TESTING FOR STOCHASTIC DOMINANCE WITH DIVERSIFICATION POSSIBILITIES

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Abstract	We derive empirical tests for stochastic dominance that allow for diversification between choice alternatives. The tests can be computed using straightforward linear programming. Bootstrapping techniques and asymptotic distribution theory can approximate the sampling properties of the test results and allow for statistical inference. Our results could provide a stimulus to the further proliferation of stochastic dominance for the problem of portfolio selection and evaluation (as well as other choice problems under uncertainty that involve diversification possibilities). An empirical application for US stock market data illustrates our approach.	
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Testing for Stochastic Dominance with Diversification Possibilities

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* Post is with the department of Finance of the Erasmus University Rotterdam in the Netherlands. This study forms part of a research program on nonparametric analysis of economic behavior. Details on the program are available at the program homepage: *http://www.few.eur.nl/few/people/gtpost/program.htm*. This study is the first step towards extending the program towards analyzing investor behavior. We would like to express our appreciation to Martijn van den Assem, Timo Kuosmanen and Jaap Spronk and, as well as the editor, Rick Green, and two anonymous Journal of Finance referees, for providing helpful comments and stimulating discussion. We gratefully acknowledge financial support by Tinbergen Institute and Erasmus Research Institute of Management.

ABSTRACT

We derive empirical tests for stochastic dominance that allow for diversification between choice alternatives. The tests can be computed using straightforward linear programming. Bootstrapping techniques and asymptotic distribution theory can approximate the sampling properties of the test results and allow for statistical inference. Our results could provide a stimulus to the further proliferation of stochastic dominance for the problem of portfolio selection and evaluation (as well as other choice problems under uncertainty that involve diversification possibilities). An empirical application for US stock market data illustrates our approach.

THE THEORY OF STOCHASTIC DOMINANCE (SD; Hadar and Russell, 1969, Hanoch and Levy, 1969, Rothschild and Stiglitz, 1970, and Whitmore, 1970) gives a systematic framework for analyzing economic behavior under uncertainty. SD has seen considerable theoretical development and empirical application in the last decades, in various areas of economics, finance and statistics (see e.g. Levy, 1992). It is useful both for positive analysis (where the objective is to analyze the decision rules actually used by decision-makers) as well as in normative analysis (where the objective is to support practical decision making). The theoretical attractiveness of SD lies in its nonparametric orientation. SD criteria do not require a full parametric specification of decision-maker preferences, but rather rely on general preference assumptions.

For applying SD criteria to empirical data, simple crossing algorithms have been developed that check the difference of the empirical distribution functions (EDFs) of the choice alternatives (e.g. Levy and Hanoch, 1970, Porter *et al.*, 1973). This approach is computationally very efficient; one first constructs the EDFs from the ordered outcomes of the alternatives and next checks if the EDFs cross for any of the observed values. Unfortunately, practical application of this approach is restricted to pairwise comparison of a finite number of choice alternatives, and SD cannot be applied for problems where full diversification across alternatives is allowed. The problem is that the ordering of the outcomes (and hence the EDF) of a diversified portfolio of alternatives cannot be determined in a straightforward way from the orderings of the individual alternatives. Therefore, the ordering of each portfolio has to be determined individually. This is computationally impossible if infinitely many portfolios have to be considered, as is true if we allow for full diversification. Also, in case of more than two choice alternatives, pairwise dominance is not a necessary condition for SD inefficiency. Bawa *et al.* (1985) provide algorithms for applying the concept of convex stochastic dominance (Fishburn, 1974), which does provide a necessary and sufficient condition if more than two choice alternatives are compared. However, also this approach considers only a finite number of choice alternatives, and it can not account for diversification between the alternatives.

The selection and evaluation of investment portfolios is an important application area where this problem arises; in many cases the portfolio possibilities consist of infinitely many weighted averages of available assets. (For simplicity, we will formulate in terms of this problem. However, our arguments apply with equal strength for alternative applications of SD.) This subject is of interest both for empirical tests of theoretical asset pricing models and for practical portfolio management applications. The focus of the research in this area has predominantly been on mean-variance analysis (e.g. Kandel and Stambaugh, 1987, 1989, Gibbons *et al.*, 1989, MacKinlay and Richardson, 1991, Wang, 1998, and Britten-Jones, 1999), and extensions of that framework (e.g. Yamakazi and Konno, 1991, and Young, 1998). The inability to deal with diversification can help explain why SD has not seen the proliferation that one might expect based on the theoretical benefits of SD.

Some authors suggest ways to test whether a given portfolio is SD efficient relative to all possible portfolios. The Marginal Conditional Stochastic Dominance (MCSD) framework by Shalit and Yitzhaki (1994) provides tests for identifying assets whose portfolio weights should be altered in order to construct alternative portfolios that SD dominate the evaluated portfolio. These tests give necessary but not sufficient conditions for SD efficiency, and they may fail to identify inefficient portfolios. By contrast, Kuosmanen (2001) provides necessary *and* sufficient tests. He includes the ordering of the return observations as model variables and provides a linear programming (LP) relaxation of the resulting mixed integer linear programming model. Unfortunately, the number of model variables increases progressively with the number of observations (for second-order SD, the number of observations is a second-order polynomial; for third-order SD the order is four). The computational complexity of LP problems (as measured by the number of arithmetic operations, run time and working memory requirements) increases progressively with the number of model variables. Hence, the Kuosmanen model is computationally very demanding for real-life problems (especially if the analysis is complemented with computer intensive bootstrap techniques; see below). We therefore conclude that there currently are no computationally tractable necessary and sufficient tests for SD with diversification. The purpose of this study is to fill this gap; we develop tractable LP algorithms for testing the SD efficiency of a given portfolio relative to all possible portfolios created from a set of assets.

Our tests are derived from the optimality conditions for portfolio optimization in the expected utility framework. The expected utility framework has a number of well-known limitations (see e.g. Machina, 1987). However, SD criteria are also economically meaningful for a whole range of non-expected utility theories of choice behavior under uncertainty (see e.g. Starmer, 2000), like Yaari's (1987) dual theory of risk (see e.g. Wang and Young, 1998). Our tests rely on constructing utility functions in a nonparametric fashion, and on testing if the evaluated portfolio is optimal relative to those utility functions. In this respect, our approach has a strong analogy with the nonparametric approach to consumer analysis by Afriat (1967) and Varian (1982). In financial economics, Dybvig and Ross (1982), Varian (1983), and Green and Srivastava (1985, 1986), among others, have proposed similar approaches to deriving optimality conditions and to recovering investor preferences.

The SD literature involves a multitude of different criteria, associated with different sets of preference assumptions. Higher order criteria involve more discriminating power than lower order ones, because they induce a larger reduction of the set of efficient portfolios. However, that power has to be balanced against the stringency of the additional preference assumptions. In general, striking that balance requires a careful consideration of the structure and the context of the decision problem considered. For the sake of compactness, we focus on the popular criteria of second-order SD (SSD) and third-order SD (TSD). The assumptions associated with these criteria have a good economic interpretation (nonsatiation, risk aversion and skewness preference), and also empirical evidence exists to support these assumptions for many choice problems. Still, nothing excludes the generalization of our analysis towards higher order criteria (although the computational burden increases substantially for higher-order criteria). A test for first-order SD (FSD) seems more difficult, as our results critically depend on the concavity of the utility function (and FSD accounts for the possibility of risk seeking investors or non-concave utility functions).

The remainder of this paper is organized as follows. Section I recaptures the definition of SSD for pairwise comparison and discusses generalization of that definition towards the case with portfolio diversification. Section II gives our LP formulation of SSD in terms of nonparametric empirical utility functions. Section III provides an equivalent dual formulation in terms of Bowden's (2000) Ordered Mean Difference performance measure. Section IV generalizes the analysis towards the more powerful TSD criterion. Apart from the computational problems associated with portfolio diversification, another problem in practical applications of SD is the sensitivity of the results to sampling error. Section V discusses how bootstrapping techniques and asymptotic distribution theory can approximate the sampling distribution of the test results and allow for statistical inference. Section VI illustrates our approach by means of an empirical application for US stock market data. Finally, Section VII gives conclusions and suggests directions for future research. The Appendix gives the formal proofs of our theorems.

I. SECOND-ORDER STOCHASTIC DOMINANCE

Consider an investment universe consisting of N assets, associated with returns $\boldsymbol{x} \in \Re^N$.[1] Throughout the text, we will use the index set $\mathrm{I} \equiv \{1, \cdots, N\}$ to denote the different assets. In addition, we will treat the returns as serially independent and identically distributed random variables with a continuous joint cumulative distribution function (CDF) $G : \Re^N \to [0,1]$.[2] Investors may diversify between the assets, and we will use $\boldsymbol{?} \in \Re^N$ for a vector of portfolio weights. For simplicity, we will consider the case where short selling is not allowed, and the portfolio weights belong to the portfolio possibilities set $\Lambda \equiv \{\boldsymbol{?} \in \Re_+^N : \boldsymbol{?}^T \boldsymbol{e} = 1\}$. However, it is possible to generalize the analysis towards cases where short selling is allowed and cases where additional restrictions are imposed on the portfolio weights.[3]

We consider the problem of establishing whether a particular portfolio, say $\boldsymbol{t} \in \Lambda$, is optimal, i.e. whether it maximizes the expected value of the investor’s utility function $u : \Re^1 \to P$, $u \in U$, with U for the class of von Neuman-Morgenstern utility

[1] Throughout the text, we will use $\Re^m$ for an *m*-dimensional Euclidean space, and $\Re_+^m$ denotes the positive orthant. Further, to distinguish between vectors and scalars, we use a bold font for vectors and a regular font for scalars.
[2] This is a standard assumption in the SD literature. Still, there is substantial evidence that the distribution of assets returns (e.g. risk premia and volatilities) varies through time. Further research could focus on developing tests that relax the assumption that the observartions are serially IID.
[3] Our analysis is based on the optimality conditions from subdifferential calculus for optimizing a concave utility function over a convex portfolio possibilities set (see the proof to Theorem 2 in the Appendix). These conditions apply for any non-empty, closed and convex portfolio set. We may generalize our analysis by simply replacing ? by a more general polyhedron in the dual formulation (D) developed in Section III. The generalized primal formulation can then be obtained by applying linear duality theory to the generalized dual.

functions, and P for a nonempty, closed and convex subset of $\Re$. The portfolio $\boldsymbol{t}$ is optimal if and only if:

$$(1) \qquad \int u(\boldsymbol{x}\boldsymbol{t})\partial G(\boldsymbol{x}) = \max_{?\in\Lambda} \int u(\boldsymbol{x}?)\partial G(\boldsymbol{x}) .$$

In practical applications, full information about the utility function typically is not available, and this condition cannot be verified directly. This provides the rationale for using SD criteria that rely on a set of general assumptions rather than a full specification of the utility function. The SSD criterion restricts attention to the class of nonsatiable and risk-averse investors or the class of strictly increasing and concave utility functions $U_2 \subseteq U$. Note that we do not assume that the utility function is continuously differentiable. However, utility is concave and hence everywhere continuous and superdifferentiable. Throughout the text, we will denote supergradients at x by $\partial u(x)$.

Apart from the utility function, also the CDF generally is not known in practical applications. Rather, information typically is limited to a discrete set of time series observations, say $\mathbf{X} \equiv (\boldsymbol{x}_1 \cdots \boldsymbol{x}_T)^T$ with $\boldsymbol{x}_t \equiv (x_{1t} \cdots x_{Nt}) \in \Re^N$, which can be treated as independent random samples from the CDF. Throughout the text, we will use the index set $\Theta \equiv \{1, \cdots, T\}$ to denote different points in time. Using the observations, we can construct the empirical distribution function (EDF):

$$(2) \qquad F(\boldsymbol{x}) \equiv \mathrm{card}\{t \in \Theta : \boldsymbol{x}_t \leq \boldsymbol{x}\}/T .$$

In this paper, we analyze SD for the EDF rather than for the CDF, so as to focus on the computational problems encountered in practical applications. We discuss the relationship between the EDF and the CDF in Section V.

For convenience, we assume that the data are ranked in ascending order by the return of the evaluated portfolio, i.e. $\boldsymbol{x}_1\boldsymbol{t} < \boldsymbol{x}_2\boldsymbol{t} < \cdots < \boldsymbol{x}_T\boldsymbol{t}$. Since we assume a continuous return distribution, ties do not occur. Still, the analysis can be extended in a straightforward way to cases where ties do occur e.g. due to a discrete return distribution or due to measurement problems or rounding, or if a riskless asset is included in the analysis (see the discussion at the end of Section II). Note that we rank only based on the returns of the evaluated portfolio. By contrast, the existing crossing algorithms also rank based on the benchmark portfolio, which introduces computational problems if the benchmark portfolio is not known at forehand, but rather has to be selected from infinitely many candidate portfolios.

Using the above notation, SSD for pairwise comparison can be defined as follows:

DEFINITION 1 *Portfolio* $? \in \Lambda$ *dominates portfolio* $\boldsymbol{t} \in \Lambda$ *by SSD if and only if, for all utility functions* $u \in U_2$, $?$ *has a higher expected utility than* $\boldsymbol{t}$, *i.e.*

$$(3) \qquad \int u(\boldsymbol{x}?)\partial F(\boldsymbol{x}) - \int u(\boldsymbol{x}\boldsymbol{t})\partial F(\boldsymbol{x}) = \sum_{t\in\Theta} \left(u(\boldsymbol{x}_t ?) - u(\boldsymbol{x}_t \boldsymbol{t})\right)/T > 0 \quad \forall u \in U_2 .$$

Note that this definition of SSD uses strict inequalities for all $u \in U_2$. By contrast, the traditional definition uses weak inequalities with a strict inequality for at least one $u \in U_2$. This difference is important from a theoretical perspective, and one can think of examples where the two definitions give different efficiency classifications. For example, using Definition 1, $\boldsymbol{?} \in \Lambda$ does not dominate mean-preserving spreads of $\boldsymbol{?}$, because risk neutral investors are indifferent between alternatives that have identical means. By contrast, dominance does exist using the traditional definition, because all strictly risk-averse investors do prefer $\boldsymbol{?}$ to mean-preserving spreads. However, from an empirical perspective, the definitions are indistinguishable, because arbitrary small data perturbations to the evaluated portfolio can make the classifications consistent. Related to this, data sets where this theoretical issue has a decisive impact are extremely unlikely for return distributions that are continuous by approximation.[4]

The following is a straightforward generalization of Definition 1 to the case where diversification is allowed:

DEFINITION 2 *Portfolio $\boldsymbol{t} \in \Lambda$ is SSD inefficient if and only if some portfolio $\boldsymbol{?} \in \Lambda$ SSD dominates it. Alternatively, portfolio $\boldsymbol{t} \in \Lambda$ is SSD efficient if and only if no portfolio $\boldsymbol{?} \in \Lambda$ SSD dominates it.*

Interestingly, this definition can be rephrased in terms of a minimax formulation:[5]

THEOREM 1 *Portfolio $\boldsymbol{t} \in \Lambda$ is SSD inefficient if and only if, for all utility functions $u \in U_2$, the maximum expected utility is greater than the expected utility of $\boldsymbol{t}$, i.e.*

(4) $$\min_{u\in U_2}\left\{\max_{?\in\Lambda}\left\{\int u(\boldsymbol{x}\boldsymbol{?})\partial F(\boldsymbol{x}) - \int u(\boldsymbol{x}\boldsymbol{t})\partial F(\boldsymbol{x})\right\}\right\} = \min_{u\in U_2}\left\{\max_{?\in\Lambda}\left\{\sum_{t\in\Theta}\left(u(\boldsymbol{x}_t\boldsymbol{?}) - u(\boldsymbol{x}_t\boldsymbol{t})\right)/T\right\}\right\} > 0 .$$

Alternatively, portfolio $\boldsymbol{t} \in \Lambda$ is SSD efficient if and only if it is optimal relative to some utility functions $u \in U_2$, i.e.

(5) $$\min_{u\in U_2}\left\{\max_{?\in\Lambda}\left\{\int u(\boldsymbol{x}\boldsymbol{?})\partial F(\boldsymbol{x}) - \int u(\boldsymbol{x}\boldsymbol{t})\partial F(\boldsymbol{x})\right\}\right\} = \min_{u\in U_2}\left\{\max_{?\in\Lambda}\left\{\sum_{t\in\Theta}\left(u(\boldsymbol{x}_t\boldsymbol{?}) - u(\boldsymbol{x}_t\boldsymbol{t})\right)/T\right\}\right\} = 0 .$$

[4] Similar arguments are used in production analysis to establish the empirical equivalence of strong and weak measures of productive efficiency in case input-output data are generated by a continuous distribution (see e.g. Kuntz and Scholtes, 2000).
[5] Similar minimax formulations exist for many weak measures of productive efficiency, including the well-known Debreu (1951)-Farrell (1957) measures.

II. LINEAR PROGRAMMING FORMULATION

The nonparametric approach to consumer analysis constructs piecewise-linear utility functions that are consistent with optimizing behavior in the sense that observed behavior (consumer data) is optimal relative to these utility functions (e.g. Afriat, 1967, Varian, 1982). In this spirit, we may ask ourselves if we can construct piecewise-linear utility functions $p \in U_2$ that rationalize the evaluated portfolio $\boldsymbol{t} \in \Lambda$ (i.e. for which $\boldsymbol{t}$ is the optimal choice). A piecewise-linear utility function $p \in U_2$ may be constructed from a series of T linear support lines characterized by intercept coefficients $\boldsymbol{a} \equiv (\boldsymbol{a}_1 \cdots \boldsymbol{a}_T) \in \Re^T$ and (normalized) slope coefficients $\boldsymbol{ß} \in \mathrm{B} \equiv \left\{ \boldsymbol{ß} \in \Re_+^T : \boldsymbol{ß}_1 \geq \boldsymbol{ß}_2 \geq \cdots \geq \boldsymbol{ß}_T = 1 \right\}$ as

(6) $$p(x|\boldsymbol{a},\boldsymbol{ß}) \equiv \min_{t \in \Theta} (\boldsymbol{a}_t + \boldsymbol{ß}_t x).$$

THEOREM 2 *Portfolio* $\boldsymbol{t} \in \Lambda$ *is SSD efficient if and only if* $\boldsymbol{t}$ *is optimal relative to a piecewise-linear utility function* $p \in U_2$. *We may test this condition using the SSD test statistic*

(7) $$x(\boldsymbol{t}) \equiv \min_{\boldsymbol{ß} \in B, q} \left\{ q : \sum_{t \in \Theta} \boldsymbol{ß}_t (\boldsymbol{x}_t \boldsymbol{t} - \boldsymbol{x}_{it}) / T + q \geq 0 \quad \forall i \in \mathrm{I} \right\}.$$

Specifically, portfolio $\boldsymbol{t} \in \Lambda$ *is SSD efficient if and only if* $x(\boldsymbol{t}) = 0$.

The test statistic $x(\boldsymbol{t})$ basically asks if we can find support lines for a strictly increasing and concave piecewise-linear utility function that rationalizes the evaluated portfolio. If the evaluated portfolio is efficient, then such support lines must exist, and if such support lines exist then the portfolio must be efficient. The necessary and sufficient condition can separate efficient portfolios from inefficient ones. However, we stress that the test statistic does not represent a meaningful performance measure that can be used for ranking portfolios based on the 'degree of efficiency'. For selecting the optimal portfolio from the efficient set, and for measuring the deviation from optimum, we typically need more information on investor preferences than is assumed in SD.

The test statistic $x(\boldsymbol{t})$ involves a linear objective function and linear constraints, and it can be solved using straightforward linear programming. The following is a full LP formulation for $x(\boldsymbol{t})$:

(P) $$\min_{q, \boldsymbol{ß}} q$$

$$\text{s.t.} \sum_{t \in \Theta} \boldsymbol{ß}_t (\boldsymbol{x}_t \boldsymbol{t} - \boldsymbol{x}_{it}) / T + q \geq 0 \quad i = 1, \cdots, N \quad (\boldsymbol{?}_i)$$

$$\beta_t - \beta_t \geq 0 \quad t = 1, \cdots, T\text{-}1 \quad (r_t)$$
$$\beta_t \geq 0 \quad t = 1, \cdots, T\text{-}1$$
$$\beta_T = 1$$
$$q \quad \text{free}$$

The shadow prices to the restrictions are given within brackets. This information is useful for interpreting the dual formulation (see Section III). The problem involves only T variables and N+T-1 constraints. Further, the model always has a feasible solution, as e.g. $\beta_t = 1$ for all $t \in \Theta$, and $q = \max_{i \in \mathrm{I}} \sum_{t \in \Theta} (\boldsymbol{x}_t \boldsymbol{t} - \boldsymbol{x}_{it}) / T$, necessarily satisfies all constraints. (This solution effectively represents risk neutral investors; risk neutral investors have linear utility functions and compare portfolios solely in terms of the expected return.) For small data sets up to hundreds of observations and/or assets, the problem can be solved with minimal computational burden, even with desktop PCs and standard solver software (like LP solvers included in spreadsheets). Still, the computational complexity, as measured by the required number of arithmetic operations, and hence the run time and memory space requirement, increases progressively with the number of variables and restrictions. Therefore, specialized LP solver software is recommended for large-scale problems involving thousands of observations and/or assets.[6]

In addition to the value of the test statistic, the model gives information on the shape of the optimal piecewise-linear utility function. Specifically, the optimal value for each β_t variable (say β_t^*) represents the slope of the t-th support line of the piecewise-linear utility function. We may recover a complete piecewise-linear utility function in the following manner:

(8)
$$u^*(x) = \begin{cases} \sum_{s=1}^{T-1} (\beta_{s+1}^* - \beta_s^*) z_s + \beta_1^* x & x \leq z_1 \\ \sum_{s=2}^{T-1} (\beta_{s+1}^* - \beta_s^*) z_s + \beta_2^* x & z_1 \leq x \leq z_2 \\ \vdots & \\ (1 - \beta_{T-1}^*) z_{T-1} + \beta_{T-1}^* x & z_{T-2} \leq x \leq z_{T-1} \\ x & x \geq z_{T-1} \end{cases}$$

with $z_t \equiv 0.5(\boldsymbol{x}_t + \boldsymbol{x}_{t+1})\boldsymbol{t}$ for $t \in \Theta \setminus T$ (see the proof to Theorem 2 in the Appendix for a formal proof).[7,8]

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[6] For an elaborate introduction in LP, we refer to Chvatal (1983). In practice, very large LPs can be solved efficiently by both the simplex method and interior-point methods. An elaborate guide to LP solver software can be found at the homepage of the Institute for Operations Research and Management Science (INFORMS); http://www.informs.org/.

[7] Alternatively, the intercept coefficients can be includes as variables in the LP model, provided appropriate restrictions are imposed. To minimize computational burden, we have chosen the formulation that minimizes the number of variables and constraints.

[8] We use $\Theta \setminus T$ to denote Θ excluding T, i.e. $\{1, \cdots, T-1\}$.

If the evaluated portfolio is SSD efficient, then the empirical utility function gives an example of the type of utility functions that rationalize the portfolio. Still, we stress the empirical utility function is not intended as an estimate for the utility function of the investor that holds the evaluated portfolio. One complication for that interpretation is the possibility of multiple optimal solutions for the slope coefficients. Further, for every given set of slope coefficients, the empirical utility function is not unique, as we can choose any $z_t \in \left[\boldsymbol{x}_t \boldsymbol{t}, \boldsymbol{x}_{t+1}\boldsymbol{t}\right]$. Another complication is the sensitivity of the piecewise-linear utility function for sampling error. While it is possible to perform powerful and accurate inference about the test statistic (at least in large samples; see Section V), this seems much more complicated for recovering an entire utility function; the number of slope coefficients grows with the length of the time series. We also stress that the interpretation of the empirical utility function is not clear if the evaluated portfolio is SSD inefficient. At first sight, the empirical function might be interpreted as the 'most-favorable utility function', i.e. the function that puts the evaluated portfolio in the best possible light by minimizing the shortfall in expected utility, and the test statistic might be interpreted as a measure of that minimum shortfall. However, this interpretation generally is incorrect. The test evaluates the observation $\boldsymbol{x}_{it}, i \in \mathrm{I}, t \in \Theta$, at the slope coefficient for the associated time period, i.e. $\boldsymbol{\beta}_t$. By contrast, the empirical utility function evaluates the observation at the slope coefficient for the associated return interval, i.e. $\boldsymbol{\beta}_s$ if $\boldsymbol{x}_{it}$ is contained in the *s*-th element of $\{\langle-\infty, z_1], [z_1, z_2], \cdots, [z_{T-2}, z_{T-1}], [z_{T-1}, \infty\rangle\}$. Brief, the purpose of our analysis is to test if a given portfolio is SSD efficient; the empirical utility function is a mere side-product of the analysis, and we can not derive strong results for it.

The EDFs of two different portfolios can cross *T*-1 times at maximum. In many applications, the EDFs cross only a few times (see e.g. Section VI). This is only natural, as cumulated differences are 'sticky', and a large negative difference (or alternatively a series of small negative differences) is generally required to change from a positive sign to a negative sign. Our test generalizes the crossing algorithms to the case where diversification is allowed, and it is affected in a similar way by the stickiness of cumulated differences. The piecewise linear utility function can consist of *T* different line segments and hence exhibit *T*-1 kinks. However, in many applications, large parts of the utility function will have the same slope coefficient, and only few kinks will occur (see e.g. Section VI). Section III explains this phenomenon in terms of the stickiness of cumulated differences.

Our analysis allows for including a riskless asset in the analysis, and thus covers not only SSD but the more powerful SSDR as well (see, for example, Levy, 1998, Section 4.3). Of course, the return observations for a riskless asset involve ties, and hence we have to account for multiple orderings of the observations. A simple solution to this problem is available. Specifically, if $\boldsymbol{x}_t \boldsymbol{t} = \boldsymbol{x}_{t+1}\boldsymbol{t}$ for some $t \in \{1, \cdots, T-2\}$, then we can drop the restriction $\boldsymbol{\beta}_t \geq \boldsymbol{\beta}_{t+1}$ and replace it with the restriction $\boldsymbol{\beta}_t \geq \boldsymbol{\beta}_{t+2}$ plus the restriction $\boldsymbol{\beta}_{t-1} \geq \boldsymbol{\beta}_{t+1}$ if $t \in \{2, \cdots, T-2\}$. In addition, if $\boldsymbol{x}_{T-1}\boldsymbol{t} = \boldsymbol{x}_T \boldsymbol{t}$, then we can replace the restriction $\boldsymbol{\beta}_{T-1} \geq \boldsymbol{\beta}_T = 1$ with the restriction $\boldsymbol{\beta}_{T-1}, \boldsymbol{\beta}_T \geq 1$. The following example with two time periods, one risky asset, and one riskless asset can provide some feeling for Theorem 2, as well as the treatment of ties:

	x_1	x_2
$t=1$	$r-a$	r
$t=2$	$r+b$	r

where *r* is the riskless return and *a*, *b*>0. In this case, we know that the risky asset, x_1, is SSD efficient if and only if $b \geq a$, and the risk free-asset, x_2, is SSD efficient for all *a*, *b*>0 (e.g. Arrow, 1971). Theorem 2 states that x_1 is efficient if and only if we can find $ß_1$ and $ß_2$ such that $ß_1 \geq ß_2 = 1$ and $ß_1((r-a)-r) + ß_2((r+b)-r) \geq 0$ $\Leftrightarrow bß_2 - aß_1 \geq 0$. Such $ß_1$ and $ß_2$ can be found if and only if $b \geq a$. In this case, the most favorable utility function is simply $u(x) = x$; comparison based on the average return puts x_1 in the best possible light. We now turn to testing efficiency for the riskless asset, x_2. The returns of x_2 are tied, and we therefore have to drop the restrictions $ß_1 \geq ß_2 = 1$, and replace it with $ß_1,ß_2 \geq 1$. Theorem 2 then states that x_2 is efficient if and only if we can find coefficients $ß_1,ß_2 \geq 1$ such that $ß_1(r-(r-a)) + ß_2(r-(r+b)) \geq 0$. For all *a*, *b*>0, we can find such weights, and hence x_2 is always efficient. For example, x_2 is the optimal portfolio for all investors with utility function $u(x) = \begin{cases} (1-c)r + cx & x \leq r \\ x & x \geq r \end{cases}$, with $c \geq b/a$. These results exactly conform with the known results for this particular case.

III. DUAL FORMULATION

In a recent study, Bowden (2000) introduced a novel statistic termed the Ordered Mean Difference (OMD). Using our notation, the OMD is defined as follows:

$$(9) \qquad \boldsymbol{r}_t(\boldsymbol{?},\boldsymbol{t}) \equiv \sum_{s=1}^{t} (\boldsymbol{x}_s \boldsymbol{l} - \boldsymbol{x}_s \boldsymbol{t})/t \quad t \in \Theta.$$

The OMD represents a running mean for the difference between the return of the benchmark portfolio $\boldsymbol{?}$ and the evaluated portfolio $\boldsymbol{t}$. Since the returns are ranked on the basis of the return of the evaluated portfolio, the OMD is different from the cumulated difference between the quantiles of the EDF of $\boldsymbol{?}$ and $\boldsymbol{t}$. (The quantiles of the EDF of $\boldsymbol{?}$ require ranking based on $\boldsymbol{x}_t\boldsymbol{?}$ rather than $\boldsymbol{x}_t\boldsymbol{t}$; the OMD uses a 'less favorable' ranking and hence gives an upper bound for the cumulated difference between the quantiles.)

The OMD was introduced by Bowden to measure (in pairwise fashion) the performance of the evaluated portfolio relative to a benchmark portfolio. Interestingly, SSD in case of portfolio diversification can also be defined in terms of this statistic:

THEOREM 3 *Portfolio $\boldsymbol{t} \in \Lambda$ is SSD efficient if and only if, for some $\boldsymbol{?} \in \Lambda$, the OMD $\boldsymbol{r}_t(\boldsymbol{?},\boldsymbol{t})$ is greater than or equal to zero for all $t \in \Theta$, with at least one strict inequality. We may test this condition using the test statistic*

(10) $\quad y(\boldsymbol{t}) \equiv \max_{?\in\Lambda} \left\{ r_T(\boldsymbol{?}, \boldsymbol{t}) : r_t(\boldsymbol{?}, \boldsymbol{t}) \geq 0 \quad \forall t \in \Theta \setminus T \right\}.$

Specifically, portfolio $\boldsymbol{t} \in \Lambda$ *is SSD efficient if and only if* $y(\boldsymbol{t}) = 0$.

The test statistic $y(\boldsymbol{t})$ in fact is a dual formulation of the test statistic $x(\boldsymbol{t})$. This statistic can also be computed using straightforward LP. (The Appendix gives a formal proof of the equivalence between $y(\boldsymbol{t})$ and $x(\boldsymbol{t})$, as well as a full LP formulation of $y(\boldsymbol{t})$).

In addition to the value of the SSD test statistic, the dual identifies a solution portfolio. Specifically, the optimal values of the dual variables $\boldsymbol{?}_i$, $i \in \mathrm{I}$, represent the shadow prices for the constraints $\sum_{t\in\Theta} \boldsymbol{ß}_t(\boldsymbol{x}_t\boldsymbol{t} - \boldsymbol{x}_{it})/T + q \geq 0, i \in \mathrm{I}$. These constraints are binding for the assets that constitute the portfolio that maximizes the value of the test statistic, and the optimal $\boldsymbol{?}$ gives the composition of this portfolio. Since the solution portfolio is found by maximizing the highest order OMD subject to restrictions on the lower order OMDs, the portfolio will be efficient in many cases. However, the solution portfolio is selected using the ranking for the evaluated portfolio rather than the less favorable ranking of the solution portfolio itself. For this reason, the solution portfolio can in fact be inefficient. In addition, even if the solution portfolio is efficient, it need not SSD dominate the evaluated portfolio, i.e. an efficient portfolio need not be preferred to the evaluated portfolio for every utility function $u \in U_2$. In our opinion, efficiency or dominance of the solution portfolio is not a very important issue. It is relatively simple to identify efficient or even dominating portfolios. For example, one can take a specific utility function and find the portfolio that maximizes the expected utility for this function (see e.g. Yitzhaki, 1982, Kroll *et al.*, 1984, and the discussion in Levy, 1998, Section 14.1). Further, Shalit and Yitzhaki (1994) suggest an iterative approach to finding efficient portfolios that dominate the evaluated portfolio. However, even if a portfolio is efficient and even if it dominates the evaluated portfolio, then it is only one element of the SSD efficient set, and that there is no prior reason to prefer this portfolio to other efficient portfolios. For selecting the optimal portfolio, one typically needs more information than is assumed in SD. Hence, as is true for the empirical utility function discussed in section II, the solution portfolio is used as an instrument for testing if the evaluated portfolio is efficient, not as a portfolio that is efficient, dominant, or even optimal for a given investor.

The dual formulation can help explain why large parts of the piecewise linear utility function typically have the same slope coefficient (see Section II), i.e. $\boldsymbol{ß}_t = \boldsymbol{ß}_{t+1}$ for many $t \in \Theta \setminus T$. It follows from the proof to Theorem 3 that $tr_t(\boldsymbol{?}, \boldsymbol{t})/T$ represents the shadow price for the primal restriction $\boldsymbol{ß}_t - \boldsymbol{ß}_{t+1} \geq 0$, $t \in \Theta \setminus T$. Duality implies that $\boldsymbol{ß}_t - \boldsymbol{ß}_{t+1} > 0$ only if $r_t(\boldsymbol{?}, \boldsymbol{t}) = 0$, and $\boldsymbol{ß}_t - \boldsymbol{ß}_{t+1} = 0$ if $r_t(\boldsymbol{?}, \boldsymbol{t}) > 0$. The OMD is a running mean, and it is very 'sticky', as $r_{t+1}(\boldsymbol{?}, \boldsymbol{t}) = tr_t(\boldsymbol{?}, \boldsymbol{t})/(t+1) +$ $\boldsymbol{x}_{t+1}(\boldsymbol{?} - \boldsymbol{t})/(t+1)$. Therefore, if the evaluated portfolio is inefficient i.e. $r_T(\boldsymbol{?}, \boldsymbol{t}) > 0$, then there typically exist few $t \in \Theta \setminus T$ for which $r_T(\boldsymbol{?}, \boldsymbol{t}) = 0$ and hence $\boldsymbol{ß}_t - \boldsymbol{ß}_{t+1} > 0$.

IV. THIRD-ORDER STOCHASTIC DOMINANCE

The SSD criterion relies on the assumptions of non-satiation and risk aversion solely. By imposing minimal assumptions, the criterion can involve low discriminating power, i.e. the efficient set can be large. Fortunately, the above analysis can be extended in a straightforward way towards more powerful higher-order SD criteria. In this section, we will discuss the generalization towards TSD. For the sake of compactness, we will focus on the primal formulation in terms of empirical utility functions only; linear duality theory can obtain a dual formulation by analogy to our treatment of SSD in Section III.

For analytical simplicity, we now assume that the utility function is once continuously differentiable, and we will use $u'(x)$ for the gradient or 'marginal utility function' at x, and $u'(\boldsymbol{t}) \equiv (u'(\boldsymbol{x}_1\boldsymbol{t})\cdots u'(\boldsymbol{x}_T\boldsymbol{t}))$ for the gradient vector. TSD focuses on the class of non-satiable and risk-averse investors that prefer positively skewed distributions (more probability in the right tail).[9] Interestingly, empirical evidence suggests that investors display this kind of skewness preference (e.g. Arditti, 1967, Kraus and Litzenberger, 1976, Cooley, 1977, and Friend and Westerfield, 1980). The TSD investors can be represented by the class of von Neuman-Morgenstern utility functions with a strictly positive, decreasing and convex marginal utility function, or $U_3 \subseteq U$. Note that we do not assume that the marginal utility function is continuously differentiable. However, marginal utility is convex and hence everywhere continuous and subdifferentiable. Throughout the text, we will denote subgradient vectors of the marginal utility function by $\partial u'(\boldsymbol{t}) \equiv (\partial u'(\boldsymbol{x}_1\boldsymbol{t})\cdots\partial u'(\boldsymbol{x}_T\boldsymbol{t}))$.

By analogy to Definition 2 and Theorem 1, TSD can be defined as follows:

DEFINITION 3 *Portfolio $\boldsymbol{t} \in \Lambda$ is TSD inefficient if and only if, for all utility functions $u \in U_3$, the maximum expected utility is greater than the expected utility of $\boldsymbol{t}$, i.e.*

(11) $$\min_{u\in U_3}\left\{\max_{?\in\Lambda}\left\{\int u(\boldsymbol{x?})\partial F(\boldsymbol{x}) - \int u(\boldsymbol{xt})\partial F(\boldsymbol{x})\right\}\right\} = \min_{u\in U_3}\left\{\max_{?\in\Lambda}\left\{\sum_{t\in\Theta}\left(u(\boldsymbol{x}_t\boldsymbol{?}) - u(\boldsymbol{x}_t\boldsymbol{t})\right)/T\right\}\right\} > 0.$$

Alternatively, portfolio $\boldsymbol{t} \in \Lambda$ is TSD efficient if and only if it is optimal relative to some utility functions $u \in U_3$, i.e.

(12) $$\min_{u\in U_3}\left\{\max_{?\in\Lambda}\left\{\int u(\boldsymbol{x?})\partial F(\boldsymbol{x}) - \int u(\boldsymbol{xt})\partial F(\boldsymbol{x})\right\}\right\} =$$

[9] This kind of skewness preference is strongly related to the concept of decreasing absolute risk aversion (DARA; Pratt, 1964), which underlies DARA SD (DSD; Vickson, 1975). Roughly speaking, DARA means that the dislike for absolute uncertainties decreases as the levels of the outcomes increase. Theoretically, TSD is a sufficient but not necessary condition for DARA, and TSD therefore is less powerful than DSD. However, DSD is difficult to fit to empirical data, and in addition the improvement in power is minimal. For these reasons, Vickson and Altman (1977) conclude that TSD is likely to be a suitable approximation for DSD for practical purposes.

$$\min_{u \in U_3} \left\{ \max_{? \in \Lambda} \left\{ \sum_{t \in \Theta} \left(u(\mathbf{x}_t\, ?) - u(\mathbf{x}_t \mathbf{t}\,) \right) / T \right\} \right\} = 0 \,.$$

Our approach to SSD relies on constructing *piecewise-linear* utility functions, and on checking if the evaluated portfolio is optimal relative to those utility functions. This approach cannot be extended directly towards TSD, because piecewise-linear utility functions exhibit risk neutrality on the line segments, and therefore these utility functions exhibit skewness preference only if they consist of a single line. Still, we may ask ourselves if we can construct *piecewise-quadratic* utility functions $p' \in U_3$ that 'rationalize' the evaluated portfolio $\mathbf{t} \in \Lambda$. A continuous piecewise-quadratic utility function may be constructed from intercept coefficients $\mathbf{a} \equiv (\mathbf{a}_1 \cdots \mathbf{a}_T)$, slope coefficients $\mathbf{ß} \equiv (\mathbf{ß}_1 \cdots \mathbf{ß}_T)$ and curvature coefficients $\mathbf{?} \equiv (\mathbf{?}_1 \cdots \mathbf{?}_T)$ as

(13) $p'(x | \mathbf{a}, \mathbf{ß}, \mathbf{?}) \equiv \min_{t \in \Theta} (\mathbf{a}_t + \mathbf{ß}_t x + 0.5 \mathbf{?}_t x^2)$.

Imposing appropriate restrictions on $\mathbf{a}$, $\mathbf{ß}$, and $\mathbf{?}$ can guarantee that p' exhibits monotonicity, concavity, and skewness preference.

THEOREM 4 *Portfolio* $\mathbf{t} \in \Lambda$ *is TSD efficient if and only if* $\mathbf{t}$ *is optimal relative to a piecewise-quadratic utility function* $p' \in U_3$. *We may test this condition using the test statistic*

(14) $$z\,(\mathbf{t}\,) \equiv \min_{(\mathbf{ß}, \mathbf{?}) \in \Omega, q} \left\{ q : \sum_{t \in \Theta} (\mathbf{ß}_t + \mathbf{?}_t \mathbf{x}_t \mathbf{t}\,)(\mathbf{x}_t \mathbf{t} - \mathbf{x}_{it}) / T + q \geq 0 \quad \forall i \in \mathrm{I} \right\},$$

with

(15) $$\Omega \equiv \{ (\mathbf{ß}, \mathbf{?}) \in \Re^T \times \Re^T_- : \mathbf{ß}_t + \mathbf{?}_t \mathbf{x}_t \mathbf{t} \geq \mathbf{ß}_{t+1} + \mathbf{?}_{t+1} \mathbf{x}_{t+1} \mathbf{t}\,;$$
$$\mathbf{ß}_t + \mathbf{?}_t \mathbf{x}_{t+1} \mathbf{t} \leq \mathbf{ß}_{t+1} + \mathbf{?}_{t+1} \mathbf{x}_{t+1} \mathbf{t}\,; \mathbf{ß}_t + \mathbf{?}_t \mathbf{x}_t \mathbf{t} \geq \mathbf{ß}_{t+1} + \mathbf{?}_{t+1} \mathbf{x}_t \mathbf{t}\,;$$
$$\mathbf{?}_t \leq \mathbf{?}_{t+1} \; \forall t \in \Theta \backslash T; \mathbf{ß}_T + \mathbf{?}_T \max_{\substack{i \in \mathrm{I} \\ t \in \Theta}} \mathbf{x}_{it} = 1 \} .$$

Specifically, portfolio $\mathbf{t} \in \Lambda$ *is TSD efficient if and only if* $z\,(\mathbf{t}\,) = 0$.

The test statistic $z\,(\mathbf{t}\,)$ involves a linear objective function and linear constraints, and it can be computed using straightforward linear programming. The problem involves $2T+1$ variables and $N+4T-3$ constraints. Further, the model always has a feasible solution, as e.g. $\mathbf{ß}_t = 1$, $\mathbf{?}_t = 1$ for all $t \in \Theta$ and $q = \sum_{t \in \Theta} (\mathbf{x}_{it} - \mathbf{x}_t \mathbf{t}\,) / T$ (the 'risk-neutral' solution) necessarily satisfies all constraints.

V. STATISTICAL INFERENCE

We have thus far discussed SD relative to the EDF rather than the CDF. Generally, the EDF is very sensitive to sampling variation and the test results are likely to be affected by sampling error in a non-trivial way. For example, if we compare two alternatives with the same population distribution, then we know that there exists no pairwise dominance relationship in the population. Still, the probability of finding a dominance relationship based on two independent random samples of 1000 observations can be as high as 50 percent (see Dardanoni and Forcina, 1999). In addition, the outcomes of various simulation studies (Kroll and Levy, 1980, Stein *et al.*, 1986, among others) cause serious doubt about the reliability of SD applications that rely in a naïve way on the EDF without accounting for sampling error. The applied researcher must therefore have knowledge of the sampling distribution in order to make inferences about the true efficiency classification. In the SD literature, two approaches have been developed to approximating the sampling distribution: bootstrapping and analytical asymptotic analysis.

The bootstrap method, first introduced by Efron (1979) and Efron and Gong (1983), is a well-established tool to analyze the sensitivity of empirical estimators to sampling variation in situations where the sampling distribution is difficult or impossible to obtain analytically. Bootstrapping is based on the idea of repeatedly simulating the CDF, usually through resampling, and applying the original estimator to each simulated sample or pseudo-sample so that the resulting estimators mimic the sampling distribution of the original estimator. Key to the success of the bootstrap is the selection of an appropriate approximation for the CDF. If the approximation is statistically consistent, then the bootstrap distribution gives a statistically consistent estimator for the original sampling distribution. In the context of our SD tests, the EDF is an appropriate approximation for the CDF; under the assumption that the return distribution is serially IID (see section I), the EDF is a consistent estimator of the true CDF. This suggests bootstrapping samples would be simply obtained by randomly sampling with replacement from the EDF along the lines of the 'correlation model' proposed by Freedman (1981) in a regression framework. Nelson and Pope (1991) demonstrated in a convincing way that this approach can quantify the sensitivity of the EDF to sampling variation, and that SD analysis based on the bootstrapped EDF is more powerful than comparison based on the original EDF.[10] The computational ease of the crossing algorithms allows for substituting brute computational force to overcome the analytical intractability of SD. Interestingly, the tractable LP structure of our tests suggests that it is possible also in the case with portfolio diversification to substitute brute computational force to overcome analytical intractability.

The alternative approach is to derive an analytical characterization of the asymptotic sampling distribution (see e.g. Beach and Davidson, 1983, Dardanoni and Forcina, 1999, Davidson and Duclos, 2000). The literature thus far invariantly deals with comparing a finite number of choice alternatives, and it is not immediately clear how

[10] Interestingly, bootstrapping is used successfully also for the nonparametric approach to analyzing productive efficiency, which has a similar structure as SD analysis. See Simar and Wilson (1998), among others.

to generalize the existing results towards the case with diversification possibilities.[11] The remainder of this section therefore develops an asymptotic test that is especially tailored to our SSD test statistic $x(\boldsymbol{t})$. There are various hypotheses that could serve as the null hypothesis in a test procedure. In the analysis of pairwise SD dominance, a typical null hypothesis is that the two choice alternatives are independent random variables with the same population distribution, or alternatively the choice alternatives are contemporaneously IID. We generalize this approach towards our case by using the null that all *N* assets are contemporaneously IID, and hence $G(\boldsymbol{x}) = \prod_{i \in I} H(\boldsymbol{x}_i)$ with $H : \Re \rightarrow [0,1]$ for a univariate CDF with variance $s^2 < \infty$. This is a very conservative hypothesis, because minimal sampling variation suffices to classify an efficient portfolio as inefficient. The shape of the distribution of $x(\boldsymbol{t})$ under the null generally depends on the shape of $H(x)$. Our approach will be to focus on the least favorable distribution, i.e. the distribution that maximizes the size or relative frequency of Type I error (rejecting the null when it is true). This approach stems from the desire to be protected against Type I error. For each $H(x)$, the size is always smaller than the size for the least favorable distribution. Interestingly, the least favorable distribution is relatively simple and known results can derive the asymptotic probability of exceedance or *p*-value for $x(\boldsymbol{t})$. The use of the most favorable distribution implies that we accept a high frequency of Type II error (accepting the null when it is not true) or a low power (1- the relative frequency of Type II error). Future research could focus on tests that minimize Type II error.[12]

THEOREM 5 *For the asymptotic least favorable distribution of* $x(\boldsymbol{t})$, *the p-value* $P(x(\boldsymbol{t}) \geq y | H_0)$, $y \geq 0$, *equals the integral* $1 - \int_{x \leq y\boldsymbol{e}} \partial\Phi(\boldsymbol{x})$ *with* $\Phi(\boldsymbol{x})$ *for the N-dimensional multivariate normal distribution function with zero means, variance terms*

$$(16) \qquad s_i^2 \equiv (\sum_{k \in \mathrm{I}} \boldsymbol{t}_k^2 - 2\boldsymbol{t}_i + 1)s^2/T,\ i \in \mathrm{I},$$

and covariance terms

$$(17) \qquad s_{ij} \equiv (\sum_{k \in \mathrm{I}} \boldsymbol{t}_k^2 - \boldsymbol{t}_i - \boldsymbol{t}_j)s^2/T,\ i, j \in \mathrm{I} : i \neq j.$$

The theorem shows the crucial role of the length of the time series (*T*) and the length of the cross-section (*N*); the *p*-values decrease as the time series grows, and increase as the cross section grows. For small time series and large cross-sections, the *p*-values are very large and a naïve approach to the test statistic (reject efficiency if $x(\boldsymbol{t}) > 0$) is unlikely to yield anything but noise. A more sound approach is to compare the *p*-

[11] Presumably, this is because much of the research on SD is aimed at ordering distributions of poverty, welfare and inequality, an application area that typically involves the comparison of a small set of distributions (e.g. for a set of countries) and diversification between the distributions is not allowed.

[12] SD is typically used to reduce the number of choice alternatives that are considered in a follow-up analysis that is intended to select the optimal portfolio. In this context, a Type I error is problematic because it may exclude the optimal choice alternative. By contrast, a Type II error merely increases the number of alternatives for further analysis.

value for the observed value of $x(t)$ with a predefined level of significance; we may reject efficiency if the *p*-value is smaller than or equal to the significance level. The following results are useful for implementing this approach in practice:

1. Computing the *p*-value requires the unknown population variance s^2. We may estimate this parameter in a distribution-free and consistent manner using the sample equivalent :

$$\text{(18)} \qquad \hat{s}^2 \equiv \sum_{\substack{i \in I \\ t \in \Theta}} (x_{it} - \sum_{\substack{i \in I \\ t \in \Theta}} x_{it} / NT)^2 / NT .$$

2. Many authors have addressed the problem of computing or approximating the multivariate normal integral. Somerville (1998) provides useful references as well as a general methodology (and Fortran 90 computer programs).

To assess the goodness of the above approach, we perform a simulation experiment. We analyze 25 risky alternatives with a multivariate normal return distribution. The joint population moments are equal to sample moments for the 25 Fama and French benchmark portfolios used in Section VI (see Table 1 for descriptive statistics). Figure 1 gives a mean-variance diagram with the individual benchmark portfolios (the bright dots), as well as the mean-variance frontier (the dark, curved line segment)[13]. Our test is based on a conservative null where the individual alternatives are not correlated. By contrast, the Fama and French benchmark portfolios are very highly correlated. This is reflected in the fact that the efficient frontier tightly envelops the individual benchmark portfolios. Since a high correlation generally reduces the variability for the differences between the alternatives, our test is likely to be very conservative for this application.

We analyze the SSD efficiency of two different test portfolios constructed from the benchmark portfolios. The first test portfolio (P1) is efficient; it is selected by maximizing the mean subject to the constraint that the standard deviation is smaller than or equal to 0.0725. This portfolio consists of only three components: 48.2 percent of benchmark portfolio 5, 1.3 percent of portfolio 12, and 50.5 percent of portfolio 14. The second test portfolio (P2) is found as the equally weighed average of all 25 benchmark portfolios. This portfolio is inefficient; it is possible to achieve a higher mean given the standard deviation, and to achieve a lower standard deviation given the mean. Both test portfolios are included in Figure 1 (the dark dots).

[13] Note that this is the frontier for the case without short selling. Again, our tests are developed for the case where the portfolio possibilities are described by all convex combinations of the individual assets; see Footnote 3.

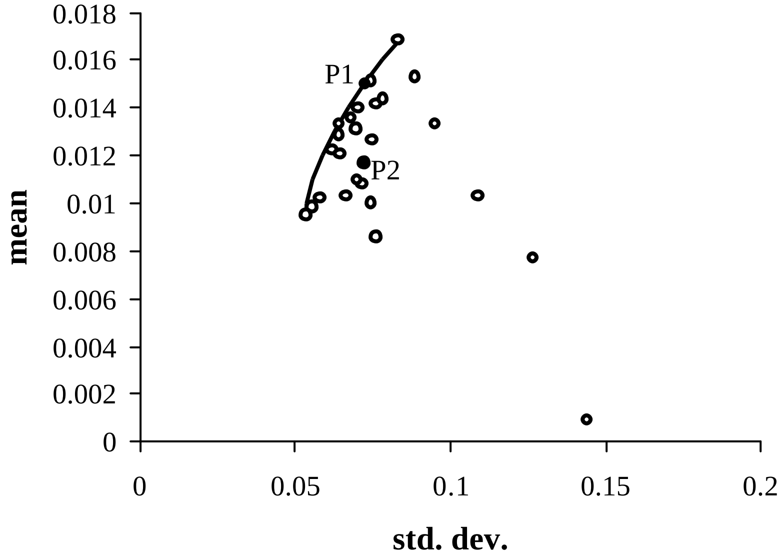

Figure 1: Mean-variance diagram of the Fama and French benchmark portfolios (the bright dots), as well as the efficient test portfolio (P1) and the inefficient equally weighted test portfolio (P2). The dark, curved line segment represents the efficient frontier.

In this case, we know that P1 is SSD efficient in the population and P2 is SSD inefficient. Sampling errors complicate the empirical determination of these known efficiency classifications. A Type I error occurs if the efficient P1 is wrongly classified as inefficient; a type II error occurs if the inefficient P2 is wrongly classified as efficient. To assess the size and the power of our test, we draw through Monte-Carlo simulation 1000 random samples from the multivariate normal population distribution, and apply the test to each random sample. This experiment is performed for samples of 25 to 5000 observations and for nominal levels of significance (a) of 2.5, 5 and 10 percent. Figure 2 gives the size of the test. For all sample sizes, the size of the test is much lower than the nominal level of significance, which reflects the conservative nature of our test (for large samples, the size even approximates zero). Figure 3 gives the power of the test. For this kind of application, we need at least 1500 to 2000 observations for a test with reasonable power. Fortunately, large data sets are available for many applications in financial economics. In addition, the test is likely to be more powerful in applications that involve individual stocks rather than stock portfolios; the correlation between individual stocks typically is much lower than for portfolios. For example, using a zero correlation in our simulation experiment for $T = 1000$ yields an increase in power that can be compared with doubling the sampling size.

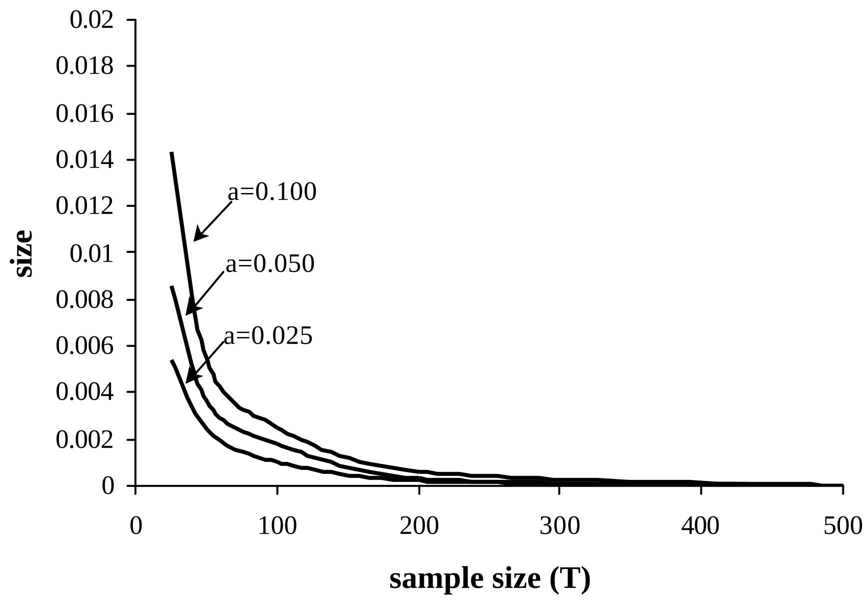

Figure 2: Size of the SSD test applied to the efficient test portfolio. The figure displays the size for nominal levels of significance of 2.5, 5 and 10 percent.

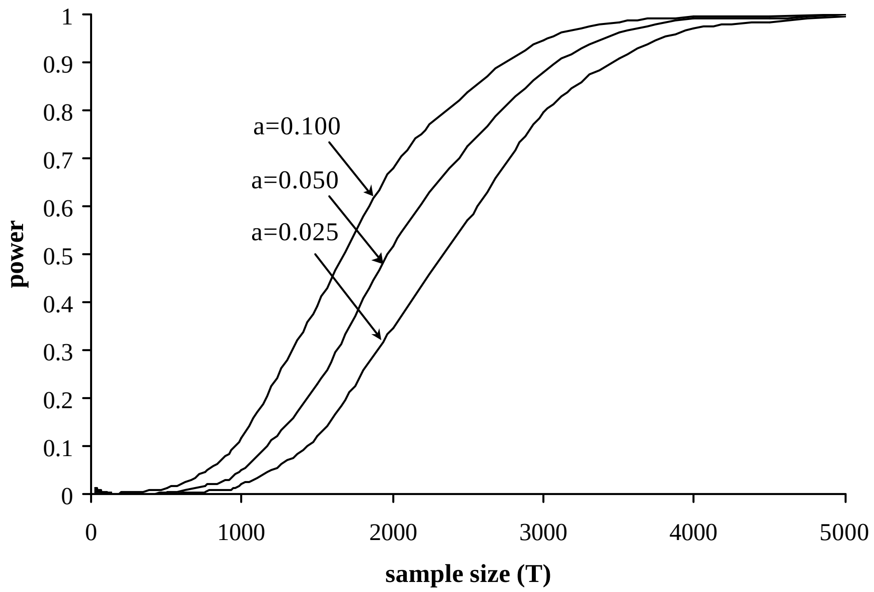

Figure 3: Power of the SSD test applied to the inefficient equally weighed test portfolio. The figure displays the power for nominal levels of significance of 2.5, 5 and 10 percent.

VI. EMPIRICAL APPLICATION

To illustrate our approach to SD with diversification, we perform an empirical application to real-life US stock market data. Specifically, we evaluate whether the Standard and Poors 500 (S&P 500) index is SSD efficient relative to all possible portfolios of the 25 Fama and French benchmark portfolios. The benchmark portfolios are the intersections of 5 portfolios formed on size (market equity) and 5 portfolios formed on the ratio of book equity to market equity (BE/ME). The benchmark portfolios are constructed from all NYSE, AMEX, and NASDAQ stocks. By contarst, the S&P 500 index is based on a much smaller set of stocks (until 1979, the index was even limited to NYSE stocks only). We use data on monthly returns (month-end to month-end) from July 1926 to December 2000 (894 observations) obtained from the data library on the homepage of Kenneth French (http://web.mit.edu/kfrench/www/). Table 1 gives some descriptive statistics for these data.

Table 1: Descriptive statistics of the monthly returns (month-end to month-end) from July 1926 to December 2000 for the S&P 500 index and the 25 Fama and French benchmark portfolios.

Portfolio				Mean	Standard Deviation	Skewness	Kurtosis
S&P 500				0.0063	0.0463	0.7735	19.1381
Fama and French benchmark portfolios	No.	BE/ME	Size				
	1	Low	Small	0.0077	0.1266	2.8118	27.9185
	2	2	Small	0.0103	0.1091	3.9491	48.6091
	3	3	Small	0.0133	0.0951	2.0006	16.8556
	4	4	Small	0.0153	0.0887	2.7616	29.3945
	5	High	Small	0.0168	0.0983	3.2252	30.4773
	6	Low	2	0.0086	0.0805	0.4236	5.1488
	7	2	2	0.0127	0.0789	1.8290	19.7922
	8	3	2	0.0136	0.0754	2.3155	24.3154
	9	4	2	0.0140	0.0768	1.8028	18.6467
	10	High	2	0.0151	0.0877	1.6950	15.9496
	11	Low	3	0.0100	0.0770	1.0103	9.9357
	12	2	3	0.0122	0.0673	0.3120	7.0884
	13	3	3	0.0129	0.0685	0.9969	13.0653
	14	4	3	0.0133	0.0691	1.2614	13.8547
	15	High	3	0.0142	0.0870	1.9309	19.2382
	16	Low	4	0.0103	0.0628	-0.1606	3.7688
	17	2	4	0.0108	0.0639	1.0598	13.6494
	18	3	4	0.0121	0.0641	1.0731	14.9477
	19	4	4	0.0131	0.0715	1.9706	21.5985
	20	High	4	0.0144	0.0927	2.1360	21.9067
	21	Low	Big	0.0099	0.0557	-0.0270	5.5123
	22	2	Big	0.0095	0.0536	-0.0703	5.2748
	23	3	Big	0.0102	0.0581	0.7875	13.6679
	24	4	Big	0.0110	0.0702	1.8283	21.3741
	25	High	Big	0.0009	0.1441	-3.9635	32.3209

We first evaluate whether the S&P 500 index is dominated by SSD by any of the individual benchmark portfolios. For this purpose, we use the crossing algorithm by Levy (1992, Appendix A), as well as the more powerful LP algorithm for convex SD by Bawa *et al.* (1985). The outcomes suggest that the S&P 500 index is efficient relative to the benchmark portfolios. Next, we apply our LP models to check whether the S&P 500 index is SSD inefficient relative to all possible portfolios of the benchmark portfolios. Interestingly, the results suggest this is indeed true; we find $x(\boldsymbol{t}) = 0.0074$. The empirical piecewise-linear utility function (as obtained from the primal solution) is given by:

$$u^*(x) = \begin{cases} 0.6559 + 6.555x & x \le -0.1376 \\ 0.0831 + 2.392x & -0.0597 \ge x \ge -0.1376 \\ x & x \ge -0.0597 \end{cases}.$$

Figure 4 displays this utility function for our data set. The empirical utility function has a strongly kinked structure. This suggests that the S&P 500 index is relatively favorable for investors that are risk neutral for a large portion of the return range, but with a strong aversion for large negative returns.[14] Still, even these investors would not select the S&P 500 index as the optimal portfolio.

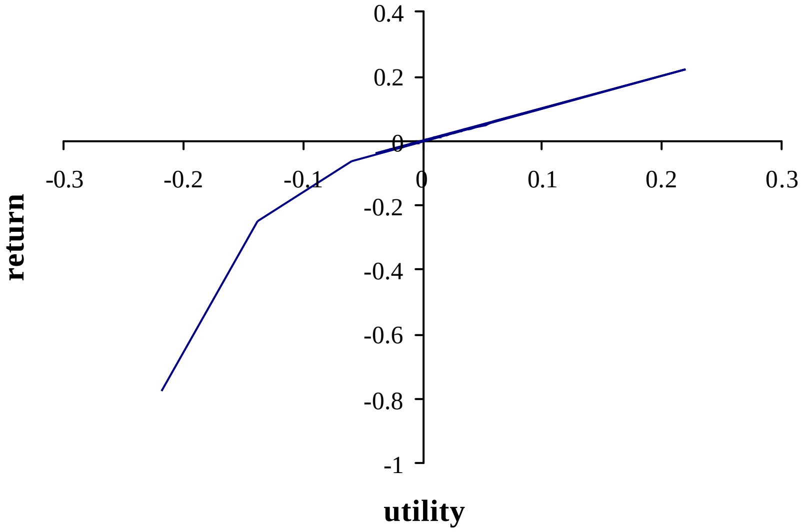

Figure 4: The empirical piecewise-linear objective function. The strongly kinked structure suggests that investors with a high aversion for large negative returns (but risk neutrality for most of the return range) judge the S&P 500 index relatively favorable. Still, even these investors do not select the S&P 500 index as the optimal portfolio.

The dual solution portfolio consists of 5.2 percent invested in benchmark portfolio 5, 86.3 percent invested portfolio 8, and 8.5 percent invested in portfolio 20. Not surprisingly, these three funds exhibited a relatively high average return during our sample period (see Table 1 above). Applying the SSD test to the solution portfolio itself, we find that the solution portfolio is SSD efficient. However, the solution portfolio does not SSD dominate the S&P 500 index, because it exhibited larger negative returns than the S&P 500 index during the sample period (especially during the stock market crash of October 1987). Again, our procedure is aimed at testing if the evaluated portfolio is SSD efficient, not at identifying an alternative portfolio that SSD dominates the evaluated portfolio.

[14] Risk neutral investors have a linear utility function and compare portfolios based on the expected return only. Indeed, $x(\mathbf{t}) = 0.0074$ comes very close to the difference in average return between the solution portfolio and the S&P 500 index. Still, a fully risk-neutral investor would invest exclusively in the fund with the maximum expected return (for our sample period: benchmark portfolio 5 with mean 0.0168).

The above analysis is based on empirical data rather than the true (unknown) return distribution, and the results are likely to be affected by sampling error in a non-trivial way. As discussed in Section V, the LP structure of our tests suggests that the brute force approach of bootstrapping can help assess the sampling properties of our results. We generate through random sampling with replacement from the EDF 1000 pseudo-samples, and compute the SSD statistic relative to each of the samples. Figure 5 gives the resulting bootstrap distribution of the SSD test statistic. In only 3 out of 1000 random pseudo-samples, the S&P 500 index is classified as efficient. In addition, the 90 percent bootstrap confidence interval (constructed using the bias-corrected and accelerated method described in Efron, 1987) is given by [0.0007, 0.0112], and it does not include the value zero. The asymptotic test developed in Section V gives another approach to sampling error. Again, the test is based on the asymptotic least favorable distribution and it involves low power for a sample of 894 observations (see Figure 3). Still, the *p*-value is as low as 8 percent in this case. These findings suggest that the S&P 500 index is SSD inefficient to a statistically significant degree.

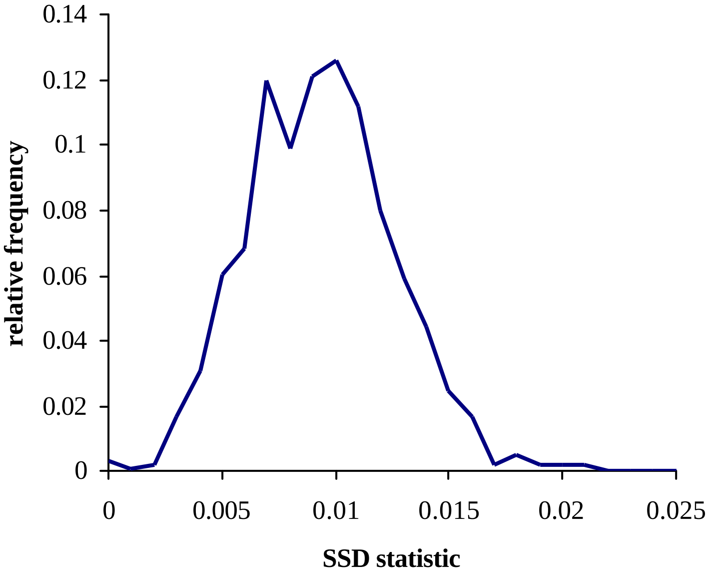

Figure 5: The bootstrap distribution of the SSD test statistic for 1000 replications. The simulations suggest that the S&P 500 index is SSD inefficient to a statistically significant degree.

Our results do not imply that investing in the S&P 500 index is irrational. For example, transactions costs not included in our model may 'rationalize' the index for some investors. The Fama and French benchmark indexes are not a real alternative for private investors that face relatively high transaction costs. Exchange-traded index funds and index trackers that follow the S&P 500 index are available and offer the diversification benefits of the index at low transaction costs. By contrast, replicating the Fama and French benchmark indexes is very costly, especially since the indexes are rebalanced on a yearly basis. However, for institutional investors that face relatively low transaction costs, our results do call into question the use of the S&P 500 index rather than a broader market benchmark for indexation. Naturally, broader indexes typically involve higher transaction costs, because they include more stocks and because they typically include relatively more small caps. For example, the

Vanguard 500 Index Fund (based on the S&P 500 index) currently has an expense ratio of 18 basis points per annum, while the Vanguard Small Cap Index (based on the Russell 2000 index) has an expense ratio of 27 basis points. Therefore, it is interesting to determine the level of additional transaction costs required to reverse the inefficiency classification. The critical value for $x(\boldsymbol{t})$ at a 10 percent significance level is 0.0070. The observed value $x(\boldsymbol{t}) = 0.0074$ therefore means that we have to lower the difference in mean return between the benchmark portfolios and the S&P 500 index by 4 basis points percent per month (or 48 basis points per annum) to classify the index as efficient with 90 percent confidence. This percentage is substantially above the level of additional transaction costs typically reported for index funds that follow broad small cap indexes. Therefore, transaction costs are unlikely to rationalize the S&P 500 index for institutional investors. Still, we stress that this application is used for the purpose of illustration. A sound empirical study requires more rigor than is possible here (e.g. accounting for possible investment restrictions not included in our model, and assessing the sensitivity of our results to the sample period and the return horizon); we leave this for further research.

VII. CONCLUDING REMARKS AND SUGGESTIONS

We have derived necessary and sufficient empirical tests for SSD efficiency in case diversification between choice alternatives is allowed. Our approach relies on building nonparametric piecewise-linear utility functions, and on checking whether the evaluated portfolio is optimal relative to these utility functions. Put differently, our approach relies on checking if we can find portfolios that satisfy dual conditions on the Ordered Mean Difference relative to the evaluated portfolio. We have also discussed how the SSD test can be generalized towards TSD: by constructing piecewise-quadratic utility functions (or alternatively by piecewise-linear marginal utility functions). Further, we have discussed how bootstrapping techniques and asymptotic distribution theory can approximate the sampling properties of the test results and allow for statistical inference.

Straightforward linear programming can apply our tests to empirical data. The primal and dual problems can be solved with minimal computational burden, even with desktop PCs. For example, the computations for the S&P application (894 observations, 25 individual portfolios) used the simplex module of Aptech Systems' GAUSS software, operated on a desktop PC with a 1700 MHz Pentium IV microprocessor and with 512 MB of working memory available. The computations required only minimal burden; the run time for the SSD tests was less than 1 minute on average. With the current exponential growth of computer power, the computational burden can be expected to drop much further in the foreseeable future. Still, the computational complexity of the model increases as the number of observations and/or the number of assets increases, and specialized LP solver software is recommended for large-scale problems involving thousands of observations and/or assets (or if higher-order criteria are used).

The computational ease is an important advantage relative to the Kuosmanen approach (see the Introduction). While the number of variables in our tests increases linearly with the number of observations, the number of variables in the Kuosmanen tests explodes as the time series grows. For example, for our S&P application, the

Kuosmanen SSD test would involve more than $8 \cdot 10^5$ model variables and the TSD test would involve more than $6 \cdot 10^{11}$ model variables, which is far beyond the computational possibilities, at present and in the foreseeable future. In our opinion, the computational ease does not involve a substantial loss of information. Kuosmanen focuses on identifying an efficient solution portfolio that SD dominates the evaluated portfolio. By contrast, our approach focuses on identifying (a set of subgradients of) a rationalizing utility function. Theorem 1 implies that both approaches are equivalent in terms of the efficiency classification (efficient or inefficient) of the evaluated portfolio. In addition, as is true for the empirical utility function and the solution portfolio identified with our approach, the solution portfolio in the Kuosmanen approach comes as a 'side-product' to the efficiency classification, and it is unlikely to be very robustness to sampling variation. Finally, the solution portfolio -even if it can be computed and if it is robust- is only one element of the SSD efficient set, and there is no prior reason to prefer this portfolio to other efficient portfolios.

From a theoretical perspective, SD is an attractive approach to analyzing economic behavior under uncertainty. In nonparametric fashion, SD allows the data 'to speak for them selves' (at least in large samples), rather than being forced to speak the idiom of prior assumptions about the return distribution. In contrast to mean-variance analysis and its extensions, SD accounts for the full distribution rather than a finite number of moments only. By extending the SD approach towards the case where diversification between choice alternatives is allowed, we hope to provide a stimulus to the further proliferation of SD for the problem of portfolio selection and evaluation (as well as other choice problems under uncertainty that involve diversification possibilities).

This paper forms the starting point for developing a full framework for SD with diversification possibilities. Further research could deal with the following subjects:

1. Our LP tests can be very useful for portfolio evaluation i.e. for evaluating whether a given portfolio is efficient. For selecting the optimal portfolio, one typically needs more information on investor preferences than is assumed in SD, and SD has to be complemented with other research instruments, like interactive Multi-Criteria Decision-Making procedures. Still, SD can be very useful as a pre-analysis screening device for reducing the number of choice alternatives. Further research could focus on obtaining a full characterization of the set of SD efficient portfolios.

2. The statistical test developed in Section V is based on the asymptotic least favorable distribution, and it involves low power in small samples. Fortunately, large data sets are available for many applications in financial economics. In addition, the test is likely to be more powerful in applications where the assets are not as highly correlated as the Fama and French benchmark portfolios. Still, it is desirable to develop a more powerful test, e.g. a test that explicitly minimizes the probability of Type II error rather than Type I error, or a test that is based on a particular class of return distributions.

APPENDIX

Proof of Theorem 1: The class of utility functions U_2 and the feasible set Λ are both convex. Therefore, Sion's (1958) minimax theorem implies that we can without harm change the order of the two optimization operators:

(19) $$\min_{u \in U_2}\left\{\max_{? \in \Lambda}\left\{\sum_{t \in \Theta}\left(u(\boldsymbol{x}_t \boldsymbol{?}) - u(\boldsymbol{x}_t \boldsymbol{t})\right)/T\right\}\right\} = \max_{? \in \Lambda}\left\{\min_{u \in U_2}\left\{\sum_{t \in \Theta}\left(u(\boldsymbol{x}_t \boldsymbol{?}) - u(\boldsymbol{x}_t \boldsymbol{t})\right)/T\right\}\right\}.$$

Hence, if $\boldsymbol{t} \in \Lambda$ is SSD inefficient, i.e. $\max_{? \in \Lambda}\left\{\sum_{t \in \Theta}\left(u(\boldsymbol{x}_t \boldsymbol{?}) - u(\boldsymbol{x}_t \boldsymbol{t})\right)/T\right\} > 0 \quad \forall u \in U_2$, then there exists a portfolio $\boldsymbol{?} \in \Lambda$ that SSD dominates $\boldsymbol{t}$, i.e. for which $\min_{u \in U_2}\left\{\sum_{t \in \Theta}\left(u(\boldsymbol{x}_t \boldsymbol{?}) - u(\boldsymbol{x}_t \boldsymbol{t})\right)/T\right\} > 0$. Similarly, if $\boldsymbol{t} \in \Lambda$ is SSD efficient i.e. $\exists u \in U_2 : \max_{? \in \Lambda}\left\{\sum_{t \in \Theta}\left(u(\boldsymbol{x}_t \boldsymbol{?}) - u(\boldsymbol{x}_t \boldsymbol{t})\right)/T\right\} = 0$, then $\min_{u \in U_2}\left\{\sum_{t \in \Theta}\left(u(\boldsymbol{x}_t \boldsymbol{?}) - u(\boldsymbol{x}_t \boldsymbol{t})\right)/T\right\} = 0$ for all $\boldsymbol{?} \in \Lambda$, and there doesn't exist a portfolio $\boldsymbol{?} \in \Lambda$ that SSD dominates $\boldsymbol{t}$.
Q.E.D.

Proof of Theorem 2: The necessary condition follows from the optimality conditions for convex problems and the properties of the solution set (see e.g. Hiriart-Urruty and Lemaréchal (1993), Thm. VII:1.1.1 and Cond. VII: 1.1.3). Specifically, $\boldsymbol{t}$ is an optimal portfolio i.e. $\boldsymbol{t} = \arg\max_{? \in \Lambda} \sum_{t \in \Theta} u(\boldsymbol{x}_t \boldsymbol{?})/T$ for $u \in U_2$ only if all portfolios $\boldsymbol{?} \in \Lambda$ are enveloped by the tangent hyperplane defined by a supergradient vector $\partial u(\boldsymbol{t}) \equiv (\partial u(\boldsymbol{x}_1 \boldsymbol{t}) \cdots \partial u(\boldsymbol{x}_T \boldsymbol{t}))$, i.e.

(20) $$\sum_{t \in \Theta} \partial u(\boldsymbol{x}_t \boldsymbol{t})(\boldsymbol{x}_t \boldsymbol{t} - \boldsymbol{x}_t \boldsymbol{?})/T \geq 0 \quad \forall \boldsymbol{?} \in \Lambda.$$

This inequality is similar to the well-known Kuhn-Tucker conditions for selecting an optimal portfolio if short selling is not allowed. The Kuhn-Tucker conditions apply for continuously differentiable utility functions; inequality (20) is a generalization towards superdifferentiable utility functions (recall that we use piecewise-linear utility functions, which generally are not continuously differentiable).

If $\boldsymbol{t}$ is optimal relative to some $u \in U_2$, then it is also optimal relative to the standardized utility function $v \equiv (u / \partial u(\boldsymbol{x}_T \boldsymbol{t})) \in U_2$. By construction, $\partial v(\boldsymbol{t})$ is a feasible solution to the primal problem, i.e. $\partial v(\boldsymbol{t}) \in \mathrm{B}$. The inequality (20) implies that this solution is associated with a solution value of zero. Hence, we find the necessary condition; $\boldsymbol{t}$ is SSD efficient only if $x(\boldsymbol{t}) = 0$.

To establish the sufficient condition, use $\boldsymbol{ß}^* \equiv (\boldsymbol{ß}_1^* \cdots \boldsymbol{ß}_T^*) \in \mathrm{B}$ for the optimal solution to the primal problem. If $x(\boldsymbol{t}) = 0$, then

(21) $$\sum_{t \in \Theta} \boldsymbol{ß}_t^* \boldsymbol{x}_t \boldsymbol{t} = \max_{? \in \Lambda} \sum_{t \in \Theta} \boldsymbol{ß}_t^* \boldsymbol{x}_t \boldsymbol{?}.$$

The piecewise-linear function $p(x|\boldsymbol{a},\boldsymbol{ß}^*)\in U_2$ connects the points $(p(x|\boldsymbol{a},\boldsymbol{ß}^*),z_t)$, $z_t\in[\boldsymbol{x}_t\boldsymbol{t},\boldsymbol{x}_{t+1}\boldsymbol{t}]$, $t\in\Theta\backslash T$, if we choose $\boldsymbol{a}_t$ such that $\boldsymbol{a}_t+\boldsymbol{ß}_t^*z_t=\boldsymbol{a}_{t+1}+\boldsymbol{ß}_{t+1}^*z_t$ for all $t\in\Theta$. This can be achieved e.g. by setting $\boldsymbol{a}_t=\boldsymbol{a}_t^*\equiv\sum_{s=t}^{T-1}(\boldsymbol{ß}_{s+1}^*-\boldsymbol{ß}_s^*)z_s$, $t\in\Theta\backslash T$, and $\boldsymbol{a}_T=0$. We then find

$$p(x|\boldsymbol{a}^*,\boldsymbol{ß}^*)=\begin{cases}\sum_{s=1}^{T-1}(\boldsymbol{ß}_{s+1}^*-\boldsymbol{ß}_s^*)z_s+\boldsymbol{ß}_1^*x & x\le z_1\\ \sum_{s=2}^{T-1}(\boldsymbol{ß}_{s+1}^*-\boldsymbol{ß}_s^*)z_s+\boldsymbol{ß}_2^*x & z_1\le x\le z_2\\ \vdots & \\ (1-\boldsymbol{ß}_{T-1}^*)z_{T-1}+\boldsymbol{ß}_{T-1}^*x & z_{T-2}\le x\le z_{T-1}\\ x & x\ge z_{T-1}\end{cases}.$$

By construction, we have $p(\boldsymbol{x}_t\boldsymbol{?}|\boldsymbol{a}^*,\boldsymbol{ß}^*)\le\boldsymbol{a}_t^*+\boldsymbol{ß}_t^*\boldsymbol{x}_t\boldsymbol{?}$ for all $\boldsymbol{?}\in\Lambda$, and $p(\boldsymbol{x}_t\boldsymbol{t}|\boldsymbol{a}^*,\boldsymbol{ß}^*)=\boldsymbol{a}_t+\boldsymbol{ß}_t^*\boldsymbol{x}_t\boldsymbol{t}$. Combining this with equality (18), we find that $\boldsymbol{t}$ is optimal relative to $p(x|\boldsymbol{a}^*,\boldsymbol{ß}^*)$ i.e.

(22) $$\sum_{t\in\Theta}p(\boldsymbol{x}_t\boldsymbol{t}|\boldsymbol{a}^*,\boldsymbol{ß}^*)=\max_{?\in\Lambda}\sum_{t\in\Theta}p(\boldsymbol{x}_t\boldsymbol{?}|\boldsymbol{a}^*,\boldsymbol{ß}^*).$$

By construction, $p(x|\boldsymbol{a}^*,\boldsymbol{ß}^*)$ is strictly increasing and concave and hence belongs to U_2. Therefore, we find the sufficient condition; portfolio $\boldsymbol{t}\in\Lambda$ is SSD efficient if $x(\boldsymbol{t})=0$. *Q.E.D.*

Proof of Theorem 3: The proof is best phrased in terms of the LP dual of (P):

(D) $$\max_{?,r_1,\cdots,r_T}\sum_{i=1}^{N}(\boldsymbol{?}_i\boldsymbol{x}_{iT}-\boldsymbol{x}_T\boldsymbol{t})/T+\boldsymbol{r}_{T-1}$$

$$\text{s.t. }\boldsymbol{r}_1+(\boldsymbol{x}_1\boldsymbol{t}-\sum_{i=1}^{N}\boldsymbol{?}_i\boldsymbol{x}_{i1})/T\le 0\quad(\boldsymbol{ß}_1)$$

$$\boldsymbol{r}_t-\boldsymbol{r}_{t-1}+(\boldsymbol{x}_t\boldsymbol{t}-\sum_{i=1}^{N}\boldsymbol{?}_i\boldsymbol{x}_{it})/T\le 0\quad t=2,\cdots,T-1\quad(\boldsymbol{ß}_t)$$

$$\sum_{i=1}^{N}\boldsymbol{?}_i=1\quad(q)$$

$$\boldsymbol{?}_i\ge 0\quad i=1,\cdots,N$$

$$\boldsymbol{r}_t\ge 0\quad t=1,\cdots,T-1$$

Again, the primal variables that correspond to each of the dual constraints are given within bracket, so as facilitate the interpretation and the relationship between the primal and the dual. The first $T-1$ inequality restrictions are always binding (the primal positivity restrictions $\boldsymbol{ß}_t\ge 0$, $t=1,\cdots,T-1$, are redundant because the primal also requires $\boldsymbol{ß}_1\ge\boldsymbol{ß}_2\ge\cdots\ge\boldsymbol{ß}_T=1$), and hence the optimal solution is

$\boldsymbol{r}_t^* = \sum_{s=1}^{t}(\boldsymbol{x}_s\, l - \boldsymbol{x}_s \boldsymbol{t})/T = t\, r_t(\boldsymbol{?}, \boldsymbol{t})/T$ for $t = 1, \cdots, T-1$. Substituting these expressions in (P), and dividing each $\boldsymbol{r}_t^*$ term by T/t, we directly obtain (10). Since (P) always has a feasible solution (see Section II), $y(\boldsymbol{t}) = x(\boldsymbol{t})$. Hence, Theorem 2 implies that portfolio $\boldsymbol{t} \in \Lambda$ is SSD efficient if and only if $y(\boldsymbol{t}) = 0$. *Q.E.D.*

Proof of Theorem 4: By analogy to the proof to Theorem 2, the necessary condition follows from the optimality conditions for optimizing a concave function over a convex set. Since *u* is continuously differentiable, we can now use the well-known Kuhn-Tucker conditions. Specifically, $\boldsymbol{t}$ is an optimal portfolio i.e. $\boldsymbol{t} = \arg\max_{?\in\Lambda} \sum_{t\in\Theta} u(\boldsymbol{x}_t \boldsymbol{?})/T$ for $u \in U_3$ if and only if all portfolios $\boldsymbol{?} \in \Lambda$ are enveloped by the tangent hyperplane, i.e.

(23) $\sum_{t\in\Theta} u'(\boldsymbol{x}_t \boldsymbol{t})(\boldsymbol{x}_t \boldsymbol{t} - \boldsymbol{x}_t \boldsymbol{?})/T \geq 0 \quad \forall \boldsymbol{?} \in \Lambda$.

If $\boldsymbol{t}$ is optimal relative to some $u \in U_3$, then it is also optimal relative to the standardized utility function $v = (u / u'(\max_{\substack{i\in I\\ t\in\Theta}} \boldsymbol{x}_{it})) \in U_3$. By definition, $(v'(\boldsymbol{t}), \partial v'(\boldsymbol{t}))$, with $\partial v'(\boldsymbol{t}) \equiv (\partial v'(\boldsymbol{x}_1 \boldsymbol{t}) \cdots \partial v'(\boldsymbol{x}_T \boldsymbol{t}))$ for a subgradient of $v'(\boldsymbol{t})$, is a feasible solution, i.e. $(v'(\boldsymbol{t}), \partial v'(\boldsymbol{t})) \in \Omega$. The inequality (23) implies that this solution is associated with a solution value of zero. Hence, we find the necessary condition; $\boldsymbol{t}$ is TSD efficient only if $z(\boldsymbol{t}) = 0$.

To establish the sufficient condition, use $(\boldsymbol{ß}^*, \boldsymbol{?}^*) \in \Omega$ for the optimal solution. From this optimal solution, we can construct the following continuous piecewise-linear marginal utility function

(24) $$p''(x \mid \boldsymbol{ß}^*, \boldsymbol{?}^*) \equiv \min_{t\in\Theta}(\boldsymbol{ß}_t^* + \boldsymbol{?}_t^* x) = \begin{cases} \boldsymbol{ß}_1^* + \boldsymbol{?}_1^* x & x \leq z_1 \\ \boldsymbol{ß}_2^* + \boldsymbol{?}_2^* x & z_1 \leq x \leq z_2 \\ \vdots & \\ \boldsymbol{ß}_{T-1}^* + \boldsymbol{?}_{T-1}^* x & z_{T-2} \leq x \leq z_{T-1} \\ \boldsymbol{ß}_T^* + \boldsymbol{?}_T^* x & x \geq z_{T-1} \end{cases}$$

with $z_t \equiv \begin{cases} \dfrac{(\boldsymbol{ß}_{t+1}^* - \boldsymbol{ß}_t^*)}{(\boldsymbol{?}_t^* - \boldsymbol{?}_{t+1}^*)} & \boldsymbol{?}_t^* \leq \boldsymbol{?}_{t+1}^* \\ 0.5(\boldsymbol{x}_t + \boldsymbol{x}_{t+1}) & \boldsymbol{?}_t^* = \boldsymbol{?}_{t+1}^* \end{cases}$, $t \in \Theta \setminus T$, for the nodes that connect the line segments. By construction, $p''(x \mid \boldsymbol{ß}^*, \boldsymbol{?}^*)$ is strictly positive over the observed return range, decreasing and convex. Integrating $\boldsymbol{ß}_t^* + \boldsymbol{?}_t^* x$, $t \in \Theta$, gives $\boldsymbol{a}_t + \boldsymbol{ß}_t^* x + 0.5\boldsymbol{?}_t^* x^2$, where $\boldsymbol{a}_t$ is free. The integrated functions can be combined to yield the continuous piecewise-quadratic utility function:

(25) $p'(x \mid \boldsymbol{a}^*, \boldsymbol{ß}^*, \boldsymbol{?}^*) \equiv \min_{t\in\Theta}(\boldsymbol{a}_t^* + \boldsymbol{ß}_t^* x + 0.5\boldsymbol{?}_t^* x^2)$

$$
= \begin{cases}
\boldsymbol{\alpha}_1^* + \boldsymbol{\beta}_1^* x + \boldsymbol{\gamma}_1^* x^2 & x \le z_1 \\
\boldsymbol{\alpha}_2^* + \boldsymbol{\beta}_2^* x + \boldsymbol{\gamma}_2^* x^2 & z_1 \le x \le z_2 \\
\vdots & \\
\boldsymbol{\alpha}_{T-1}^* + \boldsymbol{\beta}_{T-1}^* x + \boldsymbol{\gamma}_{T-1}^* x^2 & z_{T-2} \le x \le z_{T-1} \\
\boldsymbol{\alpha}_T^* + \boldsymbol{\beta}_T^* x + \boldsymbol{\gamma}_T^* x^2 & x \ge z_{T-1}
\end{cases}
$$

with $\boldsymbol{\alpha}_T^* \equiv 0$, and $\boldsymbol{\alpha}_t^* \equiv \sum_{s=t}^{T-1} ((\boldsymbol{\beta}_{s+1}^* - \boldsymbol{\beta}_s^*) z_s + 0.5(\boldsymbol{\gamma}_{s+1}^* - \boldsymbol{\gamma}_s^*) z_s^2)$, $t \in \Theta \setminus T$, to guarantee continuity, i.e. $\boldsymbol{\alpha}_t^* + \boldsymbol{\beta}_t^* z_t + 0.5\boldsymbol{\gamma}_t^* z_t^2 = \boldsymbol{\alpha}_{t+1}^* + \boldsymbol{\beta}_{t+1}^* z_t + 0.5\boldsymbol{\gamma}_{t+1}^* z_t^2$. By construction, $p'(x | \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ is strictly increasing, concave, and it exhibits a preference for positive skewness. Concavity implies for all $\boldsymbol{\lambda} \in \Lambda$:

(26) $\quad \sum_{t \in \Theta} (p'(\mathbf{x}_t \boldsymbol{\tau} | \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*) + (\boldsymbol{\beta}_t^* + \boldsymbol{\gamma}_t^* \mathbf{x}_t \boldsymbol{\tau})(\mathbf{x}_t \boldsymbol{\lambda} - \mathbf{x}_t \boldsymbol{\tau})) \ge \sum_{t \in \Theta} p'(\mathbf{x}_t \boldsymbol{\lambda} | \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$.

If $z(\boldsymbol{\tau}) = 0$, then $\sum_{t \in \Theta} (\boldsymbol{\beta}_t^* + \boldsymbol{\gamma}_t^* \mathbf{x}_t \boldsymbol{\tau}) \mathbf{x}_t \boldsymbol{\tau} = \sum_{t \in \Theta} (\boldsymbol{\beta}_t^* + \boldsymbol{\gamma}_t^* \mathbf{x}_t \boldsymbol{\tau}) \mathbf{x}_t \boldsymbol{\lambda}$. Combining this finding with (26), we find that $\boldsymbol{\tau}$ is optimal relative to $p'(x | \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ i.e.

(27) $\quad \sum_{t \in \Theta} p'(\mathbf{x}_t \boldsymbol{\tau} | \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*) = \max_{\boldsymbol{\lambda} \in \Lambda} \sum_{t \in \Theta} p'(\mathbf{x}_t \boldsymbol{\lambda} | \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$.

Hence, we find the sufficient condition; portfolio $\boldsymbol{\tau} \in \Lambda$ is TSD efficient if $z(\boldsymbol{\tau}) = 0$. *Q.E.D.*

Proof of Theorem 5: Since the unity vector is a feasible solution to the primal problem, i.e. $\mathbf{e} \in \mathrm{B}$, we know that

(28) $\quad x(\boldsymbol{\tau}) \le w(\boldsymbol{\tau}) \equiv \max_{i \in \mathrm{I}} \left\{ \sum_{t \in \Theta} (\mathbf{x}_{it} - \mathbf{x}_t \boldsymbol{\tau}) / T \right\}$.

Known results can derive the exact asymptotic sampling distribution of $w(\boldsymbol{\tau})$. Under the null, $\mathbf{x}_{it}$, $i \in \mathrm{I}$, $t \in \Theta$, are serially and contemporaneously IID with variance $\sigma^2 < \infty$. Hence, the central limit theorem implies that $\sum_{t \in \Theta} \mathbf{x}_{it} / T$, $i \in \mathrm{I}$, obey an asymptotically IID normal distribution with variance σ^2 / T, and $\sum_{t \in \Theta} (\mathbf{x}_t \boldsymbol{\tau} - \mathbf{x}_{it}) / T$, $i \in \mathrm{I}$, obey an asymptotically normal distribution with zero mean, variance $\sigma_i^2 \equiv (\sum_{k \in \mathrm{I}} \boldsymbol{\tau}_k^2 - 2\boldsymbol{\tau}_i + 1)\sigma^2 / T$ and covariance terms $\sigma_{ij} \equiv (\sum_{k \in \mathrm{I}} \boldsymbol{\tau}_k^2 - \boldsymbol{\tau}_i - \boldsymbol{\tau}_j)\sigma^2 / T$, $j \in \mathrm{I} : i \ne j$. Hence, $w(\boldsymbol{\tau})$ asymptotically behaves as the largest order statistic of N random variables with a multivariate normal distribution. Therefore, $P(w(\boldsymbol{\tau}) > y | H_0) = 1 - P(\sum_{t \in \Theta} (\mathbf{x}_t \boldsymbol{\tau} - \mathbf{x}_{it}) / T \le y \quad \forall i \in I)$ asymptotically equals the multivariate normal integral $1 - \int_{x \le y\mathbf{e}} \partial \Phi(\mathbf{x})$.

It follows directly from (28) that $P(x(\boldsymbol{\tau}) > y | H_0)$ is bounded from above by $P(w(\boldsymbol{\tau}) > y | H_0)$ for all return distributions $H(x)$. Moreover, there exist $H(x)$ for which $x(\boldsymbol{\tau})$ approximates $w(\boldsymbol{\tau})$, and therefore the asymptotic distribution of $w(\boldsymbol{\tau})$

also represents the asymptotic least favorable distribution for $x(\boldsymbol{t})$. A simple example is the binomial distribution with probability *a*, $0 < a < 1$, of finding $x = 0$ and probability (1-*a*) of finding $x = b$, $0 < b < \infty$, i.e. $H(x) = \begin{cases} 0 & x < 0 \\ a & 0 \le x < b \\ 1 & x \ge b \end{cases}$. The parameters *a* and *b* can be used to control the population variance $s^2 = a(1-a)b^2$, as well as the probability distribution of the number of observations that take a zero value, say Z, i.e. the binomial probability $P(Z = z) = \binom{NT}{z} a^z (1-a)^{(NT-z)}$, $NT \ge z \ge 0$. For any given *N*, *T*, and s^2, we may set a such that the probability of finding multiple zeros, i.e. $P(Z > 1)$ approximates zero, and we only have to consider the case with no zeros, i.e. $Z = 0$, and the case with a single zero, i.e. $Z = 1$. If no zeros occur, then $\boldsymbol{x}_{it} = b$ for all $i \in \mathrm{I}, t \in \Theta$, and hence $x(\boldsymbol{t}) = w(\boldsymbol{t}) = 0$. If one zero occurs, say $\boldsymbol{x}_{ks} = 0$, $k \in \mathrm{I}, s \in \Theta$, then we have to distinguish between the case with $\boldsymbol{t}_k = 0$ and the case with $\boldsymbol{t}_k > 0$. If $\boldsymbol{t}_k = 0$, then $\boldsymbol{x}_t \boldsymbol{t} = b \ge \boldsymbol{x}_{it}$ for all $i \in \mathrm{I}, t \in \Theta$, and hence $x(\boldsymbol{t}) = w(\boldsymbol{t}) = 0$. If $\boldsymbol{t}_k > 0$, then $\boldsymbol{x}_s \boldsymbol{t} < \boldsymbol{x}_t \boldsymbol{t} = b$ and we must set $\boldsymbol{ß}_s \ge \boldsymbol{ß}_t = 1$ for all $t \in \Theta \setminus s$ (recall the restriction $\boldsymbol{ß} \in \mathrm{B}$; see Section II). Since $\boldsymbol{x}_s \boldsymbol{t} < \boldsymbol{x}_{is} = b$ for all $i \in \mathrm{I} \setminus k$, the optimal value for $\boldsymbol{ß}_s$ is unity, and hence $x(\boldsymbol{t}) = w(\boldsymbol{t}) = b / T$.?

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