool $J+1$. Pool $J+1$ also has no quantity limit (or else Q_{J+1} is very large).

In Section 14.1 we investigate equilibrium when the delivery rate κ^{J+1} is the same for all contributors. In Section 14.2 we investigate equilibrium when pool $J+1$ can segregate contributors into subpools, conditional on the size of their primary contributions Q_j, $j \leq J$. This reflects the scenario, often seen in the real world, when secondary insurers make it a point to find out how much primary insurance has been taken out, and then charge accordingly.

We prove that with a conditional secondary pool, there is a unique equilibrium, namely the pivotal primary-secondary equilibrium defined earlier.

Primary and secondary pools constitute a primitive form of seniority, which we begin to study in greater generality in Section 15.

14.1 The Unconditional Pool

Denote $\mathcal{J} = \{1, ..., J, J+1\}$. The budget set of any household t is

$$\tilde{\Sigma}^t(K) = \{(\theta, y) \in \mathbb{R}_+^{J+1} \times \mathbb{R}_+^S : \theta_j \leq Q_j \; \forall j \in \mathcal{J},\; (\theta_1, ..., \theta_J) \in \mathbb{R}_+^J (\text{exclusive}),\; \chi^t(\theta, K) = y\}.$$

Substituting $\tilde{\Sigma}^t(K)$ for $\Sigma^t(K)$, we define refined equilibrium exactly as before.

With the onset of an unconditional pool, the separating equilibrium disappears.

Theorem 3 *Consider the model of multiple pools with exclusivity. Suppose, in addition, there is an unconditional secondary pool. Then the set of all refined supersymmetric equilibria coincide in consumption with the bottom-half of primary-secondary equilibria, starting at the pivotal equilibrium and descending all the way to the pooling equilibrium.*

The proof is in the Appendix.

14.2 The Conditional Pool: Simple Seniority

We now turn to the case where households can be segregated into secondary subpools depending on how much they contributed on their primary pool. It is simplest to think of this segregation as the creation of J secondary pools.

For each primary pool $i \in \{1, ..., J\}$ with quantity limit Q_i, we have its corresponding secondary (junior) pool $\sigma(i)$ with a large quantity limit $Q_{\sigma(i)}$. For simplicity we take $Q_{\sigma(i)} = Q_{J+1}$ for all $i \in \{1, ..., J\}$. Thus the set of pools is $\mathcal{J} = \{1, ..., J, \sigma(1), ..., \sigma(J)\}$ with the partial order: $i < \sigma(i)$ (i.e., i is senior to $\sigma(i)$). The budget set of household t now takes the form

$$\Sigma_*^t(K) = \{(\theta, y) \in \mathbb{R}_+^{\mathcal{J}} \times \mathbb{R}_+^{\mathcal{S}} : \theta_j \leq Q_j \; \forall j \in \mathcal{J},\; (\theta_1, ..., \theta_J) \in \mathbb{R}_+^J (\text{exclusive}),$$
$$\theta_i = 0 \Rightarrow \theta_{\sigma(i)} = 0 \text{ for } 1 \leq i \leq J,\; \chi^t(\theta, K) = y\}.$$

Then hierarchically-refined equilibrium is defined as before, substituting $\Sigma_*^t(K)$ for $\Sigma^t(K)$.

14.3 The Pivotal Primary-Secondary Insurance Contract

Recall our canonical insurance model with reliable and unreliable households. Let the pools $\mathcal{J}$ be defined by the simple seniority tree of two levels described in the last section. We shall show that equilibrium must be in accordance with Figure 13 below.

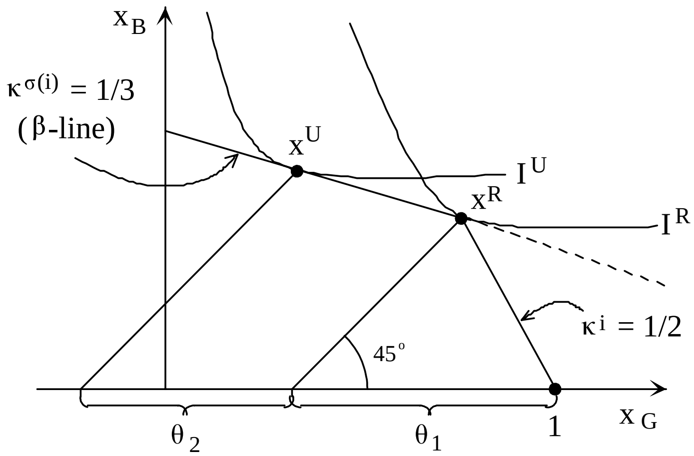

Figure 13

Let θ_1 solve

$$f(\theta) \equiv \frac{\frac{1}{3}u'\left(\frac{1}{2}\theta\right)}{\frac{2}{3}u'\left(1-\frac{1}{2}\theta\right)} - \frac{\frac{2}{3}}{\frac{1}{3}} = 0.$$

Since $f(0) = \infty$ and f is monotonically and continuously decreasing in θ and $f(1) < 0$, θ_1 is uniquely defined.

Define θ_2 to solve

$$g(\theta) \equiv \frac{\frac{2}{3}u'\left(\frac{1}{2}\theta_1 + \frac{1}{3}\theta\right)}{\frac{1}{3}u'\left(1-\frac{1}{2}\theta_1 - \frac{2}{3}\theta\right)} - \frac{\frac{2}{3}}{\frac{1}{3}} = 0.$$

Since $g(0) = 4 \times 2 - 2 = 6 > 0$, and g is monotonically and continuously decreasing, and $g(\theta)$ is near -2 for θ near $\frac{3}{2} - \frac{3}{4}\theta_1$, it follows that θ_2 is uniquely defined.

Theorem 4 *All hierarchically refined supersymmetric equilibria (of the insurance model with simple seniority) lead to the same consumption. Reliable households consume* $(x_G^R, x_B^R) = (1 - \frac{1}{2}\theta_1, \frac{1}{2}\theta_1)$ *and unreliable households consume* $(x_G^U, x_B^U) = (1 - \frac{1}{2}\theta_1 - \frac{2}{3}\theta_2, \frac{1}{2}\theta_1 + \frac{1}{3}\theta_2)$*. An equilibrium which sustains this consumption has one primary pool* i *and one secondary pool* $\sigma(i)$ *as its only active pools, and*

$$\varphi_i^R = \varphi_i^U = Q_i \equiv \theta_1;\ \varphi_{\sigma(i)}^R = 0,\ \varphi_{\sigma(i)}^U = \theta_2 \le Q_{J+1};\ \kappa^i = \tfrac{1}{2},\ \kappa^{\sigma(i)} = 1/3.$$

The proof is in the Appendix.

15 The General Seniority Tree

In many cases, insurance companies cannot prohibit a household from going elsewhere to obtain additional insurance, or stop the other companies from giving it to him. At most, the companies can reveal to each other how much insurance the household has with each of them.

We return to the situation in which households can contribute to as many pools as they wish. We shall now suppose that contributions are public information. A household or corporation getting insurance must (in some cases) declare from where each dollar of insurance comes. Catastrophe insurance might cover the first \$100,000 of damage; the next \$200,000 might come from some other (re)insurer. The second insurer is aware of the primary coverage, and will charge a rate depending on how much primary insurance there is. We incorporate this notion of seniority by supposing that pools limit entry to households based upon their history of contributions elsewhere.

Let $\mathbb{Q}$ be a collection of quantities $\mathbb{Q} \subset \mathbb{R}_{++}$, $\mathbb{Q} = \{q_1 < \cdots < q_m < \cdots < q_M\}$. We now introduce seniority levels $\ell = 1, 2, 3, ...$ on pools. The pools $\mathcal{J}$ are defined by the nodes of a rooted tree (except for its root), which in turn is defined as follows. Starting from any node, the branches that issue out correspond to the elements of $\mathbb{Q}$. A pool or node of level ℓ is therefore denoted by $j \equiv m_1 m_2 ... m_\ell$, which describes the unique path from the root to that node. It has quantity limit $Q_j = q_{m_\ell}$. This pool is *junior* to precisely all its predecessor nodes (pools) in the tree, namely the $\ell - 1$ nodes $(m_1), (m_1 m_2), ..., (m_1 m_2 ... m_{\ell-1})$. A node i is said to be *senior* to j if j is junior all the nodes in the infinite tree of which it is the root.

Seniority means that each household's contributions must lie on one path in the tree. We further assume that he can make only a finite (albeit arbitrarily large) number of contributions. A household in effect chooses a node, and then chooses nonnegative, feasible contributions to pools along the path from the root to the node.

The fact that the tree is infinite, and that there is no upper bound on the number of positive contributions, means that a household, regardless of his history of pools joined, is always free to join a new pool. But that pool is cognizant of his history and indeed is open only to households with the same history.

If the tree has only one level, then seniority is identical to exclusivity.

15.1 Equilibrium

Let $\mathcal{J}$ denote the (partially ordered) set of all nodes of the infinite tree, except for its root. (Thus $\mathcal{J}$ represents the hierarchically arranged pools potentially available to the households.) And let $K \in \mathbb{R}_+^{\mathcal{J} \times \mathcal{S}}$ be the anticipated delivery rates from these pools.

The budget set of household t is then given by

$$\begin{aligned}\hat{\Sigma}^t(K) = \{(\theta, y) \in \mathbb{R}_+^{\mathcal{J}} \times \mathbb{R}_+^{\mathcal{S}} :\ & \text{(i) } \theta_j \leq Q_j\ \forall j \in \mathcal{J}, \\ & \text{(ii) } \exists j^* \in \mathcal{J} \text{ such that } \theta_i > 0 \text{ only if } i \text{ is senior to } j^*, \text{ (iii) } y = \chi^t(\theta, K)\}.\end{aligned}$$

Notice that, on account of (ii), $\chi^t(\theta, K) \equiv e^t + \sum_{j \in \mathcal{J}} \theta_j (K^j - e^t)$ is well defined, since θ_j is zero except possibly for finitely many j. We restrict attention to maps $\varphi \in \mathbb{R}_+^{\mathcal{J} \times (0,H]}$ for which there is a $B < \infty$ such that, a.a.t, $\varphi_j^t = 0$ if the level of j is bigger than B. With this proviso, we define hierarchically refined equilibrium as before, using the budget sets $\hat{\Sigma}^t(K)$.

Theorem 5 *In the insurance model with a general seniority tree, there always exists a hierarchically refined equilibrium which yields the same consumption as the pivotal primary-secondary equilibrium, and which moreover has just two active pools, one from the first level and one of its juniors from the second level.*

The proof is in the Appendix.

16 General Existence of Equilibrium

We were able to exploit the special structure of the insurance economy (like the single crossing property) to construct equilibrium and to verify it to be so. In general, constructing equilibrium can be quite difficult, but there is no question of its existence (except, perhaps, for the general seniority tree with its infinite set of pools). Theorem 6 assures us that equilibrium exists even without special structure, for example, even if reliable households have utility $v \neq u$ so that the v and u indifference curves cross more than once.

Theorem 6 *Consider any of our perfectly competitive pooling models with a finite set $\mathcal{J}$ of pools. Then a refined (and, if $\mathcal{J}$ is partially ordered, a hierarchically refined) equilibrium exists.*

This is a corollary of Theorem 7 in [1]. It is worth noting that equilibria, in the general setting of Theorem 6, need *not* be type-symmetric on account of the nonconvexity of budget sets (brought on by exclusivity or seniority). It is a happy circumstance that they turn out to be so in our canonical insurance framework.

References

[1] Dubey, P., J. Geanakoplos and M. Shubik, 2001. "Default and Punishment in General Equilibrium with Incomplete Markets," Cowles Foundation Discussion Paper No. 1304.

[2] Dubey, P. and J. Geanakoplos, 2001. "Signalling and Default: Rothschild–Stiglitz Reconsidered," Cowles Foundation Discussion Paper No. 1305.

[3] Rothschild, M. and J. Stiglitz, 1976. "Equilibrium in Complete Insurance Markets: An Essay on the Economics of Imperfect Information," *Quarterly Journal of Economics*, 90: 629–650.

[4] Selten, R., 1975. "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*, 4: 25–55.

Appendix

Proof of Theorem 3 Let (κ, φ, x) be a refined, supersymmetric equilibrium. Suppose the reliable households (who are all making the same choice, since φ is type-symmetric by assumption) have contributed on primary pool j. Then all unreliable households must also have chosen primary pool j. For suppose they chose primary pool $i \neq j$. Then $1/3 = \kappa^i < \kappa^j = 2/3$. Let $\kappa \geq 1/3$ be the delivery rate on pool $J+1$. (κ cannot be less by our refinement.) Then any household t of type U can do strictly better by switching to pool i and then (if necessary) contributing more on pool $J+1$. This is a contradiction, proving that consumption at (κ, φ, x) must be in accordance with one of the primary–secondary equilibria.

We next show that no primary-secondary equilibrium, which lies strictly above the pivotal, survives the test of refinement.

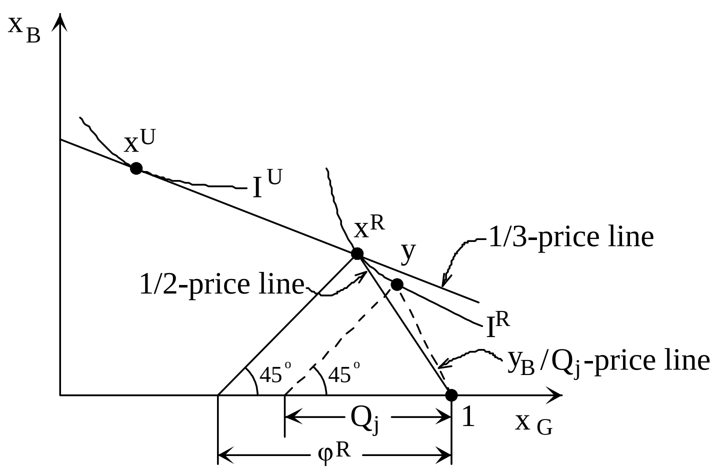

Figure 14

Consider any refined supersymmetric equilibrium (κ, φ, x) with consumption as in Figure 14 above. Let j be a primary pool with Q_j just below φ^R. Suppose $(\kappa(n), \varphi(n), x(n), \varepsilon(n))_{n=1}^{\infty}$ refines (κ, φ, x). Let $U(n)$, $R(n)$ be the sets of households of U, R type who contribute positively on pool j in $\varphi(n)$. Then there exists $c > 0$ such that

$$\lambda(U(n)) > c\lambda(R(n)) \tag{$*$}$$

for all large enough n, where $\lambda \equiv$ Lebesgue measure. Otherwise, by our refinement $\kappa^j(n) \to \kappa^j \geq 2/3$, and each R household could asymptotically achieve much more utility than his equilibrium level I^R, by contributing Q_j on pool j. For the same reason, denoting by y the intersection of the Q_j-quantity line and the I^R indifference curve,

$$\kappa^j \leq \frac{y_B}{Q_j}. \tag{$**$}$$

By $(*)$ and $(**)$, there is a non-null set of U-households, each of whom can consume (via his contribution $\varphi_j^t(n)$ on j) only some point arbitrarily close to the triangle formed by y, $(1-Q_j,0)$, $(1,0)$. But after this consumption, t can do further trade only via pool $J+1$ at the rate $\kappa^{J+1}(n) \to \kappa^J = 1/3$. All such trade keep t bounded strictly below I^U, contradicting that he was maximizing.

This proves that the equilibrium cannot be refined.

Finally we must show that any primary–secondary equilibrium of the bottom half is in fact a refined equilibrium of $((e^h,u^h)_{h\in H}(Q_j)_{j\in\mathcal{J}})$.

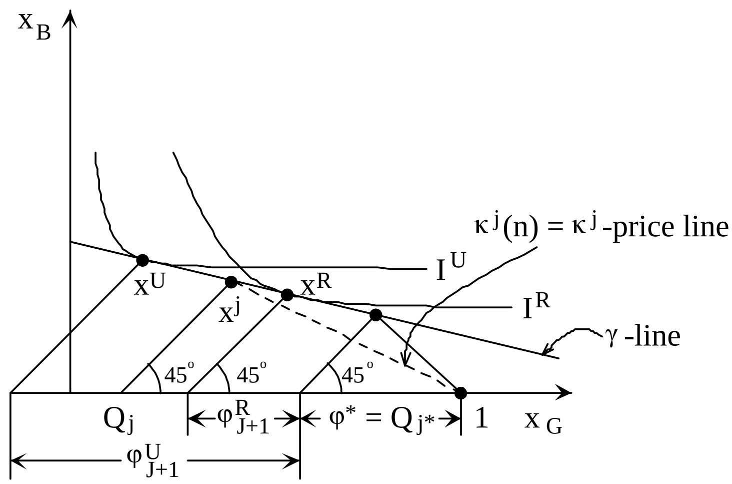

Figure 15

Let the primary–secondary equilibrium be as pictured above, with both household types contributing $\varphi^* = Q_{j^*}$ on primary pool j^*; and φ^R_{J+1}, φ^U_{J+1} respectively on pool $J+1$. We must assign κ to all the pools and show that they can be justified via refinement.

Let "γ-line" name the common tangent to I^U (at x^U) and I^R (at x^R). For any $j \in \mathcal{J}\backslash\{j^*, J+1\}$, let $x^j \equiv$ intersection of the Q_j-quantity line with the γ-line. If $Q_j \le \varphi^* + \varphi^U_{J+1}$, define $\kappa^j = x^j_B/Q_j$. If $Q_j > \varphi^* + \varphi^U_{J+1}$, define $\kappa^j = x^U_B/(\varphi^* + \varphi^U_{J+1})$.

Now we construct the refinement $E_e(n) = (\kappa(n), \varphi(n), x(n), \varepsilon(n))_{n=1}^{\infty}$. First set $e = x^1_B/Q_1$ and $\kappa^j(n) = \kappa^j$ for all $j \in \mathcal{J}$ and all n. Take a (symmetric) set $W(n)$ of U-households of measure $1/n$, and divide it into disjoint sets $W^j(n)$, $j \in \{1, ..., J\}$, of equal measure. In $\varphi(n)$, the households in $W(n)$ will deviate from their equilibrium actions, while all other households will stick to their equilibrium actions. If $Q_j \le \varphi^* + \varphi^U_{J+1}$, each $t \in W^j(n)$ sets $\varphi^t_j(n) = Q_j$ to reach x^j at the anticipated delivery rate $\kappa^j(n) = \kappa^j$. Then he further contributes $\varphi^U_{J+1} - Q_j$ on $J+1$ to advance from x^j to x^U at the anticipated delivery rate $\kappa^{J+1}(n) = \kappa^{J+1}$. If $Q_j > \varphi^* + \varphi^U_{J+1}$, let $t \in W^j(n)$ contribute $\varphi^* + \varphi^U_{J+1}$ on j to advance directly to x^U at the anticipated delivery rate $\kappa^j(n) = \kappa^j = x^U_B/(\varphi^* + \varphi^U_{J+1})$. Finally we define $\varepsilon_j(n)$ for all $j \in \mathcal{J}\backslash\{j^*, J+1\}$ to make $\kappa^j(n)$ become real with the e-external agent. (Needless to say, take $\varepsilon_j(n) = 0$ for $j = j^*$ or $j = J+1$ throughout). Thus let $\varepsilon_j(n)$ be the unique ε which solves the equation

$$\kappa^j = \frac{\varepsilon e_s + \lambda(W^j(n))\theta_j(n)}{\varepsilon + \lambda(W^j(n))\theta_j(n)}$$

where we have denoted

$$\theta_j(n) = \begin{cases} Q_j & \text{if } Q_j \leq \varphi^* + \varphi^U_{J+1} \\ \varphi^* + \varphi^U_{J+1} & \text{if } Q_j > \varphi^* + \varphi^U_{J+1} \end{cases}.$$

It can be easily checked that our sequence refines the equilibrium. ■

Proof of Uniqueness in Theorem 4 First let us notice an implication of hierarchical refinement which will be useful for us in the proof, and which may also be of some independent interest. We say that K^j_s is *boosted* on the refining sequence $E_e(n) = (K(n), \varphi(n), n(n), \varepsilon(n))$ if either

(1) $\bar{\varphi}_j(n) > 0$ for all large n and $K^j_s > \limsup_n \frac{1}{\bar{\varphi}_j(n)} \int_0^H \varphi^t_j(n) e^t_s dt$

or

(2) $\bar{\varphi}_j(n) = 0$ infinitely often (in which case $K^j_s = e$).

If (2) holds we say that K^j_s is *superboosted* on $E_e(n)$. Let us also define pool j to be *non-negligible* compared with pool i in the sequence $E_e(n)$ if $\liminf_n \{\int_0^H \varphi^t_j(n) - B \int_0^H \varphi^t_i(n) dt\} \geq 0$ for some $B > 0$. Then, if E is hierarchically refined by $E_e(n)$, it is easy to see that the following property will hold:

$$\left\{ \begin{array}{l} i < j; \\ j \text{ is non-negligible in } E_e(n) \text{ compared with } i; \\ K^i_s \text{ is not superboosted for some } s \end{array} \right\} \Rightarrow K^j_s \text{ is not boosted } \forall s \in \mathcal{S}.$$

Let us now turn to the proof of uniqueness. Arguing exactly as in the proof of Theorem 3, we see that consumption must correspond to one of the primary-secondary equilibria. Consider any such consumption allocation below the pivotal equilibrium, in which reliable households are active in the secondary pool. Let θ denote the total of reliable contributions on both pools (see Figure 16). And let κ^* denote the delivery rate of the active secondary pool. Since reliable households are contributing on the secondary pool, $\kappa^* > 1/3$.

Label by I^U and I^R (as usual) the indifference curves of the U and R types through x^U and x^R.

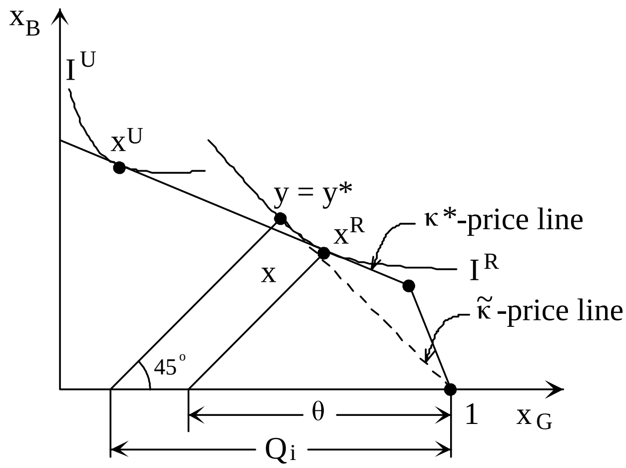

Figure 16

Let i denote the primary pool with quantity limit Q_i just above θ. Let y denote the intersection of the Q_i-quantity line with I^R, with associated price line $\tilde{\kappa} \equiv y_B/Q_i$.

First note that $\kappa^i \leq \tilde{\kappa} < e$, otherwise any reliable household could obtain higher utility than I^R by contributions on pool i. Suppose (κ, φ, x) is an equilibrium with active pools as in Figure 16, refined by the sequence $E_e(n) = (\kappa(n), \varphi(n), x(n), \varepsilon(n))_{n=1}^{\infty}$. Then $\kappa^{\sigma(i)}(n) \leq \kappa^i(n)$, else nobody acts on i and so by refinement $\kappa^i(n) = e$, a contradiction.

Let $U(n)$, $R(n)$ be the sets of U, R households who contribute positively to pool i in $\varphi(n)$. Then, as in the proof of Theorem 3,

$$\exists c > 0 \text{ such that } \lambda(U(n)) > c\lambda(R(n)) \text{ for all } n. \qquad (\star)$$

Observe that every $t \in U(n)$ must be contributing a nonvanishing amount on $\sigma(i)$, since contributing at most Q_i at $\kappa^i(n) \leq \tilde{\kappa}$ will not get him close to I^U. Combining this with $(\star)$ (and noting that all contributions on i are bounded by Q_i), we conclude that $\sigma(i)$ is non-negligible compared with i in $E_e(n)$. Also, since $\kappa^i < e$, κ^i is not superboosted. Thus, by hierarchical refinement, $\sigma(i)$ is not boosted.

Note that if no reliable household contributes to $\sigma(i)$, then $\kappa^{\sigma(i)}(n) \approx 1/3$ (because $\sigma(i)$ is not boosted). But then the unreliable households cannot get near I^U, contradicting $U(n) \neq \phi$. Hence it suffices to show that for large enough n, no reliable household contributes to $\sigma(i)$.

Next observe that we cannot have $\kappa^{\sigma(i)}(n) = \kappa^i(n)$, for any n. Otherwise the common price line from (1,0) would (nearly) be tangent to I^U, and therefore far from I^R, implying that no reliable households act on $\sigma(i)$.

Thus $\kappa^{\sigma(i)}(n) < \kappa^i(n)$. From this it follows that any reliable household contributing on $\sigma(i)$ must have contributed the full Q_i on i. If he does act on $\sigma(i)$, and via $\kappa^{\sigma(i)}(n)$ gets close to I^R, then $\kappa^{\sigma(i)}(n)$ is above (or at worst infinitesimally below) the |slope| of I^R at y. But with such a high $\kappa^{\sigma(i)}(n)$, the unreliable could do much better than I^U, since $Q_{\sigma(i)} \equiv Q_{J+1}$ is huge. Thus for all large n, no reliable household acts on $\sigma(i)$, and we are done. ■

Finally we must also rule out consumption above the pivotal equilibrium. Consider one such, figured below.

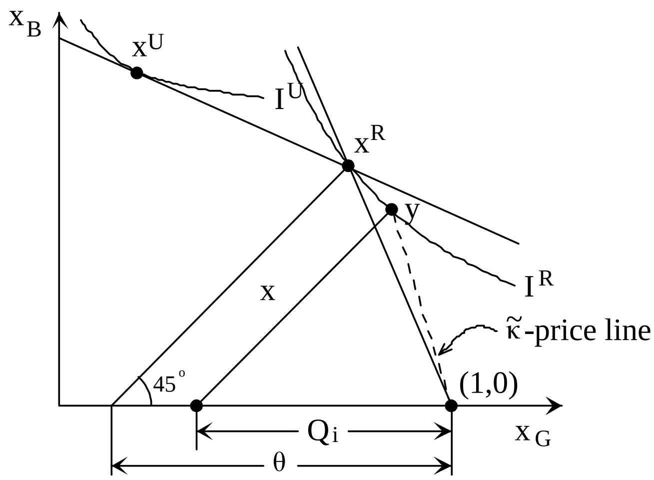

Figure 17

Consider i with Q_i just below θ. Using the same notation as before, notice $\kappa^i(n) \leq \tilde{\kappa} < e$, where $\tilde{\kappa}$ is the |slope| of the line joining $(1,0)$ to y, and $y \equiv$ intersection of the Q_i-quantity line with the I^R curve. We can now show as before that the contributions of $R(n)$ on $\sigma(i)$ are zero for large n. But, $\lambda(U(n)) > c\lambda(R(n))$ for all n as before, and all $t \in U(n)$ need to make nonvanishing contributions to $\sigma(i)$ to get to I^U, contradicting hierarchical refinement as before. ■

Proof of Existence This is identical to the proof of Theorem 5, replacing $\mathcal{J}(i)$ by $\{\sigma(i)\}$ throughout; and, in the refinement, putting the aggregate contributions on $\mathcal{J}(i) =$ the contribution on $\sigma(i)$. (As a result, the refinement here becomes simpler than in Theorem 5.) ■

Proof of Theorem 5 We begin by verifying that there is indeed an equilibrium of the type we claim.

Consider a pivotal primary-secondary equilibrium as described in Figure 13. Suppose $q_{m^*} = \theta_1$. Then define $\varphi^t_{m^*} = \theta_1$ for all $t \in (0,6]$, $\varphi^t_j = 0$ for $j \in \mathcal{J}\backslash\{m^*\}$ and $t \in (0,3]$, $\varphi^t_{m^*M} = \theta_2$ for $t \in (3,6]$, $\varphi^t_j = 0$ for $j \in \mathcal{J}\backslash\{m^*, m^*M\}$ and $t \in (3,6]$. This clearly leads to the consumption shown in Figure 13.

First we assign κ_j to all pools j other than m^* and m^*M in a way that leads all households to take the equilibrium actions.

Next we show that we can find a sequence $K(n) \to K$ such that given *any* node $j \equiv m_1 m_2 ... m_\ell$ in the tree, unreliable households could obtain the same utility by positively contributing on all pools senior to j (with possibly further contributions to j and its juniors).

Third, we specify $\varphi(n)$ by taking a small population of unreliable households, and assigning their choices differently, so that every node in the tree has a positive amount of contributions.

Lastly, we check that the $K^j(n)$ satisfy condition (3*).

For each primary pool m_1 (i.e., a pool m_1 of level 1), we shall describe κ^{m_1}, as well as a uniform $\kappa = \kappa^{m_1 m_2 \ldots m_\ell}$ for all pools $m_1 m_2 \ldots m_\ell$ that are junior to m_1.

Consider the following diagram and focus on the β-line, which is the common tangent to I^U (at x^U) and I^R (at x^R).

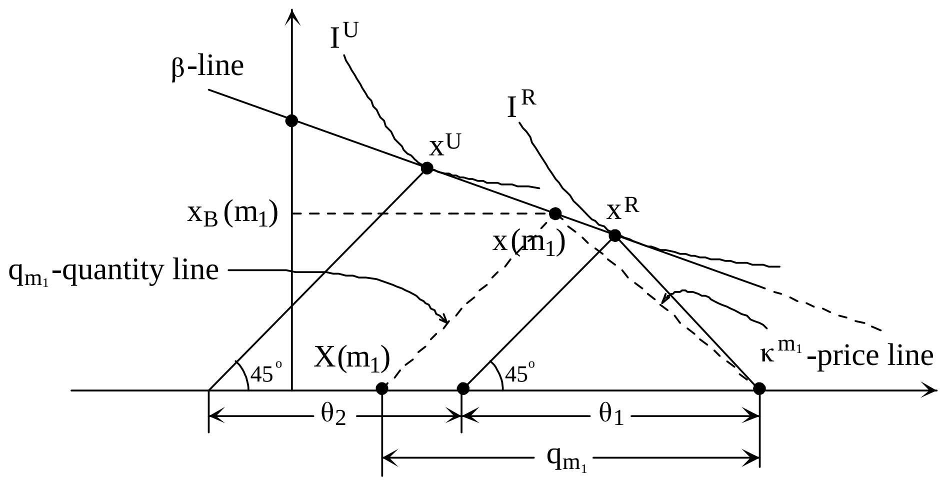

Figure 18

Suppose $q_{m_1} \leq \theta_1 + \theta_2$. Find the point $x(m_1)$ which lies at the intersection of the 45° line starting at $X(m_1) \equiv (1 - q_{m_1}, 0)$ and the β-line. Define

$$\kappa^{m_1} \equiv \frac{x_B(m_1)}{q_{m_1}}.$$

To check this formula, note that $\kappa^{m_1}/(1-\kappa^{m_1})$ should be the |slope| of the $\kappa_1^{m_1}$ price line, so $\kappa^{m_1}/(1-\kappa^{m_1}) = x_B(m_1)/(1-x_G(m_1)) = x_B(m_1)/(q_{m_1} - x_B(m_1))$, since the q_{m_1} quantity line rises at 45°. Inverting both sides gives $1/\kappa^{m_1} - 1 = q_{m_1}/x_B(m_1) - 1$ and hence our formula.

For any pool $m_1 m_2 \ldots m_\ell$ junior to m_1, define $\kappa^{m_1 m_2 \ldots, m_\ell} = 1/3$.

Next, let z be the unique point on I^U such that the ray from (1,0) through z is tangent to I^U. Let $(1-\theta_3, 0) \equiv Z$ be the unique point on the horizontal G-axis such that the 45° line moving northeast from Z intersects z.

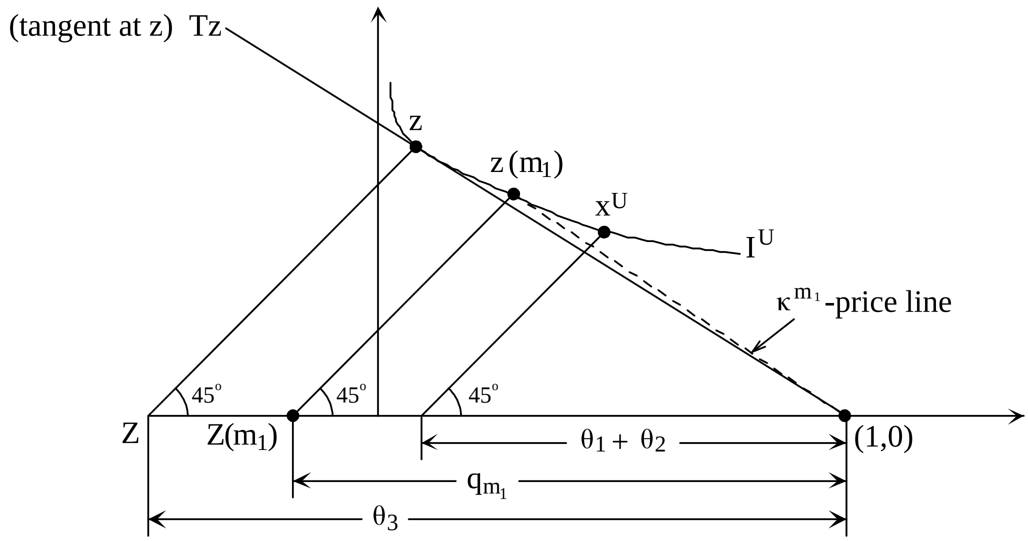

Figure 19

Let $\theta_1 + \theta_2 < q_{m_1} \leq \theta_3$. Proceed northeast at 45° from the point $Z(m_1) = (1 - q_{m_1}, 0)$ until the point $z(m_1)$ is hit on the I^U indifference curve. Define

$$\kappa^{m_1} = \frac{z_B(m_1)}{q_{m_1}}.$$

As before, $\kappa^{m_1}/(1-\kappa^{m_1})$ describes the |slope| of the dotted price line from (1,0) to $z(m_1)$. A household contributing q_{m_1} units to pool m_1, and anticipating delivery rate κ^{m_1} would end up at $z(m_1)$.

Consider the unique $\bar{\eta}$ such that $\bar{\eta}/(1-\bar{\eta})$ is the absolute value of the slope of the I^U indifference curve at $z(m_1)$. For all pools $m_1 m_2 ... m_\ell$ junior to m_1, define

$$\kappa^{m_1 m_2 ... m_\ell} = \bar{\eta}.$$

Finally, for $q_{m_1} > \theta_3$, put

$$\kappa^{m_1} = \frac{z_B}{\theta_3},$$

and also for every pool $m_1 m_2 ... m_\ell$ junior to m_1, put

$$\kappa^{m_1 m_2 ... m_\ell} = \frac{z_B}{\theta_3} = \kappa^{m_1}.$$

We must now show that with these K, we have a hierarchically-refined equilibrium. First note that our assignment of κ constitutes an equilibrium, for it is evident that no agent can improve by trading through any of these pools.

Let us now define a sequence of perturbations $E_e(n) \equiv (K(n), \varphi(n), x(n), \varepsilon(n)) \to E$. First we specify e. Recall that the smallest positive quantity constraint is $q_1 > 0$. Let x be the point on the β-line which intersects the 45° line proceeding northeast from $(1 - q_1, 0)$. Recall that we put $\kappa^1 = x_B/q_1$. Fix e with $e > \kappa^1$.

Next we specify $K(n)$. It will be convenient to consider the same three cases.

Case 1 $\quad q_{m_1} < \theta_1 + \theta_2$.

Let $\kappa^{m_1}(n) = \kappa^{m_1}$, and $\kappa^{m_1 m_2 \ldots m_\ell}(n) = \kappa^{m_1 m_2 \ldots m_\ell} = 1/3$ for all n, and all pools $m_1 m_2 \ldots m_\ell$ junior to m_1.

Case 2 $\quad \theta_3 > q_{m_1} \geq \theta_1 + \theta_2$.

Let $z(m_1, n)$ be the point on the line joining $Z(m_1)$ to $z(m_1)$, such that $||z(m_1) - z(m_1, n)|| = 1/n$. (See Figure 20.) Put

$$\kappa^{m_1}(n) = \frac{z_B(m_1, n)}{q_{m_1}}.$$

Next let $\alpha(m_1, n)$ be the absolute value of the slope of the straight line through $z(m_1, n)$ that is tangent to the I^U curve ($\tilde{z}(m_1, n) \equiv$ point of tangency in Figure 9). Let $\eta(m_1, n)$ be the unique solution of the equation $\eta/(1 - \eta) = \alpha(m_1, n)$, and put

$$\kappa^{m_1 m_2 \ldots m_\ell}(n) = \eta(m_1, n)$$

for all n, and all pools $m_1 m_2 \ldots m_\ell$ junior to m_1. Notice that $\kappa^{m_1}(n) \to \kappa^{m_1}$ and $\kappa^{m_1 m_2 \ldots m_\ell}(n) \to \kappa^{m_1 m_2 \ldots m_\ell}$ as $n \to \infty$.

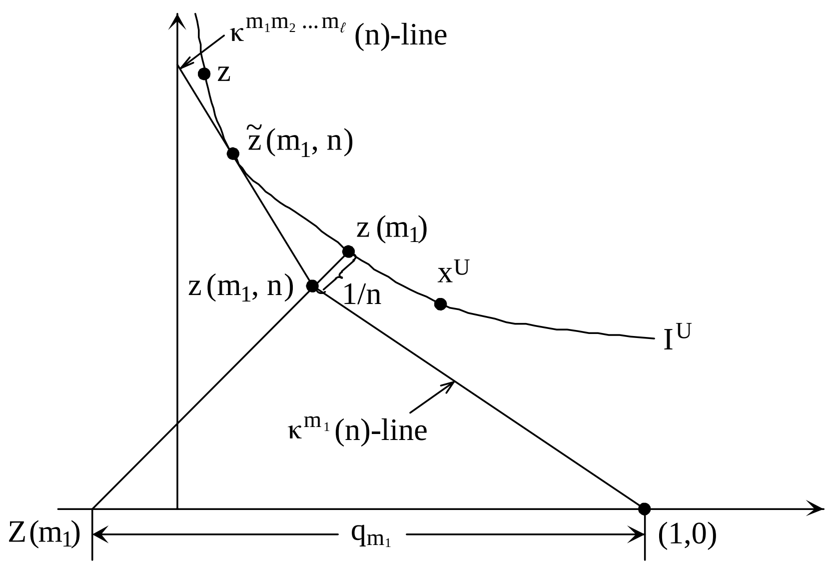

Figure 20

Case 3 $\quad q_{m_1} \geq \theta_3$.

Recall that $Z \equiv (1 - \theta_3, 0)$ and $z =$ the intersection of I^U with the 45° line moving northeast from Z. Define $\kappa^{m_1}(n)$, $\kappa^{m_1 m_2 \ldots m_\ell}(n)$ exactly as in Case 2 with Z, z substituted for $Z(m_1)$, $z(m_1)$ respectively. Once again note that $\kappa^{m_1}(n) \to \kappa^{m_1}$ and $\kappa^{m_1 m_2 \ldots m_\ell}(n) \to \kappa^{m_1 m_2 \ldots m_\ell}$.

This finishes the description of $K(n)$ in $E(n)$. Next we specify $(\varphi(n), x(n), \varepsilon(n))$. To this end, let $\mathcal{J}^* = \{1, \ldots, M\} \backslash \{m^*\}$ be the set of all inactive pools of level 1; and for each $m_1 \in \mathcal{J}^*$, let $\mathcal{J}(m_1)$ denote the set of all pools that are junior to m_1. We shall take disjoint sets $W_n(m_1)$ of unreliable households, for $m_1 \in \mathcal{J}^*$, such that each $W_n(m_1)$ has measure $1/n$. (Needless to say, $W_n(m_1)$ is symmetrically distributed over the three unreliable subtypes 4, 5, 6.) All $t \in W_n(m_1)$ will deviate in $\varphi(n)$ from their equilibrium actions in E, to actions on the pools $\{m_1\} \cup \mathcal{J}(m_1)$ in the manner about to be described. All other households will stick to their equilibrium actions.

Case 1 $q_{m_1} < \theta_1 + \theta_2$.

Each t in $W_n(m_1)$ contributes q_{m_1} to m_1 at the presumed anticipation $\kappa^{m_1}(n) = \kappa^{m_1}$ to arrive at the point $x(m_1)$. We set $\varepsilon_{m_1}(n)$ to be the unique ε which solves

$$\kappa^{m_1} = \frac{\varepsilon e + \lambda(W_n(m_1))\frac{1}{3}q_{m_1}}{\varepsilon + \lambda(W_n(m_1))q_{m_1}} = \frac{\varepsilon e + \left(\frac{1}{n}\right)\frac{1}{3}q_{m_1}}{\varepsilon + \left(\frac{1}{n}\right)q_{m_1}}.$$

Letting $\mu = \frac{1}{n}q_{m_1}$, this gives

$$\varepsilon = \frac{\left(\kappa^{m_1}(n) - \frac{1}{3}\right)\mu}{e - \kappa^{m_1}(n)}.$$

Next we must show how the households $t \in W_n(m_1)$ continue their actions on pools in $\mathcal{J}(m_1)$. All these $t \in W_n(m_1)$ will wind up with consumption x^U. The rest of their trades will therefore net them $x^U - x(m_1)$. Since all their trades in these $j \in \mathcal{J}(m_1)$ will be conducted at the same $\kappa^j = 1/3$, and since $(1/3)/(1-1/3) = 1/2$, it follows that each t will contribute $\theta(m_1) \equiv 2(x_G^U - x_G(m_1))$ across all their pools. But we shall have them split their contributions differently.

Take L so big that $\theta(m_1)/L < q_1$. Splitting their contributions evenly over at least L pools enables them to achieve θ, and their contribution to any one pool is less than q_1.

There are a countable number of pools in $\mathcal{J}_+^{L+1}(m_1) = \{j \in \mathcal{J}(m_1) : j$ is of level at least $L+1\}$. Hence we can divide $W_n(m_1)$ into countable disjoint sets $W_n(m_1m_2..., m_\ell)$ of positive measure corresponding one to one with nodes $m_1m_2...m_\ell$, $\ell \geq L+1$, in $\mathcal{J}_+^{L+1}(m_1)$. Let each household $t \in W_n(m_1m_2..., m_\ell)$ contribute $\theta/(\ell - 1) < q_1$ to each pool $m_1m_2, m_1m_2m_3, ..., m_1m_2...m_\ell$.

Thus every pool in $\mathcal{J}(m_1)$ is activated in $E_e(n)$. We use the freedom, allowed by our definition of hierarchical refinement, to put $\varepsilon_j(n) = 0$ for all $j \in \mathcal{J}(m_1)$.

This brings each $t \in W_n(m_1)$ to the consumption x^U.

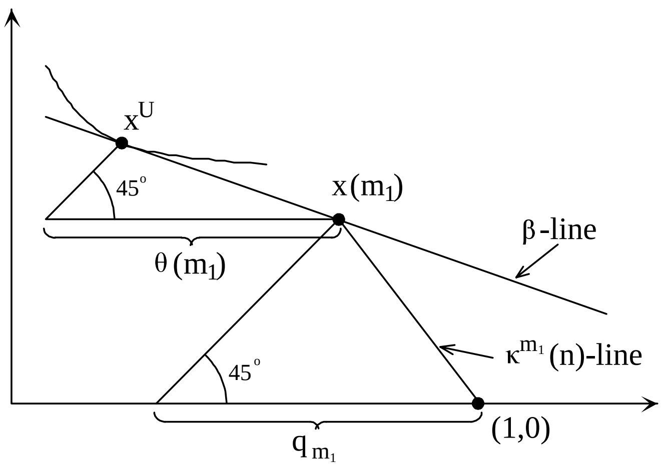

Figure 21

Case 2 $\theta_3 > q_{m_1} \geq \theta_1 + \theta_2$.

Each t in $W_n(m_1)$ contributes q_{m_1} to m_1 at the presumed anticipation $\kappa^{m_1}(n)$ to arrive at the point $z(m_1, n)$. Since all $t \in W_n(m_1)$ must ultimately end up with final consumption $\tilde{z}(m_1, n)$, and since all pools $j \in \mathcal{J}(m_1)$ junior to m_1 have the same $\kappa^j = \eta(m_1, n)$, each t must contribute a total of $\theta(m_1, n)$ across all these pools $j \in \mathcal{J}(m_1)$. (See Figure 22.) We shall again need to split these contributions differently for different t.

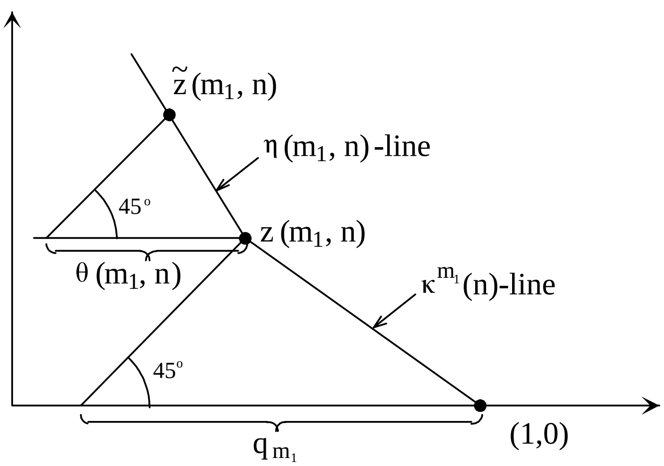

Figure 22

Note first that we may assume that $\theta(m_1, n) < q_1$, for all m_1 with $\theta_1 + \theta_2 < q_{m_1} \leq \theta_3$, since $\theta(m_1, n) \to 0$ as $n \to \infty$. For each node $j = m_1 m_2 ... m_\ell \in \mathcal{J}(m_1)$ junior to m_1, define a disjoint subset $W_n(m_1 m_2 ..., m_\ell) \subset W_n(m_1)$ of measure $(1/M)^{\ell-1}(1/n)^\ell B(n)$, where $B(n)$ is chosen so that $B(n)\sum_{\ell=2}^{\infty}(1/n)^\ell =$ measure $(W_n(m_1)) = 1/n$. (We restrict to $n \geq 2$.) Each $t \in W_n(m_1 m_2 ..., m_\ell)$ contributes $\theta(m_1, n)/(\ell - 1)$ on each pool $m_1 m_2, m_1 m_2 m_3, ..., m_1 m_2 ..., m_\ell$. on the path from $m_1 m_2$ to $m_1 m_2 ..., m_\ell$. Notice that for $n \geq 2$, $\sum_{\ell=2}^{\infty}(1/n)^\ell / \sum_{\ell=1}^{\infty}(1/n)^\ell = 1/n$.

Finally, we need to check that we can find $\varepsilon_j(n)$ satisfying our hierarchical refinement. So define $\varepsilon_{m_1}(n)$ as the unique ε solving

$$\kappa^{m_1}(n) = \frac{\varepsilon e + \lambda(W_n(m_1))\frac{1}{3}q_{m_1}}{\varepsilon + \lambda(W_n(m_1))q_{m_1}}.$$

This gives

$$\varepsilon_{m_1} = \frac{\left(\kappa^{m_1}(n) - \frac{1}{3}\right)}{e - \kappa^{m_1}(n)}\lambda(W_n(m_1))q_{m_1}.$$

For each $j = m_1 m_2 ... m_\ell \in \mathcal{J}(m_1)$, define $\varepsilon_j(n)$ as the unique ε to solve

$$\eta(m_1, n) \equiv \kappa^{m_1 m_2 \ldots m_\ell} = \frac{\varepsilon e + \left[\sum_{\tilde{\ell} \geq \ell} \sum_{m_1 \ldots m_\ell \ldots m_{\tilde{\ell}}} \lambda(W_n(m_1 ... m_\ell ... m_{\tilde{\ell}}))\frac{1}{3}\frac{\theta}{\tilde{\ell} - 1}\right]}{\varepsilon + \sum_{\tilde{\ell} \geq \ell} \sum_{m_1 \ldots m_\ell \ldots m_{\tilde{\ell}}} \lambda(W_n(m_1 ... m_\ell ... m_{\tilde{\ell}}))\frac{\theta}{\tilde{\ell} - 1}}$$

$$\equiv \frac{\varepsilon e + \mu_j(n)\frac{1}{3}}{\varepsilon + \mu_j(n)},$$

giving

$$\varepsilon_j(n) = \frac{(\eta(m_1, n) - \frac{1}{3})}{(e - \eta(m_1, n))}\mu_j(n).$$

Observe that for fixed n, if j is at level $\ell + 1$ and is an immediate successor of i, then $\varepsilon_j(n)/\varepsilon_i(n) = \mu_j(n)/\mu_i(n)$. Since households associated with pools at higher levels contribute less to each pool, this ratio is no more than

$$\frac{\sum_{\tilde{\ell} \geq \ell+1} \sum_{m_1 \ldots m_{\ell+1} \ldots m_{\tilde{\ell}}} \lambda(W_n(m_1 ... m_{\ell+1} ... m_{\tilde{\ell}})}{\sum_{\tilde{\ell} \geq \ell} \sum_{m_1 \ldots m_\ell \ldots m_{\tilde{\ell}}} \lambda(W_n(m_1 ... m_\ell ... m_{\tilde{\ell}})}$$

$$= \frac{\lambda(W_n(m_1)) \sum_{\tilde{\ell} \geq \ell+1}^{\infty} M^{\tilde{\ell}-1}\left(\frac{1}{M}\right)^{\tilde{\ell}-1}\frac{1}{n^{\tilde{\ell}}}B(n)}{\lambda(W_n(m_1)) \sum_{\tilde{\ell} \geq \ell}^{\infty} M^{\tilde{\ell}-1}\left(\frac{1}{M}\right)^{\tilde{\ell}-1}\frac{1}{n^{\tilde{\ell}}}B(n)} = \frac{1}{n}.$$

Thus we have constructed $\varepsilon_j(n)$ satisfying hierarchical refinement.

Case 3 $q_{m_1} \geq \theta_3$.

This is entirely analogous to Case 2. (Reread Figure 11 substituting Z for $Z(m_1)$ and z for $z(m_1)$.) Put $\varphi^t(m_1) = \theta_3$ at the anticipated rate $\kappa^{m_1}(n)$, to reach $z(m_1, n)$ and again let t spread contributions over all $j \in \mathcal{J}(m_1)$ as in Case 2 to total $\theta(m_1, n)$ and to advance from $z(m_1, n)$ to $\tilde{z}(m_1, n)$ on the I^U curve.

It is easy to check that the sequence $(K(n), \varphi(n), x(n), \varepsilon(n))_{n=1}^{\infty}$ validates $E = (K, \varphi, x)$ as a hierarchically-refined equilibrium.