Spanning and Intersection: A Stochastic Dominance Approach

Thierry Post

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Email address corresponding author	gtpost@few.eur.nl
Address	Erasmus Research Institute of Management (ERIM) Rotterdam School of Management / Faculteit Bedrijfskunde Erasmus Universiteit Rotterdam P.O. Box 1738 3000 DR Rotterdam, The Netherlands Phone: + 31 10 408 1182 Fax: + 31 10 408 9640 Email: info@erim.eur.nl Internet: www.erim.eur.nl

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Spanning and Intersection:
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THIERRY POST[*]

ABSTRACT

We propose linear programming tests for spanning and intersection based on stochastic dominance rather than mean-variance analysis. An empirical application investigates the diversification benefits to US investors from emerging equity markets.

SPANNING AND INTERSECTION are useful concepts for research in financial economics. Given a set of assets, spanning occurs if no investor in particular class of investors benefits from a particular expansion of the investment possibilities; intersection occurs if some but not all investors benefit from additional investment possibilities. These concepts are useful for numerous problems. For example, they are useful for analyzing the impact of the introduction of new assets (e.g. introduced via IPOs) or the relaxation of investment restrictions for existing assets (e.g. liberalization in emerging markets).

Thus far, the literature on spanning and intersection predominately focused on mean-variance analysis (MVA). MVA is useful for at least two reasons. First, MVA is analytically tractable and mean-variance spanning and intersection can be tested using basic regression techniques (see e.g. Huberman and Kandel (1987) and De Roon *et al.* (2001)). Second, MVA imposes structure on the data and hence can help to limit sampling error. However, in many cases it is difficult to demonstrate that MVA is also economically meaningful. For example, it is well known that MVA is consistent with the expected utility theory (EUT) only under relatively restrictive assumptions about the investor preferences and/or the statistical distribution of the investment returns (see e.g. Bigelow (1993) for necessary and sufficient conditions).

The possible lack of economic meaning of MVA provides a powerful argument for using the rules of stochastic dominance (SD; see e.g. Levy (1992) for an elaborate survey). SD uses only minimal prior assumptions with respect to investor preferences and the return distribution. It is consistent with a broad range of economic theories of choice under uncertainty, including EUT and non-expected utility theories like Yaari's (1987) dual theory of risk (see e.g. Wang and Young (1998) and Starmer (2000)). Still, SD has not seen the proliferation in applied research that one might expect based on theoretical considerations. This is presumably caused by two practical limitations:

1. SD is based on the full empirical distribution function (EDF), rather than a finite set of sample statistics. In many cases, the EDF is a statistically consistent

[*] Post is with the department of Finance of the Erasmus University Rotterdam in the Netherlands. This study forms part of a research program on nonparametric analysis of economic behavior. Details on the program are available at the program homepage*:* *http://www.few.eur.nl/few/people/gtpost/program.htm*. We gratefully acknowledge financial support by Tinbergen Institute and Erasmus Research Institute of Management.

estimator for the true cumulative distribution function (CDF). However, in small samples, the EDF generally is very sensitive to sampling variation. The outcomes of various simulation studies (e.g. Kroll and Levy (1980), among others) cause serious doubt about the reliability of SD applications that rely in a naïve way on the EDF without accounting for sampling error.

2. For applying SD to empirical data, simple crossing algorithms have been developed that check in a pairwise fashion the difference between the EDFs of the choice alternatives (e.g. Levy (1992), App. A). Unfortunately, these algorithms are unable to deal with cases that involve infinitely many choice alternatives, such as cases where diversification between choice alternatives is allowed. This limitation substantially reduces the possible application areas. Most notably, it excludes the important case of selecting investment portfolios, a case that typically allows for diversification between assets.

Recent research has dealt with these problems:

1. Various approaches have been developed to approximating the sampling distribution of SD results. Knowledge of the sampling distribution allows for constructing confidence intervals and for testing hypothesis. First, Nelson and Pope (1990) have convincingly demonstrated how bootstrapping techniques can approximate the sampling properties. Second, various authors have derived analytical characterizations of the asymptotic sampling distribution (see e.g. Beach and Davidson (1983), Dardanoni and Forcina (1999) and Davidson and Duclos (2000)).

2. Post (2001) recently developed linear programming (LP) tests for SD that do fully account for diversification. The LP structure implies that bootstrapping remains a computationally tractable approach to analyzing the sampling properties. Further, Post derived the asymptotic sampling distribution for the test results.

These recent developments could provide a strong stimulus for the further proliferation of SD. This paper aims at developing a framework for spanning and intersection based on SD rather than MVA.

The SD literature involves a multitude of different criteria, associated with different sets of preference assumptions. Higher order criteria involve more discriminating power than lower order ones, because they induce a larger reduction of the set of efficient portfolios. However, that power has to be balanced against the stringency of the additional preference assumptions. In general, striking that balance requires a careful consideration of the structure and the context of the decision problem considered. For the sake of compactness, we focus on the popular criterion of second-order SD (SSD). The assumptions associated with these criteria have a good economic interpretation (nonsatiation, risk aversion), and also empirical evidence exists to support these assumptions for many choice problems. Still, nothing excludes the generalization of our analysis towards higher order criteria (see e.g. Post (2001) for a generalization towards third-order SD).

The remainder of this paper is structured as follows. Section I introduces preliminary notation and definitions of SSD. Section II and III present our tests for spanning and intersection respectively. These tests are illustrated by means of a simple example in section VI. Section V presents an empirical application to assess if US investors benefit from international diversification using emerging markets. Finally, Section VI gives concluding remarks and suggestions for further research.

I. PRELIMINARIES

Consider an investment universe consisting of a risky benchmark asset (possibly a market portfolio) and a riskless benchmark asset with riskless return r. We analyze the effects on investment behavior of introducing N-2,$N \geq 3$, additional assets. The extended universe exists of N assets, associated with returns $x \in \Re^N$.[1] Throughout the text, we will use the index set $\mathrm{I} \equiv \{1, \cdots, N\}$ to denote the different assets. The risky benchmark asset is denoted by $m \in I$, and the riskless benchmark asset by $f \in I \setminus m$. In addition, we will treat the returns as serially independent and identically distributed (IID) random variables with a continuous joint cumulative distribution function (CDF), $G(x)$. Investors may construct portfolios as convex combinations of the existing assets. Throughout the text, we will denote portfolio weights by $l \in \Re^N$. The initial portfolio possibilities are represented by $\Lambda_1 \equiv \{kd_m + (1-k)d_f : k \in [0,1]\}$, where $d_i, i \in \mathrm{I}$, denotes a ($N \times 1$) coordinate vector of zeros with a unity value for the i-th element. The extended possibilities are represented by $\Lambda_2 \equiv \{l \in \Re_+^N : l^T e = 1\}$, where e denotes a ($N \times 1$) unity vector. These sets assume that short sales are not allowed and that no additional restrictions are imposed on the portfolio weights. Still, the analysis can be extended towards a general polyhedral portfolio possibilities set, and hence it is possible to introduce short selling and to include additional investment restrictions (see point 1 in the Conclusions).

Investors choose portfolios to maximize the expected value of their utility function $u : \Re^1 \to P$, $u \in U$, with U for the class of von Neuman-Morgenstern utility functions, and P for a nonempty, closed and convex subset of $\Re$. Specifically, an investor with preferences $u \in U$ and possibilities $\Lambda_i, i \in \{1,2\}$, chooses his portfolio as the solution to $\max_{l \in \Lambda_i} \int u(xl) \partial G(x)$.

In practical applications, full information about investor preferences typically is not available, and one generally can not determine the optimal portfolio. This provides the rationale for using SD rules that rely on a set of general preference assumptions only. The SSD criterion restricts attention to the class of nonsatiable and risk-averse investors or the class of strictly increasing and concave utility functions $U_2 \subseteq U$. (Note that we do not assume that the utility function is continuously differentiable. However, utility is concave and hence everywhere continuous and superdifferentiable.)

Apart from investor preferences, the CDF generally is not known in practical applications. Rather, information typically is limited to a discrete set of time series

[1] Throughout the text, we will use $\Re^m$ for an m-dimensional Euclidean space, and $\Re_+^m$ denotes the positive orthant.

observations, say $X \equiv (x_1 \cdots x_T)^T$ with $x_t \equiv (x_{1t} \cdots x_{Nt}) \in \Re^N$, which can be treated as a random sample from the CDF. Throughout the text, we will use the index set $\Theta \equiv \{1, \cdots, T\}$ to denote different points in time. Using the observations, we can construct the empirical distribution function (EDF) $F(x) \equiv \text{card}\{t \in \Theta : x_t \leq x\}/T$. In this paper, we analyze SD for the EDF rather than for the CDF, so as to derive ready empirical tests; a detailed treatment of the relationship between the EDF and the CDF is beyond the scope of this paper. Still, in the application section, we do use the bootstrap techniques to assess the sensitivity of our results with respect to sampling variation. Further, to simplify notation, we assume that the data are ranked in ascending order by the return of the risky benchmark portfolio, i.e. $x_{m1} < x_{m2} < \cdots < x_{mT}$. Since we assume a continuous return distribution, ties do not occur and the ranking is unique. Still, the analysis can be extended in a straightforward way to cases where ties do occur e.g. due to a discrete return distribution or due to measurement problems or rounding (see Post, 2001).

Following Post (2001), we may define SSD as follows:

DEFINITION 1 *Portfolio* $t \in \Lambda_i, i \in \{1,2\}$, *is SSD efficient if and only if it maximizes the expected value of some utility functions* $u \in U_2$, *i.e.*

(1) $$\max_{l \in \Lambda_i} \min_{u \in U_2} \left\{ \int (u(xl) - u(xt)) \partial F(x) \right\} = \max_{l \in \Lambda_i} \min_{u \in U_2} \left\{ \sum_{t \in \Theta} \left(u(x_t l)) - u(x_t t) \right) / T \right\} = 0.$$

For simplicity, we assume that the risk free return exceeds the minimum return for the risky benchmark asset, and that it falls below the average return for the risky asset, i.e. $\min_{t \in \Theta} x_{mt} < r < \sum_{t \in \Theta} x_{mt} / T$. Under this assumption, the efficient set for Λ_1 is represented by $\{l \in \Lambda_1 : l_m > 0\}$. Note that the efficient set does not include full investment in the riskless asset. Since $r < \sum_{t \in \Theta} x_{mt} / T$, all investors invest at least some part of their wealth in the risky asset. This finding reflects Arrow's theorem - 'A risk averter takes no part of an unfavorable or barely fair game; on the other hand, *he always takes some part of a favorable gamble*' (Arrow, 1970, p. 100, italics as in the original text). For Λ_2, the efficient set generally is more complex. To the best of our knowledge, a full characterization of the SD efficient set is not available. Still, it is possible to test if a given portfolio is included in the efficient set by using the LP tests by Post (2001).

II. SPANNING

DEFINITION 2 *The extended investment possibilities* Λ_2 *are SSD spanned by the initial possibilities* Λ_1 *if and only if no risk-averse investor is better off by investing part of his wealth in the additional assets, i.e.*

(2) $$\max_{l \in \Lambda_2} \sum_{t \in \Theta} u(x_t l)/T = \max_{l \in \Lambda_1} \sum_{t \in \Theta} u(x_t l)/T \quad \forall u \in U_2 .$$

Spanning is a useful concept, because assets that are spanned can be ignored for the purpose of portfolio selection. This property is also useful for asset pricing, as it can be used to determine a critical price at which no rational investor would invest in a new asset. Of course, we generally do not have return observations to construct the EDF for a new asset. However, this approach can be useful for pricing derivatives based on an existing asset. Interestingly, Levy (1985) took this approach to derive upper and lower bounds on option prices. As discussed in the Conclusions, our test can strengthen the Levy bounds.

THEOREM 1 *SSD spanning can be tested using the test statistics*

(3) $$y_P \equiv \inf_{i \in \mathrm{I}} \inf_{b \in B} \left\{ \sum_{t \in \Theta} b_t (x_{mt} - x_{it})/T : \sum_{t \in \Theta} b_t (x_{mt} - r)/T \geq 0 \right\},$$

with $B \equiv \left\{ b \in \Re_+^T : b_1 \geq b_2 \geq \cdots \geq b_T \geq 1 \right\}$, *and*

(4) $$y_D \equiv \inf_{i \in \mathrm{I}} \sup_{r \geq 0} \left\{ \sum_{s=1}^{T} ((1-r)x_{ms} - x_{is} + rr)/T : \sum_{s=1}^{t} ((1-r)x_{ms} - x_{is} + rr)/T \geq 0 \quad \forall t \in \Theta \right\}.$$

Specifically, spanning occurs if and only if $y_P = y_D \geq 0$.

Test statistic y_P checks an optimality condition for supporting hyperplanes for the return vectors of the benchmark assets and test assets (see the proof in the Appendix). The alternative statistic y_D gives a dual formulation of y_P. It checks an optimality condition for the running mean of the differences between the returns of the benchmark assets and the additional assets. The test statistics y_P and y_D can be computed by solving the embedded LP problems for each $i \in \mathrm{I} \setminus \{m, f\}$ (we can without harm ignore *m* and *f*, because these assets can never yield a strictly negative solution value). Full LP formulations are included in the proof in the Appendix. If spanning does not occur, then the primal model may be unbounded and the dual model may be infeasible. For example, the primal is unbounded and the dual infeasible if $\sum_{t\in\Theta} x_{it}/T > \sum_{t\in\Theta} x_{mt}/T$ for some $i \in \mathrm{I}\setminus\{m, f\}$. In such cases, the test statistics take the value minus infinity and spanning does not occur.

III. INTERSECTION

DEFINITION 3 *The efficient sets for the initial investment possibilities* Λ_1 *and the extended possibilities* Λ_2 *SSD intersect if and only if some but not all risk-averse investors are better off by investing part of their wealth in the additional assets, i.e. spanning does not occur and*

$$(5) \qquad \max_{l\in\Lambda_2} \sum_{t\in\Theta} u(x_t l)/T = \max_{l\in\Lambda_1} \sum_{t\in\Theta} u(x_t l)/T \quad u \in U_2 .$$

If spanning and intersection do not occur, then every investor will invest in the new assets. In this respect, the concept is related to Marginal Conditional Stochastic Dominance (MCSD, Shalit and Yitzhaki, 1994). MCSD states the conditions under which all risk-averse individuals, given a portfolio of assets, prefer to substitute one risky asset for another, while keeping the core of the portfolio constant. MCSD is an important concept, because many investors face investment restrictions that allow only for marginal changes to the portfolio. As discussed in the Conclusions, it is possible to generalize our approach towards general polyhedral portfolio possibilities. The generalized model includes MCSD as a special case.

THEOREM 2 *SSD intersection can be tested using the Post (2001) test statistics*

$$(6) \qquad x_P \equiv \min_{b\in B} \left\{ q : \sum_{t\in\Theta} b_t (x_{mt} - x_{it})/T + q \geq 0 \quad \forall i \in \mathrm{I} \right\},$$

$$(7) \qquad x_D \equiv \max_{l\in\Lambda_2} \left\{ \sum_{s=1}^{T} (x_t l - x_{mt})/T : \sum_{s=1}^{t} (x_t l - x_{mt})/T \geq 0 \quad \forall t \in \Theta \right\}.$$

Specifically, intersection occurs if and only if spanning does not occur and $x_P = x_D = 0$.

Like y_P, x_P checks an optimality condition for the supporting hyperplanes for the return vectors of the benchmark and the additional assets. Like y_D, x_D checks an optimality condition for the running mean of the differences between the returns of the benchmark assets and the additional assets. The test statistics x_P and x_D can be computed using straightforward LP; full LP formulations are provided in Post (2001).

IV. EXAMPLE ILLUSTRATION

To illustrate the tests for intersection and spanning, consider a simple example with two periods, $\Theta = \{1,2\}$, and three assets, $\mathrm{I} = \{1,2,3\}$. The initial portfolio possibilities include two assets, the risky benchmark asset m and the riskless benchmark asset f. The extended portfolio possibilities include the additional risky asset n.

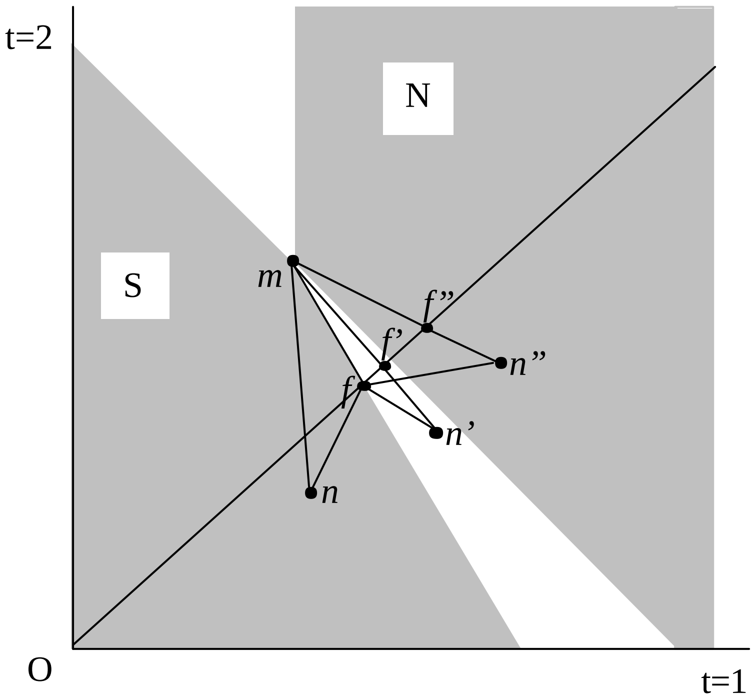

Fig. 1 Two period, three assets example. Adding a third asset to *m* and *f* expands the investment possibilities. Introducing *n* does not affect the efficient set *mf* (excluding *f*), and hence spanning occurs. Introducing *n'* changes the efficient set to *mf'* (excluding *f'*) and intersection occurs. Finally, introducing *n'* changes the efficient set to *f"n"* (excluding *f")* and spanning and intersection do not occur.

The initial portfolio possibilities set is *mf*, and the efficient set is *mf* excluding *f*. Introducing the additional asset can change the possibilities set and the efficient set. The graph displays three different cases, labeled *n*, *n*' and *n"*, associated with extended possibilities *mfn*, *mfn*' and *mfn"* respectively. In the first case, with new asset *n*, spanning occurs; *mf* (excluding *f*) remains the efficient set. In fact, spanning occurs for all new assets included in the gray area

$S \equiv \left\{ x \in \Re^2 : x_1 + x_2 \leq (1-r)(x_{m1} + x_{m2}) + 2rr, x_1 \leq (1-r)x_{m1} + rr \quad r \geq 0 \right\}$.

In the second case, with new asset *n*', intersection occurs. The efficient set changes from *mf* to *mf'* (excluding the riskless *f'*); some investors invest in *m*, and some invest in portfolios including *n'*. Intersection occurs if the new asset is not included in $N \equiv \left\{ x \in \Re^2 : x_1 \geq x_{m1}, x_1 + x_2 \geq x_{m1} + x_{m2} \right\}$. If the new asset is included in *N*, then spanning and intersection do not occurs and all investors invest at least part of their wealth in the new asset. This is what happens in the third case, with new asset n''. In this case, the efficient set is $f''n''$ (excluding the riskless f'').

V. EMPIRICAL APPLICATION

To further illustrate our approach, we analyze the diversification benefits for US investors from investing in emerging equity markets. We use the MSCI USA index as a proxy for US investors that hold a well-diversified domestic equity portfolio.[2] The riskless return equals the one-month US Treasury bill rate (from Ibbotson Associates). Further, to capture the investment possibilities in emerging markets, we use the MSCI emerging market (EM) indexes for (1) Latin America, (2) Asia and (3) Europe and Middle East. For all indexes, we use monthly dividend adjusted returns in US dollars for the period January 1988 to September 2001 (165 observations). Table 1 gives some descriptive statistics for the data set.

Table 1 Descriptive statistics. Monthly dividend adjusted returns in US dollars for the period January 1988 to September 2001 (Source: www.msci.com)

	MSCI EM Latin America	MSCI EM Asia	MSCI EM Europe and Middle East	MSCI USA
Mean	0.0170	0.0034	0.0040	0.0097
Std. Dev.	0.0928	0.0802	0.0866	0.0409
Skewness	-0.5488	0.0859	0.4573	-0.4047
Kurtosis	1.2152	0.5212	3.3253	0.5978
Minimum	-0.3557	-0.2189	-0.3104	-0.1403
Maximum	0.2422	0.2453	0.4142	0.1115

The tests discussed above focus on the case without short-selling selling. As discussed in the Conclusions, our model can be extended in a straightforward manner to account for (bounded) short selling. In this application, we consider two cases: (1) the case where it is possible to sell short a maximum of 100 percent of the net value of the portfolio, and (2) the case without short selling.

As discussed in the Introduction, analysis based on the EDF is likely to be affected by sampling error in a non-trivial way. To approximate the sampling distribution of our results, we use the bootstrap method. Bootstrapping, first introduced by Efron (1979) and Efron and Gong (1983), is based on the idea of repeatedly simulating the CDF, usually through resampling, and applying the original estimator to each simulated sample or pseudo-sample so that the resulting estimators mimic the sampling distribution of the original estimator. Key to the success of the bootstrap is the selection of an appropriate approximation for the CDF. If the approximation is statistically consistent, then the bootstrap distribution gives a statistically consistent estimator for the original sampling distribution. In the context of our tests, the EDF is an appropriate approximation for the CDF; under the assumption that the return distribution is serially IID (see Section I), the EDF is a consistent estimator of the true CDF. This suggests bootstrapping samples would be simply obtained by randomly

[2] Similar results are obtained for US investors that hold a well-diversified international portfolio of stocks in mature markets (USA, Europe and Japan), as proxied by the MSCI World index.

sampling with replacement from the EDF along the lines of the 'correlation model' proposed by Freedman (1981) in a regression framework.

Figure 2 gives the bootstrap distribution resulting from this approach, for the case with bounded short selling. In only 10 out of 1000 random pseudo-samples, the MSCI USA index is classified as SSD efficient. These results suggest that the index is SSD inefficient to a statistically significant degree. Hence, spanning and intersection do not occur; *all* risk-averse US domestic investors benefit from international diversification using emerging market equity.

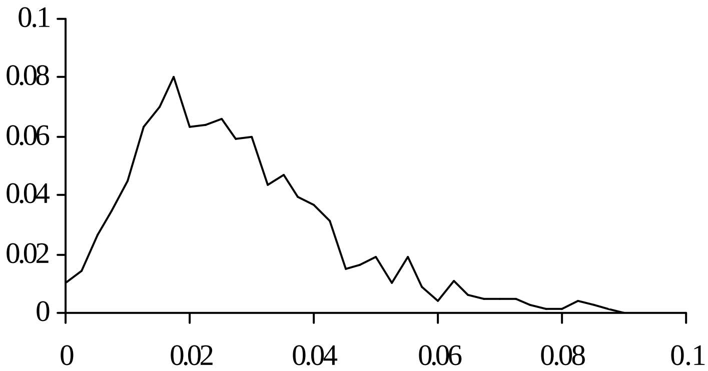

Fig. 2: Bootstrap distribution for the MSCI USA index with bounded short selling, based on 1000 replications. The index is classified as SSD efficient in only 10 out of 1000 pseudo-samples, which suggests that the index is SSD inefficient to a statistically significant degree.

The evidence in favor of diversification benefits disappears if short selling is not allowed. Figures 3 gives the results for the case without short selling. In 399 out of 1000 random pseudo-samples, the MSCI USA index is classified as SSD efficient. This suggests that the index is not inefficient to a statistically significant degree. Still, the index does not span the MSCI EM indexes in nay of the pseudo-samples (the average return on the MSCI EM Latin America index always exceeds that on the MSCI USA index). Hence, for some (but not all) risk-averse domestic market investors, the benefits associated with emerging market equity disappear if short selling is not allowed.

These results are consistent with the results as reported for studies that use MVA rather than SD (e.g. Bekaert and Urias, 1996, and de Roon *et al.*, 2001). Still, we stress that this application is used for the purpose of illustration. A sound empirical study requires more rigor than is possible here. For example, we have not analyzed the effects of transaction costs and investment restrictions other than short selling restrictions. Also, we have not analyzed the sensitivity of our results to the return horizon and the sample period (e.g. considering the effect of the emerging market liberalizations in the early 1990s). We leave these issues for further research.

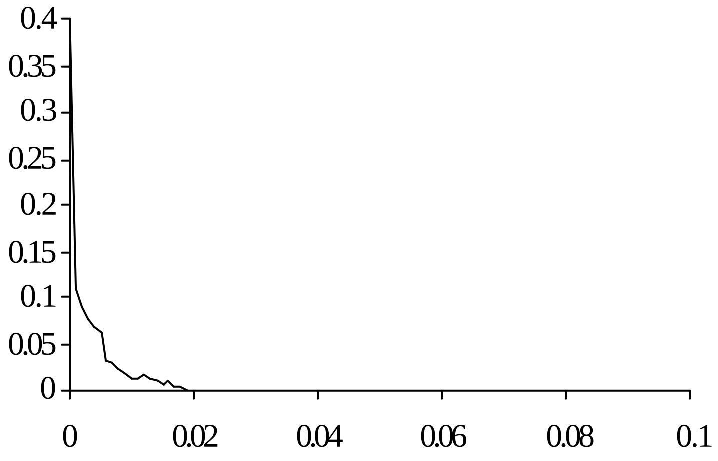

Fig. 3: Bootstrap distribution for the MSCI USA index without short selling, based on 1000 replications. The index is classified as SSD efficient in 399 out of 1000 pseudo-samples, which suggests that the index is not SSD inefficient to a statistically significant degree.

VI. CONCLUDING REMARKS

1. Straightforward linear programming can apply our tests to empirical data. For small data sets up to hundreds of observations and/or assets, the problem can be solved with minimal computational burden, even with desktop PCs and standard solver software (like LP solvers included in spreadsheets). For example, the computations for our emerging markets application (165 observations, 4 individual portfolios and a riskless fund) used the simplex module of Aptech Systems' GAUSS software, operated on a desktop PC with a 1700 MHz Pentium IV microprocessor and with 512 MB of working memory available. The computations required only minimal burden; the run time for the SSD tests was less than 1 second on average. With the current exponential growth of computer power, the computational burden can be expected to drop much further in the foreseeable future. Still, the computational complexity, as measured by the required number of arithmetic operations, and hence the run time and memory space requirement, increases progressively with the number of variables and restrictions. Therefore, specialized LP solver software is recommended for large-scale problems involving thousands of observations and/or assets.[3]

2. We stress that the SD tests are not intended to replace the MVA tests. SD uses minimal prior preference and distribution assumptions and it therefore involves less Type I error (wrongly classifying an efficient portfolio as inefficient) than MVA. However, by imposing prior structure on the data MVA involves more

[3] For an elaborate introduction in LP, we refer to Chvatal (1983). In practice, very large LPs can be solved efficiently by both the simplex method and interior-point methods. An elaborate guide to LP solver software can be found at the homepage of the Institute for Operations Research and Management Science (INFORMS); http://www.informs.org/.

power (or less Type II error; wrongly classifying an inefficient portfolio as efficient) than SD.

3. For the sake of simplicity, we have thus far assumed that short sales are not allowed and that no additional restrictions are imposed on the portfolio weights. Our analysis is based on the optimality conditions from subdifferential calculus for optimizing a concave utility function over a convex portfolio possibilities set (see the proof to Thm 1). In principle, the analysis can be extended for any nonempty, closed and convex portfolio possibilities set. However, to preserve the LP structure of our tests, we need linear restrictions on the portfolio weights, and the portfolio possibilities set needs to take a polyhedral shape. Generalizations that account for short sales or additional restrictions are best phrased in terms of the dual formulation (D) developed in Section III; one can simply replace Λ by the appropriate polyhedron $\mathrm{K} \equiv \left\{ l \in \Re^N : Al \leq b \right\}$, with A for a $(M \times N)$ matrix and b for a $(N \times 1)$ vector representing M linear inequality restrictions. The general polyhedral gives much freedom to model investor restriction and can give a more realistic approximation for real-life investment problems than the basic model without short sales and without further restrictions. Interestingly, MCSD (Shalit and Yitzhaki (1994)) is included as a special case. Specifically, it is easy to verify that MCSD is equivalent to testing if the given portfolio d_m is SSD efficient relative to $\Lambda = \{d_m + kd_j - kd_i : k \in [0,e]\}$, if we use $i \in \mathrm{I}$ and $j \in \mathrm{I}$ for the assets that can be changed, and $e > 0$.

4. As suggested in section II, one possible application of our spanning tests is in derivative prices. Derivative securities are new assets for which we can derive the return distribution from the distribution of the underlying asset. The tests can determine an upper bound for the value of a derivative security as the critical price at which the existing assets span the derivative security. Similarly, a lower bound is obtained as the critical price at which the underlier is spanned. Levy (1985) used this approach to deriving option bounds. Our analysis can strengthen the Levy bounds in three ways. First, the Levy bounds are based on the assumption that the investor chooses either the option or a portfolio of the underlier and a second fund (which can be either a riskless fund or a risky fund). Our tests can obtain tighter bounds by accounting for multiple assets. Second, Levy used necessary conditions and sufficient conditions for SD, but not the exact necessary and sufficient conditions, while we use exact necessary and sufficient conditions. Finally, Levy assumed that short-selling and riskless borrowing is not allowed, while expanded investment possibilities can be included in a straightforward in our tests (see point 3 above).

5. We focussed on the case with two benchmark assets: a risky asset and a riskless asset. This is useful e.g. if we can identify a meaningful market portfolio or alternatively if we analyze a given portfolio of a single investor. It would be interesting to extend our analysis to the case with multiple risky benchmark assets, e.g. cases where the market portfolio is not known or alternatively where different investors hold different portfolios of risky assets. Unfortunately, introducing additional risky benchmark assets introduces substantial computational complexity. In our model, all portfolios of the risky asset and a riskless asset have the same ordering for the returns. In case of multiple risky benchmark assets,

many different rankings generally occur. Determining all different rankings is not easy and enumerating all possible rankings may involve substantial computational burden. Still, it may be possible to find necessary conditions that are computationally less demanding than the full necessary and sufficient condition. For example, a (very weak) sufficient condition for spanning that applies irrespective of the ranking is that all additional assets are strictly dominated (i.e. there exist portfolios that always give a higher return). We leave this route for further research.

APPENDIX

Proof to Theorem 1 We first consider the sufficient condition, i.e. spanning occurs if $y_P \geq 0$. If spanning does not occur, we have

(i) $\exists u \in U_2 : \sum_{t\in\Theta} u(x_t t(u,\Lambda_2)) > \sum_{t\in\Theta} u(x_t t(u,\Lambda_1))$,

with $t(u,\Lambda_i)$ for the optimal solution to $\max_{l\in\Lambda_i} \sum_{t\in\Theta} u(x_t l)$, $i \in \{1,2\}$. Using $\partial u(l) \equiv (\partial u(x_1 l)\cdots\partial u(x_T l))$ for a supergradient of u at Xl, the optimality conditions for convex problems (see e.g. Hiriart-Urruty and Lemaréchal (1993), Thm. VII:1.1.1 and Cond. VII: 1.1.3) require:

(ii) $\sum_{t\in\Theta} u(x_t t(u,\Lambda_1)) = \sum_{t\in\Theta} \partial u(x_t t(u,\Lambda_1)) x_{mt} \geq \sum_{t\in\Theta} \partial u(x_t t(u,\Lambda_1)) r \quad \forall u \in U_2$.

Concavity of u implies

(iii) $\sum_{t\in\Theta} u(x_t t(u,\Lambda_2)) \leq \sum_{t\in\Theta} \partial u(x_t t(u,\Lambda_1)) x_t t(u,\Lambda_2) \leq \max_{i\in I} \sum_{t\in\Theta} \partial u(x_t t(u,\Lambda_1)) x_{it}$

Combining (i), (ii), and (iii), we find

(iv) $\exists u \in U_2 : \max_{i\in I} \sum_{t\in\Theta} \partial u(x_t t(u,\Lambda_1)) x_{it} > \sum_{t\in\Theta} \partial u(x_t t(u,\Lambda_1)) x_{mt} \geq \sum_{t\in\Theta} \partial u(x_t t(u,\Lambda_1)) r$.

If these inequalities apply for $u \in U_2$, then they also apply for the standardized utility function $v \equiv u/\partial u(x_T t(u,\Lambda_1)) \in U_2$. By construction, $\partial v(t(u,\Lambda_1))$ is a feasible solution, i.e. $\partial v(t(u,\Lambda_1)) \in B$ (recall that all portfolios of x_1 and x_2 have the same ranking as x_1). The inequality (iv) imply that this solution is associated with a strictly negative solution value. Hence, spanning does not occur only if $y_P < 0$, or alternatively spanning occurs if $y_P \geq 0$.

We next consider the necessary condition, i.e. spanning occurs only if $y_P \geq 0$. If $y_P < 0$, then

(v) $\sum_{t\in\Theta} b_t^*(x_{mt} - x_{i^*t})/T < 0; \sum_{t\in\Theta} b_t^*(x_{mt} - r)/T \geq 0$,

with $b^* \in B$, $i^* \in I \setminus \{m, f\}$ for the optimal solution.

We can then always find $k \in \Lambda_2 : k_{i^*} > 0, k_m \geq 0, k_j = 0 \quad \forall j \in I \setminus \{i^*, m\}$ such that the ranking of x_m is preserved, i.e. $x_1 k < x_2 k < \cdots < x_T k$, and

(vi) $\sum_{t\in\Theta} b_t^*(x_{mt} - x_t k)/T < 0; \sum_{t\in\Theta} b_t^*(x_{mt} - r)/T \geq 0$.

Now consider the piecewise linear utility function $p(x) \equiv \min_{t\in\Theta}(a_t + b_t^* x)$, with $a_t \equiv 0.5\sum_{s=t}^{T-1}(b_{s+1}^* - b_s^*)(x_t k + x_{t+1} k)$. By construction, this function is monotone increasing and concave and hence $p(x) \in U_2$. It is easy to verify that $\sum_{t\in\Theta} p(x_t k)/T = \sum_{t\in\Theta}(a_t + b_t^* x_t k)/T$ and $\sum_{t\in\Theta} p(x_t t(p,\Lambda_1))/T \le \sum_{t\in\Theta}(a_t + b_t x_t t(p,\Lambda_1))/T$.
Combining this with (vi), we find that $y_P < 0$ implies that

(vii) $\quad \sum_{t\in\Theta} p(x_t k)/T > \sum_{t\in\Theta} p(x_t t(p,\Lambda_1))/T$.

Hence, if $y_P < 0$, then (i) is satisfied and spanning does not occur. Alternatively, spanning occurs only if $y_P \ge 0$.

Finally, we consider the alternative test statistic y_D. This statistic is obtained by applying linear duality theory to y_P. Specifically, the following is a full LP formulation for the embedded problem for $i \in \mathrm{I}$:

(P) $\quad \min \sum_{t=1}^{T} b_t (x_{mt} - x_{i1})/T$

$$\text{s.t.} \sum_{t=1}^{T} b_t (x_{mt} - r)/T \ge 0 \quad (s_T)$$
$$b_t - b_{t+1} \ge 0 \quad t = 1,\cdots,T-1 \quad (s_t)$$
$$b_T \ge 1 \quad (r)$$
$$b_t \text{ free} \quad t = 1,\cdots,T$$

The shadow prices to the restrictions are given within brackets. This information is useful for interpreting the LP dual of (P):

(D) $\quad \max s_T$

$$\text{s.t. } r(x_{m1} - r)/T + s_1 = (x_{m1} - x_{i1})/T \quad (b_1)$$
$$r\sum_{s=1}^{t}(x_{ms} - r) + s_t - s_{t-1} = \sum_{s=1}^{t}(x_{ms} - x_{is})/T \quad t = 2,\cdots,T \quad (b_T)$$
$$s_t \ge 0 \quad t = 1,\cdots,T$$
$$r \ge 0$$

Again, the primal variables that correspond to each of the dual constraints are given within bracket, so as facilitate the interpretation and the relationship between the primal and the dual. The equality restrictions can be satisfied only by setting $s_t = s_t^* \equiv \sum_{s=1}^{t}((1-r)x_{ms} - x_{is} + rr)/T$. Substituting s_t^* for s_t in (D) gives y_D. If spanning does not occur, then (P) can be unbounded and (D) can be infeasible. However, if spanning does occur, both problems are feasible and bounded and duality implies $y_P = y_D \ge 0$. *Q.E.D.*

Proof to Theorem 2 Spanning and intersection do not occur if and only if all portfolios $l \in \Lambda_1$ are SSD inefficient relative to Λ_2. The riskless benchmark asset f is inefficient by Arrow's Theorem (see Section I). If the risky benchmark asset m is inefficient, then there exists $l^* \in \Lambda_2$ that SSD dominates m, i.e. $\sum_{t\in\Theta} u(x_t l^*)/T \geq \sum_{t\in\Theta} u(x_{mt})/T$ for all $u \in U_2$ (see Definition 1, and Post (2001), Thm. 1). Every $l \in \Lambda_1 \setminus d_f$ can be expressed as $l = kd_m + (1-k)d_f$ for some $k \in \langle 0,1]$, and if m is inefficient, then l is SSD dominated by $(kl^* + (1-k)d_f) \in \Lambda_2$. Therefore, all portfolios $l \in \Lambda_1$ are inefficient if and only if m is inefficient.
It follows from Post (2001, Thm. 2 and Thm. 3) that m is inefficient if and only if $x_P = x_D > 0$. Hence, spanning and intersection do not occur if $x_P = x_D > 0$, and intersection occurs if and only if spanning does not occur and $x_P = x_D = 0$. *Q.E.D.*

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