General licensing schemes for a cost-reducing innovation

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Abstract

Two general forms of standard licensing policies are considered for a non-drastic cost-reducing innovation: (a) combination of an upfront fee and uniform linear royalty, and (b) combination of auction and uniform linear royalty. It is shown that in an oligopoly, the total reduction in the cost due to the innovation for the pre-innovation competitive output forms the lower bound of the payoffs of both outsider and incumbent innovators. Further, the private value of the patent is increasing in the magnitude of the innovation, while the Cournot price and the payoff of any other firm fall below their respective pre-innovation levels. Sufficiently significant innovations from an outsider innovator are licensed exclusively to a single firm. Otherwise, all other firms, except perhaps one, become licensees. The dissemination of the innovation is generally higher with an incumbent innovator compared to an outsider. For both outsider and incumbent innovators, the monopoly does not provide the highest incentive to innovate; for sufficiently insignificant innovations, it is the duopoly that does so, and, the industry size that provides the highest incentive increases with the magnitude of the innovation. Finally, it is argued that significant innovations are more likely to occur when the innovator is an incumbent firm.

Keywords: Non-drastic innovation, outsider innovator, incumbent innovator, FR policy, AR policy.

JEL Classification: D21, D43, D45.

1 Introduction

Patent licensing by means of a combination of upfront fee and royalty is one of the most commonly observed licensing policies in practice [see, e.g., Taylor and Silberstone (1973), Rostoker (1984)]. This paper considers optimal combination of upfront fee and royalty for licensing of a cost-reducing innovation and discusses its impact on the price and the structure of the market, payoffs of the agents, and incentives and dissemination of innovation in case of both outsider and incumbent innovators. The formal analysis of patent licensing was initiated by Arrow (1962). Considering licensing of a cost-reducing innovation by means of uniform linear royalty only, he concluded that the innovator's licensing rent in a perfectly competitive industry exceeds that when the same innovation is sold to a monopolist. However, Arrow (1962) did not consider the aspect of strategic interaction among the firms, which plays a crucial role in an oligopoly. The effect of innovations in such an industry depends, among other factors, on whether the innovator is an outsider or an incumbent firm.

A cost-reducing innovation is said to be *drastic* [Arrow (1962)] if the monopoly price under the new technology does not exceed the competitive price under the old technology; otherwise, it is *non-drastic*. Clearly, if an incumbent innovator is a monopolist, or if she is endowed with a drastic cost-reducing innovation, she extracts the entire monopoly profit with the new technology. The same fact is true for an outsider innovator when the industry size is at least two. When an outsider innovator faces a monopolist, then irrespective of whether the innovation is drastic or not, the innovator obtains the difference between the respective monopoly profits with the new and the old technology. Thus, the issue of patent licensing is non-trivial only in case the innovation is non-drastic. For this reason, we shall only consider non-drastic innovations.

The interaction of an outsider innovator and the firms was first studied in a formal game-theoretic setting independently by Kamien and Tauman (1984, 1986) and Katz and Shapiro (1985, 1986). The literature mainly considers three standard policies of licensing, namely, (1) a flat pre-determined upfront fee, (2) a uniform per-unit linear royalty payment, and, (3) auctioning off a limited number of licenses through a first-price sealed-bid auction, where the highest bidders pay their bids and get licenses. In what follows, we provide a brief overview of the literature. We refer to Kamien (1992) for an excellent survey on patent licensing. See also Reinganum (1989) for a comprehensive survey on various aspects of innovation, including licensing.

Katz and Shapiro (1985) have considered a three-stage game of an asymmetric duopoly, where in the first stage, an outsider innovator sells the patent to one of the firms by auction. In the second stage, the licensee decides either to exclude his rival, or to share the license through a licensing policy based exclusively on upfront fee and in the final stage, the firms are engaged in strategic competition. Investigating the pattern of licensing and exclusion, Katz and Shapiro (1985) have found that while minor innovations will be licensed when firms are equally efficient, exclusion will occur in case of major innovations. Considering the auction policy as the licensing scheme of the patent, Katz and Shapiro (1986) have shown that while the seller's incentive to develop the innovation may be very high, the incentive to disseminate it may be too low. Considering licensing by means of either an exclusive upfront fee or an exclusive linear royalty, Kamien and Tauman (1986) have shown that for linear demand, both for the innovator, and from social point of view, licensing by upfront fee is better than royalty. Further, like Arrow (1962), they have shown that with royalty licensing, the perfectly competitive industry provides the highest incentive for innovation. For upfront fee licensing, however, there is no sharp conclusion regarding the relation between the industry size and the incentive to innovate, and, it depends on the magnitude of the innovation. Kamien, Oren and Tauman (1992) have extended these results for general demand and have shown that among the three standard policies, licensing by means of royalty is inferior to the other two while upfront fee is inferior to auction, both for consumers and the innovator. Erutku and Richelle (2000) have shown that in an oligopoly with at least two firms, an outsider innovator can always design an upfront fee plus royalty policy which enables her to extract the entire monopoly profit with the new technology.[1] This result is obtained with a non-linear royalty which depends on the total output of the industry and individual outputs of potential licensees. However, to the best of our knowledge, such policies are not observed in practice.

Although the theoretical literature shows the superiority of both auction and upfront fee to royalty, as licensing schemes, royalties and combination of upfront fee and royalty policies are more prevalent than other standard forms of licensing. In the oft-quoted survey of Rostoker (1984) of corporate licensing, upfront fee plus royalty policy was observed in 46% of cases, whereas licensing by means of exclusive royalty was observed in 39% of the firms surveyed. An attempt to bridge the discrepancy between empirical observations and theoretical predictions was made by Wang (1998), who considered a model of Cournot duopoly where the innovator is not an outsider, but one of the firms.[2] In this framework, licensing by means of royalty yields better payoff to the innovator than upfront fee licensing. Extending the work of Wang (1998), Kamien and Tauman (2002) have shown that in a Cournot oligopoly, for sufficiently significant (but non-drastic) innovations, licensing by means of linear royalty is superior to both auction and upfront fee policies for an incumbent innovator.[3]

[1]For other selling mechanisms that enable the innovator to obtain the maximum industry profit with the innovation, see Kamien, Oren and Tauman (1992) and Sen (2002a).

[2]Katz and Shapiro (1985) have considered a duopoly where one of the firms is a patent holder. However, they only dealt with upfront fee licensing.

[3]The role of asymmetric information has also been considered to explain the prevalence of royalty licensing. See, e.g., Beggs (1992), Sen (2002b).

In this paper, we merge this new line of enquiry with the standard literature and consider both outsider and incumbent innovators. We propose licensing schemes that are more general than the standard policies examined in the literature.[4] Specifically, we consider two policies: (i) the "upfront fee plus uniform linear royalty" (FR) policy, where each licensee pays a pre-determined upfront fee, and a uniform royalty per-unit of production;[5] and, (ii) the "auction plus uniform linear royalty" (AR) policy, where the innovator announces a uniform linear royalty and the number of licenses to be sold, say m; then, each firm bids for the license and m highest bidders win the license, and, pay their respective bids in addition to the announced uniform royalty per-unit of production. Clearly, these policies encompass the three standard licensing policies, viz., upfront fee, royalty, and auction. It should be mentioned that for both outsider and incumbent innovators, if the number of licensees is less than the industry size, then the AR policy is superior to the FR policy, and the FR policy is superior otherwise.[6]

It is shown that the total reduction of cost due to the innovation for the pre-innovation competitive quantity forms the lower bound of the payoff of both outsider and incumbent innovators. Further, in both cases, the payoff of the innovator is increasing in the magnitude of the innovation, while the Cournot price and the payoff of any other firm fall below their respective pre-innovation levels. Thus, consumers are better off and firms are worse off as a result of the innovation. Other results of this paper depend on whether the innovator is an incumbent firm or not.

When an outsider innovator faces a monopolist, she extracts the difference

[4] With the possible exception of Erutku and Richelle (2000).

[5] This licensing policy was first examined by Kamien and Tauman (1984), but only asymptotic results (with respect to the industry size) were obtained.

[6] This holds unless the auction includes a floor price, in which case, the two policies coincide. However, note that the FR policy is more attractive as it is simpler to implement.

between the respective monopoly profits with the new and the old technology through an upfront fee. When the industry size is at least two, then sufficiently significant innovations are sold to an exclusive license, which is consistent with Firestone (1971) and Caves, Crookell and Killing (1983), where it was pointed out that patents from independent innovators are often licensed exclusively. Relatively less significant innovations are sold to all firms, except perhaps one. For an incumbent innovator, sufficiently significant innovations are sold to all firms, whereas less significant innovations are sold to all firms, except one. Thus, the dissemination of the innovation is generally higher with an incumbent innovator. The results can be summarized in the following table.[7]

−[Insert Table 1]−

Regarding the incentives to innovate, the conclusions are qualitatively the same for both outsider and incumbent innovators. We show that for both cases, the industry size that provides the highest incentive to innovate is increasing in the magnitude of the innovation. In particular, for both cases, the monopoly never provides the highest incentive. When the innovation is sufficiently insignificant, the duopoly does so, and, as the innovation becomes more significant, a more competitive industry provide the highest incentive.

Finally, keeping the number of firms other than the innovator fixed, we argue that significant innovations are more likely to occur from an incumbent innovator. The comparison between outsider and incumbent innovators is conceptually problematic, since the two innovators cannot be embedded in the same model. One way of dealing with this problem is by providing an outsider innovator with the option of entering the industry. It is shown that in the presence of a negligible but positive cost of entry, when the innovation is sufficiently significant, an outsider innovator is better off entering the industry

[7]The table and figures are available from the first author upon request.

and selling the license to every firm. Otherwise, the innovator remains outside. In other words, the private value of an innovation is higher for an incumbent innovator when the innovation is sufficiently significant.

The rest of the paper is organized as follows. In Section 2, we present the model and the licensing schemes. In Section 3, we discuss about the optimal licensing schemes. The relation between the market structure and incentives to innovate is discussed in Section 4. In Section 5, we compare the incentives for outsider and incumbent innovators by providing an outsider innovator with the strategy of entry. All proofs have been relegated to the appendix.

2 The Model

Let us describe the model with an outsider innovator in detail. The model with an incumbent innovator will be similar, except for some obvious modifications. We consider a Cournot oligopoly with n firms producing the same product, where $N_1 = \{1, \dots, n\}$ is the the set of firms. For $i \in N_1$, let q_i be the quantity produced by firm i and let $Q = \sum_{i \in N_1} q_i$. The inverse demand function of the industry is linear and is given by $Q = a - p$, for $p \leq a$ and $Q = 0$, otherwise. With the old technology, all n firms produce at the identical marginal cost c, where $0 < c < a$. An outsider innovator has granted a patent on a new technology which reduces the marginal cost from c to $c - \varepsilon$, where $0 < \varepsilon < c$. The innovator decides to license the new technology to some or all firms of the industry.[8] Consider the following three-stage licensing game. In stage 1, the innovator announces a licensing policy. In stage 2, the firms in N_1 simultaneously decide whether to accept the policy or not. The set of licensees become commonly known at the end of stage 2. In stage 3, all n firms compete in quantities, where the licensees have the new technology while other firms

[8]For this section, entering the industry is not a feasible strategy of the innovator. The entry strategy is considered in Section 5.

operate under the old technology. The licensees pay the innovator according to the licensing policy.

For an incumbent innovator, the model is the same as in the last paragraph except for that we now have $n+1$ firms, where $N_2 = \{0\} \cup N_1$ is the set of firms and firm 0 is the innovator. The licensing game is also the same except that in the final stage, apart from the licensees, the innovator also produces with the new technology and competes with all other firms.

2.1 The FR Policy

A typical FR policy is given by $\langle m, r, f \rangle$, where m, $1 \leq m \leq n$, is the number of firms to whom the policy is offered, $r \in \Re$ is the per-unit uniform royalty, and $f \in \Re$ is the upfront fee that each licensee has to pay. The three-stage licensing game associated with the FR policy (denoted by G^O_{FR} for an outsider innovator and G^I_{FR} for an incumbent one), can be described as follows. In the first stage, the innovator announces a triplet $\langle m, r, f \rangle$, chooses m firms from N_1 and offers each one of them a license. The way these m firms are chosen (randomly or not) does not affect our results. In the next stage, these m firms decide simultaneously but independently whether to accept the offer or not. The firms who accept the offer become licensees. The set of licensee firms becomes commonly known at the end of the second stage. In the third stage, all firms (including the innovator in case she is an incumbent firm) compete in quantities. If a licensee firm produces q, it pays the innovator $f + rq$.

2.2 The AR Policy

A typical AR policy is given by $\langle m, r \rangle$, where m, $1 \leq m \leq n$, is the number of firms to whom the policy is offered and $r \in \Re$ is the per-unit uniform royalty that each licensee has to pay. The three-stage licensing game associated with the AR policy (denoted by G^O_{AR} and G^I_{AR} respectively), can be described as follows. In first stage, the innovator announces a pair $\langle m, r\rangle$. In the second stage, firms in N_1 simultaneously decide whether to bid for the license, and, how much to bid. If the number of bidders is less than or equal to m, all bidders win the license with their respective bids. If the number is strictly more than m, then m highest bidders win the license. Ties are resolved at random. The set of licensees become commonly known at the end of the second stage. In the third stage, all firms compete in quantities. If a firm wins the license with bid b, and produces q, it pays the innovator $b + rq$.

2.3 The Games G^O and G^I

The licensing game with an outsider innovator, denoted by G^O, is described as follows. In the beginning (say stage 0), the innovator chooses either of the following licensing policies: a specific FR policy, or, a specific AR policy. If the innovator chooses an FR policy, the game G^O_{FR} is played; otherwise, it is the game G^O_{AR} that is played. We define the game G^I for an incumbent innovator similarly, where G^I_{FR} and G^I_{AR} form the corresponding subgames.

Clearly, in equilibrium, if the innovator announces the AR policy $\langle m, r\rangle$, then the royalty rate r will support m licensees, that is, more than m firms will bid for the license.[9] Similarly, if the FR policy $\langle m, r, f\rangle$ is offered, then, in equilibrium, m is supported by r and f. Consequently, for the analysis of the equilibrium of G^O and G^I, we can assume without any loss of generality that when the innovator announces the FR policy $\langle m, r, f\rangle$, or, the AR policy $\langle m, r\rangle$, then there will be m licensees.

Consider the game G^J for $J \in \{O, I\}$. In every subgame-perfect equilibrium

[9] If the number of bidders is at most m, then every bidder is better off reducing his bid, since the innovator is committed to sell m licenses.

of G^J, in the last stage, all firms produce the Cournot quantities. Hence, we can refer to G^J_{FR} and G^J_{AR} as games involving only two stages, where at the end of the second stage, the firms choose their respective Cournot outputs. For $J \in \{O, I\}$, when there are m licensees and the rate of royalty is r, any licensee firm in N_1 produces with marginal cost $c - \varepsilon + r$ and any non-licensee firm produces with marginal cost c. Let us denote by $\Phi^J_L(m, r)$ and $\Phi^J_N(m, r)$ the Cournot profits of a licensee and a non-licensee respectively. Similarly, we use the letters q and Π, with suitable subscripts and superscripts, to denote the Cournot quantity and the total payoff respectively. Now consider the triplet $\langle m, r, f\rangle$ of the FR policy. Then, we have

$$\left\{\begin{array}{l} \Pi^J_L(\langle m, r, f\rangle) = \Phi^J_L(m, r) - f \text{ and} \\ \Pi^J_N(\langle m, r, f\rangle) = \Phi^J_N(m, r). \end{array}\right\} \tag{1}$$

Next, consider the the AR policy $\langle m, r\rangle$ where the license has been won by m firms with the common winning bid b. Then, the payoffs are given by

$$\left\{\begin{array}{l} \Pi^J_L(\langle m, r\rangle, b) = \Phi^J_L(m, r) - b \text{ and} \\ \Pi^J_N(\langle m, r\rangle, b) = \Phi^J_N(m, r). \end{array}\right\} \tag{2}$$

Let us consider the payoff of the innovator. Note that for the game G^I, the incumbent innovator produces with marginal cost $c-\varepsilon$. Using similar notations as in the last paragraph, with subscript I standing for the innovator, we have

$$\left\{\begin{array}{l} \Pi^O_I(\langle m, r, f\rangle) = mrq^O_L(m, r) + mf \text{ and} \\ \Pi^I_I(\langle m, r, f\rangle) = \Phi^I_I(m, r) + mrq^I_L(m, r) + mf. \end{array}\right\} \tag{3}$$

$$\left\{\begin{array}{l} \Pi^O_I(\langle m, r\rangle, b) = mrq^O_L(m, r) + mb \text{ and} \\ \Pi^I_I(\langle m, r\rangle, b) = \Phi^I_I(m, r) + mrq^I_L(m, r) + mb. \end{array}\right\} \tag{4}$$

For $J \in \{O, I\}$, consider a subgame-perfect equilibrium outcome of the game G^J_{FR}. Then, by (1), for every m $(1 \leq m \leq n)$, and $r \in \Re$, it follows that the upfront fee that the innovator charges any licensee is given by

$$f^J(m, r) = \Phi^J_L(m, r) - \Phi^J_N(m - 1, r). \tag{5}$$

Consequently, an outsider innovator will choose m and r that maximize

$$\Pi^O_{FR}(m,r) \equiv \Pi^O_I(\langle m,r,f^O(m,r)\rangle) =$$

$$mrq^O_L(m,r) + m[\Phi^O_L(m,r) - \Phi^O_N(m-1,r)]. \tag{6}$$

An incumbent innovator will choose m and r to maximize

$$\Pi^I_{FR}(m,r) \equiv \Pi^I_I(\langle m,r,f^I(m,r)\rangle) =$$

$$\Phi^I_I(m,r) + mrq^I_L(m,r) + m[\Phi^I_L(m,r) - \Phi^I_N(m-1,r)]. \tag{7}$$

As for the AR policy, note that in equilibrium, the innovator never chooses $\langle m,r\rangle$ such that $m=n$. The innovator may choose $\langle n,r\rangle$ only with some minimum bid, say $\underline{b}$. In that case, no firm will bid above $\underline{b}$. This modified AR policy will be equivalent to the FR policy $\langle n,r,\underline{b}\rangle$ but the latter is simpler to implement. Hence, we only consider AR policies $\langle m,r\rangle$ with $1 \le m \le n-1$. Consider a subgame-perfect equilibrium outcome of the game G^J_{AR}. Then, for every m ($1 \le m \le n-1$), and $r \in \Re$, when the AR policy $\langle m,r\rangle$ is announced, at least $m+1$ firms will bid, and the highest $m+1$ bids will be $b^J(m,r)$ for $J \in \{O,I\}$, where

$$b^J(m,r) = \Phi^J_L(m,r) - \Phi^J_N(m,r). \tag{8}$$

In contrast to (5), here we subtract $\Phi^J_N(m,r)$ instead of $\Phi^J_N(m-1,r)$, because a potentially deviant licensee knows that irrespective of whether he is a licensee or not, there will always be m licensees. Consequently, for the AR policy, an outsider innovator will choose m and r that maximize

$$\Pi^O_{AR}(m,r) \equiv \Pi^O_I(\langle m,r\rangle, b^O(m,r)) =$$

$$mrq^O_L(m,r) + m[\Phi^O_L(m,r) - \Phi^O_N(m,r)], \tag{9}$$

while an incumbent innovator will choose m and r to maximize

$$\Pi^I_{AR}(m,r) \equiv \Pi^I_I(\langle m,r\rangle, b^I(m,r)) =$$

$$\Phi_I^I(m,r) + mrq_L^I(m,r) + m[\Phi_L^I(m,r) - \Phi_N^I(m,r)]. \tag{10}$$

In view of (6)-(7) and (9)-(10), the innovator has to optimally choose the licensing policy, the number of licenses to be sold and the rate of royalty. The following lemma states that when the innovator does not sell the license to all firms (i.e., $m \leq n-1$), then the AR policy always yields better payoff than the FR policy. The underlying reason for this result is simple. From (5) and (8), one can observe that the difference between the maximum willingness to pay for each licensee for the FR and AR policies is the difference between the profits of a non-licensee when there are $m-1$ and m licensees respectively. Note that when the rate of royalty does not exceed the magnitude of the innovation, then a licensee firm is at least as efficient as a non-licensee firm. In that case, the profit of a non-licensee decreases in the number of licensees. Thus, the only non-trivial part of the proof is to show that the optimal choice of the rate of royalty does not exceed ε.

Lemma 1. *For $1 \leq m \leq n-1$ and $J \in \{O, I\}$,*

$$\max_{r\in\Re} \Pi_{FR}^J(m,r) \leq \max_{r\in\Re} \Pi_{AR}^J(m,r)$$

where $\Pi_{FR}^J(m,r)$ and $\Pi_{FR}^J(m,r)$ are defined in (6)-(7) *and* (9)-(10).

3 Optimal Licensing Schemes

To begin with, we state the results that hold for both outsider and incumbent innovators.

Proposition 1. *Consider the licensing of a non-drastic cost-reducing innovation. Then, both G^O and G^I have a unique subgame-perfect equilibrium outcome. In both of these outcomes, the following hold.*

[i] *In an oligopoly with at least two firms, the innovator obtains at least $(a-c)\varepsilon$.*

[ii] *The private value of the patent is increasing in ε.*

[iii] *The Cournot price and the payoff of any other firm fall below their respective pre-innovation levels.*

The term $(a-c)\varepsilon$ is the total reduction due to the innovation in the production cost of the pre-innovation competitive output. The private value of the patent is the rent that the innovator can obtain from the licensees. This is an increasing function in the magnitude of the innovation in case of both outsider and incumbent innovators (see Figures 1.1 and 1.2 below).

−[Insert Figures 1.1 and 1.2]−

The respective Cournot prices for outsider and incumbent innovators, $p^O(\varepsilon)$ and $p^I(\varepsilon)$, are given in figures 2.1 and 2.2. Note that both $p^O(\varepsilon)$ and $p^I(\varepsilon)$ are less than the respective pre-innovation Cournot prices. It can be noted that $p^O(\varepsilon)$ is discontinuous at $\varepsilon = (a-c)/y(n)$ and $\varepsilon = (a-c)/q(n)$, while $p^I(\varepsilon)$ is discontinuous at $\varepsilon = (a-c)/h(n)$. These discontinuities arise due to a change in the licensing policy at these points. Observe that for any given policy, both of these functions are decreasing in ε. The terms $y(n)$, $q(n)$ and $h(n)$ appears in Propositions 2 and 3, and their expressions appear in the appendix. At these points, the innovator changes her licensing policy.

−[Insert Figures 2.1 and 2.2]−

Since the Cournot price falls below the pre-innovation level, the consumers are better off for any $\varepsilon > 0$. Every firm, except for the innovator, is worse off relative to the pre-innovation case.[10] The post-innovation payoff of any firm other than the innovator can be seen from Figures 3.1 and 3.2 below. Note that there are similar discontinuities as in the case of price.

[10]This is consistent with Kamien and Tauman (1984) and Katz and Shapiro (1986), where less general policies (upfront fee or auction without royalty) were considered.

−[Insert Figures 3.1 and 3.2]−

3.1 An outsider innovator

In this subsection, we discuss the properties of the equilibrium of the game G^O that are specific to an outsider innovator.

Proposition 2. *Consider the game G^O involving an outsider innovator of a non-drastic cost-reducing innovation and suppose there is a negligible but positive cost of contracting for every licensee. Then, the subgame-perfect equilibrium of G^O has the following properties.*

[i] *In case of a monopoly, the innovator earns the difference between the respective monopoly profits with the new and the old technology through an upfront fee. This payoff is less than $(a-c)\varepsilon$.*

[ii] *For every $n \geq 2$, there is a positive number $q(n) \leq 2$ such that when $\varepsilon \geq (a-c)/q(n)$, the innovator sells the license to only one firm and sets a rate of royalty so that the monopoly price equals the pre-innovation competitive price c. As a consequence, all non-licensee firms drop out of the market and the innovator obtains $(a-c)\varepsilon$. When $\varepsilon < (a-c)/q(n)$, the innovator sells the license to at least $n-1$ firms, the Cournot price is above c, and all firms continue to operate. The innovator obtains more than $(a-c)\varepsilon$.*

In case of a monopoly, an outsider innovator obtains $(a-c+\varepsilon)^2/4-(a-c)^2/4$, which is smaller than $(a-c)\varepsilon$. The lack of competition and the resulting higher bargaining power of the monopolist is the reason why the innovator obtains less in a monopoly.

Let $n \geq 2$ and suppose that the innovation is relatively significant, namely, $\varepsilon \geq (a-c)/q(n)$. Auctioning off only one license is the unique optimal policy only in the presence of a (negligible) positive cost of contracting for every licensee. In the absence of such contracting cost, the innovator has multiple optimal policies. In fact, any AR policy $\langle m, r(m) \rangle$, where $1 \leq m \leq n-1$ and $r(m) = \varepsilon - (a-c)/m$ is an optimal policy, the payoff of the innovator is $(a-c)\varepsilon$ in each of these policies and the Cournot price equals the pre-innovation Cournot price in each of these cases.[11]

Next consider the other case when $n \geq 2$ and the innovation is less significant, that is, $\varepsilon < (a-c)/q(n)$. In that case, there will be at least $n-1$ licensees. The optimal licensing policy depends on the industry size and the magnitude of the innovation.

3.2 An incumbent innovator

Let us consider the game G^I involving an incumbent innovator.

Proposition 3. *Consider an incumbent innovator of a non-drastic cost-reducing innovation. The game G^I has a unique subgame-perfect equilibrium outcome. It has the following properties.*

[i] *In case of a duopoly, i.e., when $N_2 = \{0, 1\}$, the innovator pays the other firm to stay out of the market.*

[ii] *When there are at least two firms other than the innovator ($n \geq 2$), there is a number $h(n) > n$ such that when $\varepsilon \geq (a-c)/h(n)$, the innovator sells the license to all other firms using the FR policy. When $\varepsilon < (a-c)/h(n)$, then the innovator sells the license to all other firms, except one, using the AR policy. In both cases, all firms, including the innovator, produce positive quantity.*

In case of a duopoly, firm 1 (the firm other than the innovator) can always ensure a payoff of $(a-c-\varepsilon)^2/9$, which is the Cournot profit of a non-licensee firm who competes with the innovator in a duopoly. Thus, the upper-bound

[11]This excludes the negligible cost of contracting.

for the innovator's payoff is $(a-c+\varepsilon)^2/4-(a-c-\varepsilon)^2/9$. The innovator can achieve this upper-bound with the FR policy $\langle 1, r, f\rangle$, where $f=-(a-c-\varepsilon)^2/9$ and r is sufficiently large to ensure that firm 1 finds it optimal not to produce. Thus, in case of a duopoly, firm 0, the innovator, acquires firm 1 and becomes a monopolist.[12] However, if the innovator is not allowed to charge negative upfront fee, then the optimal licensing policy for the innovator is the royalty policy with rate of royalty ε, which coincides with Wang (1998). The innovator is worse off, while the other firm and consumers are better off compared to the unrestricted case. Measuring the welfare as the sum of consumers' surplus, the innovator's payoff and the other firms' payoff, it is easy to verify that a non-negative restriction on the upfront fee is welfare-improving. Hence, acquisition of the other firm should not be allowed.

For $n \geq 2$, when the innovation is sufficiently significant, namely, $\varepsilon \geq (a-c)/h(n)$, then the optimal licensing policy is the FR policy with rate of royalty $r_F(n)$, and all firms become licensees. It can be shown that $r_F(2)=\varepsilon$, but $0<r_F(n)<\varepsilon$ for $n \geq 3$ and $r_F(n)$ approaches ε as n increases indefinitely. Further $h(n)$ increases indefinitely with n. When $\varepsilon \leq (a-c)/h(n)$, then the optimal policy for the innovator is the AR policy where all firms, except one, become licensees.

4 The Incentives to Innovate

For both outsider and incumbent innovators, if the innovation is non-drastic, the industry that provides the innovator with the highest incentive to innovate is never the monopoly. For an outsider innovator, this follows directly from part [i] of Proposition 2. The incremental payoff of an incumbent innovator due to the innovation is the difference between the payoff from the optimal

[12] The payoff of the innovator is strictly increasing in r when $r \leq (a-c+\varepsilon)/2$ and it is constant thereafter.

licensing policy and the pre-innovation Cournot profit. When the innovator is a monopolist, this is given by

$$\Delta(1) = \frac{(a-c+\varepsilon)^2}{4} - \frac{(a-c)^2}{4}.$$

When there is an additional firm, then the incremental payoff is

$$\Delta(2) = \left[\frac{(a-c+\varepsilon)^2}{4} - \frac{(a-c-\varepsilon)^2}{9}\right] - \frac{(a-c)^2}{9}.$$

It is easy to verify that $\Delta(2) > \Delta(1)$, and thus the monopoly does not provide the innovator with the highest incentive to innovate. In general, the highest incentive to innovate is induced by a market size that is increasing in the magnitude of the innovation.

Proposition 4. *The monopoly does not provide the innovator with the highest incentive to innovate. It is the duopoly that does so for relatively insignificant innovations and the industry size that provides the highest incentive is increasing in the magnitude of the innovation.*

5 Significant Innovations and Incumbent Innovators

In this section, we make an attempt to find out who is in a better position to innovate: an outsider innovator or an incumbent one? There is no definite way to answer this question since we cannot consider the two types of innovators in the same model. However, we can shed some light on this issue by assuming that an outsider innovator could not enter the industry with the old technology (e.g., due to a patent on the old technology), but equipped with the new technology, she can enter the industry by incurring a negligible but positive entry cost. Consider the four-stage game G, where in the first stage, the innovator decides whether to enter the industry, or not. Her decision becomes commonly known at the end of the first stage. If the innovator decides to enter the industry, from stage 2 onwards, the game G^I is played. Otherwise, the game G^O is played. In other words, when the innovator decides to enter the industry, we have the case of a Cournot oligopoly in $n+1$ firms with an incumbent innovator. Otherwise, it is the case of a Cournot oligopoly in n firms with an outsider innovator. In either cases, the innovator is allowed to license the new technology to some or all of the other firms.

First consider the case when $n = 1$, that is, when the outsider innovator faces a monopolist. In that case, the payoff of the outsider innovator is given by

$$A \equiv \frac{(a-c+\varepsilon)^2}{4} - \frac{(a-c)^2}{4}.$$

If the innovator decides to enter the industry, her payoff becomes

$$B \equiv \frac{(a-c+\varepsilon)^2}{4} - \frac{(a-c-\varepsilon)^2}{9}.$$

It can be easily verified that $B > A$. Hence, in this case, the innovator is best off entering the industry.

Proposition 5. *Consider the subgame-perfect equilibrium of the game G. Then,*

[i] *Suppose* $n = 1$. *Then the innovator enters the industry whether or not the acquisition of the incumbent monopolist is allowed..*

[ii] *When* $n \geq 2$, *the innovator enters the industry when* $\varepsilon \geq (a-c)/h(n)$ *and sells the license to all firms. Otherwise, the innovator does not enter the industry.*

[iii] *If there is no entry fee, then entry is always the best action for the innovator.*

Proposition 5 essentially asserts that the innovator will enter the industry when the innovation is sufficiently significant.[13] Otherwise, she decides to remain as an outsider. This suggests that the extent of rent provided by significant innovations is higher for an incumbent innovator compared to an outsider. Hence, significant innovations are perhaps more likely to be originated from incumbent firms. As for less significant innovations, the result is not as sharp. If there is no cost of entry, then payoffs from both cases are equal. However, in the presence of a positive entry cost, such innovations are more likely to be generated from outsider innovators. The intuition behind this result is as follows. Consider a Cournot oligopoly in n firms who are symmetric in that they produce with an identical cost. Suppose there is another firm, A, who is outside the industry and has a superior technology as compared to the incumbent firms. Now consider the total industry profit in the $n+1$-firm oligopoly when firm A enters the industry. Comparing this profit with the total industry profit in the earlier n-firm oligopoly, one observes that the former is higher if and only if the technology of firm A is sufficiently superior to the others. When an incumbent innovator has a sufficiently significant innovation, then she licenses it to all other firms, but by setting a high rate of royalty the innovator still maintains a competitive edge over others. Hence, the total industry profit is increased, and moreover, the profit of every non-licensee (hence the alternative profit of every licensee) is decreased. Since every licensee is left with the profit of a sole non-licensee, and the innovator obtains the difference between the total industry profit and the total payoff of all licensees, the innovator is better off by entering the industry when the innovation is sufficiently significant.

[13]When the innovation is drastic, the innovator will enter the industry, since in that case she earns the monopoly profit with the new technology. Also observe that the conclusion of Proposition 5 will continue to hold qualitatively in the presence of higher entry costs, as long as entry is not prohibitively costly.

Appendix

Notations. Recall that the superscripts O and I refer to 'outsider' and 'incumbent' respectively and the subscripts L, N and I refer to 'licensee', 'non-licensee' and 'innovator' respectively. We denote by q and Φ, with suitable superscripts and subscripts, the Cournot output and the Cournot profit respectively, e.g., $q_L^O(m, r)$ denotes the Cournot output of a licensee when there are m licensees and the rate of royalty is r in case of an outsider innovator. For $J \in \{O, I\}$, we denote

$$f^J(m, r) = \Phi_L^J(m, r) - \Phi_L^J(m - 1, r) \text{ and}$$

$$b^J(m, r) = \Phi_L^J(m, r) - \Phi_L^J(m, r).$$

Throughout the appendix, we shall denote $x \equiv (a - c)/\varepsilon$.

Proof of Proposition 1. The proof of Proposition 1 in case of an outsider innovator will be given at the end of the proof of Proposition 2, and the corresponding proof for an incumbent innovator will be given after the proof of Proposition 3. ■

Proof of Proposition 2. We shall use the result of Lemma 1 to prove this proposition. The proof of this lemma will be given at the end of the appendix. Since in this proposition, we are only considering an outsider innovator, the superscript O is suppressed. Part [i] has been shown in the main text, so consider part [ii] and let $n \geq 2$. The results of the following lemmas will be used to prove the proposition. The proofs are straightforward, and hence omitted.

Lemma A2.1. *Let* $0 < \varepsilon < min\{a - c, c\}$. *For* $m \in \{1, \ldots, n\}$, *let*

$$\beta_1(m) = \frac{m\varepsilon - (a - c)}{m}, \beta_2(m) = \frac{a - c + (n - m + 1)\varepsilon}{n - m + 1}.$$

Then, for $m \in \{2, \ldots, n\}$, $\beta_1(m - 1) < \beta_1(m) < \varepsilon < \beta_2(m - 1) < \beta_2(m)$.

Lemma A2.2. *Suppose there are m licensees, $1 \leq m \leq n-1$. Then, the following hold.*

[1] *If $r \leq \beta_1(m)$, then*

$$q_L(m,r) = \frac{a-c+\varepsilon-r}{m+1}, \qquad q_N(m,r) = 0.$$

[2] *If $r \in [\beta_1(m), \beta_2(m)]$, then*

$$q_L(m,r) = \frac{a-c+(n-m+1)(\varepsilon-r)}{n+1}, \qquad q_N(m,r) = \frac{a-c-m(\varepsilon-r)}{n+1}.$$

[3] *If $r \geq \beta_2(m)$, then*

$$q_L(m,r) = 0, \qquad q_N(m,r) = \frac{a-c}{n-m+1}.$$

For all cases, $\Phi_J(m,r) = [q_J(m,r)]^2$ for $J \in \{L, N\}$.

Lemma A2.3. *For $m = n$, $q_L(n,r) = (a-c+\varepsilon-r)/(n+1)$ if $r \leq \beta_2(n)$, and $q_L(n,r) = 0$ otherwise. Again, $\Phi_L(m,r) = [q_L(m,r)]^2$.*

Lemma A2.4. *Let $1 \leq m \leq n$ and $r \geq \beta_2(m)$. Then, the innovator's payoff from any FR or AR policy when there are m licensees and the rate of royalty is r, is at most zero.*

We now proceed to prove the proposition. Recall that the respective payoffs of the innovator from FR and AR polices are given as follows.

$$\Pi_{FR}(m,r) = mrq_L(m,r) + m[\Phi_L(m,r) - \Phi_N(m-1,r)]. \tag{11}$$

$$\Pi_{AR}(m,r) = mrq_L(m,r) + m[\Phi_L(m,r) - \Phi_N(m,r)]. \tag{12}$$

We consider two possible cases.

Case I. All firms are licensees, that is, $m = n$. Due to Lemma 1, it is enough to consider the FR policy for this case. Due to Lemma A2.4, we can restrict $r \leq \beta_2(n)$. It is easy to verify that when $r \leq \beta_1(n-1)$, then the payoff of the innovator, which is a quadratic function in r, is maximized at $r = \beta_1(n-1)$.

On the other hand, when $r \geq \beta_2(n-1)$ it is maximized at $r = \beta_2(n-1)$. Thus, it is enough to consider $r \in [\beta_1(n-1), \beta_2(n-1)]$. In that case, from Lemma A2.2 and (11), we have

$$\Pi_{FR}(n,r) = nr\left[\frac{a-c+\varepsilon-r}{n+1}\right] + n\left[\frac{a-c+\varepsilon-r}{n+1}\right]^2 - n\left[\frac{a-c-(n-1)(\varepsilon-r)}{n+1}\right]^2.$$

The maximum is attained at $r = \bar{r}(n) \in (\beta_1(n-1), \beta_2(n-1))$, where

$$\bar{r}(n) \equiv \frac{(n-1)[(2n-1)\varepsilon-(a-c)]}{2(n^2-n+1)}. \tag{13}$$

The maximized payoff of the innovator is given by

$$\Pi_{FR}(n,\bar{r}(n)) = \frac{n[(n-1)^2(a-c)^2 + 2(n+1)(2n^2-n+1)(a-c)\varepsilon + (n+1)^2\varepsilon^2]}{4(n+1)^2(n^2-n+1)}. \tag{14}$$

Conclusion O1. *For $m = n$, the optimal licensing policy for the innovator is the FR policy $\langle n, \bar{r}, f(n, \bar{r}(n))\rangle$, where $\bar{r}(n)$ is given by the* (13). *The payoff of the innovator is given by* (14).

Case II. Consider the case where there is at least one non-licensee, that is, $1 \leq m \leq n-1$. In view of Lemma 1, we only consider AR policy for this case. When $r \leq \beta_1(m)$, the maximum is attained at $r = \beta_1(m)$, while the payoff is at most zero for $r \geq \beta_2(m)$. So, we can restrict $r \in [\beta_1(m), \beta_2(m)]$. In that case, from Lemma A2.2 and (12), we have

$$\Pi_{AR}(m,r) = mr\left[\frac{a-c+(n-m+1)(\varepsilon-r)}{n+1}\right]$$
$$+m\left[\frac{a-c+(n-m+1)(\varepsilon-r)}{n+1}\right]^2 - m\left[\frac{a-c-m(\varepsilon-r)}{n+1}\right]^2. \tag{15}$$

The unconstrained maximum is attained at $r = \tilde{r}(m)$, where

$$\tilde{r}(m) \equiv \frac{-(n+1-3m)\varepsilon-(a-c)}{2m}.$$

It is easy to verify that $\tilde{r}(m) < \varepsilon < \beta_2(m)$. Since

$$\tilde{r}(m) - \beta_1(m) = \frac{a-c-(n-m+1)\varepsilon}{2m},$$

we have

$$\tilde{r}(m) \leq \beta_1(m) \Leftrightarrow (a-c) \leq (n-m+1)\varepsilon. \tag{16}$$

Noting that $2 \leq n-m+1 \leq n$, we consider the following three subcases.

Subcase (a). $\varepsilon \geq (a-c)/2$. For this case, $(a-c) \leq (n-m+1)\varepsilon$ for all m such that $1 \leq m \leq n-1$. Hence, by (16), for every m, the maximum is attained at $\beta_1(m)$. By replacing $r = \beta_1(m)$ in (15) we find that the maximized payoff of the innovator is $(a-c)\varepsilon$.

Subcase (b). $(a-c)/n < \varepsilon < (a-c)/2$. Then, there is $\bar{m}$, $2 \leq \bar{m} \leq n$ such that $(a-c) \leq (n-m+1)\varepsilon$ for m when $1 \leq m \leq n-\bar{m}$ and $(a-c) > (n-m+1)\varepsilon$ for m when $n-\bar{m}+1 \leq m \leq n-1$. Thus, for all m such that $1 \leq m \leq n-\bar{m}$, the optimal choice of r is $\beta_1(m)$, and the payoff of the innovator is given by $(a-c)\varepsilon$. Now consider m such that $n-\bar{m}+1 \leq m \leq n-1$. For such an m, the optimal choice of r is $\tilde{r}(m)$. The payoff of the innovator for the AR policy $\langle m, \tilde{r}(m) \rangle$ is given by

$$\Pi_{AR}(m, \tilde{r}(m)) = \frac{(a-c)^2 + 2(n+m+1)(a-c)\varepsilon + (n-m+1)^2\varepsilon^2}{4(n+1)}. \tag{17}$$

From (17), it is easy to verify that $\Pi_{AR}(m, \tilde{r}(m))$ is increasing in m when $a-c > (n-m+1)\varepsilon$, so that for $n-\bar{m}+1 \leq m \leq n-1$, the payoff is maximized when $m = n-1$. Noting that

$$\tilde{r}(n-1) = \frac{2(n-2)\varepsilon - (a-c)}{2(n-1)}, \tag{18}$$

the payoff of the innovator is given by

$$\Pi_{AR}(n-1, \tilde{r}(n-1)) = \frac{(a-c)^2 + 4n(a-c)\varepsilon + 4\varepsilon^2}{4(n+1)}. \tag{19}$$

Comparing this with $(a-c)\varepsilon$, we get

$$\Pi_{AR}(n-1, \tilde{r}(n-1)) - (a-c)\varepsilon = \frac{(a-c-2\varepsilon)^2}{4(n+1)} \geq 0,$$

which implies that the optimal payoff of the innovator in this case is given by (19).

Subcase (c). $\varepsilon \leq (a-c)/n$. For this case, $a-c \geq (n-m+1)\varepsilon$ for all $1 \leq m \leq n-1$. From similar argument as in the previous case, we conclude that in this case also, the optimal payoff of the innovator is given by (19). We have the following conclusion.

Conclusion O2. *Consider AR policies $\langle m, r\rangle$ for $1 \leq m \leq n-1$. Then, the following hold.*

[i]*If $\varepsilon \geq (a-c)/2$, then the innovator's payoff is maximized for any policy $\langle m, \beta_1(m)\rangle$, where $1 \leq m \leq n-1$, and her payoff is $(a-c)\varepsilon$. However, in the presence of a negligible but positive cost of contracting for every licensee, the optimal policy of the innovator is $\langle 1, \beta_1(1)\rangle$, where the license is sold exclusively to a firm.*

[ii]*If $\varepsilon \leq (a-c)/2$, then $\langle n-1, \tilde{r}(n-1)\rangle$ is the optimal policy, and the corresponding payoff of the innovator is given by*

$$\Pi_{AR}(n-1, \tilde{r}(n-1)) = \frac{(a-c)^2 + 4n(a-c)\varepsilon + 4\varepsilon^2}{4(n+1)}.$$

To complete the proof of Proposition 2, we need to compare the results of Conclusions O1 and O2. We consider the following two cases.

Case 1. $\varepsilon \geq (a-c)/2$. It can be easily verified that $\Pi_{FR}(n, \bar{r}(n)) \geq (a-c)\varepsilon$ iff $g_1(x,n) \geq 0$, where

$$g_1(x,n) = n(n-1)^2x^2 - 2(n+1)(n^2-n+2)x + n(n+1)^2,$$

and $x \equiv (a-c)/\varepsilon$. It is easy to verify that for $2 \leq n \leq 6$, $\Pi_{FR}(n, \bar{r}(n))$ is less than $(a-c)\varepsilon$, while for $n \geq 7$, there exists $1 < q(n) < 2$, such that $\Pi_{FR}(n, \bar{r}(n)) \leq (a-c)\varepsilon$ when $\varepsilon \geq (a-c)/q(n)$ and the converse holds when $(a-c)/2 \leq \varepsilon < (a-c)/q(n)$. Defining $q(n) = 2$ for $2 \leq n \leq 6$, it can be shown that

$$q(n) = \begin{cases} 2 & \text{for } 2 \leq n \leq 6 \\ \dfrac{(n+1)(n^2-n+2+2\sqrt{n^2-n+1})}{n(n-1)^2} & \text{for } n \geq 7 \end{cases} \tag{20}$$

Hence, it follows that when $\varepsilon \geq (a-c)/q(n)$, the innovator sells the license to only one firm.

Case 2. $\varepsilon \leq (a-c)/2$. For this case, $\Pi_{FR}(n, \bar{r}(n)) \geq \Pi_{AR}(n-1, \tilde{r}(n-1))$ iff $g_2(x,n) \leq 0$, where

$$g_2(x,n) = (2n^2-n+1)x^2 - 2n(n-1)(n+1)x + (n+1)(3n^2-5n+4). \tag{21}$$

It can be shown that for all x, $g_2(x,n) > 0$ for $2 \leq n \leq 5$. For $n = 6$, there are constants $2 < x(6) < y(6)$ such that $g_2(x,6) < 0$ for $x \in (x(6), y(6))$, and $g_2(x,n) \geq 0$ otherwise. Finally, for $n \geq 7$, there is $y(n) > 2$ such that $g_2(x,n) \leq 0$ for $x \in [2, y(n)]$, and it is positive for $x > y(n)$, where

$$y(n) = \frac{n(n-1)(n+1) - \sqrt{(n+1)(n^2-n+1)(n^3-6n^2+5n-4)}}{2n^2-n+1}.$$

Conclusion O3. [i] *Let* $2 \leq n \leq 5$. *When* $\varepsilon \geq (a-c)/2$, *the innovator sells the license to only one firm using the AR policy and obtains* $(a-c)\varepsilon$. *Otherwise, the license is sold to* $n-1$ *firms using the AR policy and the payoff of the innovator is given by* (19).

[ii]*Let* $n = 6$. *There are numbers* $y(6) > x(6) > 2$ *such that when* $\varepsilon \geq (a-c)/2$, *then the innovator sells the license to only one firm using the AR policy and obtains* $(a-c)\varepsilon$. *When* $\varepsilon \in ((a-c)/y(6), (a-c)/x(6))$, *then the license is sold to all firms through the FR policy and the payoff of the innovator is given by* (14). *Otherwise, the license is sold to* $n-1$ *firms using the AR policy, and the payoff is given by* (19).

[iii]*Let* $n \geq 7$. *There are numbers* $y(n) > 2 > q(n)$ *such that the innovator sells only one license using the AR policy and earns* $(a-c)\varepsilon$ *when* $\varepsilon \geq (a-c)/q(n)$.

When $\varepsilon \leq (a-c)/y(n)$, the license is sold to $n-1$ firms using the AR policy and the payoff is given by (19). *When $(a-c)/y(n) \leq \varepsilon \leq (a-c)/q(n)$, the license is sold to n firms through the FR policy and the payoff is given by* (14).

Then, Proposition 2 follows from Conclusion O3. ■

Remark. The function $q(n)$ is decreasing in n, and $q(n) \to 1$ as n increases indefinitely. For $n \geq 6$, $y(n)$ is strictly increasing in n and $y(n) \to \infty$ as n increases indefinitely.

Proof of Proposition 1 in case of an outsider innovator. Observe from (14) and (19) that both $\Pi_{FR}(n, \bar{r}(n))$ and $\Pi_{AR}(n-1, \tilde{r}(n-1))$ are increasing in ε. Clearly, $(a-c)\varepsilon$ is increasing in ε. Since the payoff of the innovator is continuous in ε, we conclude that the payoff is increasing in ε, which proves [i] of Proposition 1. Further note that for every ε, both $\Pi_{FR}(n, \bar{r}(n))$ and $\Pi_{AR}(n-1, \tilde{r}(n-1))$ are at least $(a-c)\varepsilon$, which proves [ii] of Proposition 1.

To prove [iii], first consider the payoff of any firm. For the FR policy $\langle n, \bar{r}(n), f(n, \bar{r}(n)) \rangle$, the payoff of every firm is given by

$$\tilde{\Pi}_{FR} = \frac{[(n^2+1)(a-c) - (n^2-1)\varepsilon]^2}{4(n+1)^2(n^2-n+1)^2},$$

while for the AR policy $\langle n-1, \tilde{r}(n-1) \rangle$, the payoff of every firm is given by

$$\tilde{\Pi}_{AR} = \frac{(a-c-2\varepsilon)^2}{4(n+1)^2}.$$

When the license is sold to only firm, then we know that every firm earns zero payoff. Note that the pre-innovation Cournot profit of any firm is given by $\tilde{\Pi} = (a-c)^2/(n+1)^2$. It can be easily verified that $\tilde{\Pi}$ is more than both $\tilde{\Pi}_{FR}$ and $\tilde{\Pi}_{AR}$, implying that every firm is worse off due to the innovation.

To complete the proof of [iii] of Proposition 1, we need to show that the consumers are better off due to the innovation. For the FR policy $\langle n, \bar{r}(n), f(n, \bar{r}(n)) \rangle$, the total industry output is given by

$$q_{FR} = \frac{n[(2n^2 - n + 1)(a - c) + (n + 1)\varepsilon]}{2(n + 1)(n^2 - n + 1)},$$

so that the Cournot price is $p_{FR} = a - q_{FR}$. For the AR policy $\langle n-1, \tilde{r}(n-1)\rangle$, the total industry output is given by

$$q_{AR} = \frac{(2n + 1)(a - c) + 2\varepsilon}{2(n + 1)},$$

and the Cournot price is $p_{AR} = a - q_{AR}$. When the license is sold to only one firm, then we know that the Cournot price is c. The pre-innovation Cournot price is given by $\tilde{p} = (a + nc)/(n + 1)$. It can be easily verified that p_{FR}, p_{AR} and c are all less than $\tilde{p}$, which implies that the post-innovation Cournot price is always less than the pre-innovation price, so that the consumers are better off due to the innovation. This completes the proof of Proposition 1 in case of an outsider innovator. ■

Remark. It can be seen that both $\tilde{\Pi}_{FR}$ and $\tilde{\Pi}_{AR}$ are decreasing in ε. However, for $n \geq 6$, Π_{FR} is more than Π_{AR} at $\varepsilon = (a - c)/y(n)$ (the point where the optimal policy changes from AR policy to FR policy), implying that there is a jump at $\varepsilon = (a - c)/y(n)$ (see Figure 2.1). Similarly, both p_F and p_A are decreasing in ε. However, for $n \geq 6$, at $\varepsilon = (a - c)/y(n)$, p_{FR} is more than p_{AR}, that is, the price function has an upward jump at this point (see Figure 3.1).

Proof of Proposition 3. Consider an incumbent innovator. For notational simplicity, the superscript I, which stands for 'incumbent', is dropped. Note that the subscript I which appears, stands for 'innovator'. The following lemmas will be used to prove this proposition. The proofs are easy, and hence omitted.

Lemma A3.1. *Let* $0 < \varepsilon < min\{a - c, c\}$. *For* $m \in \{1, \ldots, n\}$, *let*

$$\theta_0(m) = \frac{c - a - (n - m + 1)\varepsilon}{m}, \qquad \theta_1(m) = \frac{c - a - \varepsilon}{m},$$

$$\theta_2(m) = \frac{c - a + (m+1)\varepsilon}{m}, \qquad \theta_3(m) = \frac{a - c + (n-m+1)\varepsilon}{n-m+2}.$$

Then, for $m \in \{2, \ldots, n\}$, $\theta_2(m-1) < \theta_2(m)$, *and*

$$\theta_0(m-1) < \theta_0(m) < \theta_1(m) < \theta_2(m) < \theta_3(m-1) < \theta_3(m).$$

Further, for all $m \in \{1, \ldots, n\}$, $\theta_1(m) < 0 < \theta_3(m)$.

Lemma A3.2. *Suppose there are* m *licensees,* $1 \leq m \leq n-1$. *Then, the following hold.*

[1] *If* $r \leq \theta_1(m)$, *then*

$$q_I(m,r) = 0, \qquad q_L(m,r) = \frac{a - c + \varepsilon - r}{m+1}, \qquad q_N(m,r) = 0.$$

[2] *If* $r \in [\theta_1(m), \theta_2(m)]$, *then*

$$q_I(m,r) = \frac{a - c + \varepsilon + mr}{m+2}, \qquad q_L(m,r) = \frac{a - c + \varepsilon - 2r}{m+2}, \qquad q_N(m,r) = 0.$$

[3] *If* $r \in [\theta_2(m), \theta_3(m)]$, *then*

$$q_I(m,r) = \frac{a - c + (n-m+1)\varepsilon + mr}{n+2},$$

$$q_L(m,r) = \frac{a - c + (n-m+1)\varepsilon - (n-m+2)r}{n+2},$$

$$q_N(m,r) = \frac{a - c - (m+1)\varepsilon + mr}{n+2}.$$

[4] *If* $r \geq \theta_3(m)$, *then*

$$q_I(m,r) = \frac{a - c + (n-m+1)\varepsilon}{n-m+2}, \qquad q_L(m,r) = 0, \qquad q_N(m,r) = \frac{a - c - \varepsilon}{n-m+2}.$$

In all cases, $\Phi_J\langle m,r\rangle = [q_J\langle m,r\rangle]^2$ *for* $J \in \{I, L, N\}$,

Lemma A3.3. *Suppose there are* n *licensees. Then the following hold.*

[1] *If* $r \leq \theta_1(n)$, *then*

$$q_I(n,r) = 0, \qquad q_L(n,r) = \frac{a - c + \varepsilon - r}{n+1}.$$

[2] *If* $r \in [\theta_1(n), \theta_3(n)]$, *then*

$$q_I(n,r) = \frac{a-c+\varepsilon+nr}{n+2}, \qquad q_L(n,r) = \frac{a-c+\varepsilon-2r}{n+2}.$$

[3] *If* $r \geq \theta_3(n)$, *then*

$$q_I(n,r) = \frac{a-c+\varepsilon}{2}, \qquad q_L(n,r) = 0.$$

Again, $\Phi_J\langle m,r\rangle = [q_J\langle m,r\rangle]^2$ *for* $J \in \{I, L, N\}$,

Lemma A3.4. *For* $1 \leq m \leq n-1$, *if* $r \leq \theta_1(m)$, *then the payoff of the innovator for any FR or AR policy is at most zero.*

Note that the payoffs of an incumbent innovator at FR and AR policies are given as follows.

$$\Pi_{FR}(m,r) = \Phi_I(m,r) + mrq_L(m,r) + m[\Phi_L(m,r) - \Phi_N(m-1,r)]. \tag{22}$$

$$\Pi_{AR}(m,r) = \Phi_I(m,r) + mrq_L(m,r) + m[\Phi_L(m,r) - \Phi_N(m,r)]. \tag{23}$$

Lemmas A3.1-A3.4, together with (22) and (23) allow us to derive $\Pi_{FR}(m,r)$ and $\Pi_{AR}(m,r)$ for all m and r. Before proceeding further, consider the FR policy $\langle m,r,f\rangle = \langle n,\varepsilon,0\rangle$. This is a policy based on royalty only, and, this is the optimal one among all licensing policies that are based exclusively on royalty. For the proof, and other details, we refer to Kamien and Tauman (2002). The payoff of the innovator from this policy, $\Pi_{FR}(n,\varepsilon)$, is given by

$$\Pi_{FR}(n,\varepsilon) = \frac{(a-c)^2 + (n^2+4n+2)(a-c)\varepsilon + \varepsilon^2}{(n+2)^2}. \tag{24}$$

This policy will be used since it dominates some other relevant policies,[14] as we show in the following lemmas. The proofs are easy, and hence omitted.

[14] In particular, $\Pi_{FR}(n,\varepsilon)$ is more than $[a-c+(n+1)\varepsilon]^2/(n+2)^2$, which is the payoff of the incumbent innovator when she does not sell license. This implies that the innovator will sell the new technology.

Lemma A3.5. *Suppose either* $r \leq \max\{\theta_1(n), \theta_2(n-1)\}$, *or* $r \geq \theta_3(n-1)$. *Then,* $\Pi_{FR}(n,r) \leq \Pi_{FR}(n,\varepsilon)$.

Lemma A3.6. *For* $1 \leq m \leq n-1$, *suppose either* $r \leq \theta_2(m)$, *or* $r \geq \theta_3(m)$. *Then,* $\Pi_{AR}(m,r) \leq \Pi_{FR}(n,\varepsilon)$.

We proceed now to prove Proposition 3. We consider the following two cases.

Case I. $m = n$. For this case, we consider $r \in [\max\{\theta_1(n), \theta_2(n-1)\}, \theta_3(n-1)]$, and show that the maximum payoff in this interval is at least $\Pi_{FR}(n,\varepsilon)$. Then, by Lemma A3.5, it follows that this maximum is also the global maximum. For this interval, from Lemma A3.3 and (22), we have

$$\Pi_{FR}(n,r) = \left[\frac{a-c+\varepsilon+nr}{n+2}\right]^2 + nr\left[\frac{a-c+\varepsilon-2r}{n+2}\right]$$

$$+n\left[\frac{a-c+\varepsilon-2r}{n+2}\right]^2 - n\left[\frac{a-c-n\varepsilon+(n-1)r}{n+2}\right]^2.$$

It can be verified that the maximum of $\Pi_{FR}(n,r)$ is attained at $r = r_F(n)$, where $r_F(n) \in (\max\{\theta_1(n), \theta_2(n-1)\}, \theta_3(n-1))$, and it is given by

$$r_F(n) = \frac{n(2n-1)\varepsilon - (n-2)(a-c)}{2(n^2-n+1)}. \tag{25}$$

The maximized payoff of the innovator is given by

$$\Pi_{FR}(n, r_F(n)) = \frac{(n^3+4)(a-c+\varepsilon)^2 + 4n^2(n+1)^2(a-c)\varepsilon}{4(n+2)^2(n^2-n+1)}. \tag{26}$$

Further, $\varepsilon \in [\max\{\theta_1(n), \theta_2(n-1)\}, \theta_3(n-1)]$, so that $\Pi_{FR}(n, r_F(n))$ at least as much as $\Pi_{FR}(n,\varepsilon)$. Then from Lemma A3.5, the following is concluded.

Conclusion I1. *Let* $m = n$. *Then, the payoff of the innovator is maximized at the FR policy* $\langle n, r_F(n), f(n, r_F(n))\rangle$ *and the payoff is given by* (26).

Case II. $1 \leq m \leq n-1$. For this case, in view of Lemma 1, it is enough to consider AR policy. Further, due to Lemma A3.6, we can restrict

$r \in [\theta_2(m), \theta_3(m)]$. For this case, from Lemma A3.2 and (23), we have

$$\Pi_{AR}(m,r) = \left[\frac{a-c+(n-m+1)\varepsilon+mr}{n+2}\right]^2$$
$$+mr\left[\frac{a-c+(n-m+1)\varepsilon-(n-m+2)r}{n+2}\right]$$
$$+m\left[\frac{a-c+(n-m+1)\varepsilon-(n-m+2)r}{n+2}\right]^2 - m\left[\frac{a-c-(m+1)\varepsilon+mr}{n+2}\right]^2.$$

The unrestricted maximum is attained at $r_A(m)$, where

$$r_A(m) \equiv \frac{-[n(a-c)+(n^2+n-4m-3mn)\varepsilon]}{2m(n+1)}. \tag{27}$$

Observe that $r_A(m) < \theta_3(m)$. Since

$$r_A(m) - \theta_2(m) = \frac{(n+2)[a-c-(n-m+1)\varepsilon]}{2m(n+1)},$$

we have

$$r_A(m) \geq \theta_2(m) \Leftrightarrow a-c \geq (n-m+1)\varepsilon. \tag{28}$$

Then, from (27) and (28) we conclude that when $r \in [\theta_2(m), \theta_3(m)]$, $\Pi_{AR}(m,r)$ is maximized at $r = r_A(m)$ if $(a-c) \geq (n-m+1)\varepsilon$. If $(a-c) \leq (n-m+1)\varepsilon$, it is maximized at $r = \theta_2(m)$. We already know that $\Pi_{AR}(m, \theta_2(m)) = (a-c)\varepsilon$. We have

$$\Pi_{AR}(m, r_A(m)) = \frac{(a-c)^2 + 2(n+m+1)(a-c)\varepsilon + (n-m+1)^2\varepsilon^2}{4(n+1)}. \tag{29}$$

Since $1 \leq m \leq n-1$, we have $2 \leq n-m+1 \leq n$. Considering the three similar subcases as in the case of an outsider innovator [see the subcases (a)-(c), pp. 24-25 of the proof of Proposition 2], it can be shown that when $\varepsilon \geq (a-c)/2$, for any $1 \leq m \leq n-1$, $\Pi_{AR}(m,r)$ is maximized at $r = \theta_2(m)$ and, the maximized value is given by $(a-c)\varepsilon$, which is less than $\Pi_{FR}(n,\varepsilon)$. When $\varepsilon < (a-c)/2$, then out of all AR policies, the maximum payoff of the innovator is attained at the policy $\langle n-1, r_A(n-1)\rangle$, and it is given by the following, which we find by replacing m by $(n-1)$ at (29).

$$\Pi_{AR}(n-1, r_A(n-1)) = \frac{(a-c)^2 + 4n(a-c)\varepsilon + 4\varepsilon^2}{4(n+1)}. \tag{30}$$

Observe that

$$\Pi_{AR}(n-1, r_A(n-1)) - (a-c)\varepsilon = \frac{(a-c-2\varepsilon)^2}{4(n+1)} \geq 0.$$

Summarizing our observations from Case (II), we have the following.

Conclusion I2. *Let* $m \in \{1, \ldots, n-1\}$. *If* $r \in [\theta_2(m), \theta_3(m)]$ *and* $\varepsilon < (a-c)/2$, *then the innovator's payoff is maximized at the AR policy* $\langle n-1, r_A(n-1)\rangle$, *and it is given by* (30). *Otherwise, any FR or AR policy gives the innovator less payoff than the FR policy* $\langle n, \varepsilon, 0\rangle$.

From Conclusions I1 and I2, it follows that when $\varepsilon \geq (a-c)/2$, then the FR policy $\langle n, r_F(n), f(n, r_F(n))\rangle$ is the optimal one. To complete the proof of Proposition 3, we need to consider the case where $\varepsilon < (a-c)/2$ and compare $\Pi_{FR}(n, r_F(n))$ and $\Pi_{AR}(n-1, r_A(n-1))$. It can be shown that $\Pi_{FR}(n, r_F(n)) \geq \Pi_{AR}(n-1, r_A(n-1))$ iff $g(x, n) \leq 0$, where

$$g(x, n) = (2n^3 + n^2 - 4n)x^2 +$$

$$(2n^3 + n^2 - 4n)x^2 - 2(n^4 + 5n^3 + 2n^2 - 4n + 4)x + 3n^4 + 11n^3 + 4n^2 - 4n + 12.$$

Note that we are considering $x > 2$. It can be shown that $g(x, n)$ has only one root above 2: $h(n)$, given by

$$h(n) = \frac{n^4 + 5n^3 + 2n^2 - 4n + 4 + (n+2)\sqrt{(n+1)(n^2 - n + 1)(n^3 + 4)}}{2n^3 + n^2 - 4n}. \tag{31}$$

Further, $g(x, n) < 0$ for $2 < x < h(n)$ and $g(x, n) > 0$ for $x > h(n)$. Thus, the optimal policy is the FR policy when $\varepsilon \geq (a-c)/h(n)$, and it is the AR policy otherwise. This completes the proof of Proposition 3. ■

Remark. It can be shown that $h(n) > n$. Further, $h(4) < h(2) < h(5)$, $h(n+1) > h(n)$ for $n \geq 3$ and $\lim_{n\to\infty} h(n) = \infty$.

Proof of Proposition 1 in case of an incumbent innovator. We have already shown while proving Proposition 3 that both $\Pi_{FR}(n, r_F(n))$ and $\Pi_{AR}(n-$

$1, r_A(n-1))$ are at least $(a-c)\varepsilon$. From (26) and (30), it can be easily verified that both $\Pi_{FR}(n, r_F(n))$ and $\Pi_{AR}(n-1, r_A(n-1))$ are increasing in ε. Since the payoff of the innovator is continuous in ε, we conclude that the payoff is increasing in ε. This proves parts [i] and [ii] of Proposition 1 in case of an incumbent innovator.

To prove [iii] of Proposition 1, first consider the payoff of any firm other than the innovator. For the FR policy, this is given by

$$\tilde{\Pi}_{FR} = \frac{n^2(n+1)^2(a-c-\varepsilon)^2}{4(n+2)^2(n^2-n+1)^2},$$

while for the AR policy, the payoff of any firm other than the innovator is given by

$$\tilde{\Pi}_{AR} = \frac{(a-c-2\varepsilon)^2}{4(n+1)^2}.$$

Note that the pre-innovation Cournot profit of any firm is given by $\tilde{\Pi} = (a-c)^2/(n+2)^2$. It can be easily verified that the both $\tilde{\Pi}_{FR}$ and $\tilde{\Pi}_{AR}$ are less than $\tilde{\Pi}$, which proves that every firm other than the innovator is worse off due to the innovation.

Finally, to show that the consumers are better off, note that for the FR policy, the total industry output is given by

$$q_{FR} = \frac{(2n^3+n^2-2n+2)(a-c)+(n^2+2)\varepsilon}{2(n+2)(n^2-n+1)},$$

so that the Cournot price is given by $p_{FR} = a - q_{FR}$. For the AR policy, the total industry output is given by

$$q_{AR} = \frac{(2n+1)(a-c)+2\varepsilon}{2(n+1)}$$

and the price is given by $p_{AR} = a - q_{AR}$. Noting that the pre-innovation Cournot price is given by $\tilde{p} = [a+(n+1)c]/(n+2)$, it can be easily verified that both p_{FR} and p_{AR} are less than $\tilde{p}$, which proves that the consumers are better off due to the innovation. This completes the proof of Proposition 1 in case of an incumbent innovator. ■

Remark. It can be further verified that $\Pi_{FR}(n, r_F(n))$ is decreasing in ε. Also, when $\varepsilon < (a-c)/h(n)$, that is, when the AR policy is optimal, then $\Pi_{AR}(n-1, r_A(n-1))$ is decreasing in ε. However, at $\varepsilon = (a-c)/h(n)$, (the point where the optimal policy changes from AR policy to FR policy), $\Pi_{FR} > \Pi_{AR}$. Thus, when $\varepsilon = (a-c)/h(n)$, there is an upward jump for the payoff of any firm other than the innovator (see Figure 2.2). Regarding the post-innovation Cournot price, it can be seen that the price is more than c. Further, both p_{FR} and p_{AR} are decreasing in ε, and there is an upward jump at $\varepsilon = (a-c)/h(n)$ (see Figure 3.2).

Proof of Proposition 4. First, we prove Proposition 4 in case the innovator is an outsider. Then, we prove this proposition in case of an incumbent innovator.

An outsider innovator. Let us denote by $N^O(\varepsilon)$ the industry size that gives the outsider innovator the highest incentive to innovate when the magnitude of innovation is ε. Denoting the payoff of the outsider innovator in an industry of n firms by $\Pi(n)$, $N^O(\varepsilon)$ is the industry size where $\Pi(n)$ is maximum. We have already shown that $\Pi(1) < \Pi(2)$, so that it is enough to consider $n \geq 2$.

To begin with, consider $\varepsilon \geq (a-c)/2$. Denoting $x \equiv (a-c)/\varepsilon$, for this case, we have $x \in (1, 2]$. Recall from the proof of Proposition 2 that when $\varepsilon \geq (a-c)/2$, the optimal payoff for the innovator is $(a-c)\varepsilon$ for $n \leq 6$. Since the FR policy $\langle n, \bar{r}(n), f(n, \bar{r}(n))\rangle$ is not the optimal policy for that case, we have $\Pi_{FR}(n, \bar{r}(n)) \leq (a-c)\varepsilon$ for $n \leq 6$. For $n \geq 7$, by similar reasoning, it follows that there is $1 < q(n) < 2$ such that $\Pi_{FR}(n, \bar{r}(n)) \leq (a-c)\varepsilon$ when $x \in [1, q(n)]$ and $\Pi_{FR}(n, \bar{r}(n)) > (a-c)\varepsilon$ when $x \in (q(n), 2]$. It can be easily verified that

$$q(n+1) < q(n) \text{ for all } n \geq 7 \text{ and } \lim_{n\to\infty} q(n) = 1.$$

Hence for every $x \in (1, 2]$, there exists N such that $x \in (q(N+1), q(N)]$. This implies that $\Pi_{FR}(n, \bar{r}(n)) \leq (a-c)\varepsilon$ for $n \leq N$ and $\Pi^{FR}(n, \bar{r}(n)) > (a-c)\varepsilon$

for $n \geq N+1$. Hence $N^O(\varepsilon) \geq N+1$. It can be easily verified that $\Pi_{FR}(n+1, \bar{r}(n+1)) \geq \Pi_{FR}(n, \bar{r}(n))$ iff $\gamma(x,n) \leq 0$, where

$$\gamma(x,n) = n(n^2+n-1)(n^3-2n^2-3n-4)x^2 -$$

$$2(n+2)(n+1)(n^4+4n^2+3n+2)x + (n^2+n-1)(n+2)^2(n+1)^2. \quad (32)$$

It can be verified that $\gamma(x,n)$ has two real roots, $x_1(n) < 1 < x_2(n)$ and $\gamma(x,n) < 0$ for $x \in (1, x_2(n))$ while $\gamma(x,n) > 0$ for $x > x_2(n)$. Further,

$$q(n) < x_2(n) \text{ and } x_2(n+1) < x_2(n) \text{ for all } n \geq 7; \lim_{n\to\infty} x_2(n) = 1. \quad (33)$$

Thus, for $x \in (q(N+1), q(N)]$, there exists $\bar{N} > N+1$ such that $x_2(n) \geq x$ for $N+1 \leq n \leq \bar{N}$ and $x_2(n) < x$ for $n \geq \bar{N}+2$. This implies that the maximum of $\Pi_{FR}(n, \bar{r}(n))$ for $n \geq N+1$ is attained at $n = \bar{N}+1$, so that $N^O(\varepsilon) = \bar{N}+1$ Since both $q(n)$ and $x_2(n)$ are decreasing in n, and both converge to 1 as n increases indefinitely, we conclude the following lemma.

Lemma A4.1. *$N^O(\varepsilon)$ is increasing in ε when $\varepsilon \geq (a-c)/2$ and $N^O(\varepsilon) \to \infty$ as $\varepsilon \to a-c$.*

In view of Lemma A4.1, to complete the proof of Proposition 4 in case of an outsider innovator, it remains to be shown that (i) there is an x_0 such that $N^O(\varepsilon) = 2$ for $\varepsilon \leq (a-c)/x_0$, and (ii) $N^O(\varepsilon)$ is increasing when $\varepsilon \leq (a-c)/2$.

Let $\varepsilon < (a-c)/2$. Recall from Proposition 2 that for $n \leq 5$, the payoff of the outsider innovator for this case is given by $\Pi_{AR}(n-1, \tilde{r}(n-1))$. For $n \geq 6$, it is $\Pi_{FR}(n, \bar{r}(n)$ if $x \in [x(n), y(n)]$ and it is $\Pi_{AR}(n-1, \tilde{r}(n-1))$ otherwise. Recall that $x(6) > 2$, $x(n) = 2$ for $n \geq 7$, $y(n)$ is increasing for $n \geq 6$ and $\lim_{n\to\infty} y(n) = \infty$. For any given magnitude of innovation, let us partition the set $\{2, 3, \ldots\} = N_A \cup N_F$, where N_A is the set of all integers n such that with industry size n, the AR policy is the optimal policy for the incumbent innovator, and N_F is the similar set for the FR policy. Hence, when

$x \in [y(N), y(N+1))$, then $N_A = \{2, 3, \ldots, N-1\}$ and $N_F = \{N, N+1, \ldots\}$. Next, note that

$$\Pi_{AR}(n, \tilde{r}(n)) - \Pi_{AR}(n-1, \tilde{r}(n-1)) = \frac{-(a-c-2\varepsilon)^2}{4(n+1)(n+2)},$$

so that $\Pi_{AR}(n-1, \tilde{r}(n-1))$ is decreasing in n, implying that over the set N_A, it is maximized at $n = 2$, i.e., at the policy $\langle n-1, \tilde{r}(n-1)\rangle = \langle 1, \tilde{r}(1)\rangle$. Note that

$$\Pi_{AR}(1, \tilde{r}(1)) = \frac{(a-c)^2 + 8(a-c)\varepsilon + 4\varepsilon^2}{12}.$$

From (32) and the last two statements of (33), it follows that for sufficiently large x, $\Pi_{FR}(n, \bar{r}(n))$ is decreasing in n, so that the maximum over N_F is attained at $n = N$. Next, note that $\Pi_{AR}(1, \tilde{r}(1)) \geq \Pi_{FR}(n, \bar{r}(n))$ iff $\tau(x, n) \geq 0$, where

$$\begin{aligned} \tau(x,n) &= (n^4 - 2n^3 - 2n + 1 + 6n^2)x^2 \\ &-2(n+1)(2n^3 - 3n^2 + 3n - 4)x + (4n^2 - 7n + 4)(n+1)^2. \end{aligned} \tag{34}$$

It can be verified that for sufficiently large n, $\tau(x, n)$ has two real roots, $\tau_1(n) < 2 < \tau_2(n)$, and $\tau(x, n) > 0$ for all $x > \tau_2(n)$. Further, $\lim_{n\to\infty} \tau_2(n) = 2$. All these facts imply that there is a sufficiently large x_0 such that $\Pi_{AR}(1, \tilde{r}(1)) > \Pi_{FR}(n, \bar{r}(n))$ for all sufficiently large n. Since $y(n)$ is increasing in n and $\lim_{n\to\infty} y(n) = \infty$, one can choose $x_0 \in [y(N), y(N+1))$ for a sufficiently large N such that $\Pi_{AR}(1, \tilde{r}(1)) > \Pi_{FR}(N, \bar{r}(N))$ for all $x > x_0$. This implies that $N^O(\varepsilon) = 2$ for all $x > x_0$.

Lemma A4.2. *There is $x_0 > 2$ such that $N^O(\varepsilon) = 2$ when $\varepsilon \leq (a-c)/x_0$.*

In view of Lemmas A4.1 and A4.2, it only remains to be shown that $N^O(\varepsilon)$ is increasing in ε for $\varepsilon \in ((a-c)/x_0, (a-c)/2]$. In this regard, we have the following lemma, which follows from certain basic properties of the quadratic functions $g_2(x, n)$ and $\gamma(x, n)$, given by (21) and (32) respectively. The proof is standard, but long and tedious as it proceeds through a series of observations. This proof is omitted here. It is available from the authors by request.

Lemma A4.3. *There are numbers* $2 < x_3 < x_2 < x_1 < x_0$ *such that*

$$N^O(\varepsilon) = \begin{cases} 2 & \textit{when } \varepsilon \leq (a-c)/x_0 \\ 9 & \textit{when } \varepsilon \in [(a-c)/x_0, (a-c)/x_1] \\ 10 & \textit{when } \varepsilon \in [(a-c)/x_1, (a-c)/x_2] \\ 11 & \textit{when } \varepsilon \in [(a-c)/x_2, (a-c)/x_3] \\ 12 & \textit{when } \varepsilon \in [(a-c)/x_3, (a-c)/2] \end{cases} \tag{35}$$

Further, $\lim_{\varepsilon \to (a-c)/2-} N^O(\varepsilon) < \lim_{\varepsilon \to (a-c)/2+} N^O(\varepsilon)$.

Then, Proposition 4 for the case of an outsider innovator follows from Lemmas A4.1-A4.3.

An incumbent innovator. Let $N^I(\varepsilon)$ denote the industry size that provides the highest incentive to innovate to an incumbent innovator when the magnitude of the innovation is ε. Further, let $\Delta(n+1)$ denote the incremental payoff of the incumbent innovator in an industry of size $n+1$, so that for a given magnitude of innovation, the industry size that provides the incumbent innovator the highest incentive to innovate is the one where $\Delta(n+1)$ is maximum. We have already shown that $\Delta(2) > \Delta(1)$. So, we can consider $n \geq 1$. Recall that

$$\Delta(2) = \frac{(a-c+\varepsilon)^2}{4} - \frac{(a-c-\varepsilon)^2}{9} - \frac{(a-c)^2}{9}. \tag{36}$$

Now consider an industry of size $n+1$ with $n \geq 2$. Let us denote by $\Delta^{FR}(n+1)$ the incremental payoff of the innovator due to the innovation when the optimal policy is the FR policy $\langle n, r_F(n), f(r_F(n), n) \rangle$, that is,

$$\Delta^{FR}(n+1) = \Pi_{FR}(n+1) - \tilde{\Pi}(n+1),$$

where $\tilde{\Pi}(n+1)$ is the pre-innovation Cournot profit of a firm in an industry of $n+1$ firms, given by $\tilde{\Pi}(n+1) = (a-c)^2/(n+2)^2$. Then, from the proof of Proposition 3, it follows that

$$\Delta^{FR}(n+1) = \frac{(n^3+4)(a-c+\varepsilon)^2 + 4n^2(n+1)^2(a-c)\varepsilon}{4(n+2)^2(n^2-n+1)} - \frac{(a-c)^2}{(n+2)^2}. \tag{37}$$

Next, consider the case when the AR policy $\langle r_A(n-1), n-1\rangle$ is the optimal one. Using similar notation, again from the proof of Proposition 3, we conclude that

$$\Delta^{AR}(n+1) = \frac{(a-c)^2 + 4n(a-c)\varepsilon + 4\varepsilon^2}{4(n+1)} - \frac{(a-c)^2}{(n+2)^2}. \tag{38}$$

Recall from Proposition 3 that the FR policy is the optimal policy for the incumbent innovator if $\varepsilon \geq (a-c)/h(n)$, and, the AR policy is the optimal one otherwise, where $h(n)$ is given by (31). Thus, for fixed $a > c > 0$, and for a given magnitude of ε, we can partition the set $\{2, 3, \ldots\} = N_A \cup N_F$, where N_A is the set of all integers n such that with industry size $n+1$, the AR policy $\langle n-1, r_A(n-1)\rangle$ is the optimal policy for the incumbent innovator, and N_F is the similar set for the FR policy $\langle n, r_F(n), f(r_F(n), n)\rangle$. Further, let

$$\bar{N}_A = \{n \in N_A | \Delta^{AR}(n+1) \geq \Delta^{AR}(m+1) \text{ for all } m \in N_A\}$$

and define the set $\bar{N}_F$ similarly for the FR policy. To determine $N^I(\varepsilon)$, we need to compare $\Delta(2)$, $\Delta^{AR}(n_A+1)$ and $\Delta^{FR}(n_F+1)$ for $n_A \in \bar{N}_A$ and $n_F \in \bar{N}_F$. To begin with, we show that when ε is sufficiently large, then $N^I(\varepsilon)$ is increasing in ε and $N^I(\varepsilon) \to \infty$ as $\varepsilon \to a-c$. Recall that $h(3) < h(n)$ for all $n \neq 3$, so that when $\varepsilon \geq (a-c)/h(3)$, then for all $n \geq 2$, the FR policy $\langle n, r_F(n), f(r_F(n), n)\rangle$ is the optimal policy. Hence for this case, it is enough to compare $\Delta^{FR}(n+1)$ with $\Delta(2)$. Denoting $x \equiv (a-c)/\varepsilon$, it can be verified that $\Delta(2) \geq \Delta^{FR}(n+1)$ iff $\zeta(x, n) \geq 0$, where

$$\zeta(x, n) = (n^3 - 5n^2 + 32n - 4)x^2 - (5n^3 + 11n^2 + 16n + 16)(2x - 1). \tag{39}$$

Noting that $x > 1$, we can consider $x \in (1, h(3)]$. It can be easily verified by replacing n by 3 in (39) that $\zeta(x, 3)$ is negative for all $x \in (1, h(3)]$, so that $\Delta(2) < \Delta^{FR}(4)$, implying that for this region, $N^I(\varepsilon) = n_F+1$ for $n_F \in \bar{N}_F$. To determine $\bar{N}_F$, we note that $\Delta^{FR}(n+2) \geq \Delta^{FR}(n+1)$ iff $\phi(x, n) \leq 0$, where

$$\phi(x, n) = (n^6 - 5n^5 - 18n^4 + 7n^3 + 27n^2 + 40n - 4)x^2 - \\ (n+1)^2(n^4 + n^3 - n^2 + 16n + 16)(2x - 1). \tag{40}$$

It is easy to check that $\phi(x,n) < 0$ for all $x > 1$ when $n \leq 7$, implying that $\Delta^{FR}(n+2) > \Delta^{FR}(n+1)$ for n such that $2 \leq n \leq 7$. Hence

$$\max_{n\in\{2,\ldots,7\}} \Delta^{FR}(n+1) = \Delta^{FR}(9). \tag{41}$$

For $n \geq 8$, it can be verified that $\phi(x,n)$ has two real roots, $\phi_1(n)$, and, $\phi_2(n)$, where $\phi_1(n) < 1 < \phi_2(n)$. Further, $\phi_2(n+1) < \phi_2(n)$ for all $n \geq 8$, $\phi_2(12) < h(3) < \phi_2(11)$, and

$$\lim_{n\to\infty} \phi_1(n) = \lim_{n\to\infty} \phi_2(n) = 1. \tag{42}$$

Also note that $\phi(x,n) \leq 0$ for $x \in [1, \phi_2(n)]$, and $\phi(x,n) > 0$ for $x > \phi_2(n)$. All these facts imply that for every $x \in [1, h(3)]$, there is an integer $n(x) \geq 12$ such that $\phi(x,n) \leq 0$ for $n \leq n(x)$ and $\phi(x,n) > 0$ for $n \geq n(x)+1$, so that $N^I(\varepsilon) = n(x)+2$. Since $\phi_2(n+1)$ is decreasing in n, it is concluded that $n(x)$ is decreasing in x. Due to (42), we further conclude that $n(x) \to \infty$ as $x \to 1$. Then, we have the following lemma.

Lemma A4.4. *$N^I(\varepsilon)$ is increasing in ε when $\varepsilon \in [(a-c)/h(3), a-c]$ and $N^I(\varepsilon) \to \infty$ as $\varepsilon \to a-c$.*

To complete the proof of Proposition 4, it remains to be shown that (i) there is a constant $y_0 > 1$ such that when $\varepsilon \leq (a-c)/y_0$, then $N^I(\varepsilon) = 2$ and (ii) $N^I(\varepsilon)$ is increasing in ε for $\varepsilon < (a-c)/h(3)$. To prove (i), we note that $\Delta(2) \geq \Delta^{AR}(n+1)$ iff $\rho(x,n) \geq 0$, where

$$\rho(x,n) = (n^3-4n^2+8n+4)x^2 - 2(5n-13)(n+2)^2x + (5n-31)(n+2)^2. \tag{43}$$

It can be verified that $\rho(x,n)$ has two real roots $\rho_1(n) < 1 < \rho_2(n)$, and $\rho(x,n) > 0$ for all $x > \rho_2(n)$. Further, the sequence $\{\rho_2(n)\}$ is bounded, since $\lim_{n\to\infty} \rho_2(n) = 5 + 2\sqrt{5}$. All these facts imply that there is a positive number y_1 such that for every $n \geq 2$, $\rho(x,n) > 0$ for all $x > y_1$. This implies that when $x > y_1$, then $\Delta(2) \geq \Delta^{AR}(n+1)$ for all $n \geq 2$. Next, recall from (39) that $\Delta(2) \geq \Delta^{FR}(n+1)$ iff $\zeta(x,n) \geq 0$. In view of (41), we can consider $n \geq 8$. It can be verified that $\zeta(x,n)$ has two real roots, $\zeta_1(n) < 1 < \zeta_2(n)$ and $\lim_{n\to\infty} \zeta_2(n) = 5 + 2\sqrt{5}$. Hence we conclude that there is a positive number y_1 such that when $x > y_1$, then $\Delta(2) \geq \Delta^{FR}(n+1)$ for all $n \geq 2$. Thus, there is a sufficiently large positive number y_0 such that when $x > y_0$, then $\Delta(2)$ is more than both $\Delta^{AR}(n+1)$ and $\Delta^{FR}(n+1)$ for all $n \geq 2$, which implies the following lemma.

Lemma A4.5. *There is positive number y_0 such that when $\varepsilon < (a-c)/y_0$, then $N^I(\varepsilon) = 2$.*

In view of Lemmas A4.4 and A4.5, it only remains to be shown that $N^I(\varepsilon)$ is increasing in ε when $\varepsilon[(a-c)/h(3), (a-c)/y_0]$. It can be verified that $\Delta^{AR}(n+2) \geq \Delta^{AR}(n+1)$ iff $\sigma(x,n) \leq 0$, where

$$\sigma(x,n) = (n^3 - 7n - 2)x^2 - 4(n+2)(n+3)^2(x-1). \tag{44}$$

From (39), (40), (43) and (44), it follows that the functions that determine the sets $\bar{N}_A$ and $\bar{N}_F$ are all quadratic in x. The following lemma completes the proof of Proposition 4. The proof of this lemma long and tedious, but standard in that it relies on the basic properties of the quadratic functions encountered. The proof is available from the authors by request.

Lemma A4.6. *There are constants $h(3) < y_4 < y_3 < y_2 < y_1 < y_0$ such that*

$$N^I(\varepsilon) = \begin{cases} 2 & \textit{when } \varepsilon \leq (a-c)/y_0 \\ 7 & \textit{when } \varepsilon \in [(a-c)/y_0, (a-c)/y_1] \\ 8 & \textit{when } \varepsilon \in [(a-c)/y_1, (a-c)/y_2] \\ 10 & \textit{when } \varepsilon \in [(a-c)/y_2, (a-c)/y_3] \\ 11 & \textit{when } \varepsilon \in [(a-c)/y_3, (a-c)/y_4] \\ 12 & \textit{when } \varepsilon \in [(a-c)/y_4, (a-c)/h(3)] \end{cases} \tag{45}$$

Further, $\lim_{\varepsilon\to(a-c)/h(3)-} N^I(\varepsilon) < \lim_{\varepsilon\to(a-c)/h(3)+} N^I(\varepsilon)$.

Then, Proposition 4 in case of an incumbent innovator follows from Lemmas A4.4-A4.6. ■

Proof of Proposition 5. To prove this proposition, we consider the following two cases.

Case 1. $\varepsilon \geq (a-c)/2$. Note from Conclusion O3 of the proof of Proposition 2 that when $n \leq 6$, then an outsider innovator sells an exclusive license and earns $(a-c)\varepsilon$. For $n \geq 7$, there is $q(n) > 2$ such that the same is true if $\varepsilon \geq (a-c)/q(n)$. When $(a-c)/2 \leq \varepsilon \leq (a-c)/q(n)$, then the optimal policy for an outsider innovator is the FR policy, and the payoff is given by

$$\Pi^O_{FR}(n, \bar{r}(n)) = \frac{n[(n-1)^2(a-c)^2 + 2(n+1)(2n^2-n+1)(a-c)\varepsilon + (n+1)^2\varepsilon^2]}{4(n+1)^2(n^2-n+1)}.$$

Since $h(n) > 2$, from Proposition 3 we conclude that in an oligopoly with $n+1$ firms, the payoff of an incumbent innovator is $\Pi^I_{FR}(n, r_F(n))$ when $\varepsilon \geq (a-c)/2$, where

$$\Pi^I_{FR}(n, r_F(n)) = \frac{(n^3+4)(a-c+\varepsilon)^2 + 4n^2(n+1)^2(a-c)\varepsilon}{4(n+2)^2(n^2-n+1)}. \tag{46}$$

It can be easily verified that $\Pi^I_{FR}(n, r_F(n))$ is more than both $(a-c)\varepsilon$ and $\Pi^O_{FR}(n, \bar{r}(n))$. Hence, an outsider innovator earns more payoff if she enters the industry instead of being an outsider. Thus, when $\varepsilon \geq (a-c)/2$, then an outsider innovator enters the industry and sells the license to all firms.

Case 2. $\varepsilon \leq (a-c)/2$. Observe that the payoff of an outsider innovator from the AR policy $\langle n-1, \tilde{r}(n-1)\rangle$ in an oligopoly of n firms is the same as the payoff of an incumbent innovator from the AR policy $\langle n-1, r_A(n-1)\rangle$ in an oligopoly of $n+1$ firms. Indeed,

$$\Pi^O_{AR}(n-1, \tilde{r}(n-1)) = \Pi^I_{AR}(n-1, r_A(n-1)) = \frac{(a-c)^2 + 4n(a-c)\varepsilon + 4\varepsilon^2}{4(n+1)}. \tag{47}$$

We consider the following two subcases.

Subcase (a). $n \leq 5$. Recall from Conclusion O3 of the proof of Proposition 2 that the payoff of the outsider innovator is given by $\Pi^O_{AR}(n-1, \tilde{r}(n-1)$. From Proposition 3, it follows that when $(a-c)/h(n) < \varepsilon \leq (a-c)/2$, the payoff of an incumbent innovator is $\Pi^I_{FR}(n, r_F(n))$, given by (46) and $\Pi^I_{FR}(n, r_F(n)) > \Pi^I_{AR}(n-1, r_A(n-1))$, implying that $\Pi^I_{FR}(n, r_F(n)) > \Pi^I_{AR}(n-1, \tilde{r}(n-1))$. Thus, for this case, an outsider innovator will enter the industry when $(a-c)/h(n) < \varepsilon \leq (a-c)/2$. For $\varepsilon \leq (a-c)/h(n)$, the payoff of an outsider innovator is $\Pi^O_{AR}(n-1, \tilde{r}(n-1))$ while that of an incumbent innovator is $\Pi^I_{AR}(n-1, r_A(n-1))$. Then, from (47), we conclude that an outsider innovator is indifferent between entering the industry or otherwise. However, taking the negligible but positive cost of entry into account, she will not enter.

Subcase (b). $n \geq 6$. For this case, the payoff of an outsider innovator is $\Pi^I_{FR}(n, \bar{r}(n)$ when $\varepsilon \in [(a-c)/x(n), (a-c)/y(n)]$, and it is $\Pi^I_{AR}(n-1, \tilde{r}(n-1)$ otherwise. It can be verified that $h(n) > x(n)$ for all $n \geq 6$. Thus, for $\varepsilon > (a-c)/h(n)$, depending on the interval where ε lies, the payoff of an outsider innovator is either $\Pi^I_{FR}(n, \bar{r}(n)$, or $\Pi^I_{AR}(n-1, \tilde{r}(n-1)$, and we have already shown that both of these are less than $\Pi^I_{FR}(n, r_F(n))$, so that for this case, an outsider innovator will enter the industry. When $\varepsilon \geq (a-c)/h(n)$, then from (47), it follows that the innovator is indifferent between entering or otherwise, and taking the negligible but positive cost of entry into account, we conclude that she will remain outside. This completes the proof of this proposition. ■

Proof of Lemma 1. We prove Lemma 1 for the following two cases.

An outsider innovator. For this case, Lemma 1 is proved by showing that for $1 \leq m \leq n-1$ and for every r, there exists an r' such that $\Pi_{FR}(m, r) \leq \Pi_{AR}(m, r')$. Note that

$$\Pi_{FR}(m, r) - \Pi_{AR}(m, r) = m[q_N(m, r)]^2 - m[q_N(m-1, r)]^2,$$

so that $\Pi_{FR}(m,r) \leq \Pi_{AR}(m,r)$ iff $q_N(m,r) \leq q_N(m-1,r)$. When $r \leq \beta_1(m)$, from Lemma A2.2, it follows that $q_N(m,r) = 0 \leq q_N(m-1,r)$, so that $\Pi_{FR}(m,r) \leq \Pi_{AR}(m,r)$ for every r for this case. When $r \geq \beta_2(m)$, then from Lemma A2.4, it follows that the payoff at any FR policy is at most zero. When $r \in [\beta_2(m-1), \beta_2(m)]$, then the maximum of $\Pi_{FR}(m,r)$ is attained at $r = \beta_2(m-1)$. Thus, to prove the lemma, it is enough to consider $r \in [\beta_1(m), \beta_2(m-1)]$. In what follows, we show that for this case

$$\max_r \Pi_{FR}(m,r) = \Pi_{FR}(m,r^*) \leq \Pi_{AR}(m,r^*). \tag{48}$$

For this case, from Lemma A2.1, it can be seen that

$$q_N(m-1,r) - q_N(m,r) = \frac{\varepsilon - r}{n+1} \geq 0 \Leftrightarrow r \leq \varepsilon.$$

To prove (48), thus, it is enough to show that $r^* \leq \varepsilon$. It can be easily verified that $\Pi_{FR}(m,r)$ is decreasing at $r = \varepsilon$. Noting that $\Pi_{FR}(m,r)$ is quadratic in r and $\varepsilon > \beta_1(m)$, it is concluded that when $r \in [\beta_1(m), \beta_2(m-1)]$, the maximum of $\Pi_{FR}(m,r)$ is attained at some $r^* \leq \varepsilon$. This completes the proof of Lemma 1 in case of an outsider innovator.

An incumbent innovator. As in the previous case, the lemma is proved for this case by showing that for every r, there is an r' such that $\Pi_{FR}(m,r) \leq \Pi_{AR}(m,r')$. For $r \leq \theta_1(m)$, from Lemma A3.4, it follows that the payoff of the innovator from any FR policy is at most zero, so that one can consider $r \geq \theta_1(m)$. Note that $\Pi_{FR}(m,r) \leq \Pi_{AR}(m,r)$ iff $q_N(m,r) \leq q_N(m-1,r)$. For $r \in [\theta_1(m), \theta_2(m)]$, from Lemma A3.2, it follows that $q_N(m,r) = 0 \leq q_N(m-1,r)$. For $r \geq \theta_3(m)$, $\Pi_{FR}(m,r)$ is maximized at $r = \theta_3(m)$, while for $r \in [\theta_3(m-1), \theta_3(m)]$, it is maximized at $r = \theta_3(m-1)$. Thus, to prove Lemma 1, it is enough to consider $r \in [\theta_2(m), \theta_3(m-1)]$. For this case, we show that

$$\max_r \Pi_{FR}(m,r) = \Pi_{FR}(m,r^*) \leq \Pi_{AR}(m,r^*). \tag{49}$$

It can be verified from Lemma A3.2 that

$$q_N(m-1,r) - q_N(m,r) = \frac{\varepsilon - r}{n+2} \geq 0 \Leftrightarrow r \leq \varepsilon.$$

To prove (49), thus, it is enough to show that $r^* \leq \varepsilon$. It can be verified that $\Pi_{FR}(m,r)$ is strictly decreasing at $r = \varepsilon$. Noting that $\Pi_{FR}(m,r)$ is quadratic in r, and $\varepsilon > \theta_2(m)$, we conclude that for $r \in [\theta_2(m), \theta_3(m-1)]$, the maximum of $\Pi_{FR}(m,r)$ is attained at some $r^* < \varepsilon$. This completes the proof of Lemma 1 in case of an incumbent innovator. ■

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