TESTING FOR THIRD-ORDER STOCHASTIC DOMINANCE WITH DIVERSIFICATION POSSIBILITIES

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Testing for Third-Order Stochastic Dominance with Diversification Possibilities

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http://www.few.eur.nl/few/people/gtpost/stochastic_dominance.htm.
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ABSTRACT

We derive an empirical test for third-order stochastic dominance that allows for diversification between choice alternatives. The test can be computed using straightforward linear programming. Bootstrapping techniques and asymptotic distribution theory can approximate the sampling properties of the test results and allow for statistical inference. Our approach is illustrated using real-life US stock market data.

The theory of stochastic dominance (SD; see e.g. Levy, 1998) gives a systematic framework for analyzing economic behavior under uncertainty. The theoretical attractiveness of SD lies in its nonparametric orientation. SD criteria do not require a full parametric specification of the preferences of the decision-maker and the statistical distribution of the choice alternatives, but rather they rely on a set of general assumptions. One possible application area is the problem of selecting and evaluating investment portfolios. This problem is of interest both for empirical tests of theoretical asset pricing models and for practical portfolio management applications. The focus of the research in this area has predominantly been on mean-variance analysis (MVA; e.g. Kandel and Stambaugh, 1987, 1989, and Gibbons *et al.*, 1989). Unfortunately, MVA is not always economically meaningful. Roughly speaking, MVA is consistent with expected utility theory only if investor utility is quadratic or if asset returns obey a normal distribution (see e.g. Bigelow, 1993, for exact conditions). An important reason why SD has not seen the proliferation that one might expect is the inability of traditional empirical tests to properly account for the possibility to diversify between the choice alternatives. Post (2001) developed a tractable linear programming (LP) test for second-order SD (SSD) efficiency of a given portfolio relative to all possible portfolios created from a set of assets. This test could provide a stimulus towards the further proliferation of SD for portfolio selection and evaluation (as well as other choice problems under uncertainty that involve diversification possibilities).

The SSD criterion relies on the assumptions of non-satiation and risk-aversion solely. By imposing minimal assumptions, the criterion can involve low discriminating power; i.e. the efficient set can be large. Imposing additional preference assumptions could help to remedy this problem. Third-order SD (TSD; Whitmore, 1970) complements the SSD assumptions with the additional assumption that investors prefer positively skewed return distributions. Interestingly, empirical evidence suggests that investors indeed display this kind of skewness preference (e.g. Arditti, 1967, Kraus and Litzenberger, 1976, Cooley, 1977, Friend and Westerfield, 1980, and Harvey and Siddique, 2000). This paper extends Post's SSD analysis towards TSD so as to develop a test with more discriminating power. Section I recaptures the definition of TSD. Section II gives a LP test for TSD efficiency. The test relies on constructing piecewise-quadratic utility functions and on testing if the evaluated portfolio is optimal relative to those utility functions. Apart from the computational problems associated with portfolio diversification, another problem in practical applications of SD is the sensitivity of the results to sampling error. Section III discusses how bootstrapping techniques and asymptotic distribution theory can approximate the sampling distribution of the test results and allow for statistical inference. Section IV applies our approach to test if the Fama and French market portfolio is efficient relative to the Fama and French benchmark indexes. Finally, Section V gives conclusions and suggests directions for future research. The Appendix gives the formal proofs of our theorems.

I. THIRD-ORDER STOCHASTIC DOMINANCE

Consider an investment universe consisting of N assets, associated with returns $\boldsymbol{x} \in \Re^N$.[1] Throughout the text, we will use the index set $\mathrm{I} \equiv \{1,\cdots,N\}$ to denote the different assets. In addition, we will treat the returns as serially independent and identically distributed (IID) random variables with a continuous joint cumulative distribution function (CDF) $G : \Re^N \to [0,1]$. Investors may diversify between the assets, and we will use $\boldsymbol{?} \in \Re^N$ for a vector of portfolio weights. For simplicity, we will consider the case where short selling is not allowed, and the portfolio weights belong to the portfolio possibilities set $\Lambda \equiv \{\boldsymbol{?} \in \Re_+^N : \boldsymbol{?}^T \boldsymbol{e} = 1\}$, with $\boldsymbol{e}$ for a unity vector with dimensions conforming to the rules of matrix algebra.[2]

We consider the problem of establishing whether a particular portfolio, say $\boldsymbol{t} \in \Lambda$, is optimal, i.e. whether it maximizes the expected value of the investor's utility function $u : \Re \to P$, $u \in U$, with U for the class of von Neuman-Morgenstern utility functions, and P for a nonempty, closed and convex subset of $\Re$. The portfolio $\boldsymbol{t}$ is optimal if and only if:

$$(1) \qquad \int u(\boldsymbol{xt})\partial G(\boldsymbol{x}) = \max_{?\in\Lambda} \int u(\boldsymbol{x?})\partial G(\boldsymbol{x}) .$$

In practical applications, full information about the utility function typically is not available, and this condition cannot be verified directly. This provides the rationale for using SD criteria that rely on a set of general assumptions rather than a full specification of the utility function. The TSD criterion restricts attention to the class of non-satiable and risk-averse investors that prefer positively skewed distributions (more probability in the right tail).[3] The TSD investors can be represented by the class of von Neuman-Morgenstern utility functions with a strictly positive, decreasing and convex marginal utility function, or $U_3 \subseteq U$. For analytical simplicity, we assume that the utility function is once continuously differentiable, and we will use $u'(x)$ for the gradient or 'marginal utility function' at x, and $u'(\boldsymbol{t}) \equiv (u'(\boldsymbol{x}_1\boldsymbol{t})\cdots u'(\boldsymbol{x}_T\boldsymbol{t}))$ for a gradient vector. Note that we do not assume that the marginal utility function is continuously differentiable. However, marginal utility is convex and hence everywhere continuous and subdifferentiable. Throughout the text, we will denote

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[1] Throughout the text, we will use $\Re^m$ for an m-dimensional Euclidean space, and $\Re_+^m$ denotes the positive orthant. Further, to distinguish between vectors and scalars, we use a bold font for vectors and a regular font for scalars.

[2] It is possible to generalize the analysis towards cases where short selling is allowed and cases where additional restrictions are imposed on the portfolio weights. Our analysis is based on the optimality conditions from subdifferential calculus for optimizing a concave utility function over a convex portfolio possibilities set (see the proof to Theorem 1). These conditions apply for any non-empty, closed and convex portfolio set, and we may therefore generalize our analysis towards a more general polyhedral portfolio possibilities set.

[3] This kind of skewness preference is strongly related to the concept of decreasing absolute risk aversion (DARA; Pratt, 1964), which underlies DARA SD (DSD; Vickson, 1975). Roughly speaking, DARA means that the dislike for absolute uncertainties decreases as the levels of the outcomes increase. Theoretically, TSD is a sufficient but not necessary condition for DARA, and TSD therefore is less powerful than DSD. However, DSD is difficult to fit to empirical data, and in addition the improvement in power is minimal. For these reasons, Vickson and Altman (1977) conclude that TSD is likely to be a suitable approximation for DSD for practical purposes.

subgradient vectors of the marginal utility function by $\partial u'(\boldsymbol{t}) \equiv (\partial u'(\boldsymbol{x}_1\boldsymbol{t}) \cdots \partial u'(\boldsymbol{x}_T\boldsymbol{t}))$.

Apart from the utility function, also the CDF generally is not known in practical applications. Rather, information typically is limited to a discrete set of time series observations, say $\mathrm{X} \equiv (\boldsymbol{x}_1 \cdots \boldsymbol{x}_T)^T$ with $\boldsymbol{x}_t \equiv (x_{1t} \cdots x_{Nt}) \in \Re^N$. For convenience, we assume that the data are ranked in ascending order by the return of the evaluated portfolio, i.e. $\boldsymbol{x}_1\boldsymbol{t} < \boldsymbol{x}_2\boldsymbol{t} < \cdots < \boldsymbol{x}_T\boldsymbol{t}$.[4] Using the observations, we can construct the empirical distribution function (EDF) $F(\boldsymbol{x}) \equiv \mathrm{card}\{t \in \Theta : \boldsymbol{x}_t \leq \boldsymbol{x}\}/T$ with $\Theta \equiv \{1, \cdots, T\}$. In this paper, we will analyze SD for the EDF rather than the CDF, so as to focus on the computational problems encountered in practical applications. Under the maintained assumption that the return observations are serially IID random variables, the EDF gives a statistically consistent estimator for the CDF.[5] Section III discusses the role of sampling error.

Using the above notation and assumptions, TSD can be defined as follows:

DEFINITION 1 *Portfolio* $\boldsymbol{t} \in \Lambda$ *is TSD inefficient if and only if, for all utility functions* $u \in U_3$*, the maximum expected utility is greater than the expected utility of* $\boldsymbol{t}$ *, i.e.*

(2) $$\min_{u \in U_3} \left\{ \max_{? \in \Lambda} \left\{ \int u(\boldsymbol{x}?) \partial F(\boldsymbol{x}) - \int u(\boldsymbol{x}\boldsymbol{t}) \partial F(\boldsymbol{x}) \right\} \right\} = \min_{u \in U_3} \left\{ \max_{? \in \Lambda} \left\{ \sum_{t \in \Theta} \left(u(\boldsymbol{x}_t ?) - u(\boldsymbol{x}_t \boldsymbol{t}) \right)/T \right\} \right\} > 0 .$$

Alternatively, portfolio $\boldsymbol{t} \in \Lambda$ *is TSD efficient if and only if it is optimal relative to some utility functions* $u \in U_3$*, i.e.*

(3) $$\min_{u \in U_3} \left\{ \max_{? \in \Lambda} \left\{ \int u(\boldsymbol{x}?) \partial F(\boldsymbol{x}) - \int u(\boldsymbol{x}\boldsymbol{t}) \partial F(\boldsymbol{x}) \right\} \right\} = \min_{u \in U_3} \left\{ \max_{? \in \Lambda} \left\{ \sum_{t \in \Theta} \left(u(\boldsymbol{x}_t ?) - u(\boldsymbol{x}_t \boldsymbol{t}) \right)/T \right\} \right\} = 0 .$$ [6]

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[4] Since we assume a continuous return distribution, ties do not occur. Still, the analysis can be extended in a straightforward way to cases where ties do occur e.g. due to a discrete return distribution or due to measurement problems or rounding, or if a riskless asset is evaluated (see Post, 2001).

[5] There is substantial evidence that the distribution of assets returns (e.g. interest rates, risk premia, volatilities and correlation coefficients) varies through time. This problem is especially relevant for applications that use data for long time periods. In such cases, the observations generally are not serially IID random variables and the EDF is not a statistically consistent estimate for the CDF. One possible approach to account for time variation is to use econometric time series estimation techniques to estimate a conditional CDF. Our empirical tests can then be applied to random samples from the estimated CDF rather than the EDF.

[6] Note that this definition of TSD uses strict inequalities for all $u \in U_3$. By contrast, the traditional definition uses weak inequalities with a strict inequality for at least one $u \in U_3$. This difference is important from a theoretical perspective, and one can think of examples where the two definitions give different efficiency classifications. However, from an empirical perspective, the definitions are indistinguishable, because arbitrary small data perturbations to the evaluated portfolio can make the classifications consistent. Related to this, data sets where this theoretical issue has a decisive impact are extremely unlikely for return distributions that are continuous by approximation.

II. LINEAR PROGRAMMING FORMULATION

Post's SSD test relies on constructing piecewise-linear utility functions that 'rationalize' the evaluated portfolio, i.e. for which the evaluated portfolio is optimal. Interestingly, we may extend this approach towards TSD efficiency by asking if we can construct piecewise-*quadratic* utility functions $p \in U_3$ that 'rationalize' the evaluated portfolio. A continuous piecewise-quadratic utility function may be constructed from intercept coefficients $\boldsymbol{a} \equiv (\boldsymbol{a}_1 \cdots \boldsymbol{a}_T)$, slope coefficients $\boldsymbol{ß} \equiv (\boldsymbol{ß}_1 \cdots \boldsymbol{ß}_T)$ and curvature coefficients $\boldsymbol{?} \equiv (\boldsymbol{?}_1 \cdots \boldsymbol{?}_T)$ as

(4) $$p(x|\boldsymbol{a},\boldsymbol{ß},\boldsymbol{?}) \equiv \min_{t\in\Theta} (\boldsymbol{a}_t + \boldsymbol{ß}_t x + 0.5\boldsymbol{?}_t x^2).$$

Imposing appropriate restrictions on the slope and curvature coefficients can guarantee that the piecewise-quadratic utility function exhibits monotonicity, concavity, and skewness preference. Specifically, for p to belong to U_3, $\boldsymbol{ß}$ and $\boldsymbol{?}$ need to be elements of the following set:

(5) $$\Omega \equiv \{(\boldsymbol{ß},\boldsymbol{?}) \in \Re^T \times \Re^T_- : \boldsymbol{ß}_t + \boldsymbol{?}_t \boldsymbol{x}_t \boldsymbol{t} \geq \boldsymbol{ß}_{t+1} + \boldsymbol{?}_{t+1} \boldsymbol{x}_{t+1} \boldsymbol{t};$$
$$\boldsymbol{ß}_t + \boldsymbol{?}_t \boldsymbol{x}_{t+1} \boldsymbol{t} \leq \boldsymbol{ß}_{t+1} + \boldsymbol{?}_{t+1} \boldsymbol{x}_{t+1} \boldsymbol{t};\ \boldsymbol{ß}_t + \boldsymbol{?}_t \boldsymbol{x}_t \boldsymbol{t} \geq \boldsymbol{ß}_{t+1} + \boldsymbol{?}_{t+1} \boldsymbol{x}_t \boldsymbol{t};$$
$$\boldsymbol{?}_t \leq \boldsymbol{?}_{t+1}\ \forall t \in \Theta\backslash T; \boldsymbol{ß}_T + \boldsymbol{?}_T \max_{\substack{i\in \mathrm{I}\\ t\in\Theta}} \boldsymbol{x}_{it} = 1\}.$$

The constraints $\boldsymbol{ß}_t + \boldsymbol{?}_t \boldsymbol{x}_t \boldsymbol{t} \geq \boldsymbol{ß}_{t+1} + \boldsymbol{?}_{t+1} \boldsymbol{x}_{t+1} \boldsymbol{t}$ for all $t \in \Theta\backslash T$ and the normalizing constraint $\boldsymbol{ß}_T + \boldsymbol{?}_T \max_{\substack{i\in \mathrm{I}\\ t\in\Theta}} \boldsymbol{x}_{it} = 1$ restrict p to be strictly increasing. Further, $\boldsymbol{ß}_t + \boldsymbol{?}_t \boldsymbol{x}_{t+1} \boldsymbol{t} \leq \boldsymbol{ß}_{t+1} + \boldsymbol{?}_{t+1} \boldsymbol{x}_{t+1} \boldsymbol{t}$ and $\boldsymbol{ß}_t + \boldsymbol{?}_t \boldsymbol{x}_t \boldsymbol{t} \geq \boldsymbol{ß}_{t+1} + \boldsymbol{?}_{t+1} \boldsymbol{x}_t \boldsymbol{t}$ for all $t \in \Theta\backslash T$ guarantees concavity. Finally, $\boldsymbol{?}_t \leq \boldsymbol{?}_{t+1}$ for all $t \in \Theta\backslash T$ guarantees skewness preference.

THEOREM 1 *Portfolio $\boldsymbol{t} \in \Lambda$ is TSD efficient if and only if $\boldsymbol{t}$ is optimal relative to a piecewise-quadratic utility function $p \in U_3$. We may test this condition using the test statistic*

(6) $$z(\boldsymbol{t}) \equiv \min_{(\boldsymbol{ß},\boldsymbol{?})\in\Omega, q} \left\{ q : \sum_{t\in\Theta} (\boldsymbol{ß}_t + \boldsymbol{?}_t \boldsymbol{x}_t \boldsymbol{t})(\boldsymbol{x}_t \boldsymbol{t} - \boldsymbol{x}_{it})/T + q \geq 0 \quad \forall i \in \mathrm{I} \right\}.$$

Specifically, portfolio $\boldsymbol{t} \in \Lambda$ is TSD efficient if and only if $z(\boldsymbol{t}) = 0$. Alternatively, portfolio $\boldsymbol{t} \in \Lambda$ is TSD inefficient if and only if $z(\boldsymbol{t}) > 0$.

The test statistic $z(\boldsymbol{t})$ involves a linear objective function and linear constraints, and it can be solved using straightforward linear programming. The following is a full LP formulation for $z(\boldsymbol{t})$:

(P) $$\min_{q,\boldsymbol{ß},g} q$$

$$\text{s.t.} \sum_{t\in\Theta}(\beta_t + \gamma_t x_t \tau)(x_t \tau - x_{it})/T + \theta \geq 0 \quad i = 1,\cdots,N$$

$$\beta_t + \gamma_t x_t \tau \geq \beta_{t+1} + \gamma_{t+1} x_{t+1}\tau \quad t = 1,\cdots,T-1$$

$$\beta_t + \gamma_t x_{t+1}\tau \leq \beta_{t+1} + \gamma_{t+1} x_{t+1}\tau \quad t = 1,\cdots,T-1$$

$$\beta_t + \gamma_t x_t \tau \geq \beta_{t+1} + \gamma_{t+1} x_t \tau \quad t = 1,\cdots,T-1$$

$$\gamma_t \leq \gamma_{t+1} \ t = 1,\cdots,T$$

$$\beta_T + \gamma_T \max_{\substack{i\in I\\ t\in\Theta}} x_{it} = 1$$

$$\beta_t \quad \text{free} \quad t = 1,\cdots,T$$

$$g_t \geq 0 \quad t = 1,\cdots,T$$

$$\theta \quad \text{free}$$

The problem involves only 2T+1 variables and N+4T-2 constraints. Further, the model always has a feasible solution, as e.g. $\beta_t = 1$ and $\gamma_t = 0$ for all $t \in \Theta$, and $\theta = \max_{i\in I} \sum_{t\in\Theta}(x_{it} - x_t \tau)/T$, necessarily satisfies all constraints. (This solution effectively represents risk neutral investors; risk neutral investors have linear utility functions and compare portfolios solely in terms of the expected return.) For small data sets up to hundreds of observations and/or assets, the problem can be solved with minimal computational burden, even with desktop PCs and standard solver software (like LP solvers included in spreadsheets). Still, the computational complexity, as measured by the required number of arithmetic operations, and hence the run time and memory space requirement, increases progressively with the number of variables and restrictions. Therefore, specialized hardware and solver software is recommended for large-scale problems involving thousands of observations and/or assets.[7]

Two disclaimers apply for interpreting the test results. First, the test statistic can separate efficient portfolios from inefficient ones. However, we stress that the test statistic does not represent a meaningful performance measure that can be used for ranking portfolios based on the 'degree of efficiency'. For selecting the optimal portfolio from the efficient set, and for measuring the deviation from the optimum, we typically need more information on investor preferences than is assumed in SD. Second, we may recover a complete utility function from the optimal slope and curvature coefficients (see the proof to Theorem 1). However, we stress that the piecewise-quadratic utility functions are mere instruments for our analysis, and we cannot derive strong results for these instruments. For example, there typically exist multiple optimal solutions for the slope and curvature coefficients if the evaluated portfolio is TSD inefficient, and in addition the interpretation of the utility function is not clear if the evaluated portfolio is TSD inefficient. Another complication stems from sampling error. While it is possible to perform powerful and accurate inference about the test statistic (at least in large samples; see Section III), this is much more complicated for recovering an entire utility function.

[7] For an elaborate introduction in LP, we refer to Chvatal (1983). In practice, very large LPs can be solved efficiently by both the simplex method and interior-point methods. An elaborate guide to LP solver software can be found at the homepage of the Institute for Operations Research and Management Science (INFORMS); http://www.informs.org/.

III. STATISTICAL INFERENCE

We have thus far discussed SD relative to the EDF rather than the CDF. Generally, the EDF is very sensitive to sampling variation and the test results are likely to be affected by sampling error in a non-trivial way. The applied researcher must therefore have knowledge of the sampling distribution in order to make inferences about the true efficiency classification. In the SD literature, two approaches have been developed to approximate the sampling distribution: bootstrapping and analytical asymptotic analysis. As discussed in Post (2001), the tractable LP structure suggests that bootstrapping can substitute brute computational force to overcome analytical intractability for the SSD test statistic. This conclusion applies with equal strength to our TSD test. The alternative approach is to derive an analytical characterization of the asymptotic sampling distribution. In this spirit, Post (2001) derived an asymptotic distribution for the SSD test statistic. The distribution is based on the conservative null hypothesis that all assets $i \in \mathrm{I}$ are contemporaneously IID, and hence $G(\boldsymbol{x}) = \prod_{i \in I} H(\boldsymbol{x}_i)$ with $H : \Re \to [0,1]$ for a univariate CDF with variance $s^2 < \infty$.[8] In addition, the analysis is based on the least favorable distribution, i.e. the distribution that maximizes the size or relative frequency of Type I error (rejecting the null when it is true). Interestingly, the asymptotic least favorable distribution of the SSD test statistic also applies for our TSD test statistic:

THEOREM 2 *For the asymptotic least favorable distribution of* $z(\boldsymbol{t})$, *the p-value* $P(z(\boldsymbol{t}) \geq y | H_0)$, $y \geq 0$, *equals the integral* $1 - \int_{x \leq ye} \partial\Phi(\boldsymbol{x})$ *with* $\Phi(\boldsymbol{x})$ *for the N-dimensional multivariate normal distribution function with zero means, variance terms*

$$(7) \qquad s_i^2 \equiv (\sum_{k \in \mathrm{I}} \boldsymbol{t}_k^2 - 2\boldsymbol{t}_i + 1)s^2/T, \; i \in \mathrm{I},$$

and covariance terms

$$(8) \qquad s_{ij} \equiv (\sum_{k \in \mathrm{I}} \boldsymbol{t}_k^2 - \boldsymbol{t}_i - \boldsymbol{t}_j)s^2/T, \; i, j \in \mathrm{I} : i \neq j.$$

We may use this theorem by comparing the *p*-value for the observed value of $z(\boldsymbol{t})$ with a predefined level of significance; we may reject efficiency if the *p*-value is smaller than or equal to the significance level. Computing the *p*-value requires the unknown population variance s^2. We may estimate this parameter in a distribution-free and consistent manner using the sample equivalent:

$$(9) \qquad \hat{s}^2 \equiv \sum_{\substack{i \in \mathrm{I} \\ t \in \Theta}} (\boldsymbol{x}_{it} - \sum_{\substack{i \in \mathrm{I} \\ t \in \Theta}} \boldsymbol{x}_{it} / NT)^2 / NT.$$

[8] This null is conservative, because it gives a sufficient but not necessary condition for efficiency. In fact, under the null, all portfolios **?** $\in \Lambda$ are efficient, and minimal sampling variation suffices to classify an efficient portfolio as inefficient.

The use of the most favorable distribution implies that we accept a high frequency of Type II error (accepting the null when it is not true) or a low power (1- the relative frequency of Type II error) in small samples. Fortunately, large data sets are available for many applications in financial economics. Further, we could apply econometric time series techniques to obtain an estimate for the CDF that is more efficient than the EDF. We could then apply our test to a large random sample from the estimated CDF rather than the raw data. This approach effectively uses prior distribution information to generate artificial return observations. Still, future research could focus on tests that minimize Type II error.

IV. EMPIRICAL APPLICATION

To illustrate our approach to TSD with diversification, we perform an empirical application to real-life US stock market data. Specifically, we evaluate whether the Fama and French market index is SSD efficient relative to all possible portfolios of the 25 Fama and French benchmark portfolios. The market index is the value-weighted average of all NYSE, AMEX, and NASDAQ stocks. The benchmark portfolios are the intersections of 5 portfolios formed on size (market equity) and 5 portfolios formed on the ratio of book equity to market equity (BE/ME). We use data on monthly returns (month-end to month-end) from July 1926 to December 2000 (894 observations) obtained from the data library on the homepage of Kenneth French.[9] Table 1 gives some descriptive statistics for the data.

[INSERT TABLE 1 ABOUT HERE]

We perform four different tests. First, we use the Post (2001) test for SSD efficiency. Post (2001) used the S&P 500 index to proxy the market index, and found the index to be inefficient to a statistically significant degree. Since the S&P 500 index is based on a much smaller set of stocks than the Fama and French market index (until 1979, the S&P 500 index was even limited to NYSE stocks only), our results may differ from those in Post (2001). Second, we apply our test for TSD efficiency. Again, the TSD test involves more discriminating power, because it adds the assumption that investors prefer positive skewness. More discriminating power can be obtained by adding a riskless asset to the 25 benchmark portfolios. The third and fourth tests test for SSD and TSD with a riskless asset (SSDR and TSDR). We use the one-month US Treasury bill as the riskless asset.

Our data set involves observations for a 75-year period and it is not realistic to assume that the observations are serially IID over the sample period. To account for time variation of the return distribution, we correct the raw return observations for changes in the riskless rate. In addition, we split the sample into two non-overlapping subsamples of equal size (447 observations) and analyze the robustness of our results by applying the tests both subsamples. The first subsample contains the observations from July 1926 to September 1963). The second subsample contains the remaining observations (October 1963 to December 2000).

Table 2 displays our results. Overall, there is strong evidence that the market portfolio is inefficient. For the least powerful SSD test, the *p*-value is 0.015. Introducing a riskless asset and skewness preference increases the discriminating power and lowers the *p*-values. Most notably, introducing skewness preference has a substantial impact; the *p*-values for TSD and TSDR are less than 0.001. TSD and

[9] The data library is found at *http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/*.

TSDR penalize negative skewness and the market index involves less positive skewness than most of the benchmark indexes do (see Table 1). The results are remarkably robust across the two subsamples. Still, the subsamples are relatively small, and the *p*-values increase substantially. In the second subsample, the market index can not be demonstrated to be SSD or SSDR inefficient at conventional levels of significance. However, the market index remains TSD and TSDR inefficient with 90 percent confidence.

[INSERT TABLE 2 ABOUT HERE]

V. CONCLUDING REMARKS AND SUGGESTIONS

We have derived necessary and sufficient empirical tests for TSD efficiency in case diversification between choice alternatives is allowed. Our approach relies on building nonparametric piecewise-quadratic utility functions, and on checking whether the evaluated portfolio is optimal relative to these utility functions. Further, we have discussed how bootstrapping techniques and asymptotic distribution theory can approximate the sampling properties of the test results and allow for statistical inference.

APPENDIX

Proof of Theorem 1: The necessary condition follows from the well-known Kuhn-Tucker conditions for optimizing a concave function over a convex set. Specifically, $\boldsymbol{t}$ is an optimal portfolio i.e. $\boldsymbol{t} = \arg\max_{?\in\Lambda} \sum_{t\in\Theta} u(\boldsymbol{x}_t \boldsymbol{?})/T$ for $u \in U_3$ if and only if all portfolios $\boldsymbol{?} \in \Lambda$ are enveloped by the tangent hyperplane, i.e.

$$(10)\qquad \sum_{t\in\Theta} u'(\boldsymbol{x}_t \boldsymbol{t})(\boldsymbol{x}_t \boldsymbol{t} - \boldsymbol{x}_t \boldsymbol{?})/T \geq 0 \quad \forall \boldsymbol{?} \in \Lambda .$$

If $\boldsymbol{t}$ is optimal relative to some $u \in U_3$, then it is also optimal relative to the standardized utility function $v = (u / u'(\max_{\substack{i\in \mathrm{I}\\ t\in\Theta}} \boldsymbol{x}_{it})) \in U_3$. By definition, $(v'(\boldsymbol{t}), \partial v'(\boldsymbol{t}))$, with $\partial v'(\boldsymbol{t}) \equiv (\partial v'(\boldsymbol{x}_1 \boldsymbol{t}) \cdots \partial v'(\boldsymbol{x}_T \boldsymbol{t}))$ for a subgradient of $v'(\boldsymbol{t})$, is a feasible solution, i.e. $(v'(\boldsymbol{t}), \partial v'(\boldsymbol{t})) \in \Omega$. The inequality (10) implies that this solution is associated with a solution value of zero. Hence, we find the necessary condition; $\boldsymbol{t}$ is TSD efficient only if $z(\boldsymbol{t}) = 0$.

To establish the sufficient condition, use $(\boldsymbol{ß}^*, \boldsymbol{?}^*) \in \Omega$ for the optimal solution. From this optimal solution, we can construct the following continuous piecewise-linear marginal utility function

$$(11)\quad p'(x \,|\, \boldsymbol{ß}^*, \boldsymbol{?}^*) \equiv \min_{t\in\Theta} (\boldsymbol{ß}_t^* + \boldsymbol{?}_t^* x) = \begin{cases} \boldsymbol{ß}_1^* + \boldsymbol{?}_1^* x & x \leq z_1 \\ \boldsymbol{ß}_2^* + \boldsymbol{?}_2^* x & z_1 \leq x \leq z_2 \\ \vdots & \\ \boldsymbol{ß}_{T-1}^* + \boldsymbol{?}_{T-1}^* x & z_{T-2} \leq x \leq z_{T-1} \\ \boldsymbol{ß}_T^* + \boldsymbol{?}_T^* x & x \geq z_{T-1} \end{cases}$$

with $z_t \equiv \begin{cases} \dfrac{(ß^*_{t+1} - ß^*_t)}{(?^*_t - ?^*_{t+1})} & ?^*_t \le ?^*_{t+1} \\ 0.5(\boldsymbol{x}_t + \boldsymbol{x}_{t+1}) & ?^*_t = ?^*_{t+1} \end{cases}$, $t \in \Theta \setminus T$, for the nodes that connect the line segments. By construction, $p'(x|ß^*, ?^*)$ is strictly positive over the observed return range, decreasing and convex. Integrating $ß^*_t + ?^*_t x$, $t \in \Theta$, gives $\boldsymbol{a}_t + ß^*_t x + 0.5?^*_t x^2$, where $\boldsymbol{a}_t$ is free. The integrated functions can be combined to yield the continuous piecewise-quadratic utility function:

(12) $p(x|\boldsymbol{a}^*, ß^*, ?^*) \equiv \min_{t\in\Theta} (\boldsymbol{a}^*_t + ß^*_t x + 0.5?^*_t x^2)$

$$= \begin{cases} \boldsymbol{a}^*_1 + ß^*_1 x + ?^*_1 x^2 & x \le z_1 \\ \boldsymbol{a}^*_2 + ß^*_2 x + ?^*_2 x^2 & z_1 \le x \le z_2 \\ \vdots & \\ \boldsymbol{a}^*_{T-1} + ß^*_{T-1} x + ?^*_{T-1} x^2 & z_{T-2} \le x \le z_{T-1} \\ \boldsymbol{a}^*_T + ß^*_T x + ?^*_T x^2 & x \ge z_{T-1} \end{cases}$$

with $\boldsymbol{a}^*_T \equiv 0$, and $\boldsymbol{a}^*_t \equiv \sum_{s=t}^{T-1} ((ß^*_{s+1} - ß^*_s) z_s + 0.5(?^*_{s+1} - ?^*_s) z_s^2)$, $t \in \Theta \setminus T$, to guarantee continuity, i.e. $\boldsymbol{a}^*_t + ß^*_t z_t + 0.5?^*_t z_t^2 = \boldsymbol{a}^*_{t+1} + ß^*_{t+1} z_t + 0.5?^*_{t+1} z_t^2$. By construction, $p(x|\boldsymbol{a}^*, ß^*, ?^*)$ is strictly increasing, concave, and it exhibits a preference for positive skewness. Concavity implies for all $? \in \Lambda$:

(13) $\sum_{t\in\Theta} (p(\boldsymbol{x}_t \boldsymbol{t} | \boldsymbol{a}^*, ß^*, ?^*) + (ß^*_t + ?^*_t \boldsymbol{x}_t \boldsymbol{t})(\boldsymbol{x}_t ? - \boldsymbol{x}_t \boldsymbol{t})) \ge \sum_{t\in\Theta} p(\boldsymbol{x}_t ? | \boldsymbol{a}^*, ß^*, ?^*)$.

If $z(\boldsymbol{t}) = 0$, then $\sum_{t\in\Theta} (ß^*_t + ?^*_t \boldsymbol{x}_t \boldsymbol{t}) \boldsymbol{x}_t \boldsymbol{t} = \sum_{t\in\Theta} (ß^*_t + ?^*_t \boldsymbol{x}_t \boldsymbol{t}) \boldsymbol{x}_t ?$. Combining this finding with (13), we find that $\boldsymbol{t}$ is optimal relative to $p(x|\boldsymbol{a}^*, ß^*, ?^*)$ i.e.

(14) $\sum_{t\in\Theta} p(\boldsymbol{x}_t \boldsymbol{t} | \boldsymbol{a}^*, ß^*, ?^*) = \max_{?\in\Lambda} \sum_{t\in\Theta} p(\boldsymbol{x}_t ? | \boldsymbol{a}^*, ß^*, ?^*)$.

Hence, we find the sufficient condition; portfolio $\boldsymbol{t} \in \Lambda$ is TSD efficient if $z(\boldsymbol{t}) = 0$. *Q.E.D.*

Proof of Theorem 2: As discussed in Post (2001), $w(\boldsymbol{t}) \equiv \max_{i\in I} \left\{ \sum_{t\in\Theta} (\boldsymbol{x}_{it} - \boldsymbol{x}_t \boldsymbol{t}) / T \right\}$ asymptotically behaves as the largest order statistic of N random variables with a multivariate normal distribution, and $P(w(\boldsymbol{t}) > y | H_0) =$ $1 - P(\sum_{t\in\Theta} (\boldsymbol{x}_t \boldsymbol{t} - \boldsymbol{x}_{it}) / T \le y \quad \forall i \in I)$ asymptotically equals the multivariate normal integral $1 - \int_{x \le ye} \partial\Phi(\boldsymbol{x})$. One possible solution to (P) is to set $ß_t = 1$ and $?_t = 0$ for all $t \in \Theta$ and $q = w(\boldsymbol{t})$. Therefore, $z(\boldsymbol{t}) \le w(\boldsymbol{t})$ and $P(z(\boldsymbol{t}) > y | H_0)$ is bounded from above by $P(w(\boldsymbol{t}) > y | H_0)$ for all return distributions $H(x)$. Moreover, there exist $H(x)$ for which $w(\boldsymbol{t})$ approximates $z(\boldsymbol{t})$, and therefore the asymptotic distribution of $w(\boldsymbol{t})$ also represents the asymptotic least favorable distribution for $z(\boldsymbol{t})$. *Q.E.D.*

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Table 1: Descriptive Statistics

Monthly returns (month-end to month-end) from July 1926 to December 2000 for the Fama and French market index and benchmark portfolios. The data are obtained from the data library on Kenneth French's homepage *http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/*.

Portfolio				Mean	Standard Deviation	Skewness	Kurtosis
Fama and French market index				0.0099	0.0550	0.192	7.908
Fama and French benchmark portfolios	No.	BE/ME	Size				
	1	Low	Small	0.0077	0.1266	2.812	27.919
	2	2	Small	0.0103	0.1091	3.949	48.609
	3	3	Small	0.0133	0.0951	2.001	16.856
	4	4	Small	0.0153	0.0887	2.762	29.395
	5	High	Small	0.0168	0.0983	3.225	30.477
	6	Low	2	0.0086	0.0805	0.424	5.149
	7	2	2	0.0127	0.0789	1.829	19.792
	8	3	2	0.0136	0.0754	2.316	24.315
	9	4	2	0.0140	0.0768	1.803	18.647
	10	High	2	0.0151	0.0877	1.695	15.950
	11	Low	3	0.0100	0.0770	1.010	9.936
	12	2	3	0.0122	0.0673	0.312	7.088
	13	3	3	0.0129	0.0685	0.997	13.065
	14	4	3	0.0133	0.0691	1.261	13.855
	15	High	3	0.0142	0.0870	1.931	19.238
	16	Low	4	0.0103	0.0628	-0.161	3.769
	17	2	4	0.0108	0.0639	1.060	13.649
	18	3	4	0.0121	0.0641	1.073	14.948
	19	4	4	0.0131	0.0715	1.971	21.599
	20	High	4	0.0144	0.0927	2.136	21.907
	21	Low	Big	0.0099	0.0557	-0.027	5.512
	22	2	Big	0.0095	0.0536	-0.070	5.275
	23	3	Big	0.0102	0.0581	0.788	13.668
	24	4	Big	0.0110	0.0702	1.828	21.374
	25	High	Big	0.0009	0.1441	-3.964	32.321

Table 2: Test Results

The table gives the observed value for the test statistics, as well as the asymptotic least favorable *p*-values. Panel A gives the results for the full sample (July 1926 to December 2000). Panel B gives the results for the first subsample (July 1926 to September 1963). Panel C gives the results for the second subsample (October 1963 to December 2000).

Panel A: July 1926 to December 2000		
	Statistic	*p*-value
SSD	0.0075	0.008
SSDR	0.0081	0.003
TSD	0.0092	0.000
TSDR	0.0095	0.000
Panel B: July 1926 to September 1963		
	Statistic	*p*-value
SSD	0.0088	0.058
SSDR	0.0095	0.029
TSD	0.0109	0.006
TSDR	0.0109	0.006
Panel C: October 1963 to December 2000		
	Statistic	*p*-value
SSD	0.0070	0.272
SSDR	0.0070	0.272
TSD	0.0084	0.085
TSDR	0.0084	0.085

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