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A METHOD TO GENERATE STRUCTURAL IMPULSE-RESPONSES FOR MEASURING THE EFFECTS OF SHOCKS IN STRUCTURAL MACRO MODELS

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In 2006 all ECB publications will feature a motif taken from the €5 banknote.

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Abstract

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Non Technical Summary

We develop a technique for analyzing the response dynamics of economic variables to structural shocks in linear rational expectations models. Our work differs from standard SVARs since we allow expectations of future variables to enter structural equations. We show how to estimate the variance-covariance matrix of fundamental and non-fundamental shocks and we construct point estimates and confidence bounds for impulse response functions. Our technique can handle both determinate and indeterminate equilibria. We provide an application to U.S. monetary policy under pre and post Volcker monetary policy rules.

1 Introduction

This paper introduces a technique for analyzing the dynamic effects of structural shocks in linear rational expectations models. It is analogous to the method used to construct impulse response functions in structural VARs. However, our suggested technique differs from identification schemes commonly used in the SVAR literature because the models we study contain future expectations as explanatory variables. Typically, equations of this kind arise as linearized equilibrium or first-order conditions in representative agent dynamic stochastic general equilibrium (DSGE) models. If the model is sufficiently identified, the parameters of these equations will be truly structural and the impulse responses that we compute may be used in comparative exercises to assess the effects of shocks across different regimes.

We demonstrate our method along an empirical application to U.S. monetary policy that is closely related to the large literature, surveyed by Christiano, Eichenbaum and Evans [12], on the use of structural VAR's to estimate the monetary transmission mechanism. Prominent contributions to this literature include e.g. papers by Bernanke and Mihov [3] and Christiano et. al. [13].[1] Bernanke and Mihov estimate models of regime changes and find that the best indicator of monetary policy stance differs across regimes. Our own analysis complements their approach. Although we consider only a single indicator of monetary policy, the fed funds rate, our estimates are potentially of more use to the policy maker since, by including expectations as explanatory variables, we claim to identify parameters of the "deep structure" that are, conceptually, invariant to changes in the policy rule. This issue was first pointed to by Keating [22] who argued that most standard SVAR identifica-

[1] See also the debate between Chari, Kehoe and McGratten [11] and Christiano, Eichenbaum and Vigfussen [14], [15] over the effects of a productivity shock in an SVAR and the paper by Ferndandez-Villaverde, Rubio-Ramirez and Sargent [18].

tion schemes will yield inconsistent estimates since they confound structural parameters with expectational effects. Our estimates are not subject to the "Keating critique" since we explicitly account for expectations in our structural estimates.

One important issue that arises in our work is the possibility that structural parameter estimates may be associated with either a determinate or an indeterminate model. For some points in the parameter space, a structural linear rational expectations model has a unique determinate equilibrium that is driven solely by shocks to preferences, endowments and technology. But for other points in the parameter space there may exist multiple indeterminate equilibria and in this case non-fundamental or 'sunspot' shocks may also play a role such that there would be room for belief shocks, due to the self-fulfilling revisions of expectations, to add additional components to the variances of economic variables of interest. We confront this possibility by presenting a method for estimating the variance-covariance matrix of structural shocks that may be applied in both the determinate and the indeterminate case. From that perspective we find the SVAR approach to identification of structural shocks problematic since, if the nature of the shocks driving the economy can change when economic policy changes, fixed identification schemes in an SVAR cannot be considered "structural".

To illustrate our method, we apply it to a New-Keynesian model estimated using U.S. data before and after 1979. We chose this data set since Clarida et. al. ([16], CGG) have argued that U.S. data in the period from 1960 through 1979 is well characterized by an indeterminate equilibrium but that after 1979 the characteristics of the data changed and the period since that date is described by a single determinate rational expectations equilibrium. CGG suggested that the observed reduction in volatility in inflation, unemployment and the output gap, following 1979, may be attributed to the implementation of a monetary policy that caused a switch from an indeterminate to a determinate equilibrium. Those findings, which have been substantiated by Lubik-Schorfheide (LS) [24] and Boivin-Giannoni [9], suggest that there may be a high payoff to a method that can disentangle the direct effects of fundamental impulses to the structural equations with their indirect effects on expectations.

Our work is most closely related to the paper by LS [24] who present a Bayesian maximum likelihood approach to compute posterior odds ratios for determinate versus indeterminate regions of the parameter space of a DSGE model. In contrast, we use a classical method and suggest a system GMM estimator which has the advantage that the researcher needs only to specify certain moment conditions rather than the full density of the errors. As a practical matter, our method is relatively easy to implement in particular in larger models and we provide MATLAB code that is easily transportable to a range of environments. A corresponding disadvantages of our method, if the true structural model is known, is that since GMM is a limited information estimator it is less efficient than full information maximum likelihood. Moreover, GMM estimates may suffer from weak instruments, but that can be tested and can, in practice, often be avoided.

Our method involves four steps. First, estimate a linear rational expectations model. For reasons that we motivate further in Section 2, we prefer a limited information method such as system-GMM. Second, compute the reduced form of the model by applying a complex Schur decomposition to the structural parameter estimates. Third, construct an estimate of the variance-covariance matrix of the fundamental and non-fundamental shocks. Fourth, order the fundamental and non-fundamental shocks and compute impulse responses to the reduced form using a Choleski decomposition. We provide MATLAB code to implement our method.[2]

[2] http://farmer.sscnet.ucla.edu

The remainder of this paper is organized as follows. Section 2 outlines briefly the class of linear rational expectation models we deal with and motivates a suitable method for model estimation. Section 3 discusses the meaning of structural shocks within the framework of linear rational expectations models. Section 4 presents an application of our Impulse-Response method to a standard New-Keynesian model and the final Section concludes. Appendices A and B present details about our solution algorithm and Appendix C contains a consistency proof for our proposed estimator of the variance-covariance matrix.

2 Estimating a structural linear rational expectations model

In this section we discuss the use of GMM to obtain consistent estimates of the parameters of A, F, B and C in the structural model:

$$AY_t + FE_t\left[Y_{t+1}\right] = BY_{t-1} + C + V_t, \tag{1}$$

$$E_t\left[V_t V_s'\right] = \begin{cases} \Omega_{vv}, & t = s, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

In this notation A, F, and B are $n \times n$ matrices of coefficients, C is an $n \times 1$ vector of constants, E_t is a conditional expectations operator, Y_t is an n-dimensional vector of endogenous variables, and $\{V_t\}$ is a weakly stationary i.i.d. stochastic process with covariance matrix Ω_{vv} and mean zero.[3] We

[3] Without loss of generality, we focus on the case of one lead and one lag. Our method can easily be expanded to include additional lags or leads.

maintain the convention that coefficients of endogenous variables appear on the left side of each equation with positive signs and explanatory variables appear on the right side of equations with positive signs.

The first issue we face is that of identification. Each of the n equations in (1) contains $2n$ endogenous variables since the expectations terms $E_t(Y_{t+1})$ are endogenous variables to be determined at date t. Application of order and rank conditions should be checked for the entire system, but identification does not pose insurmountable complications over standard structural models. In Beyer and Farmer [6] we present an algorithm, implemented in MATLAB, that finds equivalence classes of exactly identified models. This algorithm can easily be adapted to check for system-wide identification in a DSGE model. The main complication introduced by the presence of expectations variables is that the validity of the instruments will depend on the degree of determinacy of the solution. (See Pesaran [26]).

In order to estimate the parameters of model (1), we propose an estimator based on a system GMM approach. This method, originally suggested by McCallum [25], replaces unobserved expectations $E_t(Y_{t+1})$ by their realizations Y_{t+1} and rewrites Equation (1) as a linear model that includes future values of the observed endogenous variables with moving average error terms:

$$AY_t + FY_{t+1} = BY_{t-1} + C + \Psi_v V_t + \Psi_w W_{t+1}. \tag{3}$$

The vector W_{t+1} represents one-step-ahead forecast errors. Let the joint variance-covariance matrix of forecast errors W_t and fundamental shocks V_t be:

$$\Omega = E[V_t, W_t][V_t, W_t]'.$$

When the model has a unique rational expectations equilibrium the non-fundamental errors W_t will be exact functions of the fundamental shocks V_t.

In this case the $2n \times 2n$ covariance matrix:

$$\Omega = \begin{bmatrix} \Omega_{vv} & \Omega_{vw} \\ \Omega_{wv} & \Omega_{ww} \end{bmatrix}, \tag{4}$$

has rank n. When the model has an indeterminate equilibrium of degree r, the variance-covariance matrix Ω has rank $n + r > n$. In this case one can pick a particular rational expectations equilibrium by imposing the assumption that the elements of Ω_{ww} and Ω_{wv} are time invariant. In either case, estimation of Equation (3) must take account of the fact that the errors have an MA(1) structure. This is taken care of in GMM by estimating (3) using a heteroskedastic-autocorrelation-consistent (HAC) estimator for the optimal weighting matrix.

A prominent alternative approach to estimate the parameters of (1) is full information maximum likelihood (FIML). This approach is discussed in Anderson *et al* [1] and has been implemented in models similar to ours by, amongst others, Lindé [21], and LS [23]. We chose not to use FIML since it requires the econometrician to take a prior stand on the determinacy properties of the equilibrium. To construct the likelihood function one must be prepared to specify the joint probability distribution of the errors and to make assumptions about the covariance matrix Ω. The rank of Ω can change across regions of the parameter space, depending on whether the rational expectation equilibrium is determinate or indeterminate. In case of indeterminacy the rank of Ω depends also on the degree of indeterminacy. A likelihood based estimation technique requires the econometrician to estimate a different theoretical model for every such region of the parameter space. Although it is possible to construct a piecewise likelihood function to tackle this issue, as in LS [24], in practice their approach is not particularly easy to implement and has been applied only in simple examples. When applying Bayesian estimation techniques the researcher faces a couple of decision problems given that different types of equilibria might require different identification schemes. He might be forced to decide either on the choice of different priors corresponding to different characteristics of the equilibria. Alternatively, he might chose flat priors for a set of chosen parameters. In any case a rather firm stand on the choice of the priors and a firm view on which parameters to be identified is required. This might lead to serious identification problems as pointed out by Canova and Sala [10] in a recent study. In that respect the advantages of a system GMM estimator are obvious: the moment conditions that define the estimation procedure do not depend on the rank of Ω, hence the same estimator can be used for both determinate and indeterminate models. As a consequence, GMM does not require to break up the parameter space according to different types of equilibria such that different identification schemes for different types of equilibria can be avoided. Nevertheless, as it is well known, a potential disadvantage of GMM is that the estimator may suffer from weak instruments. There are, however, methods available that test for weak instruments and, in practice, help avoiding them to be used for estimation (see e.g. Stock *et al.* [31]).

3 Accounting for Shocks

In this section we discuss the problem of disentangling the dynamic effects of different kinds of shocks. This problem involves first, estimating Ω, the variance-covariance matrix of the fundamental and non-fundamental shocks and second, attributing the effects of these shocks to the reduced form equations. When equilibria are indeterminate the impact effects of alternative shocks must be attributed to fundamental and non-fundamental sources.

We begin by constructing an estimator of Ω using a two-step approach. First, we obtain estimates of the structural parameters $\hat{A}, \hat{F}, \hat{B}, \hat{C}, \hat{\Psi}_v$ and

$\hat{\Psi}_w$ in (3) by GMM; second, we use these parameter estimates to construct reduced form residuals from which we estimate Ω. This two-step procedure involves a complication which we discuss in the following section. It arises from the fact that the reduced form of the model obtained by standard solution algorithms will not generally be free of unobserved expectations.[4]

3.1 Finding an Observable Reduced Form

To compute the reduced form of our structural model we define a vector:

$$X_t = \begin{bmatrix} Y_t \\ E_t\left[Y_{t+1}\right] \end{bmatrix},$$

which consists of observable variables Y_t and (possibly unobserved) expectations variables $E_t\left[Y_{t+1}\right]$. Our procedure for computing the reduced form of the model uses an algorithm, `SysSolve,`which returns a VAR(1) in this augmented state vector. When the equilibrium is determinate, the system can be broken down into two separate subsystems. One is a VAR(1) in the observable variables Y_t and the other is a static function that determines $E_t\left[Y_{t+1}\right]$ as a function of Y_t. When the equilibrium is indeterminate, however, it is not generally possible to carry out this decomposition. The following example, taken from Beyer-Farmer [5], illustrates this problem for a one variable model and proposes a solution that can be generalized to the case of n variables.

Consider the single equation model

$$p_t = \frac{1}{\alpha} E_t\left[p_{t+1}\right] + v_t,$$

where p_t is observable and v_t is a fundamental error. This model can be

[4] We compute the reduced form of the model with a QZ decomposition. Our algorithm, `SysSolve`, is described in Appendix A. It is based on code by Sims [29] as amended by LS [23] to account for the possibility that there may be multiple indeterminate solutions.

written as follows,

$$\begin{bmatrix} 1 & -\frac{1}{\alpha} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_t \\ E_t\left[p_{t+1}\right] \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{t-1} \\ E_{t-1}\left[p_t\right] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_t.$$

When $|\alpha| < 1$, the solution is indeterminate and `SysSolve` returns the solution

$$p_t = E_{t-1}\left[p_t\right] + w_t, \tag{5}$$

$$E_t\left[p_{t+1}\right] = \alpha E_{t-1}\left[p_{t-1}\right] - \alpha v_t + \alpha w_t. \tag{6}$$

Although this solution is valid, both equations contain unobservable expectations and for some purposes it might be helpful to have an alternative dynamic representation that involved a single stochastic difference equation in the observable variable p_t. In this example one can find such a representation by rearranging Equation (5) to find $E_{t-1}\left[p_t\right]$ as a function of p_t and w_t and substituting this solution at dates $t-1$ and $t-2$ into Equation (6). This process leads to the expression

$$p_t = \alpha p_{t-1} - \alpha v_{t-1} + w_t. \tag{7}$$

Equation (7) is a VARMA$(1,1)$ in the observable variable p_t and the vector of shocks $(v_t, w_t)'$.

The coexistence of VAR and VARMA representations, exhibited in this example, carries over to more general DSGE models when the solution is indeterminate. The QZ solution method suggested by Sims and implemented in `SysSolve`, leads to a reduced form expression of the form

$$X_t = \Gamma^* X_{t-1} + C^* + \Psi_V^* V_t + \Psi_W^* W_t.$$

Since Γ^* is generally singular, there will be more than one way to partition X_t into two subsets (X_t^1, X_t^2) such that X_t^1 forms an autonomous VARMA$(1,k)$ model that is independent of X_t^2 and k is the number of zero eigenvalues of

Γ^*. One or more of these representations will be in terms of the observable variables Y_t and Y_{t-1}, but these observable representations will not generally reduce to a VAR(1). An exception, is the case of a determinate equilibrium when the solution is unique and, in this case, the rank of Γ^* equals n. In Appendix B, we provide an algorithm to generalize the above example to the case of an $n-$dimensional DSGE and we provide MATLAB code, `Arrange`, that implements our algorithm by rearranging the output from the QZ decomposition provided by `SysSolve`.

3.2 Computing an Estimate of Ω

In this section we provide a method to recover consistent estimates of the population variance-covariance matrix Ω. First, we write the reduced form as a VARMA$(1,k)$ in the observable variables Y_{1t} and the unobserved shocks $\eta_t = (V_t, W_t^1)'$,

$$Y_{1t} = \Gamma_1^* Y_{1t-1} + C_1^* + \sum_{j=0}^{k+1} \Psi_j^* \eta_{t-j}. \tag{8}$$

We assume that the econometrician can obtain consistent estimates of the population parameters Γ_1^*, C_1^*, and Ψ_j^* which we refer to as $\hat{\Gamma}_1^*, \hat{C}_1^*$, and $\hat{\Psi}_j^*$.

Let e_t be a vector of sample residuals defined as follows;

$$e_t = Y_{1t} - \hat{\Gamma}_1^* Y_{1t-1} - \hat{C}_1^*, \tag{9}$$

where Y_{1t} are observable variables and $\hat{\Gamma}_1^*$ and $\hat{C}_1^*$ are consistent estimates of the parameters of the VARMA$(1,k)$ representation of the reduced form. Define the sample autocorrelations $\hat{S}_0$ and $\hat{S}_j$ as follows;

$$\hat{S}_0 = \frac{1}{T}\sum_{t=2}^{T}\left(Y_{1t} - \hat{\Gamma}_1^* Y_{1t-1} - \hat{C}_1^*\right)\left(Y_{1t} - \hat{\Gamma}_1^* Y_{1t-1} - \hat{C}_1^*\right)', \tag{10}$$

$$\hat{S}_j = \frac{1}{T}\sum_{t=j+2}^{T}\left(Y_{1t} - \hat{\Gamma}_1^* Y_{1t-1} - \hat{C}_1^*\right)\left(Y_{1t-j} - \hat{\Gamma}_1^* Y_{1t-j-1} - \hat{C}_1^*\right)',\ j = 1, ...k. \tag{11}$$

In Appendix C, we show that one can obtain consistent estimates of the elements of Ω by finding a solution to the equations

$$\underset{n\times n}{\hat{S}} = \underset{n\times(n+r)}{\hat{\Psi}} \hat{\Omega} \underset{(n+r)\times n}{\hat{\Psi}'}, \tag{12}$$

where

$$\hat{S} = \hat{S}_0 + \sum_{j=1}^{k} \left(\hat{S}_j + \hat{S}_j'\right), \tag{13}$$

$$\hat{\Psi} = \sum_{j=0}^{k} \hat{\Psi}_j^*, \tag{14}$$

and $\hat{\Psi}_j^*$ are consistent estimates of Ψ_j^* for $j = 1, ...k$.

It is important to notice that Equation (12) cannot be solved uniquely for the elements of $\hat{\Omega}$ since it consists of $n(n+1)/2$ independent equations in $(n+r)(n+r+1)/2$ unknowns.[5] The non-uniqueness of the solution to Equation (12) means that, when equilibrium is indeterminate, the econometrician cannot distinguish between fundamental and non-fundamental disturbances to the economy.

For the purposes of examining the dynamic properties of the model, the inability to distinguish between determinate and indeterminate shocks is not a problem as long as the variance-covariance matrix Ω remains time invariant - it is simply a question of how we choose to name the observed disturbances to each equation.[6] For the purposes of constructing impulse response

[5] As an example, consider the case when there are two equations and one degree of indeterminacy. In this case $\hat{S}$ is a known symmetric 2×2 matrix and $\hat{\Psi}$ is a known 2×3 matrix both of which are functions of the data. For this example, Equation (12) consists of 4 equations in 9 unknowns. Since $\hat{S}$ is symmetric only 3 of these equations are independent and since $\hat{\Omega}$ is symmetric only 6 elements of Ω need to be independently calculated. Only three linear combinations of the variance-covariance parameters Ω are identified from the data.

[6] The question becomes more interesting if we observe data from different regimes since then one might ascribe a change in the observed variance of the data to the additional contribution of sunspots as suggested by CGG [16].

functions in the New-Keynesian model that we describe below, we chose to ascribe all shocks in the indeterminate regime to fundamentals by setting the elements of Ω_{ww} and Ω_{wv} to zero.

4 Application to the New-Keynesian Model

In this section we describe a New-Keynesian model that puts together simplified versions of specifications of the representative agent's Euler equation by Fuhrer and Rudebusch [19], the Phillips curve by Galí-Gertler [20], and the Central Bank reaction function by CGG [16].

4.1 A Description of the Model

The model we estimate consists of the following three equations.

$$y_t = \alpha_0 + \alpha_1 E_t\left[y_{t+1}\right] + \alpha_2(i_t - E_t\left[\pi_{t+1}\right]) + \alpha_3 y_{t-1} + v_t^1, \tag{15}$$

$$\pi_t = \beta_0 + \beta_1 E_t\left[\pi_{t+1}\right] + \beta_2 y_t + \beta_3 \pi_{t-1} + v_t^2, \tag{16}$$

$$i_t = \gamma_0 + \gamma_1\left(1-\gamma_3\right) E_t\left[\pi_{t+1}\right] + \gamma_2\left(1-\gamma_3\right) y_t + \gamma_3 i_{t-1} + v_t^3. \tag{17}$$

The variable y_t is a measure of the output gap, we used the same one-sided HP-filtered series as in Beyer et. al., (BFHM [8]), π_t is the GDP deflator, i_t is the Federal Funds rate and E_t is again a conditional expectations operator. Equation (15) is an output equation derived from the representative agent's Euler equation, Equation (16) is a hybrid New-Keynesian Phillips curve, and Equation (17) is a Central Bank reaction function, (also referred to as a Taylor rule after the work of Taylor [30]).

4.2 Parameter Estimates

In this section we report the results of estimating Equations (15)–(17) by GMM on the full system. In Beyer-Farmer [6] we show that parameters associated with the unstable roots of DSGE models are typically not identified in the absence of additional restrictions. Table 1 reports these estimates using two lags of the endogenous variables as instruments.[7] The reduced form of our model contains one lag if the solution is determinate. In the case of an indeterminate solution one or more second lags of the variables may appear as additional explanatory variables. Since we do not take a prior stand on whether the solution is determinate or indeterminate we included two lags as instruments in our GMM estimation.

Since there is evidence of parameter instability across the full sample, particularly in the policy rule, we split the data in 1979. This follows the lead of CGG [16], who suggest that the rule followed in the pre-Volcker period (1960:4–1979:3), has very different properties from that during the Volcker-Greenspan years. We discarded the quarters 1979:4–1982:4 since this was a period of considerable instability in which the Fed followed a money targeting rule that was quickly abandoned. Our second sub-sample consists of the years 1983:1–1999:3.

[7] Beyer et. al. [8] estimate this model on the same data set that we use here. They report results from a number of alternative estimation methods and show how to obtain more efficient parameter estimates using factors as instruments. The reader is referred to their work for a more complete description of the robustness properties of the system GMM estimator and for a discussion of parameter stability across different subsamples.

TABLE 1:		GMM PARAMETER ESTIMATES UNDER RESTRICTIONS							
		Sample 60:4 79:3				Sample 83:1 99:3			
Eqn	**Param**	**Coeff**	**Std Err**	**t-stat**	**p-val**	**Coeff**	**Std Err**	**t-stat**	**p-val**
Euler Eq.	$gap_{t+1}(\alpha 1)$	0. 503***	0. 025	19.69	0.00	0. 487***	0. 030	15.80	0.00
	$ri_t(\alpha 2)$	*-0.02*	n.a.			*-0.02*	n.a.		
	$gap_{t-1}(\alpha 3)$	0.514***	0. 023	19.68	0.00	0. 516***	0. 026	19.81	0.00
Phil. Curve	$\pi_{t+1}(\beta 1)$	0. 618***	0.056	11.00	0.00	0.616***	0. 081	7.52	0.00
	$gap_t(\beta 2)$	*0.025*	n.a.			*0.025*	n.a.		
	$\pi_{t-1}(\beta 3)$	0. 366***	0.058	6.30	0.00	0. 331**	0. 051	6.42	0.00
Pol. Rule	$\pi_{t+1}(\gamma 1)$	0.789***	0. 189	4.17	0.00	1.794***	0. 598	3.00	0. 002
	$gap_t(\gamma 2)$	0.759**	0.316	2.40	0.016	0. 294	0. 184	1.59	0. 11
	$i_{t-1}(\gamma 3)$	0. 867***	0.046	18.72	0.00	0. 877***	0. 047	18.40	0.00
		J-stat = 10.98 **p-val** = 0.94				**J-stat** = 10.31 **p-val** = 0.96			

* (**) (***) denotes significance at 10% (5%) (1%) level

Table 1 is divided into three sections, one for each equation of the New-Keynesian model. The table is further divided into two halves reporting estimates, in the left panel, for the sub-sample from 1960:4–1979:3 and in the right panel, for the sub-sample 1983:1–1999:3. For each sub-sample we were able to fit a tightly parameterized model; the equality and exclusion restrictions that we imposed to achieve identification passed Hansen's $J-$test with $p-$values of 94% and 96% for the two samples. Further, as reported in BFHM [8], the residuals for this model are consistent with the model assumptions. After removing an MA(1) component, as predicted by theory, BFHM report that the residuals passed a range of mis-specification tests including absence of ARCH effects, absence of additional serial correlation and the Jarque-Bera test for normality.

To identify the forward dynamics associated with the Euler equation, we restricted the interest rate coefficient α_2. In single agent DSGE models this parameter is obtained from linearization of a representative agent's marginal utility of consumption. Theory suggests that the absolute value of α_2 in Equation (15) should be (approximately) in the range 0.01 to 0.1, the same order of magnitude as a measure of the real interest rate. We experimented with a number of values in this range with little qualitative difference from the results reported in Table 1 which contains parameter estimates by GMM for the case when a_2 is equal to -0.02.

To identify the forward dynamics associated with the Phillips curve we restricted the output-gap coefficient, β_2 in Equation (16). We experimented with values in the range 0 to 1 but our parameter estimates led to non-existence of stationary equilibrium for values much above 0.05. In Table 1 we report the results of GMM estimates in which we restrict $\beta_2 = 0.025$. CGG choose $\beta_2 = 1$ in their calibrated model and LS set a prior mean of $\beta_2 = 0.5$. Our value of β_2 is smaller than those used in earlier studies because we estimated a hybrid Phillips curve that includes lagged inflation as a right-hand-variable and we explicitly modeled the dynamics of the model instead of adding autocorrelated disturbance terms.

As in BFHM, we find that detrended output and inflation are well described by their own future and lagged values. Coefficients on future and lagged output in the Euler equation are tightly estimated and qualitatively similar across sub-periods. Our point estimate for α_1, (the estimated coefficient on future output), is equal to 0.503 in the first sub-period and 0.487 in the second and both coefficients are significant at the 1% level using HAC standard errors. The coefficient on lagged output, a_3, is estimated as 0.514 and 0.516 in the two sub-samples and are also highly significant. The coefficients on future and lagged inflation, β_1 and β_3, are equal to 0.618 and 0.366 in the first sub-period and 0.616 and 0.331 in the second sub-period. These parameter estimates are remarkably similar across the two regimes and they provide strong support to the CGG interpretation that the change in the reduced form coefficients in 1979 can be attributed solely to a change in the Fed policy rule.[8]

Our estimates of the policy rule are similar to those reported by CGG. Like CGG, we find that the estimated coefficient on future inflation in the policy rule, γ_1, switches from 0.79 in the pre-Volcker period to 1.79 in the Volcker-Greenspan years.[9] This is an important coefficient since, when the parameters of the Phillips curve and the Euler equation are calibrated to values suggested by economic theory, γ_1 regulates the determinacy of equilibrium. If γ_1 is less than one, the Fed responds to expected future inflation by lowering the real rate of interest; a policy of this kind is called *passive*. If γ_1 is greater than one, the Fed responds to expected inflation by raising the real interest rate; a policy of this kind is called *active*.

4.3 Dynamics Implied by the Unrestricted Parameter Estimates

Our next step was to compute VARMA$(1,1)$ representations of the reduced form for each regime using the `SysSolve` and `Arrange` algorithms described in Appendices A and B. In Table 2 we report the absolute values of the generalized eigenvalues of the companion forms for the first and second sub-samples. In the first sub-sample our point estimates suggest an indeterminate equilibrium with two unstable roots, and for the second sub-sample, a determinate equilibrium with three unstable roots. These findings are consistent with the reported results of CGG [16], LS [23] and Boivin-Giannoni [9].

[8] In Beyer-Farmer [4] we report estimates of a DSGE model in which we *identify* the coefficients of the private sector equations by assuming that these coefficients remain stable across the break in 1979.

[9] CGG used GMM in a single equation framework and used a larger instrument set. Our findings for the policy rule are, however, qualitatively the same as theirs for the unrestricted model.

TABLE 2:	RESTRICTED ESTIMATES OF GENERALIZED EIGENVALUES								
Sample	**Point estimates of roots**								
60:4—79:3	0.00	0.00	0.00	0.56	0.93	0.93	0.97	**Inf.**	**1.12**
83:1—99:3	0.00	0.00	0.00	0.46	0.91	0.91	**Inf.**	**1.18**	**1.10**

*Bold figures indicate unstable roots.

Estimated size of determinate, indeterminate and unstable regions of the parameter space

60:4—79:3	**83:1—99:3**
Point Estimates Imply **Indeterminate Equilibrium**	Point Estimates Imply **Determinate Equilibrium**
Percentage of Indeterminate Draws = 70.1	Percentage of Indeterminate Draws = 19.33
Percentage of Determinate Draws = 25.58	Percentage of Determinate Draws = 72.76
Percentage of Non-Existent Draws = 4.31	Percentage of Non-Existent Draws = 7.91

To check the robustness of our determinacy findings for each sub-sample we took $100,000$ parameter draws from a normal distribution centered on the point estimates of the parameters with a variance covariance matrix equal to the asymptotic estimate using HAC standard errors from the GMM estimation. For each draw, we counted the number of stable generalized eigenvalues and calculated whether the implied equilibrium was determinate, indeterminate or non-existent. The results of this exercise are reported in Table 2. For the first sub-sample we found that 70.1% of our draws were consistent with the point estimate in the sense that they fell in the indeterminate region. A further 25.6% were in the determinate region and for 4.3% of the draws stationary equilibrium did not exist. For the second sub-sample 72.8% of the the draws were determinate, (consistent with the point estimates for this sub-sample), 19.3% were indeterminate and 7.9% implied non-existence. This exercise suggests a lower degree of confidence than that reported by LS [24] who developed Bayesian techniques to determine the posterior odds ratio for the probability that any given model is associated with a determinate as opposed to an indeterminate region of the parameter space.

Our next step was to study the dynamics of the economic response of the output gap, inflation and the Fed funds rate to fundamental shocks to the system. First, we constructed an estimate of the variance-covariance matrix of the these shocks using the methods described in Section 3. Since we found an indeterminate model in the first sub-period, we were forced to take a stand on how to attribute the residuals to three equations to four possible shocks. As described in Sub-section 3.2, there is no unique solution to this problem and we chose to identify the shocks by setting the variance and covariance terms of the sunspot shock equal to zero. The result of identifying shocks with this assumption is reported in Table 3 which reports the Choleski decomposition $\hat{P}$,of the estimated variance-covariance matrix $\hat{\Omega}$ for each sub-sample, where$\tilde{P}$, defined by the following equation:

$$\hat{P}\hat{P}' = \hat{\Omega},$$

is lower triangular.

According to our estimates of the elements of P, the Phillips curve and the Euler equation were hit by uncorrelated shocks with individual variances that changed across sub-samples. The estimated cross correlation of the output gap and inflation is insignificantly different from zero in both periods. We find the estimated standard deviation of output to be roughly three times as high in the pre-Volcker period as in the Volcker-Greenspan years; the standard deviation of interest rate shocks is also slightly larger. The inflation shock is highly imprecisely estimated in the first sub-period and of comparable magnitude to that in the second period. The second period inflation shock is, however, significant.

TABLE 3: ESTIMATES OF CHOLESKI DECOMPOSITION OF VCV MATRIX OF FUNDAMENTAL SHOCKS

Sample 60:4—79:3				
X 10^{-3}	**Gap**	**Infl**	**Irate**	**Sunspot**
Gap	16.1 (1.8)			0
Infl	29.5 (24.4)	12.9 (20.9)		0
Irate	-1.0 (4.7)	4.4 (3.1)	10.7 (1.4)	0
Sunspot	0	0	0	0

Sample 83:1—99:3			
X 10^{-3}	**Gap**	**Infl**	**Irate**
Gap	5.4 (3.5)		
Infl	5.0 (4.3)	8.5 (2.3)	
Irate	4.5 (1.7)	1.0 (2.2)	7.1 (0.6)

Standard errors in parentheses from Monte Carlo simulation.

In Figures 1 and 2 we used our point estimates of the parameters to generate impulse response functions associated with the theoretical models for each sub-sample. The solid lines in each figure are impulse responses computed from the point estimates and the dashed lines are 90% confidence bounds. The upper and lower bounds were computed by simulating $100,000$ draws from the asymptotic distribution of the parameter estimates, ranking the responses for each quarter, and picking the values that delineate the $5th$ and $95th$ quantiles. In our simulations, we discarded draws for which the determinacy properties of the simulation were different from the point estimates. These confidence intervals should therefore be interpreted as conditional on the determinacy properties of the point estimates.

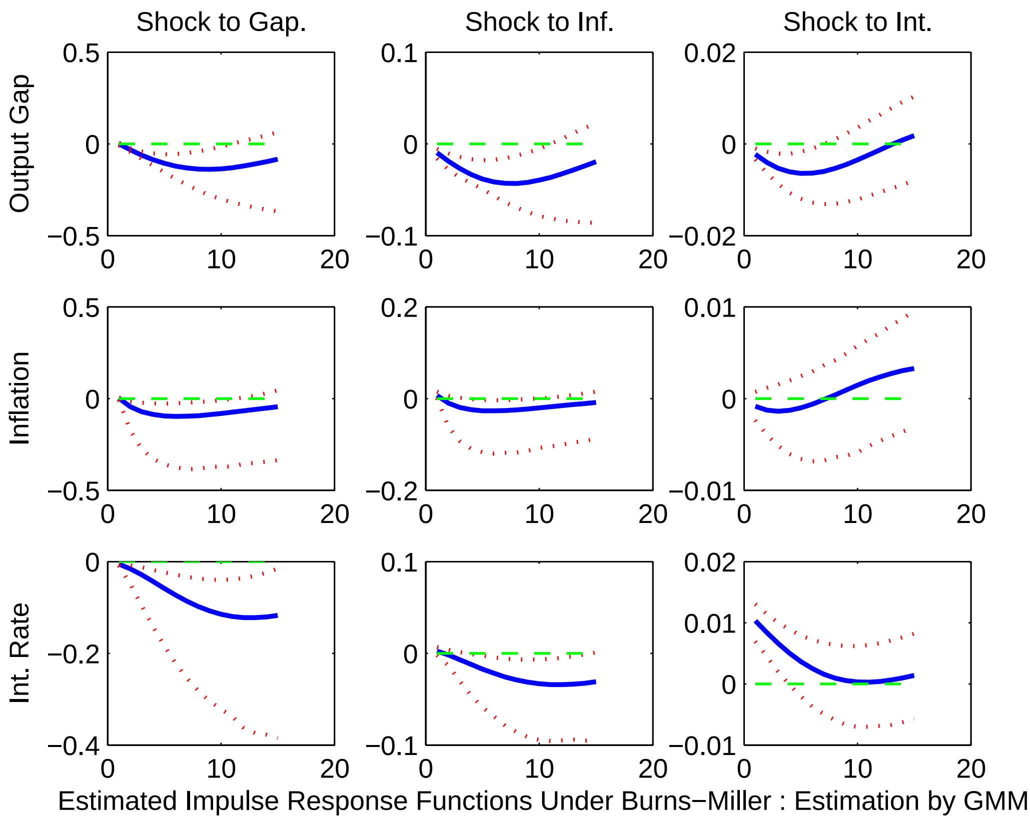

Figure 1

Our estimation procedure is capable of adding moment conditions that force the covariance of structural shocks to be zero. We did not impose this condition since we take the view that a structural model may be hit by correlated disturbances. For example, a common shock may shift both the Phillips curve and the Euler equation.

An implication of the finding that the equilibrium in the first sub-period is indeterminate is that non-fundamental sunspot shocks cannot be separately identified. The numbers reported in Table 3, and the corresponding impulse response functions reported in Figures 1 and 2, disentangle fundamental and non-fundamental shocks by imposing the identifying assumption that Ω_{ww} and Ω_{wv} are zero; that is, all of the observed shocks were caused by fundamentals. Using our methods, there are many alternative possible identification schemes, including one in which Ω_{vv} is diagonal and assumed to remain stable across regimes.

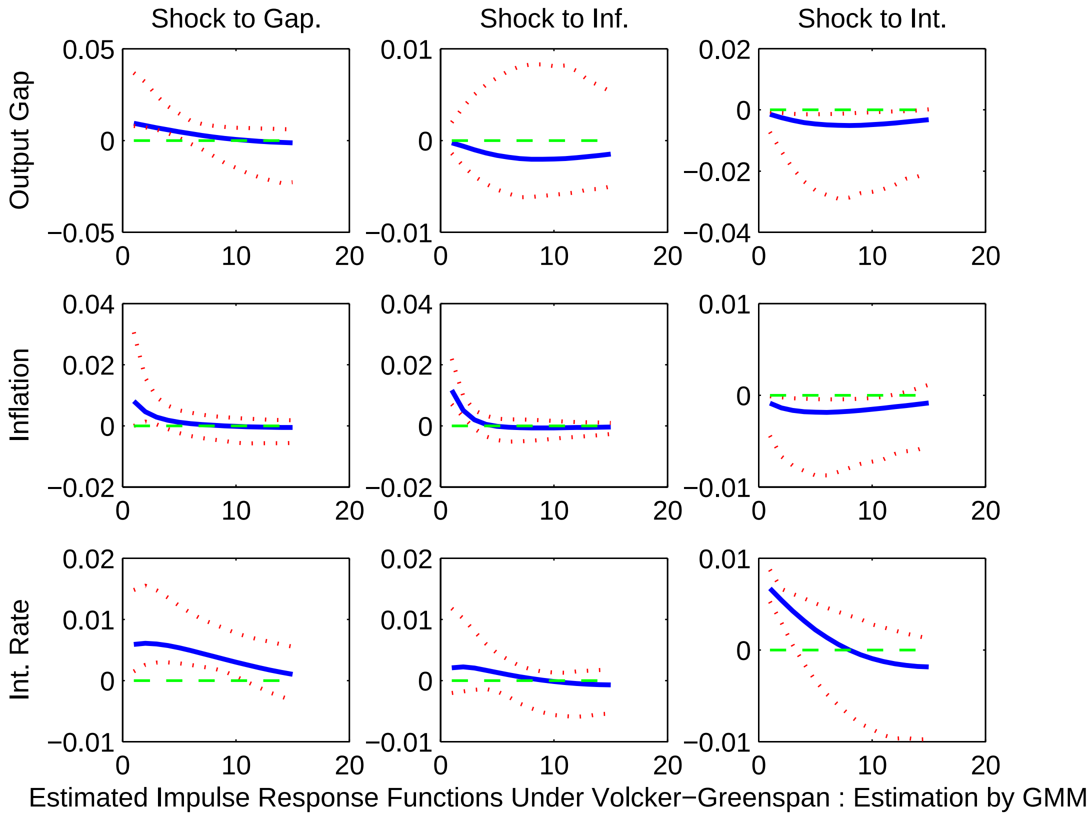

Figure 2

Although the parameter estimates of the Euler equation and the Phillips curve are almost identical across sub-periods, the policy rule has changed dramatically. This shows up in Figures 1 and 2 as qualitative difference in the impulse response functions. Nevertheless, the model provides a plausible economic interpretation of the effects of shocks and the way that their dynamic effects on the endogenous variables trace themselves through the system.

5 Conclusion

In this paper we presented a technique to construct impulse response functions in dynamic stochastic general equilibrium models. We applied our method to the example of U.S. monetary policy before and after 1979 and were able to estimate a tightly specified version of the New-Keynesian model. Our estimates of the parameters of the Euler equation and the Phillips curve remain stable across regimes but the parameters of the policy rule changed dramatically.

Our method is simple to implement and the MATLAB code is capable of being adapted to a wide range of related applications.

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Appendix A

In this section we explain how our solution algorithm `SySolve` works in the case of an indeterminate equilibrium. The reader is referred to Sims [29] for a more detailed explanation of `GENSYS`, on which our algorithm is based. The structural model has the form

$$\tilde{A}_0 X_t = \tilde{A}_1 X_{t-1} + \tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t. \tag{A1}$$

Using a QZ decomposition, write this as

$$QSZX_t = QTZX_{t-1} + \tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t. \tag{A2}$$

where $QQ' = ZZ' = I$ and S and T are upper triangular and S and T are ordered such that all unstable generalized eigenvalues are in the bottom right corner. Recall that the generalized eigenvalues are defined as the ratios of the diagonal elements of T to the diagonal elements of S. Now define,

$$x_t = ZX_t \tag{A3}$$

and

$$e_t = S^{-1}Q'\left(\tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t\right), \tag{A4}$$

and partition x_t and e_t as follows;

$$x_t = \left(x_t^1, x_t^2\right)', \quad e_t = \left(e_t^1, e_t^2\right)' \tag{A5}$$

where $x_t^1 \in \mathbb{C}^{n_1}$, $x_t^2 \in \mathbb{C}^{n_2}$ are (possibly) complex vectors and n_1 and n_2 are the numbers of stable and unstable roots. Now partition the matrices S and T

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \tag{A6}$$

and the matrices S^{-1} and Q' as;

$$S^{-1} = \begin{bmatrix} S^{11} & S^{12} \\ 0 & S^{22} \end{bmatrix}, \; Q' = \begin{bmatrix} Q^{11} & Q^{12} \\ Q^{22} & Q^{22} \end{bmatrix}. \tag{A7}$$

Using this notation write (A2) as;

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} + \begin{bmatrix} e_t^1 \\ e_t^2 \end{bmatrix}. \tag{A8}$$

In Equation (A8) the lower block acts as an autonomous unstable subsystem in the transformed variables x_t^2. For the system to exhibit a non-explosive solution, one requires that $x_t^2 = e_t^2 = 0$ for all t. This restriction requires that that the non-fundamental errors W_t be chosen to remove the influence of the fundamental errors V_t. To this end, the solution algorithm sets

$$e_t^2 = S^{22} \begin{bmatrix} Q^{21} & Q^{22} \end{bmatrix} \left(\tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t \right) = 0. \tag{A9}$$

A necessary condition for these equations to have a solution is that there are at least as many elements of W_t as there are unstable roots (the number of rows in Equation system (A9)). In the case of r degrees of indeterminacy there are r more elements of W_t than one requires to eliminate unstable roots. In this case, our algorithm transfers the first r non-fundamental shocks to the vector V_t thereby treating the elements of $W_t^1 \in R^r$ as additional fundamentals. We refer to the expanded vector of fundamentals as (V_t, W_t^1). It might appear that this solution is arbitrary since a particular solution depends on the ordering of W_t. To see that this is not the case, let Ω represent the variance-covariance matrix of the expanded fundamentals

$$\Omega = E_t \begin{bmatrix} V_t \\ W_t^1 \end{bmatrix} \begin{bmatrix} V_t \\ W_t^1 \end{bmatrix}' = \begin{bmatrix} \Omega_{vv} & \Omega_{vw} \\ \Omega_{wv} & \Omega_{ww} \end{bmatrix}. \tag{A10}$$

Since we do not place any restrictions on Ω our algorithm is capable of generating the full range of sunspot solutions. Different solutions are captured by picking different values for the variance-covariance terms Ω_{ww} and Ω_{wv}.

Appendix B

This appendix explains how to generate a VARMA model in observable variables Y_t from the VAR, (possibly involving unobservable components $E_{t-1}(Y_t)$) that is generated by the solution algorithm `SySolve`.

Consider the structural model, Equation (1), which we write in canonical form by adding a second block of identities and a third block that defines the expectational errors, W_t:

$$\overbrace{\begin{bmatrix} A & 0 & F \\ 0 & I & 0 \\ I & 0 & 0 \end{bmatrix}}^{A_0}\overbrace{\begin{bmatrix} Y_t \\ Y_{t-1} \\ E_t[Y_{t+1}] \end{bmatrix}}^{X_t} = \overbrace{\begin{bmatrix} B & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}}^{A_1}\overbrace{\begin{bmatrix} Y_{t-1} \\ Y_{t-2} \\ E_{t-1}[Y_t] \end{bmatrix}}^{X_{t-1}}$$
$$+\overbrace{\begin{bmatrix} C \\ 0 \\ 0 \end{bmatrix}}^{C_0}+\overbrace{\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}}^{\Psi_V}V_t+\overbrace{\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}}^{\Psi_W}W_t. \quad \text{(B1)}$$

The reason for adding the identity block is to give us enough observable variables to 'carry' the dynamics of the solution. Writing Equation (B1) more compactly, letting $N = 3n$, gives:

$$\underset{N\times N}{A_0}\,\underset{N\times 1}{X_t} = \underset{N\times N}{A_1}\,\underset{N\times 1}{X_{t-1}} + \underset{N\times 1}{C_0} + \underset{N\times n}{\Psi_V}\,\underset{n\times 1}{V_t} + \underset{N\times nn\times 1}{\Psi_W W_t}. \quad \text{(B2)}$$

Our goal is to find an n_1 dimensional subset of the observable variables $Y_{1t} \subset \{Y_t, Y_{t-1}\}$ that can be described by a VARMA$(1,k)$:

$$Y_{1t} = \Gamma_1^* Y_{1t-1} + C_1^* + \sum_{j=0}^{k} \Psi_j^* \eta_{t-j}, \quad \text{(B3)}$$

where

$$\eta_t = \begin{bmatrix} V_t \\ W_t^1 \end{bmatrix}, \quad \text{(B4)}$$

and W_t^1 is a subset of W_t. The W_t^1 terms represent non-fundamental errors that may be correlated with the fundamental error terms V_t and may exert an independent influence on Y_t if the equilibrium is indeterminate.

Using a complex Schur decomposition to eliminate the influence of unstable generalized eigenvalues, we first write the reduced form of (B2) as follows:[10]

$$X_t = \tilde{\Gamma} X_{t-1} + \tilde{C} + \tilde{\Psi}_V V_t + \tilde{\Psi}_W W_t^1. \tag{B5}$$

The complex Schur decomposition always exists. The matrix $\tilde{\Gamma}$ is $N \times N$ and has n_2 zero eigenvalues and n_1 non-zero eigenvalues where $n_1 + n_2 = N$. In the following analysis we restrict ourselves to the case where a stationary solution exists.[11]

By construction, all of the roots of $\tilde{\Gamma}$ are inside the unit circle and W_t^1 has dimension equal to the degree of indeterminacy. $\tilde{\Gamma}$ has at least n zero roots: these are associated with the identity block $Y_{t-1} = Y_{t-1}$. For each unstable generalized eigenvalue of $\left\{\tilde{A}_0, \tilde{A}_1\right\}$, $\tilde{\Gamma}$ has an additional zero root. It follows that

$$rank\left(\tilde{\Gamma}\right) = n_1 \leq 2n. \tag{B6}$$

Next we make a second use of the complex Schur decomposition to find orthonormal matrices Q and Z, and uppertriangular matrices D and T such that

$$Q'DZ' = I, \tag{B7}$$

$$Q'TZ' = \tilde{\Gamma}. \tag{B8}$$

[10] See Sims [29] for a description of how to use the complex Schur decomposition to find a solution to this problem in the determinate case and Lubik and Schorfheide [23] for a generalization to models with indeterminate equilibria.

[11] A necessary condition for existence is that the number of unstable generalized eigenvalues of $\{A_0, A_1\}$ is less than or equal to n; a sufficient condition is more complicated to state but is relatively easy to compute (see Sims [29]).

In general the Schur decomposition delivers an upper-triangular D but in our case, since I is the identity matrix, D is diagonal. Using these results we write the system as

$$Q'DZ'X_t = Q'TZ'X_{t-1} + \tilde{C} + \tilde{\Psi}_V V_t + \tilde{\Psi}_W W_t. \tag{B9}$$

Next we rearrange the rows of Equation (B9) so that the first n_1 rows correspond to the nonzero eigenvalues of $\tilde{\Gamma}$ and, we define the following coefficient matrices:

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \; Z' = \begin{bmatrix} z^{11} & z^{12} \\ z^{21} & z^{22} \end{bmatrix}, \tag{B10}$$

$$\begin{bmatrix} s_{11} & s_{12} \\ 0 & s_{22} \end{bmatrix} = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}^{-1} \begin{bmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{bmatrix}, \tag{B11}$$

and we define new variables $\{x_{1t}, x_{2t}\}$ and $\{\zeta_{1t}, \zeta_{2t}\}$:

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} z^{11} & z^{12} \\ z^{21} & z^{22} \end{bmatrix} \begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix}, \tag{B12}$$

$$\begin{bmatrix} \zeta_{1t} \\ \zeta_{2t} \end{bmatrix} = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}^{-1} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & \tilde{\Psi}_{1V} V_t & \tilde{\Psi}_{1W} W_t \\ \tilde{C}_2 & \tilde{\Psi}_{2V} V_t & \tilde{\Psi}_{2W} W_t \end{bmatrix}. \tag{B13}$$

Making use of this change of variables, premultiply (B9) by $D^{-1}Q$ to give

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ 0 & s_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} \zeta_{1t} \\ \zeta_{2t} \end{bmatrix}. \tag{B14}$$

The second block of (B14) reads

$$x_{2t} = \zeta_{2t} + s_{22} x_{2t-1}. \tag{B15}$$

Since s_{22} is upper triangular of dimension n_2 with zeros on the diagonal it follows that

$$(s_{22})^k = 0, \quad \text{for } k \geq n_2, \tag{B16}$$

and hence x_{2t} can be written as an n_2th degree moving average:

$$x_{2t} = p(L)\,\zeta_{2t}, \tag{B17}$$

where

$$p(L) = \sum_{k=0}^{n_2} (s_{22}L)^k\,, \tag{B18}$$

is a polynomial in the lag operator of degree, at most, n_2 (the number of zero roots of $\tilde{\Gamma}$).

Now we use Definition (B12) and Equation (B17) to find an expression for X_{2t} in terms of X_{1t}, and $p(L)\,\zeta_{2t}$:

$$z^{21}X_{1t} + z^{22}X_{2t} = p(L)\,\zeta_{2t}, \tag{B19}$$

or more compactly,

$$X_{2t} = N_1 X_{1t} + N_2(L)\,\zeta_{2t}, \tag{B20}$$

where the terms N_1 and $N_2(L)$ are defined as:

$$N_1 = -\left(z^{22}\right)^{-1} z^{21}, \tag{B21}$$

$$N_2(L) = \left(z^{22}\right)^{-1} p(L)\,. \tag{B22}$$

Turning to the first block of (B14) we can use (B17) to write:

$$x_{1t} = s_{11}x_{1t-1} + s_{12}p(L)\,\zeta_{2t-1} + \zeta_{1t}. \tag{B23}$$

Equation (B23) can be expanded using Definition (B12) to give:

$$z^{11}X_{1t} + z^{12}X_{2t} = s_{11}\left(z^{11}X_{1t-1} + z^{12}X_{2t-1}\right) + s_{12}p(L)\,\zeta_{2t-1} + \zeta_{1t}. \tag{B24}$$

Using Equations (B20) and (B24), noting that $p(L) = z^{22}N_2(L)$, (from B22), where $N_2(L)$ is a degree k polynomial in L, we can find a representation for X_{1t} as an ARMA$(1,k)$:

$$\begin{aligned} z^{11}X_{1t}+z^{12}\left(N_1X_{1t} + N_2(L)\,\zeta_{2t}\right) &= s_{11}\left(z^{11}X_{1t-1} + z^{12}\left(N_1X_{1t-1} + N_2(L)\,\zeta_{2t-1}\right)\right) \\ &\quad + s_{12}z^{22}N_2(L)\,\zeta_{2t-1} + \zeta_{1t}. \end{aligned} \tag{B25}$$

Collecting terms and writing the system more compactly leads to the expression:

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ N_1 M_1 & 0 \end{bmatrix} \begin{bmatrix} X_{1t-1} \\ X_{2t-1} \end{bmatrix} + \begin{bmatrix} M_2 \\ N_1 M_2 \end{bmatrix} \zeta_{1t} + \begin{bmatrix} M_3(L) \\ M_4(L) \end{bmatrix} \zeta_{2t}, \quad \text{(B26)}$$

where the following definitions apply:

$$N_1 = -\left(z^{22}\right)^{-1} z^{21}, \quad \text{(B27)}$$

$$N_2(L) = \left(z^{22}\right)^{-1} \sum_{k=0}^{n} s_{22}^{k} L^k, \quad \text{(B28)}$$

$$M_1 = \left(z^{11} + z^{12} N_1\right)^{-1} s_{11} \left(z^{11} + z^{12} N_1\right), \quad \text{(B29)}$$

$$M_2 = \left(z^{11} + z^{12} N_1\right)^{-1}, \quad \text{(B30)}$$

$$M_3(L) = \left(z^{11} + z^{12} N_1\right)^{-1} \left(-z^{12} + \left(s_{11} z^{12} + s_{12} z^{22}\right) L\right) N_2(L), \quad \text{(B31)}$$

$$M_4(L) = N_1 M_3(L) + N_2(L). \quad \text{(B32)}$$

To obtain Equation (B3) notice that since $n_1 \leq 2n$, we can choose $Y_{1t} = X_{1t}$ to be a subset of the observables $\{Y_t, Y_{t-1}\}$. In the determinate case, $Y_{1t} = Y_t$ In the indeterminate case Y_t must be augmented by one or more lags. The constant term in Equation (B3) is equal to

$$C_1^* = M_2 \tilde{C}_1 + M_3(1) \tilde{C}_2, \quad \text{(B33)}$$

and expressions for the coefficients of the lag polynomial $\Psi_j^*(L)$ can be computed from the definitions of η_{1t} and η_{2t} (Equation B13) and the constants and lag polynomials M_i and N_i defined in Equations (B27–B32).

Appendix C

This Appendix shows that the estimator of Ω proposed in Section 3 is consistent. Taking probability limits of (10) and (11), making use of Equation (8), the consistency of $\hat{\Gamma}_1^*$ and $\hat{C}_1^*$ and the assumption that η_t is uncorrelated with its own lags leads to the following expressions:

$$
\begin{aligned}
p\lim_{T\to\infty} \hat{S}_{0T} &= \sum_{t=2}^{T} \frac{\left(\sum_{q=0}^{k} \Psi_q^* \eta_{t-q}\right)\left(\sum_{q=0}^{k} \Psi_q^* \eta_{t-q}\right)'}{T} \\
&= p\lim_{T\to\infty} \frac{\left(\Psi_0^*\eta_t + \Psi_1^*\eta_{t-1} + \ldots \Psi_k^*\eta_{t-k}\right)\left(\Psi_0^*\eta_t + \Psi_1^*\eta_{t-1} + \ldots \Psi_k^*\eta_{t-k}\right)}{T} \\
&= \left(\Psi_0^*\right)\Omega\left(\Psi_0^*\right)' + \left(\Psi_1^*\right)\Omega\left(\Psi_1^*\right)' + \ldots \left(\Psi_k^*\right)\Omega\left(\Psi_k^*\right)',
\end{aligned}
$$

or

$$
p\lim_{T\to\infty} \hat{S}_{0T} = \sum_{q=0}^{k} \Psi_q^* \Omega \Psi_q^{*\prime}. \tag{C1}
$$

$$
\begin{aligned}
p\lim_{T\to\infty} \hat{S}_{jT} &= \sum_{t=j+2}^{T} \frac{\left(\sum_{q=0}^{k} \Psi_q^* \eta_{t-q}\right)\left(\sum_{q=0}^{k} \Psi_q^* \eta_{t-q-j}\right)}{T} \\
&= p\lim_{T\to\infty} \frac{\left(\Psi_0^*\eta_t + \Psi_1^*\eta_{t-1} + \ldots \Psi_k^*\eta_{t-k}\right)\left(\Psi_0^*\eta_{t-j} + \Psi_1^*\eta_{t-j-1} + \ldots \Psi_k^*\eta_{t-j-k}\right)}{T} \\
&= \left(\Psi_j^*\right)\Omega\left(\Psi_0^*\right)' + \left(\Psi_{j+1}^*\right)\Omega\left(\Psi_1^*\right)' + \ldots \left(\Psi_k^*\right)\Omega\left(\Psi_{k-j}^*\right)',
\end{aligned}
$$

or,

$$
p\lim_{T\to\infty} \hat{S}_{jT} = \sum_{q=j}^{k} \left(\Psi_q^*\right)\Omega\left(\Psi_{j-q}^*\right)'. \tag{C2}
$$

Now form the sum

$$
\hat{S}_T = \hat{S}_{0T} + \sum_{j=1}^{k}\left(\hat{S}_{jT} + \hat{S}_{jT}'\right). \tag{C3}
$$

Taking probability limits of (C3), using (C1) and (C2) gives

$$p\lim_{T\to\infty}\hat{S}_T = \sum_{j=0}^{k}\left(\Psi_{jT}^{*}\right)\Omega\sum_{j=0}^{k}\left(\Psi_{jT}^{*}\right)'. \tag{C4}$$

Now replace Ψ_j^* by consistent estimates $\hat{\Psi}_{jT}^*$ obtained from passing the GMM estimates of the structural parameters through `SysSolve` and `Arrange` to obtain the following system of n^2 equations in the $(n+r)^2$ unknown elements of the variance-covariance matrix Ω.

$$\underset{n\times n}{\hat{S}_T} = \hat{\Psi}_T \underset{(n+r)\times(n+r)}{\hat{\Omega}_T} \hat{\Psi}_T', \tag{C5}$$

where

$$\hat{\Psi}_T = \sum_{j=0}^{k}\hat{\Psi}_{jT}^{*}. \tag{C6}$$

Since $\hat{S}_T$ and $\hat{\Omega}_T$ are symmetric this system reduces to $n\,(n+1)\,/2$ equations in $(n+r)\,(n+r+1)\,/2$ unknowns which we write as

$$\text{vech}\left(\hat{S}_T\right) = B\left(\hat{\Psi}_T\right)\text{vech}\left(\hat{\Omega}_T\right), \tag{C7}$$

where vech is the operator that stacks the lower triangular elements of a symmetric matrix into a row vector. For $r > 1$ Equation system (C7) will have multiple solutions and we are free to choose $r\,(n+r+1)\,/2$ linear combinations. We identify a solution by adding an arbitrary $[r\,(n+r+1)\,/2] \times [(n+r)\,(n+r+1)\,/2]$ matrix R such that.

$$\text{vech}\left(\hat{S}_T\right) = R\text{vech}\left(\hat{\Omega}_T\right). \tag{C8}$$

Our estimator of $\hat{\Omega}_T$ is given by

$$\text{vech}\left(\hat{\Omega}_T\right) = \begin{bmatrix} B\left(\hat{\Psi}_T\right) \\ R \end{bmatrix}^{-1}\text{vech}\left(\hat{S}_T\right). \tag{C9}$$

Consistency follows for arbitrary R from the properties of probability limits and the fact that

$$\operatorname{vech}(S_T) = \begin{bmatrix} B(\Psi) \\ R \end{bmatrix} \operatorname{vech}(\Omega). \tag{C10}$$

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