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Uniform Topologies on Types*

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Abstract

We study the robustness of interim correlated rationalizability to perturbations of higher-order beliefs. We introduce a new metric topology on the universal type space, called *uniform weak topology*, under which two types are close if they have similar first-order beliefs, attach similar probabilities to other players having similar first-order beliefs, and so on, where the degree of similarity is uniform over the levels of the belief hierarchy. This topology generalizes the now classic notion of proximity to common knowledge based on *common p-beliefs* (Monderer and Samet (1989)). We show that convergence in the uniform weak topology implies convergence in the *uniform strategic topology* (Dekel, Fudenberg, and Morris (2006)). Moreover, when the limit is a finite type, uniform-weak convergence is also a necessary condition for convergence in the strategic topology. Finally, we show that the set of finite types is nowhere dense under the uniform strategic topology. Thus, our results shed light on the connection between similarity of beliefs and similarity of behaviors in games.

Keywords: Rationalizability, incomplete information, higher-order beliefs, strategic topology, electronic mail game.

JEL Classification: C70, C72.

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1 Introduction.

The Bayesian analysis of incomplete information games requires the specification of a *type space*, which is a representation of the players' uncertainty about fundamentals, their uncertainty about the other players' uncertainty about fundamentals, and so on, *ad infinitum*. Thus the strategic outcomes of a Bayesian game may depend on entire infinite hierarchies of beliefs. Critically, in some games this dependence can be very sensitive at the tails of the hierarchies, so that a mispecification of higher-order beliefs, even at arbitrarily high orders, can have a large impact on the predictions of strategic behavior, as shown by the Electronic Mail game of Rubinstein (1989). As a matter of fact, this phenomenon is not special to the E-mail game. Recently, Weinstein and Yildiz (2007) have shown that in any game satisfying a certain payoff richness condition, if a player has multiple actions that are consistent with *interim correlated rationalizability*—the solution concept that embodies common knowledge of rationality—then any of these actions can be made *uniquely* rationalizable by suitably perturbing the player's higher-order beliefs at any arbitrarily high order. This phenomenon raises a conceptual issue: if predictions of strategic behavior are not robust to mispecification of higher-order beliefs, then the common practice in applied analysis of modeling uncertainty using small type spaces—often finite—may give rise to spurious predictions.

A natural approach to study this robustness problem is topological. Consider the correspondence that maps each type of a player into his set of interim correlated rationalizable (ICR) actions. The fragility of strategic behavior identified by Rubinstein (1989) and Weinstein and Yildiz (2007) can be recast as a certain kind of discontinuity of the ICR correspondence in the *product topology* over hierarchies of beliefs, i.e. the topology of weak convergence of k-order beliefs, for each $k \geq 1$. While in every game the ICR correspondence is upper hemi-continuous in the product topology, lower hemi-continuity can fail even for the *strict* ICR correspondence—a refinement of ICR that requires the incentive constraints to hold with strict inequality.[1] Strictness rules out incentives that hinge on a "knife-edge," which can always be destroyed by suitably perturbing the payoffs of the game. Indeed, non-strict solution concepts are known to fail lower hemi-continuity in other contexts: e.g., in complete information games, Nash equilibrium, and, in fact, even best-reply correspondences fail to be lower hemi-continuous with respect to payoff perturbations. On the other hand, the strict Nash equilibrium and the strict best-reply correspondences *are* lower hemi-continuous. It is therefore surprising that this form of continuity breaks down when it comes to perturbations of higher-order beliefs.

[1] Here, the notion of strictness is actually quite strong: the slack in the incentive constraints is required to be bounded away from zero uniformly on a best reply set. Despite this, the strict ICR correspondence fails to be lower hemi-continuous in the product topology.

There exist, of course, finer topologies under which the ICR correspondence is upper hemi-continuous and the strict ICR correspondence is lower hemi-continuous in all games. The coarsest such topology is the *strategic topology* introduced by Dekel, Fudenberg, and Morris (2006); it embodies the minimum restrictions on the class of admissible perturbations of higher-order beliefs necessary to render rationalizable behavior continuous. Thus the strategic topology gives a tight measure of the robustness of strategic behavior: if the analyst considers any larger set of perturbations he is bound to make a non-robust prediction in some game. Given this significance, we believe the strategic topology deserves closer examination. Indeed, Dekel, Fudenberg, and Morris (2006) only define it *implicitly* in terms of proximity of behavior in games, as opposed to *explicitly* using some notion of proximity of probability measures. This leaves open the important question as to what proximity in the strategic topology means in terms of the beliefs of the players.

To address this question we introduce a new metric topology on types, called *uniform weak topology*, under which a sequence of types $(t_n)_{n\geq 1}$ converges to a type t if the k-order belief of t_n weakly converges to that of t and the rate of convergence is uniform over $k \geq 1$. More precisely, for each $k \geq 1$ we consider the *Prohorov* metric, d^k, over k-order beliefs—a standard metric that metrizes the topology of weak convergence of probability measures—and then define the uniform weak topology as the topology of convergence in the metric $d^{\mathsf{uw}} \equiv \sup_{k\geq 1} d^k$. Our first main result, Theorem 1, is that convergence in the uniform weak topology implies convergence in the *uniform strategic topology*. The latter, also introduced by Dekel, Fudenberg, and Morris (2006), is the coarsest topology on types under which the ICR correspondence is upper hemi-continuous and the strict ICR correspondence is lower hemi-continuous, where the continuity is now required to hold uniformly across all games.[2] In particular, Theorem 1 implies that convergence in the uniform weak topology is a sufficient condition for convergence in the strategic topology.

To put Theorem 1 in perspective, a comparison with a well known result of Monderer and Samet (1989) will prove useful. This early paper studies the robustness of Nash equilibrium to small amounts of incomplete information, defining proximity to complete information via the notion of *common belief*. Given a payoff-relevant parameter θ, say that a type of a player has *common p-belief in θ* if he assigns probability no smaller than p to θ, assigns probability no smaller than p to the event that θ obtains and the other players assign probability no smaller than p to θ, and so forth, *ad infinitum*. A sequence of types $(t_n)_{n\geq 1}$ has *asymptotic common certainty of θ* if for every $p < 1$, t_n has common p-belief in θ for all n large enough. Although the focus of Monderer and Samet (1989) is on the *ex ante* robustness of Nash equilibrium under common prior perturbations, their main result has the following counterpart in our interim, non-common prior, non-equilibrium framework:

[2]See section 3 for the precise definition of the modulus of continuity on which the uniformity is based.

If a sequence of types $(t_n)_{n\geq 1}$ has asymptotic common certainty of θ then, for every game, every action that is strictly interim correlated rationalizable when θ is common certainty remains interim correlated rationalizable for type t_n, for all n large enough.

It turns out that asymptotic common certainty of θ is equivalent to uniform-weak convergence to the type that has common 1-belief in θ. Thus our Theorem 1 is a generalization of Monderer and Samet's (1989) main result to environments where the limit game has incomplete information.

An important corollary of Theorem 1 is that the strategic, uniform-strategic and product topologies generate the same σ-algebra.[3] Indeed, a fundamental result of Mertens and Zamir (1985), which is the Bayesian foundation of Harsanyi's (1967-68) model of types, is that the space of hierarchies of beliefs, called the *universal type space*, exhausts all the relevant uncertainty of the players when endowed with the product σ-algebra. It is reassuring to know that this universality property remains valid when the players can reason about any strategic event.[4]

Our second main result, Theorem 2, is that uniform-weak convergence is also a necessary condition for strategic convergence when the limit is a *finite type*, i.e. a type belonging to a finite type space. Indeed, for any finite type t, and for any sequence of (possibly infinite) types $(t_n)_{n\geq 1}$ that fails to converge to t uniform-weakly, we construct a game in which an action is strictly interim correlated rationalizable for t, but not interim correlated rationalizable for t_n, infinitely often along the sequence.[5] Thus, the uniform weak topology fully characterizes the strategic topology around finite types. Moreover, the assumption that the limit is a finite type cannot be dispensed with. Under the uniform weak topology the universal type space is not *separable*, i.e. it does not contain a countable dense subset; on the other hand, Dekel, Fudenberg, and Morris (2006) show that a countable set of finite types is dense under the strategic topology.[6] This implies the existence of infinite types to which uniform-weak convergence is not a necessary condition for strategic convergence. (We explicitly construct such an example in section 4.) While this fact imposes a natural

[3]This is because uniform-weak balls are countable intersections of finite-order cylinders and the strategic topologies are sandwiched between the uniform-weak and the product topologies, by Theorem 1.

[4]Morris (2002, section 4.2) raises the question of whether the Mertens-Zamir construction is still meaningful when strategic topologies are assumed.

[5]This complements the main result of Weinstein and Yildiz (2007), who *fix a game* (satisfying a payoff-richness assumption) and a finite type t, and then *construct a sequence of types* converging to t in the product topology such that the behavior of t is bounded away from the behavior of all types in the sequence. By way of contrast, we *fix a sequence of types* which fails to converge to a finite type t in the uniform-weak topology, and then *construct a game* for which the behavior of t is bounded away from the behavior of the types in the sequence, infinitely often.

[6]While Dekel, Fudenberg, and Morris (2006) only state the weaker result that the set of *all* finite types is dense in the strategic topology, their proof actually establishes the stronger result above.

limit to our analysis, finite type spaces play a prominent role both in applied and theoretical work, so it is important to know that our sufficient condition for strategic convergence is also necessary in this case.

Finite types are also the focus of our third main result, Theorem 3. We show that, under the uniform-strategic topology, the set of finite types is *nowhere dense*, i.e. its closure has an empty interior. To understand the conceptual implications of this result, recall that Dekel, Fudenberg, and Morris (2006) have demonstrated the denseness of finite types under the non-uniform version of the strategic topology.[7] Arguably, this result provides a compelling justification for why it might be without loss of generality to model uncertainty restricting attention to finite type spaces: Irrespective of how large the “true” type space T is, for any given game there is always a finite type space T' with the property that the predictions of strategic behavior based on T' are arbitrarily close to those based on T. Our nowhere denseness result thus implies that such finite type space T' cannot be chosen independently of the game. This is particularly relevant for environments such as those of mechanism design, where the game—both payoffs and action sets—is not a priori fixed. More generally, our result implies that the uniform strategic topology is strictly finer than the strategic topology. Thus, while a priori these two notions of strategic continuity seem equally compelling, assuming one or the other can have a large impact on the ensuing theory.

The exercise in this paper is similar in spirit to that of Monderer and Samet (1996) and Kajii and Morris (1998), who, like us, consider perturbations of incomplete information games. These papers provide belief-based characterizations of strategic topologies for Bayesian Nash equilibrium in countable partition models à la Aumann (1976). However, since both of these papers assume a common prior and adopt an *ex ante* approach, while we adopt an interim approach without imposing a common prior, it is difficult to establish a precise connection.[8] Another important difference between their approach and ours is in the distinct *payoff relevance constraints* adopted: we fix the set of payoff-relevant states, so our games cannot have payoffs depending directly on players' higher-order beliefs; Monderer and Samet (1996) and Kajii and Morris (1998) have no such payoff relevance constraint.

The connection between uniform and strategic topologies first appears in Morris (2002), who studies a special class of games, called *higher-order expectation* games (HOE), and

[7] Mertens and Zamir (1985) prove the denseness of finite types under the product topology. Dekel, Fudenberg, and Morris (2006) argue that this result does not provide a sound justification for restricting attention to finite types, for strategic behavior is not continuous in the product topology.

[8] Monderer and Samet (1996) fix the common prior and consider proximity of information partitions, whereas Kajii and Morris (1998) vary the common prior on a fixed information structure. For this reason, it is already unclear what the precise connection between these papers is.

shows that the topology of uniform convergence of higher-order *iterated expectations* is equivalent to the coarsest topology under which a certain notion of strict ICR correspondence—different from the one we consider—is lower hemi-continuous in every game of the HOE class.[9] Compared to the uniform weak topology, the topology of uniform convergence of iterated expectations is neither finer nor coarser, even around finite types. We further elaborate on this relationship in section 3.3.

This paper is also related to contemporaneous work by Ely and Peski (2008). Following their terminology, a type t is *critical* if, under the product topology, the strict ICR correspondence is discontinuous at t in some game. Ely and Peski (2008) provide an insightful characterization of critical types in terms of a common belief property: a type is critical if and only if, for some $p > 0$, it has common p-belief in some closed (in product topology) proper subset of the universal type space.[10] Conceptually, this result shows that the usual type spaces that appear in applications consist almost entirely of critical types, as these type spaces typically embody nontrivial common belief assumptions. For instance, all finite types are critical and so are almost all types belonging to a common prior type space. Thus Ely and Peski's (2008) result tells us when—based on the common beliefs of the players—there will be some game and some product-convergent sequence along which strategic behavior is discontinuous, whereas we identify a condition for an arbitrary sequence to display continuous strategic behavior in all games.

The rest of the paper is organized as follows. Section 2 introduces the standard model of hierarchies of beliefs and type spaces and reviews the solution concept of ICR. Section 3 reviews the strategic and uniform-strategic topologies of Dekel, Fudenberg, and Morris (2006), introduces the uniform-weak topology and presents our two main results concerning the relationship between these topologies (Theorems 1 and 2) along with a discussion. Section 4 examines the non-genericity of finite types under the uniform-strategic and uniform-weak topologies, presenting the nowhere denseness result (Theorem 3).

[9] Morris (2002) defines his strategic topology for HOE games using a distance that makes no reference to ICR. But, as we claimed above, it can be shown that his strategic topology coincides with the coarsest topology under which a certain notion of strict ICR correspondence is continuous in every HOE game. The notion of strictness implicit in Morris's (2002) analysis, unlike ours, does not require the slack in the incentive constraints to be uniform.

[10] Moreover, they show that under the product topology the *regular* types, i.e. those types which are not critical, form a *residual* subset of the universal type space —a standard topological notion of "generic" set.

2 Preliminaries

Throughout the paper, we fix a two-player set I and a finite set Θ of payoff-relevant states with at least two elements.[11] Given a player $i \in I$, we write $-i$ to designate the other player in I. All topological spaces, when viewed as measurable spaces, are endowed with their Borel σ-algebra. For a topological space S we write $\Delta(S)$ to designate the space of probability measures over S equipped with the topology of weak convergence. Unless explicitly noted, all product spaces are endowed with the product topology and subspaces with the relative topology.

2.1 Hierarchies of beliefs and types

Our formulation of incomplete information follows Mertens and Zamir (1985).[12] Define $X^0 = \Theta$, $X^1 = X^0 \times \Delta(X^0)$, and for each $k \geq 2$ define recursively

$$X^k = \Big\{(\theta, \mu^1, \ldots, \mu^k) \in X^0 \times \mathop{\times}_{\ell=1}^{k} \Delta(X^{\ell-1}) \ : \ \mathrm{marg}_{X^{\ell-2}}\, \mu^\ell = \mu^{\ell-1} \ \forall \ell = 2, \ldots, k\Big\}.$$

By virtue of the above coherency condition on marginal distributions, each element of X^k is determined by its first and last coordinates, so we can identify X^k with $\Theta \times \Delta(X^{k-1})$. For each $i \in I$ and $k \geq 1$ we let $\mathcal{T}_i^k = \Delta(X^{k-1})$ designate the space of *k-order beliefs* of player i, so that $\mathcal{T}_i^k = \Delta(\Theta \times \mathcal{T}_{-i}^{k-1})$. The space $\mathcal{T}_i$ of *hierarchies of beliefs* of player i is

$$\mathcal{T}_i = \Big\{(\mu^k)_{k \geq 1} \in \mathop{\times}_{k \geq 1} \Delta(X^k) \ : \ \mathrm{marg}_{X^{k-2}}\, \mu^k = \mu^{k-1} \ \ \forall k \geq 2\Big\}.$$

Since Θ is finite, $\mathcal{T}_i$ is a compact metrizable space. Moreover, there is a unique mapping $\mu_i : \mathcal{T}_i \to \Delta(\Theta \times \mathcal{T}_{-i})$ which is *belief preserving*, i.e. for all $t_i = (t_i^1, t_i^2, \ldots) \in \mathcal{T}_i$ and $k \geq 1$,

$$\mu_i(t_i)\big[\theta \times (\pi_{-i}^k)^{-1}(E)\big] = t_i^{k+1}[\theta \times E] \quad \text{for all } \theta \in \Theta \text{ and measurable } E \subseteq \mathcal{T}_{-i}^k,$$

where π_i^k is the natural projection of $\mathcal{T}_i$ onto $\mathcal{T}_i^k$. Furthermore, the mapping μ_i is a homeomorphism, and so, to save on notation, we will identify each hierarchy of belief $t_i \in \mathcal{T}_i$ with its corresponding belief $\mu_i(t_i)$ over $\Theta \times \mathcal{T}_{-i}$. Similarly, for each $t_i \in \mathcal{T}_i$ we will write $t_i^k \in \mathcal{T}_i^k$ instead of the more cumbersome $\pi_i^k(t_i)$.

Hierarchies of beliefs can be implicitly represented using a *type space*, i.e. a tuple $(T_i, \phi_i)_{i \in I}$ where each T_i is a Polish space of *types* and each $\phi_i : T_i \to \Delta(\Theta \times T_{-i})$ is a

[11] We restrict attention to two-player games with finitely many payoff-relevant states for ease of notation. Our results remain valid with any finite number of players and Θ a compact metric space.

[12] An alternative, equivalent formulation is found in Brandenburger and Dekel (1993).

measurable function. Indeed, every type $t_i \in T_i$ is mapped into a hierarchy of beliefs $\nu_i(t_i) = (\nu_i^k(t_i))_{k\geq 1}$ in a natural way: $\nu_i^1(t_i) = \mathsf{marg}_\Theta\, \phi_i(t_i)$ and for $k \geq 2$,

$$\nu_i^k(t_i)[\theta \times E] = \phi_i(t_i)[\theta \times (\nu_{-i}^{k-1})^{-1}(E)] \text{ for all } \theta \in \Theta \text{ and measurable } E \subseteq \mathcal{T}_{-i}^{k-1}.$$

The type space $(\mathcal{T}_i, \mu_i)_{i\in I}$ is called the *universal type space*, since for every type space $(T_i, \phi_i)_{i\in I}$ there is a unique belief-preserving mapping from T_i into $\mathcal{T}_i$, namely the mapping ν_i above.[13] When the mappings $(\nu_i)_{i\in I}$ are injective the type space $(T_i, \phi_i)_{i\in I}$ is called *non-redundant*. In this case, $(\nu_i)_{i\in I}$ are measurable embeddings onto their images $(\nu_i(T_i))_{i\in I}$, which are measurable and can be viewed as a non-redundant type space, since we have $\mu_i(\nu_i(t_i))[\Theta \times \nu_{-i}(T_{-i})] = 1$ for all $i \in I$ and $t_i \in T_i$. Conversely, any $(T_i)_{i\in I}$ such that $T_i \subseteq \mathcal{T}_i$ and $\mu_i(t_i)[\Theta \times T_{-i}] = 1$ for all $i \in I$ and $t_i \in T_i$ can be viewed as a non-redundant type space.

2.2 Bayesian games and interim correlated rationalizability

A *game* is a tuple $G = (A_i, g_i)_{i\in I}$, where A_i is a finite set of *actions* for player i and $g_i : A_i \times A_{-i} \times \Theta \to [-M, M]$ is his *payoff* function, with $M > 0$ an arbitrary bound on payoffs that we fix throughout.[14] We write $\mathcal{G}$ to denote the set of all games, and for each integer $m \geq 1$ we write $\mathcal{G}^m$ for the set of games with $|A_i| \leq m$ for all $i \in I$.

The solution concept of *interim correlated rationalizability*, or ICR, was introduced in Dekel, Fudenberg, and Morris (2007). Given a $\gamma \in \mathbb{R}$, a type space $(T_i, \phi_i)_{i\in I}$ and a game G, for each player $i \in I$, integer $k \geq 0$ and type $t_i \in T_i$, we let $R_i^k(t_i, G, \gamma) \subseteq A_i$ designate the set of *k-order γ-rationalizable actions* of t_i. These sets are defined as follows:

$$R_i^0(t_i, G, \gamma) = A_i,$$

and recursively for each integer $k \geq 1$, $R_i^k(t_i, G, \gamma)$ is the set of all actions $a_i \in A_i$ for which there is a *conjecture*, i.e. a measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$, such that

$$\mathsf{supp}\, \sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}^{k-1}(t_{-i}, G, \gamma) \quad \text{for } \phi_i(t_i)\text{-almost every } (\theta, t_{-i}) \in \Theta \times T_{-i}$$

and for all $a_i' \in A_i$,

$$\int_{\Theta\times T_{-i}} [g_i(a_i, \sigma_{-i}(\theta, t_{-i}), \theta) - g_i(a_i', \sigma_{-i}(\theta, t_{-i}), \theta)]\, \phi_i(t_i)(d\theta \times dt_{-i}) \;\geq\; -\gamma.$$

[13] To say that ν_i is belief-preserving means that $\mu_i(\nu_i(t_i))[\theta \times E] = \phi_i(t_i)[\theta \times (\nu_{-i})^{-1}(E)]$ for all $\theta \in \Theta$ and measurable $E \subseteq \mathcal{T}_{-i}$.

[14] We will also denote by g_i the payoff function in the mixed extension of G, writing $g_i(\alpha_i, \alpha_{-i}, \theta)$ with the obvious meaning for any $\alpha_i \in \Delta(A_i)$ and $\alpha_{-i} \in \Delta(A_{-i})$.

For future reference, a conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ satisfying the former condition will be called a $(k-1)$*-order* γ*-rationalizable conjecture* for type t_i. The set of γ*-rationalizable* actions of type t_i is then defined as

$$R_i(t_i, G, \gamma) = \bigcap_{k \geq 1} R_i^k(t_i, G, \gamma).$$

Finally, following Ely and Peski (2008), an action $a_i \in A_i$ is *strictly interim correlated* γ*-rationalizable* for type t_i, and we write $a_i \in \mathring{R}_i(t_i, G, \gamma)$, if $a_i \in R_i(t_i, G, \gamma')$ for some $\gamma' < \gamma$.

As shown in Dekel, Fudenberg, and Morris (2007), $R_i(t_i, G, \gamma)$ is non-empty for every game G, type t_i and $\gamma \geq 0$.[15]

ICR has a characterization in terms of *best reply sets.* A pair of measurable functions $\varsigma_i : T_i \to 2^{A_i}$, $i \in I$, has the γ*-best reply property* if for each $i \in I$ and $t_i \in T_i$, each action $a_i \in \varsigma_i(t_i)$ is a γ-best reply for t_i to a conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ with

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq \varsigma_{-i}(t_{-i}) \quad \text{for } \phi_i(t_i)\text{-almost every } (\theta, t_{-i}) \in \Theta \times T_{-i}.$$

If $(\varsigma_i)_{i \in I}$ has the γ-best reply property then $\varsigma_i(t_i) \subseteq R_i(t_i, G, \gamma)$ for all $i \in I$ and $t_i \in T_i$. As shown in Dekel, Fudenberg, and Morris (2007), the pair $(R_i(\cdot, G, \gamma))_{i \in I}$ is the *maximal* pair of correspondences with the γ-best reply property. This means there is no other pair $(\varsigma_i)_{i \in I}$ with the γ-best reply property such that $R_i(t_i, G, \gamma) \subseteq \varsigma_i(t_i)$ for each $i \in I$ and $t_i \in T_i$, with strict inclusion for some $i \in I$ and $t_i \in T_i$. Therefore, an action is γ-rationalizable for a type t_i if and only if it is a γ-best reply to a γ*-rationalizable conjecture for* t_i, i.e. a conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}(t_{-i}, G, \gamma) \quad \text{for } \phi_i(t_i)\text{-almost every } (\theta, t_{-i}) \in \Theta \times T_{-i}.$$

Dekel, Fudenberg, and Morris (2007) also show that the set of γ-rationalizable actions of a type is determined by the induced hierarchy of beliefs. Indeed, for any $k \geq 1$, any two types (possibly belonging to different type spaces) mapping into the same k-order belief must have the same set of k-order ε-rationalizable actions. This has two implications. First, for interim correlated rationalizability it is without loss of generality to identify types with their corresponding hierarchies. Thus, in what follows we will restrict attention to type spaces $(T_i)_{i \in I}$ with $T_i \subseteq \mathcal{T}_i$ and $t_i[\Theta \times T_{-i}] = 1$ for all $i \in I$ and $t_i \in T_i$. Accordingly, we will take the universal type space $\mathcal{T}_i$ to be the domain of the correspondence $R_i(\cdot, G, \gamma) : \mathcal{T}_i \rightrightarrows A_i$. Second, in order to establish whether an action is k-order γ-rationalizable for a type t_i we can restrict attention to $(k-1)$-order γ-rationalizable conjectures σ_{-i} for t_i which are measurable with respect to $(k-1)$-order beliefs.[16]

[15] Note that for $\gamma < -2M$, we have $R_i(t_i, G, \gamma) = \emptyset$, and that for $\gamma > 2M$ we have $R_i(t_i, G, \gamma) = A_i$.

[16] This means that $\sigma_{-i}(\theta, s_{-i}) = \sigma_{-i}(\theta, t_{-i})$ for all θ and all types s_{-i}, t_{-i} with the same $(k-1)$-order beliefs.

Finally, the following result shows that, similar to rationalizability in complete information games, interim correlated rationalizability has a characterization in terms of iterated dominance, where the notion of dominance now becomes an interim one.

Proposition 1. *Fix γ, a game $G = (A_i, g_i)_{i\in I}$ and a type space $(T_i)_{i\in I}$. For each $k \geq 1$, player $i \in I$, type $t_i \in T_i$ and action $a_i \in A_i$ we have $a_i \in R_i^k(t_i, G, \gamma)$ if and only if for each $\alpha_i \in \Delta(A_i)$ and $\eta > 0$ there exists a measurable $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ with*

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \in R_{-i}^{k-1}(t_{-i}, G, \gamma) \quad \text{for } t_i\text{-almost every } (\theta, t_{-i}) \in \Theta \times T_{-i} \tag{1}$$

such that

$$\int_{\Theta\times T_{-i}} \Big[g_i(a_i, \sigma_{-i}(\theta, t_{-i}), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta, t_{-i}), \theta)\Big]\, t_i(d\theta \times dt_{-i}) \geq -\gamma - \eta. \tag{2}$$

The proof of this proposition, relegated to Appendix A, uses a separation argument analogous to the one establishing the equivalence between strictly dominated and never best reply strategies in complete information games. Here, too, the usefulness of the result comes from the fact that, in order to check whether an action is rationalizable for a type, we are able to reverse the order of quantifiers and seek a possibly different conjecture for each possible (mixed) deviation.

3 Topologies on types

The *strategic topology* introduced in Dekel, Fudenberg, and Morris (2006), or simply S-*topology*, is the coarsest topology on the universal type space $\mathcal{T}_i$ under which the ICR correspondence is upper hemi-continuous and the strict ICR correspondence is lower hemi-continuous in all games. More explicitly, following a formulation due to Ely and Peski (2008), the S-topology is the topology generated by the collection of all sets of the form

$$\Big\{t_i \in \mathcal{T}_i \ : \ a_i \notin R_i(t_i, G, \gamma)\Big\} \quad \text{and} \quad \Big\{t_i \in \mathcal{T}_i \ : \ a_i \in \mathring{R}_i(t_i, G, \gamma)\Big\},$$

where $G = (A_i, g_i)_{i\in I}$, $a_i \in A_i$ and $\gamma \in \mathbb{R}$.[17]

The S-topology on $\mathcal{T}_i$ is metrizable by the distance d_i^{S}, defined as follows.[18] For each game $G = (A_i, g_i)_{i\in I}$, action $a_i \in A_i$ and type $t_i \in \mathcal{T}_i$ let

$$h_i(t_i|a_i, G) = \inf\{\gamma \ : \ a_i \in R_i(t_i, G, \gamma)\}.$$

[17]The strategic topology can be given an equivalent definition which makes no direct reference to γ-rationalizability for $\gamma \neq 0$. Indeed, by Ely and Peski (2008, Lemma 4), a sub-basis of the strategic topology is the collection of all sets of the form $\{t_i : a_i \notin R(t_i, G, 0)\}$ and $\{t_i : a_i \in \mathring{R}_i(t_i, G, 0)\}$.

[18]Dekel, Fudenberg, and Morris (2006) define the S-topology directly using the distance d_i^{S}, rather than using the topological definition above.

Then, for each s_i and $t_i \in \mathcal{T}_i$,

$$d_i^{\mathsf{s}}(s_i, t_i) \;=\; \sum_{m\geq 1} 2^{-m} \sup_{G=(A_i,g_i)_{i\in I}\in \mathcal{G}^m} \max_{a_i\in A_i} \bigl|h_i(s_i|a_i,G) - h_i(t_i|a_i,G)\bigr|.$$

In terms of convergence of sequences, Dekel, Fudenberg, and Morris (2006) show the following: For every $t_i \in \mathcal{T}_i$ and every sequence $(t_{i,n})_{n\geq 1}$ in $\mathcal{T}_i$, we have $d_i^{\mathsf{s}}(t_{i,n}, t_i) \to 0$ if and only if for every game $G = (A_i, g_i)_{i\in I}$, action $a_i \in A_i$ and $\gamma \in \mathbb{R}$, the following upper and lower hemi-continuity properties hold: for every sequence $\gamma_n \to \gamma$,

$$a_i \in R_i(t_{i,n}, G, \gamma_n) \quad \forall n \geq 1 \quad \Longrightarrow \quad a_i \in R_i(t_i, G, \gamma), \qquad \text{(u.h.c.)}$$

and for some sequence $\gamma_n \searrow \gamma$,

$$a_i \in R_i(t_i, G, \gamma) \quad \Longrightarrow \quad a_i \in R_i(t_{i,n}, G, \gamma_n) \quad \forall n \geq 1. \qquad \text{(l.h.c.)}$$

Dekel, Fudenberg, and Morris (2006) also introduce the *uniform strategic topology*, or US-*topology* for short, which strengthens the definition of the strategic topology by requiring the convergence to be uniform over all games. More precisely, the US-topology is the topology of convergence under the metric d_i^{us}, defined as

$$d_i^{\mathsf{us}}(t_i, s_i) \;=\; \sup_{G=(A_i,g_i)_{i\in I}\in \mathcal{G}} \max_{a_i\in A_i} \bigl|h_i(t_i|a_i,G) - h_i(s_i|a_i,G)\bigr|.$$

This uniformity renders the US-topology particularly relevant for environments where the game—both payoffs and action sets—is not fixed a priori, such as in a mechanism design environment.

We now introduce a metric topology on types, which we call *uniform weak topology*, or UW-*topology*, under which two types of a player are close if they have similar first-order beliefs, attach similar probabilities to other players having similar first-order beliefs, and so on, where the degree of similarity is uniform over the levels of the belief hierarchy. Thus, unlike the S and US topologies, which are *behavior-based*, the UW-topology is a *belief-based* topology, i.e. a metric topology defined explicitly in terms of proximity of hierarchies of beliefs. The two main results of this section, Theorems 1 and 2 below, establish a connection between these behavior- and belief-based topologies.

Before we present the formal definition of the UW-topology, recall that for a complete separable metric space (S,d) the topology of weak convergence on $\Delta(S)$ is metrizable by the *Prohorov* distance ρ, defined as

$$\rho(\mu,\mu') = \inf\{\delta > 0 \;:\; \mu(E) \leq \mu'(E^\delta) + \delta \;\text{ for each measurable } E \subseteq S\}, \quad \forall \mu,\mu' \in \Delta(S),$$

where $E^\delta = \{s \in S \;:\; \inf_{s'\in S} d(s,s') < \delta\}$. The UW-*topology* is the metric topology on $\mathcal{T}_i$ generated by the distance

$$d_i^{\mathsf{uw}}(s_i, t_i) \;=\; \sup_{k\geq 1} d_i^k(s_i, t_i), \quad \forall s_i, t_i \in \mathcal{T}_i,$$

where d^0 is the discrete metric on Θ, and recursively for $k \geq 1$, d_i^k is the Prohorov distance on $\Delta(\Theta \times \mathcal{T}_{-i}^{k-1})$ induced by the metric $\max\{d^0, d_{-i}^{k-1}\}$ on $\Theta \times \mathcal{T}_{-i}^{k-1}$.

In the remainder of section 3 we explore the relationship between the UW-topology and the S- and US-topologies. First, we show that the UW-topology is finer than the US-topology (Theorem 1). Second, we prove a partial converse, namely that around *finite types*, i.e. types belonging to a finite type space, the S-topology (and hence also the US-topology) is finer than the UW-topology (Theorem 2). We conclude the section with a discussion of our results in connection with the literature.

3.1 UW-convergence implies US-convergence

Theorem 1. *For each player $i \in I$ and for all types $s_i, t_i \in \mathcal{T}_i$,*

$$d_i^{\mathsf{us}}(s_i, t_i) \;\leq\; 4M d_i^{\mathsf{uw}}(s_i, t_i).$$

Thus the UW*-topology is finer than the* US*-topology.*

This theorem is a direct implication of the following proposition, whose proof exploits the characterization of rationalizability provided in Proposition 1.

Proposition 2. *Fix a game G, $\gamma \geq 0$ and $\delta > 0$. For each integer $k \geq 1$,*

$$d_i^k(s_i, t_i) < \delta \quad \Longrightarrow \quad R_i^k(t_i, G, \gamma) \subseteq R_i^k(s_i, G, \gamma + 4M\delta) \qquad \forall i \in I,\ \forall s_i, t_i \in \mathcal{T}_i.$$

Before presenting the proof of this proposition, it is useful to discuss its basic steps. Fix $\gamma \geq 0$, $\eta > 0$, a k-order rationalizable action a_i for a type t_i and a (possibly mixed) deviation α_i. By Proposition 1 there is a $(k-1)$-order γ-rationalizable conjecture σ_{-i} such that the difference in expected payoffs between a_i and α_i is at least $-\gamma - \eta$. Now partition the space of $(k-1)$-order beliefs of player $-i$ in two ways: first, so that beliefs in the same cell have the same set of $(k-1)$-order γ-rationalizable actions; second, analogously, according to the $(k-1)$-order $(\gamma + 4M\delta)$-rationalizable actions. Then the expected payoff difference between a_i and α_i for t_i under σ_{-i} cannot decrease if instead of σ_{-i} we use an appropriate pure conjecture that is measurable with respect to the first partition—namely, a conjecture that selects for each θ and each $(k-1)$-order belief of $-i$ some pure action of $-i$ that maximizes the payoff difference between a_i and α_i under θ over all $(k-1)$-order γ-rationalizable actions of $-i$ corresponding to the given partition cell. Now take any type s_i whose k-order beliefs are δ-close to those of t_i, and define a pure conjecture for s_i in the same way, but according to the second partition and using a $(\gamma + 4M\delta)$-rationalizable action as the maximizer for each cell. To prove that a_i is $(\gamma + 4M\delta)$-rationalizable for s_i, we show that the differences in expected payoffs between a_i and α_i, computed for t_i

under the first pure conjecture and for s_i under the second one, are close. We achieve this in three final steps: First, for each partition and associated pure conjecture, we order the pairs comprising states and partition cells by the payoff difference between a_i and α_i under the pure conjecture under consideration; accordingly, we obtain two ordered partitions of $\Theta \times \mathcal{T}_{-i}^{k-1}$. Second, we use the induction hypothesis and the assumption that the k-order beliefs of t_i and s_i are close to argue that the probabilities assigned by t_i^k to the *upper-contour sets* which are measurable with respect to the first ordered partition are close to those assigned by s_i^k to the upper-contour sets which are measurable with respect to the second ordered partition.[19] Finally, using a summation-by-parts argument we prove that proximity of probabilities of upper-contour sets is indeed enough to guarantee that the expected payoff difference between a_i and α_i for s_i is at least $-\gamma - \eta - 4M\delta$. This, again by Proposition 1, delivers the desired conclusion.

Proof of Proposition 2. Fix a game $G = (A_i, g_i)_{i \in I}$, $\gamma \geq 0$ and $\delta > 0$. The proof is by induction on k. For $k = 1$, let s_i and $t_i \in \mathcal{T}_i$ be such that $d_i^1(s_i, t_i) < \delta$. Fix an arbitrary $a_i \in R_i^1(t_i, G, \gamma)$ and let us show that $a_i \in R_i^1(s_i, G, \gamma + 4M\delta)$ using Proposition 1. Fix $\eta > 0$ and $\alpha_i \in \Delta(A_i)$. By Proposition 1 there exists a conjecture $\sigma_{-i} : \Theta \to \Delta(A_{-i})$ for type t_i such that

$$\sum_{\theta \in \Theta} (g_i(a_i, \sigma_{-i}(\theta), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta), \theta)) t_i^1[\theta] \geq -\gamma - \eta. \tag{3}$$

(Note that condition (1) is trivial for $k = 1$.) Pick any function $\mathrm{a}_{-i} : \Theta \to A_{-i}$ such that

$$\mathrm{a}_{-i}(\theta) \in \operatorname*{arg\,max}_{a_{-i} \in A_{-i}} [g_i(a_i, a_{-i}, \theta) - g_i(\alpha_i, a_{-i}, \theta)] \qquad \forall \theta \in \Theta,$$

and define

$$h(\theta) = g_i(a_i, \mathrm{a}_{-i}(\theta), \theta) - g_i(\alpha_i, \mathrm{a}_{-i}(\theta), \theta) \quad \forall \theta \in \Theta,$$

so that

$$h(\theta) \geq g_i(a_i, \sigma_{-i}(\theta), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta), \theta) \qquad \forall \theta \in \Theta. \tag{4}$$

To conclude the proof for $k = 1$ we now show that $\sum_{\theta \in \Theta} h(\theta) s_i^1[\theta] \geq -\gamma - 4M\delta - \eta$. Indeed, let $\{\theta_n\}_{n=1}^N$ be an enumeration of Θ such that $h(\theta_n) \geq h(\theta_{n+1})$ for all $1 \leq n \leq N - 1$. Thus,

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[19] An *upper-contour set* induced by the first (resp. second) ordered partition is a subset of $\Theta \times \mathcal{T}_{-i}^{k-1}$ which is measurable with respect to the first (resp. second) ordered partition and contains all pairs (θ, t_{-i}^{k-1}) for which the payoff difference between a_i and α_i under the first (resp. second) conjecture is greater than some value.

it follows from $d_i^1(s_i, t_i) < \delta$ and $|h(\theta)| \le 2M$ for all θ that

$$\begin{aligned}
\sum_{\theta\in\Theta} h(\theta)(s_i^1[\theta] - t_i^1[\theta]) &= \sum_{n=1}^{N-1} (h(\theta_n) - h(\theta_{n+1})) \sum_{m=1}^{n} (s_i^1[\theta_m] - t_i^1[\theta_n]) \\
&= \sum_{n=1}^{N-1} \underbrace{(h(\theta_n) - h(\theta_{n+1}))}_{\ge 0} \underbrace{(s_i^1[\{\theta_m\}_{m=1}^n] - t_i^1[\{\theta_m\}_{m=1}^n])}_{\ge -\delta} \\
&\ge -\delta \sum_{n=1}^{N-1} h(\theta_n) - h(\theta_{n+1}) \\
&= -\delta(h(\theta_1) - h(\theta_N)) \\
&\ge -4M\delta,
\end{aligned}$$

hence

$$\begin{aligned}
\sum_{\theta\in\Theta} h(\theta)s_i^1[\theta] &= \sum_{\theta\in\Theta} h(\theta)(s_i^1[\theta] - t_i^1[\theta]) + \sum_{\theta\in\Theta} h(\theta)t_i^1[\theta] \ge -4M\delta + \sum_{\theta\in\Theta} h(\theta)t_i^1[\theta] \\
&\ge -4M\delta + \sum_{\theta\in\Theta} (g_i(a_i, \sigma_{-i}(\theta), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta), \theta))t_i^1[\theta] \ge -\gamma - 4M\delta - \eta,
\end{aligned}$$

where the penultimate inequality follows from (4) and the last inequality from (3). Thus, $a_i \in R_i^1(s_i, G, \gamma + 4M\delta)$ by Proposition 1, which proves the desired result for $k = 1$.

Proceeding by induction, we now suppose the result is valid for some $k \ge 1$ and show that it remains valid for $k+1$. Let $s_i, t_i \in \mathcal{T}_i$ be such that $d_i^{k+1}(s_i, t_i) < \delta$. Fix an arbitrary $a_i \in R_i^{k+1}(t_i, G, \gamma)$ and let us show that $a_i \in R_i^{k+1}(s_i, G, \gamma + 4M\delta)$. Fix $\eta > 0$ and $\alpha_i \in \Delta(A_i)$. By Proposition 1 there exists a k-order $(\gamma + \eta)$-rationalizable conjecture $\sigma_{-i} : \Theta \times \mathcal{T}_{-i}^k \to \Delta(A_{-i})$ for type t_i such that

$$\int_{\Theta\times\mathcal{T}_{-i}^k} (g_i(a_i, \sigma_{-i}(\theta, t_{-i}^k), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta, t_{-i}^k), \theta))\, t_i^{k+1}(d\theta \times dt_{-i}^k) \ge -\gamma - \eta. \tag{5}$$

Pick any measurable function $\mathrm{a}_{-i} : \Theta \times \mathcal{T}_{-i}^k \to A_{-i}$ such that

$$\mathrm{a}_{-i}(\theta, t_{-i}^k) \in \operatorname*{arg\,max}_{a_{-i}\in R_{-i}^k(t_{-i}^k, G, \gamma+4M\delta)} (g_i(a_i, a_{-i}, \theta) - g_i(\alpha_i, a_{-i}, \theta)) \quad \forall(\theta, t_{-i}^k) \in \Theta \times \mathcal{T}_{-i}^k.$$

By construction, a_{-i} is a k-order $(\gamma + 4M\delta)$-rationalizable conjecture for type s_i. (In fact, for any type.) Thus, By Proposition 1, to conclude that $a_i \in R_i^{k+1}(s_i, G, \gamma + 4M\delta)$ we need to show that

$$\int_{\Theta\times\mathcal{T}_{-i}^k} (g_i(a_i, \mathrm{a}_{-i}(\theta, t_{-i}^k), \theta) - g_i(\alpha_i, \mathrm{a}_{-i}(\theta, t_{-i}^k), \theta))\, s_i^{k+1}(d\theta \times dt_{-i}^k) \ge -\gamma - 4M\delta - \eta. \tag{6}$$

Let $\bar{A}_1, \ldots, \bar{A}_L$ be an enumeration of the non-empty subsets of A_{-i} and define

$$h_\ell(\theta) = \max_{a_{-i}\in\bar{A}_\ell} [g_i(a_i, a_{-i}, \theta) - g_i(\alpha_i, a_{-i}, \theta)] \qquad \forall\theta \in \Theta,\ \forall 1 \le \ell \le L.$$

Next, define a partition $\{P_1, \ldots, P_L\}$ of $\mathcal{T}_{-i}^k$ as follows:

$$P_\ell = \left\{ t_{-i}^k \in \mathcal{T}_{-i}^k \,:\, R_{-i}^k(t_{-i}^k, G, \gamma) = \bar{A}_\ell \right\} \qquad \forall 1 \le \ell \le L.$$

Since σ_{-i} is a k-order γ-rationalizable conjecture for t_i, we have

$$h_\ell(\theta) \ge g_i(a_i, \sigma_{-i}(\theta, t_{-i}^k), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta, t_{-i}^k), \theta)$$

for t_i^{k+1}-almost every $(\theta, t_{-i}^k) \in \Theta \times P_\ell$ and therefore

$$\sum_{\theta \in \Theta} \sum_{\ell=1}^{L} h_\ell(\theta) t_i^{k+1}[\theta \times P_\ell] \ \ge \int\limits_{\Theta \times \mathcal{T}_{-i}^k} [g_i(a_i, \sigma_{-i}(\theta, t_{-i}^k), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta, t_{-i}^k), \theta)]\, t_i^{k+1}(d\theta \times dt_{-i}^k). \quad (7)$$

Likewise, define a partition $\{Q_1, \ldots, Q_L\}$ as follows:

$$Q_\ell = \left\{ t_{-i}^k \in \mathcal{T}_{-i}^k \,:\, R_{-i}^k(t_{-i}^k, G, \gamma + 4M\delta) = \bar{A}_\ell \right\} \qquad \forall 1 \le \ell \le L.$$

Thus we have

$$\int\limits_{\Theta \times \mathcal{T}_{-i}^k} [g_i(a_i, \mathrm{a}_{-i}(\theta, t_{-i}^k), \theta) - g_i(\alpha_i, \mathrm{a}_{-i}(\theta, t_{-i}^k), \theta)]\, s_i^{k+1}(d\theta \times dt_{-i}^k) = \sum_{\theta \in \Theta} \sum_{\ell=1}^{L} h_\ell(\theta) s_i^{k+1}[\theta \times Q_\ell],$$

which, together with (5) and (7), implies

$$\begin{aligned}
&\int\limits_{\Theta \times \mathcal{T}_{-i}^k} [g_i(a_i, \mathrm{a}_{-i}(\theta, t_{-i}^k), \theta) - g_i(\alpha_i, \mathrm{a}_{-i}(\theta, t_{-i}^k), \theta)]\, s_i^{k+1}(d\theta \times dt_{-i}^k) \quad \ge \\
&\ge \int\limits_{\Theta \times \mathcal{T}_{-i}^k} [g_i(a_i, \sigma_{-i}(\theta, t_{-i}^k), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta, t_{-i}^k), \theta)]\, t_i^{k+1}(d\theta \times dt_{-i}^k) \\
&\qquad + \sum_{\theta \in \Theta} \sum_{\ell=1}^{L} h_\ell(\theta)(s_i^{k+1}[\theta \times Q_\ell] - t_i^{k+1}[\theta \times P_\ell]) \\
&\ge -\gamma - \eta + \sum_{\theta \in \Theta} \sum_{\ell=1}^{L} h_\ell(\theta)(s_i^{k+1}[\theta \times Q_\ell] - t_i^{k+1}[\theta \times P_\ell]),
\end{aligned}$$

Therefore, in order to prove (6) and conclude that $a_i \in R_i^{k+1}(s_i, G, \gamma + 4M\delta)$ we only need to show that

$$\sum_{\theta \in \Theta} \sum_{\ell=1}^{L} h_\ell(\theta)(s_i^{k+1}[\theta \times Q_\ell] - t_i^{k+1}[\theta \times P_\ell]) \ge -4M\delta.$$

Indeed, to prove this inequality first note that the induction hypothesis implies

$$P_\ell^\delta \subseteq \bigcup_{n:\bar{A}_n \supseteq \bar{A}_\ell} Q_n \qquad \forall 1 \leq \ell \leq L. \tag{8}$$

Next, let $N = |\Theta|L$ and consider an enumeration $\{(\theta_n, \ell_n)\}_{n=1}^N$ of $\Theta \times \{1, \ldots, L\}$ such that for all n,

$$h_{\ell_n}(\theta_n) \geq h_{\ell_{n+1}}(\theta_{n+1}),$$

and for all $m \neq n$,

$$(\bar{A}_{\ell_m} \supseteq \bar{A}_{\ell_n} \quad \text{and} \quad \bar{A}_{\ell_m} \neq \bar{A}_{\ell_n}) \quad \Longrightarrow \quad m < n.\text{[20]} \tag{9}$$

Thus, for each $n = 1, \ldots, N$,

$$\begin{aligned}
s_i^{k+1}\Big[\bigcup_{m=1}^n \theta_m \times Q_{\ell_m}\Big] &\geq s_i^{k+1}\Big[\bigcup_{m=1}^n \theta_m \times P_{\ell_m}^\delta\Big] && \text{(by (8) and (9))} \\
&\geq s_i^{k+1}\Big[\big(\bigcup_{m=1}^n \theta_m \times P_{\ell_m}\big)^\delta\Big] \\
&\geq t_i^{k+1}\Big[\bigcup_{m=1}^n \theta_m \times P_{\ell_m}\Big] - \delta, && \text{(by } d_i^{k+1}(s_i, t_i) < \delta\text{)}
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\sum_{\theta\in\Theta}\sum_{\ell=1}^L h_\ell(\theta)(s_i^{k+1}[\theta \times Q_\ell] - t_i^{k+1}[\theta \times P_\ell]) = \\
&\quad = \sum_{n=1}^N h_{\ell_n}(\theta_n)(s_i^{k+1}[\theta_n \times Q_{\ell_n}] - t_i^{k+1}[\theta_n \times P_{\ell_n}]) \\
&\quad = \sum_{n=1}^{N-1} (h_{\ell_n}(\theta_n) - h_{\ell_{n+1}}(\theta_{n+1})) \sum_{m=1}^n (s_i^{k+1}[\theta_m \times Q_{\ell_m}] - t_i^{k+1}[\theta_m \times P_{\ell_m}]) \\
&\quad = \sum_{n=1}^{N-1} \Big(\underbrace{h_{\ell_n}(\theta_n) - h_{\ell_{n+1}}(\theta_{n+1})}_{\geq 0}\Big)\underbrace{\Big(s_i^{k+1}\Big[\bigcup_{m=1}^n \theta_m \times Q_{\ell_m}\Big] - t_i^{k+1}\Big[\bigcup_{m=1}^n \theta_m \times P_{\ell_m}\Big]\Big)}_{\geq -\delta} \\
&\quad \geq -\delta \sum_{n=1}^{N-1} (h_{\ell_n}(\theta_n) - h_{\ell_{n+1}}(\theta_{n+1})) = -\delta[h_{\ell_1}(\theta_1) - h_{\ell_N}(\theta_N)] \geq -4M\delta,
\end{aligned}$$

as required. □

Corollary 1. *The Borel σ-algebras of the* UW-, US-, S- *and product topologies coincide.*

[20] To see why an enumeration of $\Theta \times \{1, \ldots, L\}$ satisfying these two properties exists, note that it follows directly from the definition of $h_\ell(\theta)$ that $\bar{A}_\ell \supseteq \bar{A}_m$ implies $h_\ell(\theta) \geq h_m(\theta)$.

Proof. Theorem 1 implies that the Borel σ-algebra of the US-topology is contained in the Borel σ-algebra of the UW-topology. Moreover, Lemma 4 in Dekel, Fudenberg, and Morris (2006) implies that the Borel σ-algebra of the strategic topology contains the product σ-algebra. Hence, it suffices to show that the product σ-algebra contains the UW-σ-algebra. In effect, every uniform-weak ball is a countable intersection of cylinders, therefore every uniform-weak ball is product-measurable, which implies that every UW-measurable set is product measurable. □

An important implication of this corollary is that the Mertens-Zamir universal type space $(\mathcal{T}_i, \mu_i)_{i\in I}$ remains a universal type space when equipped with either of the topologies S, US or UW instead of the product topology, a fact that was not known prior to this paper. Indeed these topologies leave the measurable structure unchanged, so $\mu_i : \mathcal{T}_i \to \Delta(\Theta \times \mathcal{T}_{-i})$ remains the unique belief-preserving mapping and a Borel isomorphism, albeit no longer a homeomorphism.

3.2 S-convergence to finite types implies UW-convergence

Here we provide a partial converse to Theorem 1. We show that, as far as convergence to finite types is concerned, convergence in the S-topology implies convergence in the UW-topology (and hence also in the US-topology).

Theorem 2. *Around finite types the* S*-topology is finer than the* UW*-topology, i.e. for each player $i \in I$, finite type $t_i \in \mathcal{T}_i$ and $\delta > 0$ there exists $\varepsilon > 0$ such that for each $s_i \in \mathcal{T}_i$,*

$$d_i^{\mathsf{S}}(s_i, t_i) \le \varepsilon \quad \Longrightarrow \quad d_i^{\mathsf{UW}}(s_i, t_i) \le \delta.$$

This theorem is a direct implication of Proposition 3 below, which in turn relies on the following result:

Lemma 1. *Let $(T_i)_{i\in I}$ be a finite type space. For every $\delta > 0$ there exist $\varepsilon > 0$ and a game $G = (A_i, g_i)_{i\in I}$, with $A_i \supseteq T_i$ for all $i \in I$, such that for every $i \in I$ and $t_i \in T_i$,*

$$t_i \in \operatorname*{arg\,max}_{a_i\in A_i} \sum_{\theta\in\Theta} \sum_{t_{-i}\in T_{-i}} g_i(a_i, t_{-i}, \theta)\, t_i[\theta, t_{-i}], \tag{10}$$

and for every $\psi \in \Delta(\Theta \times A_{-i})$ such that $\psi[D] < t_i[D] - \delta$ for some $D \subseteq \Theta \times T_{-i}$,

$$\min_{a_i\in A_i} \sum_{\theta\in\Theta} \sum_{a_{-i}\in A_{-i}} \left(g_i(t_i, a_{-i}, \theta) - g_i(a_i, a_{-i}, \theta)\right) \psi[\theta, a_{-i}] < -\varepsilon. \tag{11}$$

The proof of this lemma, given in Appendix A, uses a "report-your-beliefs" game embedded in a "coordination" game. More precisely, we construct a game where each player

i chooses a point in a finite grid $A_i \subseteq \Delta(\Theta \times T_{-i})$ that includes all types in T_i (viewed as probability distributions over $\Theta \times T_{-i}$). When player $-i$ chooses an action $a_{-i} \in T_{-i}$ the payoff to player i is $f_i(\theta, a_{-i}, a_i)$, where $f_i : \Theta \times T_{-i} \times \Delta(\Theta \times T_{-i}) \to [-1, 1]$ is a *proper scoring rule*.[21,22] This guarantees that truthful reporting by all types of all players in the type space $(T_i)_{i\in I}$ has the 0-best reply property, as shown in (10). If instead player $-i$ chooses an action in $A_{-i} \setminus T_{-i}$ then the payoff to player i is either $-4/\delta$ or -1, according to whether i chooses an action in T_i or $A_i \setminus T_i$, respectively. In any case, this is no greater than the minimum possible value of $f_i(\theta, a_{-i}, a_i)$, and strictly less when choosing an action in T_i. The grid A_i is chosen fine enough, and ε small enough, so no action $t_i \in T_i$ can be an ε-best reply to a conjecture $\psi \in \Delta(\Theta \times A_{-i})$ that is not δ-close to t_i (viewed as a probability over $\Theta \times A_{-i}$), as shown in (11). Indeed, either ψ assigns large probability to $-i$ choosing an action in $A_{-i} \setminus T_{-i}$, and hence on the payoff difference between t_i and any $a_i \in A_i \setminus T_i$ being $-4/\delta + 1$, or it assigns probability large enough to $\Theta \times T_{-i}$, so that the conditional $\bar{\psi} = \psi(\cdot|\Theta \times T_{-i})$ is close to ψ and hence far from t_i. Thus, in both cases the expected payoff difference under ψ between t_i and any grid point $a_i \in A_i \setminus T_i$ sufficiently close to $\bar{\psi}$ will be less than $-\varepsilon$. The proof of Proposition 3 uses Lemma 1 to show by induction that for any type $s_i \in \mathcal{T}_i$ whose k-order beliefs differ from those of a type $t_i \in T_i$ by more than δ, action t_i is not ε-rationalizable.

Proposition 3. *Let $(T_i)_{i\in I}$ be a finite type space. For each $\delta > 0$ there exist $\varepsilon > 0$ and a game G such that for each integer $k \geq 1$, each player $i \in I$ and each $(t_i, s_i) \in T_i \times \mathcal{T}_i$,*

$$d_i^k(s_i, t_i) \geq \delta \quad \Longrightarrow \quad R_i^k(t_i, G, 0) \not\subseteq R_i^k(s_i, G, \varepsilon).$$

Proof. Fix a finite type space $(T_i)_{i\in I}$ and $\delta > 0$. By Lemma 1 there exist $\varepsilon > 0$ and a game $G = (A_i, g_i)_{i\in I}$ with $A_i \supseteq T_i$ such that (10) and (11) hold for every $t_i \in T_i$ and every $\psi \in \Delta(\Theta \times A_{-i})$ such that $\psi[D] < t_i[D] - \delta$ for some $D \subseteq \Theta \times T_{-i}$. Thus, for each $(t_i, s_i) \in T_i \times \mathcal{T}_i$ and each measurable function $\sigma_{-i} : \Theta \times \mathcal{T}_{-i} \to \Delta(A_{-i})$, if for some $D \subseteq \Theta \times T_{-i}$

$$\sum_{(\theta, a_{-i})\in D} \underbrace{\int_{\mathcal{T}_{-i}} \sigma_{-i}(\theta, t_{-i})[a_{-i}]\, s_i(\theta \times dt_{-i})}_{\psi(\theta, a_{-i})} < t_i[D] - \delta,$$

[21] A *proper scoring rule* on a measurable space Ω is a measurable function $f : \Omega \times \Delta(\Omega) \to \mathbb{R}$ such that $\int f(\omega, \mu)\, \mu(d\omega) \geq \int f(\omega, \mu')\, \mu(d\omega)$ for all μ, $\mu' \in \Delta(\Omega)$, with strict inequality whenever $\mu' \neq \mu$.

[22] Dekel, Fudenberg, and Morris (2006) use a report-your-beliefs game to prove their Lemma 4, which states that for every $k \geq 1$ and $\delta > 0$ there exists $\varepsilon > 0$ such that, for all $t_i, s_i \in \mathcal{T}_i$, $d_i^S(s_i, t_i) \leq \varepsilon$ implies $d_i^k(s_i, t_i) \leq \delta$. Our assumption that t_i is finite allows us to find an ε that does not depend on k, and hence obtain Theorem 2. The game we construct in Lemma 1 differs from theirs in two respects: first, Dekel, Fudenberg, and Morris (2006) use a pure report-your-beliefs game, while we embed a report-your-beliefs game in a coordination game; second, in our game the players report infinite hierarchies of beliefs (albeit in a *finite* type space), whereas in the game of Dekel, Fudenberg, and Morris's (2006) Lemma 4, players report only their k-order beliefs.

then for some $a_i \in A_i$,

$$\int_{\Theta\times\mathcal{T}_{-i}} \left[g_i(t_i, \sigma_{-i}(\theta, t_{-i}), \theta) - g_i(a_i, \sigma_{-i}(\theta, t_{-i}), \theta)\right] s_i(d\theta \times dt_{-i}) \ < \ -\varepsilon. \tag{12}$$

We now show that for each $i \in I$,

$$t_i \in R_i(t_i, G, 0) \qquad \forall t_i \in T_i, \tag{13}$$

$$d_i^k(t_i, s_i) > \delta \quad \Longrightarrow \quad t_i \notin R_i^k(s_i, G, \varepsilon) \qquad \forall k \geq 1,\ \forall (t_i, s_i) \in T_i \times \mathcal{T}_i. \tag{14}$$

For $i \in I$ and $t_i \in T_i$ consider the conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that $\sigma_{-i}(\theta, t_{-i})[t_{-i}] = 1$ for all $(\theta, t_{-i}) \in \Theta \times T_{-i}$. Then the action t_i is a best reply to the conjecture σ_{-i} for type t_i by (10), and hence $t_i \in R_i(t_i, G, 0)$ by the characterization of ICR in terms of best reply sets, proving (13).

To prove (14) for $k = 1$, pick $s_i \in \mathcal{T}_i$ with $d_i^1(t_i, s_i) > \delta$. Then there exists $E \subseteq \Theta$ such that $s_i^1[E] < t_i^1[E] - \delta$, and hence for every $\sigma_{-i} : \Theta \to \Delta(A_{-i})$ we have

$$\sum_{\theta\in E} \sigma_{-i}(\theta)[T_{-i}]\, s_i^1[\theta] \leq s_i^1[E] < t_i^1[E] - \delta = t_i[E \times T_{-i}] - \delta.$$

It follows from (12) that $t_i \notin R_i^1(s_i, G, \varepsilon)$. Proceeding by induction, let $k \geq 2$ and assume that (14) holds for $k - 1$. Fix $i \in I$ and $t_i \in T_i$ and pick $s_i \in \mathcal{T}_i$ with $d_i^k(t_i, s_i) > \delta$. Then there exists some $E \subseteq \Theta \times T_{-i}^{k-1}$ such that

$$s_i^k[E^\delta] < t_i^k[E] - \delta. \tag{15}$$

Define $D = \{(\theta, t_{-i}) \in \Theta \times T_{-i} : (\theta, t_{-i}^{k-1}) \in E\}$, so that $s_i[D] = s_i^k[E]$. Consider an arbitrary $(k-1)$-order ε-rationalizable conjecture $\sigma_{-i} : \Theta \times \mathcal{T}_{-i} \to \Delta(A_{-i})$ for type s_i, i.e.

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}^{k-1}(t_{-i}, G, \varepsilon) \qquad \text{for } s_i\text{-almost every } (\theta, t_{-i}) \in \Theta \times \mathcal{T}_{-i}. \tag{16}$$

By the induction hypothesis, for s_i-almost every $(\theta, t_{-i}) \in \Theta \times \mathcal{T}_{-i}$ and every $(\theta, s_{-i}) \in D$, we can have $\sigma_{-i}(\theta, t_{-i})[s_{-i}] > 0$ only if $d_{-i}^{k-1}(s_{-i}, t_{-i}) \leq \delta$. Thus,

$$\begin{aligned}
&\sum_{(\theta,s_{-i})\in D} \int_{\mathcal{T}_{-i}} \sigma_{-i}(\theta, t_{-i})[s_{-i}]\, s_i(\theta \times dt_{-i}) \leq \\
&\qquad \leq \sum_{(\theta,s_{-i})\in D} \int_{(\pi_{-i}^{k-1})^{-1}(\{s_{-i}^{k-1}\}^\delta)} \sigma_{-i}(\theta, t_{-i})[s_{-i}]\, s_i(\theta \times dt_{-i}) \\
&\qquad \leq \sum_{(\theta,s_{-i})\in D} s_i[\theta \times (\pi_{-i}^{k-1})^{-1}(\{s_{-i}^{k-1}\}^\delta)] \\
&\qquad = \sum_{(\theta,s_{-i}^{k-1})\in E} s_i^k[\theta \times \{s_{-i}^{k-1}\}^\delta] = s_i^k[E^\delta] < t_i^k[E] - \delta = t_i[D] - \delta,
\end{aligned}$$

where the last inequality is (15). By (12) this implies $t_i \notin R_i^k(s_i, G, \varepsilon)$. $\square$

Theorems 1 and 2 combined yield:

Corollary 2. *The* UW-, US-, *and* S- *topologies are equivalent around finite types.*

To end this section, we note that in Theorem 2 the assumption that t_i is a finite type cannot be dispensed with. This is because the universal type space is not separable under the uniform-weak topology, whereas Dekel, Fudenberg, and Morris (2006) have shown that a countable set of finite types is dense under the strategic topology. To see why the uniform-weak topology is not separable, fix two states θ_0 and θ_1 in Θ and consider the non-redundant type space $(X_i)_{i\in I}$, where $X_i = \{0,1\}^{\mathbb{N}}$ and each type $x_i = (x_{i,n})_{n\in\mathbb{N}}$ assigns probability one to the pair $(\theta_{x_{i,1}}, L_i(x_i))$, where $L_i : X_i \to X_{-i}$ is the shift operator, i.e. $L((x_{i,1}, x_{i,2}, \ldots)) = (x_{i,2}, x_{i,3}, \ldots)$ for each $x_i = (x_{i,n})_{n\in\mathbb{N}}$. Clearly, the UW-distance between any two different types in X_i is one, and hence, under the UW-metric, X_i is a discrete subset of the universal type space. Since X_i is uncountable, it follows that the universal type space is not separable under the UW-topology.

3.3 Discussion

3.3.1 Relationship with common *p*-belief

As we mentioned in the introduction, the uniform-weak topology is related to the notion of *common p-belief* due to Monderer and Samet (1989). Fix a state $\theta \in \Theta$ and $p \in [0,1]$. For each player $i \in I$ define

$$B_i^{1,p}(\theta) = \left\{t_i^1 \in \mathcal{T}_i^1 : \ t_i^1[\theta] \geq p\right\} \quad \text{and} \quad B_i^{k,p}(\theta) = \left\{t_i^k \in \mathcal{T}_i^k : \ t_i^k[\theta \times B_{-i}^{k-1,p}(\theta)] \geq p\right\},$$

recursively for all $k \geq 2$. A type t_i has *common p-belief in* θ, and we write $t_i \in C_i^p(\theta)$, if $t_i^k \in B_i^{k,p}(\theta)$ for all $k \geq 1$. A sequence of types $(t_{i,n})_{n\geq 1}$ has *asymptotic common certainty of* θ if for every $p < 1$ we have $t_{i,n} \in C_i^p(\theta)$ for n large enough.

Monderer and Samet (1989) use this notion of proximity to common certainty, i.e. common 1-belief, to study the robustness of Nash equilibrium to small amounts of incomplete information. Their main result states that for any game and any sequence of common prior type spaces, a sufficient condition for Nash equilibrium to be robust to incomplete information (relative to the given sequence of type spaces) is that, for some sequence $p_n \nearrow 1$, the prior probability of the event that the players have common p_n-belief on the payoffs from the complete information game converges to 1 as $n \to \infty$. A related paper, Kajii and Morris (1997), shows that asymptotic common certainty is actually a necessary condition for robustness in all games. Since both results are formulated for Bayesian Nash equilibrium in common prior type spaces, to facilitate comparison with our results we report (without proof) an analogue of their results for interim correlated rationalizability and without imposing common priors:

Proposition 4. *A sequence of types* $(t_{i,n})_{n\geq 1}$ *has asymptotic common certainty of* θ *if and only if for every game and every* $\varepsilon > 0$*, every action that is rationalizable for player* i *when* θ *is common certainty remains interim correlated* ε*-rationalizable for type* $t_{i,n}$ *for all* n *large enough.*

Thus the "only if" part is an interim version of Monderer and Samet (1989) and the "if" part an interim version of Kajii and Morris (1997).

As it turns out, the uniform-weak topology can be viewed as an extension of the concept of asymptotic common certainty: these two notions of convergence coincide when the limit type has common certainty of some state. Indeed, letting $t_{i,\theta}$ designate the type of player i who has common certainty of θ, we have:

Proposition 5. *A sequence* $(t_{i,n})_{n\geq 0}$ *has asymptotic common certainty of* θ *if and only if* $d_i^{\mathsf{uw}}(t_{i,n}, t_{i,\theta}) \to 0$ *as* $n \to \infty$*.*

Proof. It suffices to show that for each $i \in I$, $p \in [0,1]$ and $k \geq 1$ we have $B_i^{k,p}(\theta) = \{t_{i,\theta}^k\}^{1-p}$. For $k = 1$ this follows directly from $t_{i,\theta}^1[\theta] = 1$. Now suppose this holds for $k-1$ and let us show that it also holds for k. Indeed,

$$B_i^{k,p}(\theta) = \left\{t_i^k \in \mathcal{T}_i^k : \; t_i^k[\theta \times B_{-i}^{k-1,p}(\theta)] \geq p\right\} = \\ = \left\{t_i^k \in \mathcal{T}_i^k : \; t_i^k[\theta \times \{t_{-i,\theta}^{k-1}\}^{1-p}] \geq p\right\} = \{t_{i,\theta}^k\}^{1-p},$$

where the second equality follows from the induction hypothesis and the third from the fact that $t_{i,\theta}^k[\theta, t_{-i,\theta}^{k-1}] = 1$. □

Thus, taken together, Theorems 1 and 2 extend Proposition 4 to perturbations of incomplete information models.[23]

3.3.2 Other uniform metrics

The Prohorov metric, on which the uniform-weak topology is based, is but one of many equivalent distances that metrize the topology of weak convergence of probability measures. For any such distance one can consider the associated uniform distance over hierarchies of beliefs. Interestingly, these metrics can generate different topologies over infinite hierarchies, even though the induced topologies over k-order beliefs coincide for each $k \geq 1$. Below we provide such an example.

[23] Note that $t_{i,\theta}$ is a finite type.

Given a metric space (S,d) let $\mathsf{BL}(S,d)$ designate the vector space of real-valued, bounded, Lipschitz continuous functions over S, endowed with the norm

$$\|f\|_{\mathsf{BL}} = \max\left\{\sup_x |f(x)|, \sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)}\right\}, \quad \forall f \in \mathsf{BL}(S,d).$$

Recall that the *bounded Lipschitz* distance over $\Delta(S,d)$ is

$$\beta(\mu,\mu') = \sup\left\{\left|\int f\,d\mu - \int f\,d\mu'\right| : f \in \mathsf{BL}(S,d) \text{ with } \|f\|_{\mathsf{BL}} \le 1\right\}, \quad \forall \mu,\mu' \in \Delta(S,d).$$

This distance metrizes the topology of weak convergence and it relates to the Prohorov metric ρ as follows:[24]

$$(2/3)\rho^2 \le \beta \le 2\rho.$$

Now define a uniform metric β_i^{uw} over hierarchies of beliefs as follows. Let β^0 denote the discrete metric over Θ and, recursively, for $k \ge 1$ let β_i^k denote the bounded Lipschitz metric on $\Delta(\Theta \times \mathcal{T}_{-i}^{k-1})$ when $\Theta \times \mathcal{T}_{-i}^{k-1}$ is equipped with the metric $\max\{\beta^0, \beta_{-i}^{k-1}\}$. Then

$$\beta_i^{\mathsf{uw}} = \sup_{k\ge 1} \beta_i^k.$$

For each $k \ge 1$ the metric β_i^k is equivalent to d_i^k, as they both induce the weak topology on k-order beliefs. However, as we will now show, β_i^{uw} is *not* equivalent to d_i^{uw}.[25] Suppose that $\Theta = \{\theta_0, \theta_1\}$, and for each $n \ge 1$ consider the type space $(T_{i,n})_{i\in I}$ where

$$T_{i,n} = \{u_{i,0}, u_{i,1}, t_{i,n}\} \qquad \forall i \in I$$

and beliefs are as follows:

$$u_{i,0}[\theta_0, u_{-i,0}] = 1, \quad u_{i,1}[\theta_1, u_{-i,1}] = 1 \qquad \forall i \in I,$$

and

$$t_{i,n}[\theta_0, u_{-i,0}] = 1/n, \qquad t_{i,n}[\theta_1, t_{-i,n}] = 1 - 1/n \qquad \forall i \in I.$$

Thus $d_i^k(t_{i,n}, u_{i,1}) = 1/n$ for all $k \ge 1$, and therefore $d_i^{\mathsf{uw}}(t_{i,n}, u_{i,1}) \to 0$ as $n \to \infty$. We will now show that $\beta_i^{\mathsf{uw}}(t_{i,n}, u_{i,1}) \not\to 0$. Let f be the indicator function of $\{\theta_1\}$, i.e. $f(\theta_m) = m$ for $m \in \{0,1\}$. Then, define the k-order *iterated expectation* of f for each $k \ge 1$ and each player i, denoted $f_i^k : \mathcal{T}_i^k \to \mathbb{R}$, as follows:

$$f_i^1(t_i^1) = \int f\,dt_i^1 = t_i^1[\theta_1] \quad \text{and} \quad f_i^k(t_i^k) = \int f_{-i}^{k-1}\,dt_i^k, \quad \text{for } k \ge 2.$$

[24] See Dudley (2002), p. 398 and p. 411.

[25] The example actually shows that the two metrics are not equivalent even around complete information types. In particular, asymptotic common certainty does not guarantee convergence under β_i^{uw}.

Thus, we have

$$\int f_{-i}^{k-1}\, du_{i,1}^{k} = 1 \quad \text{and} \quad \int f_{-i}^{k-1}\, dt_{i,n}^{k} = (1-1/n)^{k}.$$

Since it can be shown that $f_i^k \in \mathsf{BL}(\mathcal{T}_i^k, \beta_i^k)$ and $\|f_i^k\|_{\mathsf{BL}} \leq 1$, we have $\beta_i^k(t_{i,n}, u_{i,1}) \geq 1-(1-1/n)^k$, and hence $\beta_i^{\mathsf{uw}}(t_{i,n}, u_{i,1}) \geq 1$ for every $n \geq 1$.

This example is also relevant for the comparison between our work and Morris (2002), who shows that the topology of uniform convergence of iterated expectations is equivalent to the strategic topology associated with a restricted class of games, called *higher-order expectations* (HOE) games. By this result and the example above, uniform-weak convergence is not sufficient for convergence in the strategic topology for HOE games. This might seem puzzling at first, given that we have shown uniform-weak convergence to imply convergence in Dekel, Fudenberg, and Morris's (2006) strategic topology, which is defined by requiring lower hemi-continuity of the strict ICR correspondence in *all* games, not just HOE games. To reconcile these facts, we note that the notion of strict ICR correspondence implicitly used in Morris (2002) is different from the one we use, in that it does not require the slack in the incentive constraints to hold uniformly in a best reply set. Thus, for a given game, continuity of Morris's (2002) notion of strict ICR is more demanding than ours.

4 Non-genericity of finite types

Dekel, Fudenberg, and Morris (2006) show that finite types are dense under the S-topology, thus strengthening an early result of Mertens and Zamir (1985) that finite types are dense under the product topology. In contrast, in Theorem 3 below we show that under the US-topology finite types are *nowhere dense*, i.e. the closure of finite types has an empty interior.[26] An implication of this result is that the US-topology is strictly finer than the S-topology.[27]

The proof of Theorem 3 relies on Lemmas 2 and 3 below. First, Lemma 2 states that finite types are not dense under the UW-topology. To prove this, we consider an instance of the countably infinite common-prior type space from Rubinstein's (1989) E-mail game and show that none of its types can be UW-approximated by a sequence of finite types. Second, in Lemma 3 we show that any sequence of types that fails to converge to a type in the E-mail type space under the UW-topology must also fail to converge under the US-topology. Together, these lemmas imply that finite types are bounded away from the E-mail

[26] This is equivalent to saying that the complement of the set of finite types contains an open and dense set under the US-topology.

[27] Dekel, Fudenberg, and Morris (2006) state the result that the US-topology is strictly finer than the S-topology. However, as reported in Chen and Xiong (2008), the proof in that paper contains a mistake.

type space in US-distance, which we state as Proposition 6 below. This implies that the set of finite types is not dense under the US-topology. Using this result, the proof of Theorem 3 shows that every finite type can be US-approximated by a sequence of infinite types, none of which is the US-limit of a sequence of finite types, thereby establishing nowhere denseness.

In effect, consider the following instance of the E-mail type space. Let $\Theta = \{\theta_0, \theta_1\}$ and let the type space (U_1, U_2) be thus defined:[28]

$$U_1 = \{u_{1,0}, u_{1,1}, u_{1,2}, \dots\}, \qquad U_2 = \{u_{2,0}, u_{2,1}, u_{2,2}, \dots\},$$

and $u_{1,0}[\theta_0, u_{2,0}] = 1$, $u_{2,0}[\theta_0, u_{1,0}] = 2/3$, $u_{2,0}[\theta_1, u_{1,1}] = 1/3$,

$$\begin{aligned} u_{1,n}[\theta_1, u_{2,n-1}] &= 2/3, & u_{1,n}[\theta_1, u_{2,n}] &= 1/3 & \forall n &\geq 1, \\ u_{2,n}[\theta_1, u_{1,n}] &= 2/3, & u_{2,n}[\theta_1, u_{1,n+1}] &= 1/3 & \forall n &\geq 1. \end{aligned}$$

Proposition 6. $d_i^{\mathsf{us}}(t_i, u_{i,n}) \geq M/6$ *for every* $i \in I$*, finite type* $t_i \in \mathcal{T}_i$ *and* $n \geq 0$.

The proposition is a direct consequence of the following two lemmas:

Lemma 2. $d_i^{\mathsf{uw}}(t_i, u_{i,n}) \geq 1/3$ *for every* $i \in I$*, finite type* $t_i \in \mathcal{T}_i$ *and* $n \geq 0$.

Lemma 3. $d_i^{\mathsf{us}}(t_i, u_{i,n}) \geq (M/2) d_i^{\mathsf{uw}}(t_i, u_{i,n})$ *for every* $i \in I$*,* $t_i \in \mathcal{T}_i$ *and* $n \geq 0$.

In the proof of Lemma 2, given in Appendix A, we first show by induction that the UW-distance between any two distinct types of any player in the E-mail type space above is at least $2/3$.[29] Second, we use another induction to show that any finite type $t_{2,n}$ whose UW-distance from $u_{2,n}$ is less than $1/3$ must attach positive probability to (and hence implies the existence, in the same finite type space, of) a type $t_{1,n+1}$ whose UW-distance from $u_{1,n+1}$ is less than $1/3$, which in turn implies the existence in the same finite type space of some type $t_{2,n+1}$ whose UW-distance from $u_{2,n+1}$ is less than $1/3$, and so on. These two facts together imply the contradiction that the types $t_{i,1}, t_{i,2}, \dots$ are all different but belong to the same finite type space, whence the result follows.

Turning to Lemma 3, fix an arbitrary $\delta \geq 0$. The proof, also in Appendix A, constructs for each $N \geq 0$ a game such that for each player i and each $0 \leq n \leq N$ a certain action $a_{i,n}$ is rationalizable for $u_{i,n}$ but is not δ-rationalizable for any type t_i with $d_i^k(t_i, u_{i,n}) > 2\delta/M$,

[28] This type space is an instance of the E-mail type space where the more informed player 1 who received k messages attaches probability $p = 2/3$ (resp. $1 - p = 1/3$) to player 2 having received $k - 1$ (resp. k) messages, and the less informed player 2 who received k messages attaches probability p (resp. $1 - p$) to player 1 having received k (resp. $k + 1$) messages. Our choice that $p = 2/3$ is immaterial; our results hold true if we assume any other value for p.

[29] The type $u_{1,k}$ of player 1 who received k messages assigns probability $2/3$ to the other player having received $k - 1$ messages, while $u_{1,k+1}$ attaches probability zero to that event, and similarly for player 2.

	$a_{2,0}$	$b_{2,0}$	$c_{2,0}$	$a_{2,1}$	s_2
$a_{1,0}$	0, 0	-4, 1	-4, -1	-4, -4	-4, 0
$a_{1,1}$	-4, -4	-4, -4	-4, -4	-4, -4	-4, 0
$b_{1,1}$	-4, -4	-4, -4	-4, -4	-4, -4	-4, 0
$c_{1,1}$	-4, -4	-4, -4	-4, -4	-4, -4	-4, 0
s_1	0, -4	0, -4	0, -4	0, -4	0, 0

θ_0

	$a_{2,0}$	$b_{2,0}$	$c_{2,0}$	$a_{2,1}$	s_2
$a_{1,0}$	-4, -4	-4, -4	-4, -4	-4, -4	-4, 0
$a_{1,1}$	0, 0	-4, -2	-4, 2	0, 0	-4, 0
$b_{1,1}$	1, -4	-4, -4	-4, -4	-2, -4	-4, 0
$c_{1,1}$	-1, -4	-4, -4	-4, -4	2, -4	-4, 0
s_1	0, -4	0, -4	0, -4	0, 0	0, 0

θ_1

Figure 1: The game from Lemma 3 for $N = 1$ and $M = 4$.

where the order k grows with the difference $N - n$. To provide intuition we discuss the argument for the case $N = 1$. The game corresponding to this case is depicted in Figure 1, with the payoff bound normalized to $M = 4$.

It is clear that in this game, for all $i = 1, 2$ and $n = 0, 1$, action $a_{i,n}$ is rationalizable for $u_{i,n}$.[30] However, $a_{i,n}$ is weakly dominated by s_i, and the payoffs from $b_{i,n}$ and $c_{i,n}$ are such that whenever the beliefs of a type t_i are sufficiently far from those of $u_{i,n}$, then any δ-rationalizable conjecture about player $-i$ that δ-rationalize $a_{i,n}$ against s_i cannot do so against both $b_{i,n}$ and $c_{i,n}$ as well. Indeed we have

$$d_i^k(t_i, u_{i,n}) > 2\delta/M \quad \Longrightarrow \quad a_{i,n} \notin R_i^k(t_i, \delta) \qquad \forall 1 \le k \le 2 - n. \tag{17}$$

To see this for $k = 1$, first note that $a_{1,0}$ is weakly dominated by s_1, hence $a_{1,0} \notin R_1^1(t_i, \delta)$ for any type t_1 with $d_1^1(t_1, u_{1,0}) > \delta/2$. Indeed, $u_{1,0}^1[\theta_0] = 1$ and hence $d_1^1(t_1, u_{1,0}) > \delta/2$ implies $t_1^1[\theta_0] < 1 - \delta/2$, so the highest possible expected payoff for t_1 under $a_{1,0}$ is -2δ, whereas s_1 yields 0. By the same token, $a_{1,1} \notin R_1^1(t_1, \delta)$ for any t_1 with $d_1^1(t_1, u_{1,1}) > \delta/2$, and $a_{2,1} \notin R_2^1(t_2, \delta)$ for any t_2 with $d_2^1(t_2, u_{2,1}) > \delta/2$. Consider action $a_{2,0}$ now, and pick any t_2 such that $d_2^1(t_2, u_{2,0}) > \delta/2$. Since $u_{2,0}^1[\theta_1] = 1/3$, we must have either $t_2^1[\theta_1] < 1/3 - \delta/2$ or $t_2^1[\theta_0] < 2/3 + \delta/2$. Pick any conjecture σ_1 that δ-rationalizes $a_{2,0}$, so that the difference in expected payoff between s_1 and $a_{2,0}$ is at most δ. This requires the induced distribution over $\Theta \times A_1$ to satisfy

$$\mathsf{Pr}[\theta_0, a_{1,0} \mid t_2, \sigma_1] + \mathsf{Pr}[\theta_1, a_{1,1} \mid t_2, \sigma_1] \ge 1 - \delta/4,$$

hence the difference in expected payoffs between $b_{2,0}$ and $a_{2,0}$ is

$$\mathsf{Pr}[\theta_0, a_{1,0} \mid t_2, \sigma_1] - 2\mathsf{Pr}[\theta_1, a_{1,1} \mid t_2, \sigma_1] \ge -3\mathsf{Pr}[\theta_1, a_{1,1} \mid t_2, \sigma_1] + 1 - \delta/4,$$

which is greater than δ when $t_2^1[\theta_1] < 1/3 - \delta/2$. Likewise, the difference in expected

[30] The pair $(\varsigma_1, \varsigma_2)$ with $\varsigma_i(u_{i,n}) = a_{i,n}$ if $n \le 1$ and $\varsigma_i(u_{i,n}) = s_i$ if $n \ge 2$ has the best reply property.

payoffs between $c_{2,0}$ and $a_{2,0}$ is

$$-\mathsf{Pr}[\theta_0, a_{1,0} \mid t_2, \sigma_1] + 2\mathsf{Pr}[\theta_1, a_{1,1} \mid t_2, \sigma_1] \geq -3\mathsf{Pr}[\theta_0, a_{1,0} \mid t_2, \sigma_1] + 2 - \delta/2,$$

which is greater than δ when $t_2^1[\theta_0] < 2/3 + \delta/2$. Thus, in any case $a_{2,0} \notin R_2^1(t_2, \delta)$, and the proof of (17) for $k = 1$ is complete. The proof for $n = 0$ and $k = 2$ uses the arguments just given for the case $k = 1$ and is completely analogous—for instance, those arguments show that if σ_2 is a first-order δ-rationalizable conjecture that δ-rationalizes $a_{1,0}$ for a type t_1, then we must have $1 - \delta/4 \leq \mathsf{Pr}[\theta_0, a_{2,0} \mid t_1, \sigma_2] \leq t_1^2[\theta_0 \times \{u_{2,0}^1\}^{2\delta/M}]$ and hence the distance between the second-order beliefs of t_1 and $u_{1,0}$ is at most δ.

We are now ready to prove the main result of this section.

Theorem 3. *Finite types are nowhere dense under the* US- *and the* UW-*topology.*

Proof. It suffices to prove that every finite type can be UW-approximated by a sequence of infinite types, none of which is the US-limit of a sequence of finite types.[31] Fix a finite type space (T_1, T_2) and a type $t_2 \in T_2$. For each $n \geq 1$ let $\delta_n = 1/(n+1)$ and define the infinite type $t_{2,n}$ by the requirement that, for every $k \geq 1$ and every measurable $E \subseteq \Theta \times \mathcal{T}_1^{k-1}$,

$$t_{2,n}^k[E] = (1 - \delta_n)t_2^k[E] + \delta_n u_{2,0}^k[E].$$

Note that for all $n \geq 1$, $k \geq 1$ and measurable $E \subseteq \Theta \times \mathcal{T}_1^{k-1}$ we have

$$t_{2,n}^k[E] = (1 - \delta_n)t_2^k[E] + \delta_n u_{2,0}^k[E] \leq t_{2,n}^k[E^{\delta_n}] + \delta_n,$$

hence $d_2^{\mathsf{uw}}(t_{2,n}, t_2) \leq \delta_n \longrightarrow 0$.

It remains to prove that none of the types in the sequence $(t_{2,n})_{n\geq 1}$ is in the US-closure of the set of finite types, i.e. for every $n \geq 1$ there exists $\varepsilon_n > 0$ such that the US-distance between $t_{2,n}$ and every finite type in $\mathcal{T}_2$ is at least ε_n. Thus, fix $n \geq 1$, pick any $0 < \varepsilon_n < \min\{M/6, M/(3n+1)\}$, any finite type space (S_1, S_2) and any type $s_2 \in S_2$, and let us show that $d_2^{\mathsf{us}}(t_{2,n}, s_2) \geq \varepsilon_n$. Using Lemma 2 choose $N \geq 1$ large enough so that

$$d_1^{2(N+1)}(t_1, u_{1,0}) \geq 1/3 \qquad \forall t_1 \in T_1 \cup S_1 \tag{18}$$

and let $G_N = (A_{i,N}, g_{i,N})_{i=1,2}$ be the game defined in the proof of Lemma 3. Now define another game $G'_N = (A'_{i,N}, g'_{i,N})_{i=1,2}$ as follows:

$$A'_{1,N} = A_{1,N}, \qquad A'_{2,N} = A_{2,N} \times \{0,1\},$$

[31] Indeed, by Theorem 1 the sequence will also US-approximate the finite type, hence nowhere denseness in the US-topology will follow; by the same theorem, none of the types in the sequence will be the UW-limit of a sequence of finite types, thus nowhere denseness in the UW-topology will also follow.

and for all $a_1 \in A_{1,N}$, $a_2 \in A_{2,N}$, $x \in \{0,1\}$ and $\theta \in \Theta$,

$$g'_{1,N}(a_1, a_2, x, \theta) = \frac{1}{2} g_{1,N}(a_1, a_2, \theta) \tag{19}$$

$$g'_{2,N}(a_1, a_2, x, \theta) = \frac{1}{2} g_{2,N}(a_1, a_2, \theta) + \begin{cases} M/2 & \text{if } x = 1 \text{ and } a_1 = a_{1,0}, \\ -M/(3n+1) & \text{if } x = 1 \text{ and } a_1 \neq a_{1,0}, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Note that, since all payoffs in G_N are between $-M$ and M, the same is true for all payoffs in G'_N. Moreover, we have the following lemma, which is proved in Appendix A.

Lemma 4. *For all $k \geq 0$ and all $\varepsilon \geq 0$,*

$$R_1^k(t_1, G_N, 2\varepsilon) = R_1^k(t_1, G'_N, \varepsilon) \qquad \forall t_1 \in \mathcal{T}_1, \tag{21}$$

$$R_2^k(t_2, G_N, 2\varepsilon) = \mathsf{proj}_{A_{2,N}} R_2^k(t_2, G'_N, \varepsilon) \qquad \forall t_2 \in \mathcal{T}_2. \tag{22}$$

We now prove that $(a_2, 1) \in R_2(t_{2,n}, G'_N, 0)$ for some $a_2 \in A_{2,N}$, but $(a_2, 1) \notin R_2(s_2, G'_N, \varepsilon_n)$ for all $a_2 \in A_{2,N}$, reaching the desired conclusion that $d_2^{\mathsf{us}}(t_{2,n}, s_2) \geq \varepsilon_n$.

To show that $(a_2, 1) \in R_2(t_{2,n}, G'_N, 0)$ for some $a_2 \in A_{2,N}$, it suffices to construct a rationalizable conjecture σ'_1 for $t_{2,n}$ in game G'_N under which, for all $a_2 \in A_{2,N}$, actions $(a_2, 0)$ and $(a_2, 1)$ give $t_{2,n}$ the same expected payoff. Let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,N})$ be an arbitrary rationalizable conjecture for $t_{2,n}$ in G_N and define $\sigma'_1 : \Theta \times \mathcal{T}_1 \to \Delta(A'_{1,N})$ as

$$\sigma'_1(\theta, t_1)[a_1] = \sigma_1(\theta, t_1)[a_1] \qquad \forall t_1 \in \mathcal{T}_1 \setminus U_1,\ \forall a_1 \in A'_{1,N},$$

$$\sigma'_1(\theta, u_{1,k})[a_{1,k}] = 1 \qquad \forall k \geq 0.$$

From the proof of Lemma 3 it follows, using (21) with $\varepsilon = 0$, that σ'_1 is a rationalizable conjecture for $t_{2,n}$ in G'_N and also, using (18) and the fact that $\varepsilon_n < M/6$, that

$$a_{1,0} \notin R_1(t_1, G_N, \varepsilon_n) \qquad \forall t_1 \in T_1 \cup S_1. \tag{23}$$

Thus, $\sigma'_1(\theta, t_1)[a_{1,0}] = 0$ for all $\theta \in \Theta$ and $t_1 \in T_1$, hence for all $a_2 \in A_{2,N}$ we have

$$\int_{\Theta \times \mathcal{T}_1} \left[g'_{2,N}(\sigma'_1(\theta, t_1), a_2, 1, \theta) - g'_{2,N}(\sigma'_1(\theta, t_1), a_2, 0, \theta) \right] t_{2,n}(d\theta \times dt_1) =$$

$$= (2\delta_n/3)\frac{M}{2} - (1 - 2\delta_n/3)\frac{M}{3n+1} = 0.$$

This proves that $(a_2, 0)$ and $(a_2, 1)$ give type $t_{2,n}$ the same expected payoff under σ'_1 for all $a_2 \in A_{2,N}$, as was to be shown.

Turning to the proof that $(a_2, 1) \notin R_2(s_2, G'_N, \varepsilon_n)$ for all $a_2 \in A_{2,N}$, consider an arbitrary ε_n-rationalizable conjecture σ'_1 for s_2 in game G'_N. By (21) and (23), for all $\theta \in \Theta$ and $s_1 \in S_1$

we must have $\sigma_1'(\theta, s_1)[a_{1,0}] = 0$. Thus, for all $a_2 \in A_{2,N}$,

$$\sum_{(\theta,s_1)\in\Theta\times S_1} s_2[\theta, s_1]\Big[g_{2,N}'(\sigma_1(\theta, s_1), a_2, 1, \theta) - g_{2,N}'(\sigma_1(\theta, s_1), a_2, 0, \theta)\Big] = -\frac{M}{3n+1} < -\varepsilon_n,$$

which proves that $(a_2, 1)$ is not ε_n-rationalizable for s_2 in game G_N'. $\square$

A Omitted Proofs

A.1 Proof of Proposition 1

Fix $k \geq 1$ and $t_i \in T_i$. Let Σ_{-i} denote the set of equivalence classes of measurable functions $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}^{k-1}(t_{-i}, G, \gamma) \quad \text{for } t_i\text{-almost every } (\theta, t_{-i}) \in \Theta \times T_{-i},$$

where we identify pairs of functions that are equal t_i-almost surely. Notice that Σ_{-i} can be viewed as a convex subset of the real vector space L of (equivalence classes of) $\mathbb{R}^{|A_{-i}|}$-valued measurable functions over $\Theta \times T_{-i}$.

Consider the function $f : \Delta(A_i) \times \Sigma_{-i} \to \mathbb{R}$ such that

$$f(\alpha_i, \sigma_{-i}) = \int\limits_{\Theta\times T_{-i}} \Big[g_i(a_i, \sigma_{-i}(\theta, t_{-i}), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta, t_{-i}), \theta)\Big]\, t_i(d\theta \times dt_{-i}).$$

Thus, f is the restriction of a bi-linear functional on $\mathbb{R}^{|A_-|} \times L$ to the Cartesian product of the compact, convex set $\Delta(A_i)$ with the convex set Σ_{-i} (not topologized). By a minmax theorem of Fan (1953) we obtain

$$\min_{\alpha_i\in\Delta(A_i)} \sup_{\sigma_{-i}\in\Sigma_{-i}} f(\alpha_i, \sigma_{-i}) = \sup_{\sigma_{-i}\in\Sigma_{-i}} \min_{\alpha_i\in\Delta(A_i)} f(\alpha_i, \sigma_{-i}).$$

Now $a_i \in R_i^k(t_i, G, \gamma)$ if and only if the right-hand side is greater than or equal to $-\gamma$. We have thus shown that $a_i \in R_i^k(t_i, G, \gamma)$ if and only if for every $\eta > 0$ and $\alpha_i \in \Delta(A_i)$ there exists $\sigma_{-i} \in \Sigma_{-i}$ such that $f(\alpha_i, \sigma_{-i}) > -\gamma - \eta$, which is the desired result.

A.2 Proof of Lemma 1

For each $i \in I$ let ρ_i and $\|\cdot\|_i$ denote the Prohorov distance on $\Delta(\Theta \times T_{-i})$ and the Euclidean norm on $\mathbb{R}^{|\Theta||T_{-i}|}$, respectively. Also, let $f_i : \Theta \times T_{-i} \times \Delta(\Theta \times T_{-i}) \to \mathbb{R}$ be the function defined by

$$f_i(\theta, t_{-i}, \psi) = 2\psi[\theta, t_{-i}] - \|\psi\|_i^2,$$

and let $F_i : \Delta(\Theta \times T_{-i}) \times \Delta(\Theta \times T_{-i}) \to \mathbb{R}$ be the function defined by

$$F_i(\psi', \psi) = \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} f_i(\theta, t_{-i}, \psi')\psi[\theta, t_{-i}].$$

Note that $F_i(\psi, \psi) - F_i(\psi', \psi) = \|\psi - \psi'\|_i^2$ for all $\psi, \psi' \in \Delta(\Theta \times T_{-i})$, hence

$$\eta \equiv \frac{1}{2} \min \left\{ F_i(\psi, \psi) - F_i(\psi', \psi) \,:\, \psi', \psi \in \Delta(\Theta \times T_{-i}),\ \rho_i(\psi, \psi') \geq \frac{\delta}{2} \right\} > 0,$$

and also[32]

$$\rho_i(\psi, \psi') < \eta/2 \quad \Longrightarrow \quad F_i(\psi, \psi) - F_i(\psi', \psi) < \eta \qquad \forall \psi, \psi' \in \Delta(\Theta \times T_{-i}).$$

The compact set $\Delta(\Theta \times T_{-i})$ can be covered by a finite union of open balls of radius $\eta/2$. (These balls are taken according to the metric ρ_i.) Choose one point in each of these balls and let $A_i \subseteq \Delta(\Theta \times T_{-i})$ denote the finite set of selected points. Enlarge A_i, if necessary, to ensure $A_i \supseteq T_i$. (Recall that we identify each $t_i \in T_i$ with $\mu_i(t_i)$.) Thus, for every $\psi \in \Delta(\Theta \times T_{-i})$ there exists $a_i \in A_i \setminus T_i$ such that $F_i(\psi, \psi) - F_i(a_i, \psi) < \eta$.

Now define the payoff function $g_i : \Theta \times A_i \times A_{-i} \to \mathbb{R}$, as follows:

$$g_i(\theta, a_i, a_{-i}) = \begin{cases} f_i(\theta, a_{-i}, a_i) & \text{if } a_{-i} \in T_{-i}, \\ -4/\delta & \text{if } a_i \in T_i \text{ and } a_{-i} \notin T_{-i}, \\ -1 & \text{if } a_i \notin T_i \text{ and } a_{-i} \notin T_{-i}. \end{cases}$$

It follows directly from the definition of g_i and the fact that $t_i[\Theta \times T_{-i}] = 1$ that each $a_i \in A_i$ yields an expected payoff of $F_i(a_i, t_i)$ to type t_i under the conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that $\sigma_{-i}(\theta, t_{-i})[t_{-i}] = 1$ for all $(\theta, t_{-i}) \in \Theta \times T_{-i}$. Since $F_i(t_i, t_i) \geq F_i(a_i, t_i)$ for all $a_i \in A_i$, (10) follows.

Fix any $0 < \varepsilon < \min\{\eta(1 - \delta/2), \delta/2\}$. We shall prove (11) now. Fix $t_i \in T_i$ and $\psi \in \Delta(\Theta \times A_{-i})$, and assume that there exists $D \subseteq \Theta \times T_{-i}$ such that $\psi[D] < t_i[D] - \delta$. First suppose $\psi[\Theta \times T_{-i}] < 1 - \delta/2$. Pick any $a_i \in A_i \setminus T_i$. Since f_i maps into $[-1, 1]$,

$$\sum_{\theta \in \Theta} \sum_{a_{-i} \in A_{-i}} (g_i(t_i, a_{-i}, \theta) - g_i(a_i, a_{-i}, \theta))\psi[\theta, a_{-i}] \leq$$
$$\leq 2(1 - \delta/2) + (\delta/2)(-4/\delta + 1) = -\delta/2 < -\varepsilon,$$

[32] Letting $h : \Theta \times T_{-i} \to [-1, 1]$ denote the mapping $(\theta, t_{-i}) \mapsto h(\theta, t_{-i}) = \psi[\theta, t_{-i}] - \psi'[\theta, t_{-i}]$, for each $\zeta \geq 0$ we have

$$F_i(\psi, \psi) - F_i(\psi', \psi) = \|\psi - \psi'\|^2 = \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \psi[\theta, t_{-i}]h(\theta, t_{-i}) - \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \psi'[\theta, t_{-i}]h(\theta, t_{-i}) \leq 2\zeta$$

whenever $\rho_i(\psi, \psi') \leq \zeta$.

which proves (11) for the case $\psi[\Theta \times T_{-i}] < 1-\delta/2$. Now suppose that $\psi[\Theta \times T_{-i}] \geq 1-\delta/2$. Consider the conditional probability $\bar{\psi}(\cdot) \equiv \psi(\cdot|\Theta \times T_{-i})$. Then

$$\bar{\psi}[D] \geq \psi[D] = \bar{\psi}[D]\psi[\Theta \times T_{-i}] \geq \bar{\psi}[D] - \delta/2, \tag{24}$$

hence

$$\big|\bar{\psi}[D] - t_i[D]\big| \geq \big|\psi[D] - t_i[D]\big| - \big|\psi[D] - \bar{\psi}[D]\big| > \delta - \delta/2 = \delta/2,$$

which implies $F_i(\bar{\psi},\bar{\psi}) - F_i(t_i,\bar{\psi}) \geq 2\eta$ by the definition of η. Now pick any $a_i \in A_i \setminus T_i$ with $\rho_i(\bar{\psi},a_i) < \eta/2$, so that $F_i(a_i,\bar{\psi}) - F_i(\bar{\psi},\bar{\psi}) > -\eta$. Then $F_i(a_i,\bar{\psi}) - F_i(t_i,\bar{\psi}) > \eta$ and hence

$$\begin{aligned}
&\sum_{\theta\in\Theta}\sum_{a_{-i}\in A_{-i}} (g_i(t_i,a_{-i},\theta) - g_i(a_i,a_{-i},\theta))\psi[\theta,a_{-i}] = \\
&\qquad = \big(F_i(t_i,\bar{\psi}) - F_i(a_i,\bar{\psi})\big)\psi[\Theta \times T_{-i}] + \big(-4/\delta + 1\big)\big(1 - \psi[\Theta \times T_{-i}]\big) \leq \\
&\qquad\qquad \leq \big(F_i(t_i,\bar{\psi}) - F_i(a_i,\bar{\psi})\big)\psi[\Theta \times T_{-i}] < (1-\delta/2)(-\eta) < -\varepsilon,
\end{aligned}$$

which proves (11) also for the case $\psi[\Theta \times T_{-i}] \geq 1 - \delta/2$.

Finally, to ensure that the payoffs are bounded by M, multiply g_i and ε by a factor of $M\delta/4$, if necessary. This normalization does not affect the validity of (11).

A.3 Proof of Lemma 2

First we prove by induction that

$$d_i^{\mathsf{uw}}(u_{i,n},u_{i,m}) \geq 2/3 \qquad \forall i = 1,2,\ \forall n \geq 0,\ \forall m \geq 0 \ \text{ s.t. } \ m \neq n. \tag{25}$$

For all $n \geq 1$ we have $u^1_{1,0}[\theta_0] = 1$ and $u^1_{1,n}[\theta_0] = 0$, hence $d^1_1(u_{1,0},u_{1,n}) = 1 > 2/3$; moreover, $u^1_{2,0}[\theta_0] = 2/3$ and $u^1_{2,n}[\theta_0] = 0$, hence $d^1_2(u_{2,0},u_{2,n}) \geq 2/3$. Assume that we have proved $d^n_i(u_{i,n-1},u_{i,m}) \geq 2/3$ for all $i = 1,2$, some $N \geq 1$, all $1 \leq n \leq N$, and all $m \geq n$. Then, for all $m > n$, since $u_{1,n}[\theta_1 \times u_{2,n-1}] = 2/3$ and $u_{1,m}[\theta_1 \times u_{2,\ell}] = 0$ for all $\ell < n$, we obtain $u^{n+1}_{1,n}[\theta_1 \times u^n_{2,n-1}] = 2/3$ and $u^{n+1}_{1,m}[\theta_1 \times \{u^n_{2,n-1}\}^{2/3}] = 0$, hence $d^{n+1}_1(u_{1,n},u_{1,m}) \geq 2/3$. Since $u_{2,n}[\theta_1 \times u_{1,n}] = 2/3$ and $u_{2,m}[\theta_1 \times u_{1,\ell}] = 0$ for all $\ell \leq n$, we also get $u^{n+1}_{2,n}[\theta_1 \times u^n_{1,n}] = 2/3$ and $u^{n+1}_{2,m}[\theta_1 \times \{u^n_{1,n}\}^{2/3}] = 0$, hence $d^{n+1}_2(u_{2,n},u_{2,m}) \geq 2/3$. The proof of (25) is complete.

Now let (T_1,T_2) be a finite type space and for every $i = 1,2$ and every $n \geq 0$ define

$$T_{i,n} = \{t_i \in T_i \,:\, d_i^{\mathsf{uw}}(t_i,u_{i,n}) < 1/3\}.$$

We must show that each $T_{i,n}$ is empty. Note that (25) implies $T_{i,n} \cap T_{i,m} = \varnothing$ for each player i and all $n \geq 0$ and $m \geq 0$ such that $m \neq n$. Thus, it suffices to show that if $T_{i,n} \neq \varnothing$

for some player i and some $n \geq 0$, then $T_{1,m} \neq \emptyset$ and $T_{2,m} \neq \emptyset$ for all $m > n$, as this contradicts the finiteness of T_1 and T_2.

Assume that $T_{1,0} \neq \emptyset$. Pick any $t_{1,0} \in T_{1,0}$ and $1/3 > \delta > d_1^{\mathsf{uw}}(t_{1,0}, u_{1,0})$. Then

$$t_{1,0}^k\Big[\theta_0 \times \{u_{2,0}^{k-1}\}^\delta\Big] \;\geq\; u_{1,0}^k\Big[\theta_0 \times u_{2,0}^{k-1}\Big] - \delta \;=\; 1 - \delta \qquad \forall k \geq 1$$

and hence, using the fact that $\delta < 1/3$ and $t_{1,0}[\theta_0 \times T_2] = t_{1,0}[\theta_0 \times \mathcal{T}_2]$, also

$$t_{1,0}[\theta_0 \times T_{2,0}] \;\geq\; t_{1,0}[\theta_0 \times \{t_2 \in \mathcal{T}_2 \;:\; d_2^{\mathsf{uw}}(t_2, u_{2,0}) < \delta\}] \;\geq\; 1 - \delta \;>\; 0,$$

implying that $T_{2,0} \neq \emptyset$ as well. Now let $n \geq 0$ and assume $T_{2,n} \neq \emptyset$. Pick any $t_{2,n} \in T_{2,n}$ and $1/3 > \delta > d_2^{\mathsf{uw}}(t_{2,n}, u_{2,n})$. Then

$$t_{2,n}^k\Big[\theta_1 \times \{u_{1,n+1}^{k-1}\}^\delta\Big] \;\geq\; u_{2,n}^k[\theta_1 \times u_{1,n+1}^{k-1}] - \delta \;=\; 1/3 - \delta \qquad \forall k \geq 1$$

and hence, as before,

$$t_{2,n}[\theta_1 \times T_{1,n+1}] \;\geq\; t_{2,n}[\theta_1 \times \{t_1 \in \mathcal{T}_1 \;:\; d_1^{\mathsf{uw}}(t_1, u_{1,n+1}) < \delta\}] \;\geq\; 1/3 - \delta \;>\; 0,$$

so $T_{1,n+1} \neq \emptyset$. Similarly, we can show that $T_{1,n} \neq \emptyset$ implies $T_{2,n} \neq \emptyset$ for all $n \geq 1$.

A.4 Proof of Lemma 3

For any given $N \geq 1$ we construct a game G_N with action sets

$$\begin{aligned} A_{1,N} &= \Big\{a_{1,0}, a_{1,1}, b_{1,1}, c_{1,1}, \ldots, a_{1,N}, b_{1,N}, c_{1,N}, s_1\Big\}, \\ A_{2,N} &= \Big\{a_{2,0}, b_{2,0}, c_{2,0}, \ldots, a_{2,N-1}, b_{2,N-1}, c_{2,N-1}, a_{2,N}, s_2\Big\} \end{aligned}$$

such that

$$a_{i,n} \in R_i(u_{i,n}, G_N, 0) \qquad \forall i \in I,\ \forall 0 \leq n \leq N \tag{26}$$

and moreover, for every $\delta \geq 0$ and $0 \leq k \leq N$,

$$a_{1,n} \in R_1^{2(k+1)}(t_1, G_N, \delta) \;\Longrightarrow\; d_2^{2(k+1)}(t_2, u_{2,n}) \leq 2\delta/M \quad \forall n \leq N-k,\ \forall t_1 \in \mathcal{T}_1, \tag{27}$$

$$a_{2,n} \in R_2^{2k+1}(t_2, G_N, \delta) \;\Longrightarrow\; d_2^{2k+1}(t_2, u_{2,n}) \leq 2\delta/M \quad \forall n \leq N-k,\ \forall t_2 \in \mathcal{T}_2. \tag{28}$$

Indeed, this implies the statement of the lemma.

Fix $N \geq 1$. For convenience, throughout the proof let $a_{1,N+1} = s_1$ and $\theta_n = \theta_1$ for every $n \geq 2$. The payoffs in G_N are as follows. Actions s_1 and s_2 give constant payoffs:

$$g_{1,N}(\theta, s_1, a_2) = g_{2,N}(\theta, a_1, s_2) = 0 \qquad \text{for every } \theta \in \Theta,\ a_1 \in A_{1,N}, \text{ and } a_2 \in A_{2,N}.$$

Actions $a_{1,0},\dots,a_{1,N}$ and $a_{2,0},\dots,a_{2,N}$ are weakly dominated by s_1 and s_2, respectively:

$$g_{1,N}(\theta,a_{1,n},a_2)=\begin{cases}0 & \text{if } n=0 \text{ and } (\theta,a_2)=(\theta_0,a_{2,0}),\\ 0 & \text{if } n>0 \text{ and } (\theta,a_2)\in\{(\theta_1,a_{2,n-1}),(\theta_1,a_{2,n})\},\\ -M & \text{otherwise;}\end{cases}$$

$$g_{2,N}(\theta,a_1,a_{2,n})=\begin{cases}0 & \text{if } (\theta,a_1)\in\{(\theta_n,a_{1,n}),(\theta_1,a_{1,n+1})\},\\ -M & \text{otherwise.}\end{cases}$$

The payoffs for actions $b_{1,1},c_{1,1},\dots,b_{1,N},c_{1,N}$ are as follows:

$$g_{1,N}(\theta,b_{1,n},a_2)=-g_{1,N}(\theta,c_{1,n},a_2)=\begin{cases}M/4 & \text{if } (\theta,a_2)=(\theta_1,a_{2,n-1}),\\ -M/2 & \text{if } (\theta,a_2)=(\theta_1,a_{2,n}),\end{cases}$$

$$g_{1,N}(\theta,b_{1,n},a_2)=g_{1,N}(\theta,c_{1,n},a_2)=-M \quad \text{otherwise.}$$

Finally, the payoffs for $b_{2,0},c_{2,0},\dots,b_{2,N-1},c_{2,N-1}$ are

$$g_{2,N}(\theta,a_1,b_{2,n})=-g_{2,N}(\theta,a_1,c_{2,n})=\begin{cases}M/4 & \text{if } (\theta,a_1)=(\theta_n,a_{1,n}),\\ -M/2 & \text{if } (\theta,a_1)=(\theta_1,a_{1,n+1}),\end{cases}$$

$$g_{2,N}(\theta,a_1,b_{2,n})=g_{2,N}(\theta,a_1,c_{2,n})=-M \quad \text{otherwise.}$$

It is immediate to verify that (26) holds. To see this, just note that the mappings $\varsigma_i : U_i \to 2^{A_{i,N}}$ such that $\varsigma_i(u_{i,n})=a_{i,n}$ for $0\le n\le N$ and $\varsigma_i(u_{i,n})=s_i$ for $n>N$ have the best reply property.

It remains to prove that (27) and (28) hold for every $0\le k\le N$. To do this we now fix $\delta\ge 0$ and establish the following three claims. First, we show that (28) holds for $k=0$. Second, we prove that (28) implies (27) for all $0\le k\le N$. Third, we show that if (27) holds for some $0\le k<N$ then (28) holds with $k+1$ substituted for k, thus concluding the proof. To ease notation, for every player i, type $t_i\in\mathcal{T}_i$ and conjecture $\sigma_{-i}:\Theta\times\mathcal{T}_{-i}\to\Delta(A_{-i,N})$, in what follows we write $\mathsf{Pr}[\cdot\mid t_i,\sigma_{-i}]$ for the probability distribution over $\Theta\times A_{-i,N}$ induced by t_i and σ_{-i}, i.e.

$$\mathsf{Pr}[\theta,a_{-i}\mid t_i,\sigma_{-i}]=\int_{\mathcal{T}_{-i}}\sigma_{-i}(\theta,t_{-i})[a_{-i}]t_i(\theta\times dt_{-i}) \qquad \forall(\theta,a_{-i})\in\Theta\times A_{-i,N}.$$

To prove our first claim, namely that (28) is valid for $k=0$, fix any $t_2\in\mathcal{T}_2$ and $0\le n\le N$, assume that $a_{2,n}\in R_2^1(t_2,G_N,\delta)$, and let $\sigma_1:\Theta\times\mathcal{T}_1\to\Delta(A_{1,N})$ be a corresponding 0-order δ-rationalizable conjecture. Since $a_{2,n}$ is a δ-best reply to σ_1, the difference in expected payoff when choosing s_2 instead of $a_{2,n}$ under σ_1 must be at most δ, hence

$$\mathsf{Pr}[\theta_n,a_{1,n}\mid t_2,\sigma_1]+\mathsf{Pr}[\theta_1,a_{1,n+1}\mid t_2,\sigma_1]\ge 1-\delta/M. \tag{29}$$

Similarly, the difference in expected payoff when choosing $b_{2,n}$ or $c_{2,n}$ instead of $a_{2,n}$ under σ_1 must be at most δ, hence

$$-\delta \;\le\; \frac{M}{4}\mathsf{Pr}[\theta_n, a_{1,n} \mid t_2, \sigma_1] - \frac{M}{2}\mathsf{Pr}[\theta_1, a_{1,n+1} \mid t_2, \sigma_1] \;\le\; \delta.$$

The latter inequalities together with (29) imply

$$\mathsf{Pr}[\theta_n, a_{1,n} \mid t_2, \sigma_1] \ge 2/3 - 2\delta/M, \qquad \mathsf{Pr}[\theta_1, a_{1,n+1} \mid t_2, \sigma_1] \ge 1/3 - 2\delta/M, \tag{30}$$

hence $t_2^1[\theta_n] \ge 2/3 - 2\delta/M$ and $t_2^1[\theta_1] \ge 1/3 - 2\delta/M$. Moreover, if $n > 0$, then (29) implies $t_2^1[\theta_1] \ge 1 - 2\delta/M$. Thus, $d_2^1(t_2, u_{2,n}) \le 2\delta/M$, as (28) requires for $k = 0$.

To prove our second claim, namely that (28) implies (27) for all $0 \le k \le N$, fix any such k, any $t_1 \in \mathcal{T}_1$ and any $0 \le n \le N$, assume that $a_{1,n} \in R_1^{2(k+1)}(t_1, G_N, \delta)$, and let $\sigma_2 : \Theta \times \mathcal{T}_2 \to \Delta(A_{2,N})$ be a corresponding $(2k+1)$-order δ-rationalizable conjecture. First consider the case $n = 0$. Since $a_{1,0}$ is a δ-best reply to σ_2, it must give an expected payoff within δ of the one from s_1, hence

$$\mathsf{Pr}[\theta_0, a_{2,0} \mid t_1, \sigma_2] \ge 1 - \delta/M \ge 1 - 2\delta/M.$$

Since σ_2 is $(2k+1)$-order δ-rationalizable for t_1, from (28) we thus obtain

$$t_1^{2(k+1)}\left[\theta_0 \times \{u_{2,0}^{2k+1}\}^{2\delta/M}\right] \ge 1 - 2\delta/M,$$

as required by (27) when $n = 0$. Next consider the case $n > 0$. Since $a_{1,n}$ is a δ-best reply to σ_2, it must give an expected payoff within δ of the one from s_1, hence

$$\mathsf{Pr}[\theta_1, a_{2,n-1} \mid t_1, \sigma_2] + \mathsf{Pr}[\theta_1, a_{2,n} \mid t_1, \sigma_2] \ge 1 - \delta/M.$$

Similarly, comparing $a_{1,n}$ to $b_{1,n}$ and $c_{1,n}$ we must have

$$-\delta \le \frac{M}{4}\mathsf{Pr}[\theta_1, a_{2,n-1} \mid t_1, \sigma_2] - \frac{M}{2}\mathsf{Pr}[\theta_1, a_{2,n} \mid t_1, \sigma_2] \le \delta.$$

The latter three inequalities together imply

$$\mathsf{Pr}[\theta_1, a_{2,n-1} \mid t_1, \sigma_2] + \mathsf{Pr}[\theta_1, a_{2,n} \mid t_1, \sigma_2] \ge 1 - 2\delta/M, \tag{31}$$

$$\mathsf{Pr}[\theta_1, a_{2,n-1} \mid t_1, \sigma_2] \ge 2/3 - 2\delta/M, \tag{32}$$

$$\mathsf{Pr}[\theta_1, a_{2,n} \mid t_1, \sigma_2] \ge 1/3 - 2\delta/M. \tag{33}$$

Since σ_2 is $(2k+1)$-order δ-rationalizable for t_1, by (28) we have $\sigma_2(\theta_1, t_2)[a_{2,n-1}] = 0$ for all $t_2 \in \mathcal{T}_2$ such that $d_2^{2k+1}(t_2, u_{2,n-1}) > 2\delta/M$ and $\sigma_2(\theta_1, t_2)[a_{2,n}] = 0$ for all $t_2 \in \mathcal{T}_2$ such that $d_2^{2k+1}(t_2, u_{2,n}) > 2\delta/M$. By (31), (32) and (33) this implies

$$t_1^{2(k+1)}\left[\theta_1 \times \{u_{2,n-1}^{2k+1}, u_{2,n}^{2k+1}\}^{2\delta/M}\right] \ge 1 - 2\delta/M,$$

$$t_1^{2(k+1)}\left[\theta_1 \times \{u_{2,n-1}^{2k+1}\}^{2\delta/M}\right] \ge 2/3 - 2\delta/M,$$

$$t_1^{2(k+1)}\left[\theta_1 \times \{u_{2,n}^{2k+1}\}^{2\delta/M}\right] \ge 1/3 - 2\delta/M,$$

as required by (27) when $n > 0$.

There remains to prove our third claim. Assuming (27) for some $0 \le k < N$, we must show that (28) remains valid when k is replaced by $k+1$. Pick any $t_2 \in \mathcal{T}_2$ and $0 \le n \le N-k-1$, assume that $a_{2,n} \in R_2^{2(k+1)+1}(t_2, G_N, \delta)$ and let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,N})$ be a corresponding $2(k+1)$-order δ-rationalizable conjecture. Since $a_{2,n}$ is a δ-best reply to σ_1, the difference in expected payoff when choosing s_2 or $b_{2,n}$ or $c_{2,n}$ instead of $a_{2,n}$ under σ_1 must be at most δ. Thus, as before, (29) and (30) must hold. Moreover, since σ_1 is $2(k+1)$-order δ-rationalizable for t_2, by (27) we have $\sigma_1(\theta_n, t_1)[a_{1,n}] = 0$ for all $t_1 \in \mathcal{T}_1$ with $d_1^{2(k+1)}(t_1, u_{1,n}) > 2\delta/M$ and $\sigma_1(\theta_1, t_1)[a_{1,n+1}] = 0$ for all $t_1 \in \mathcal{T}_1$ with $d_1^{2(k+1)}(t_1, u_{1,n+1}) > 2\delta/M$. This implies

$$t_2^{2(k+1)+1}\Big[\theta_n \times \{u_{1,n}^{2(k+1)}\}^{2\delta/M}\Big] \ge 2/3 - 2\delta/M,$$
$$t_2^{2(k+1)+1}\Big[\theta_1 \times \{u_{1,n+1}^{2(k+1)}\}^{2\delta/M}\Big] \ge 1/3 - 2\delta/M,$$

and if $n > 0$ also

$$t_2^{2(k+1)+1}\Big[\theta_1 \times \{u_{1,n}^{2(k+1)}, u_{1,n+1}^{2(k+1)}\}^{2\delta/M}\Big] \ge 1 - 2\delta/M,$$

as required by (28) when k is replaced by $k+1$.

A.5 Proof of Lemma 4

Fix $\varepsilon \ge 0$ and note that (21) and (22) are trivially true for $k = 0$. Now we assume they are true for some $k \ge 0$ and prove that they hold for $k+1$. Note that since (22) holds for k, there exists a mapping $\xi : \mathcal{T}_2 \times A_{2,N} \to \{0,1\}$ satisfying

$$(a_2, \xi(t_2, a_2)) \in R_2^k(t_2, G'_N, \varepsilon) \qquad \forall t_2 \in \mathcal{T}_2,\ \forall a_2 \in R_2^k(t_2, G_N, 2\varepsilon). \tag{34}$$

Let us prove (21) for $k+1$ now. Fix any $t_1 \in \mathcal{T}_1$ and $a_1 \in R_1^{k+1}(t_1, G_N, 2\varepsilon)$ and let $\sigma_2 : \Theta \times \mathcal{T}_2 \to \Delta(A_{2,N})$ be a corresponding k-order 2ε-rationalizable conjecture. Define the conjecture $\sigma'_2 : \Theta \times \mathcal{T}_2 \to \Delta(A'_{2,N})$ for game G'_N as follows:

$$\sigma'_2(\theta, t_2)[a_2, \xi(t_2, a_2)] = \sigma_2(\theta, t_2)[a_2] \qquad \forall \theta \in \Theta,\ \forall t_2 \in \mathcal{T}_2,\ \forall a_2 \in A_{2,N}.$$

By (34), σ'_2 is a k-order ε-rationalizable conjecture for t_1. Moreover, the difference in expected payoff for t_1 between any $a'_1 \in A'_{1,N}$ and a_1 under σ'_2 in game G'_N is

$$\int_{\Theta\times\mathcal{T}_2^k} \Big[g'_{1,N}(a'_1, \sigma'_2(\theta, t_2), \theta) - g'_{1,N}(a_1, \sigma'_2(\theta, t_2), \theta)\Big] t_1^{k+1}(d\theta \times dt_2^k) =$$
$$= \frac{1}{2}\int_{\Theta\times\mathcal{T}_2^k} \Big[g_{1,N}(a'_1, \sigma_2(\theta, t_2), \theta) - g_{1,N}(a_1, \sigma_2(\theta, t_2), \theta)\Big] t_1^{k+1}(d\theta \times dt_2^k) \le \frac{1}{2} 2\varepsilon = \varepsilon,$$

where the inequality follows from the fact that $a_1 \in R_1^{k+1}(t_1, G_N, 2\varepsilon)$. This proves that $a_1 \in R_1^{k+1}(t_1, G'_N, \varepsilon)$, and we have thus shown that $R_1^{k+1}(t_1, G_N, 2\varepsilon) \subseteq R_1^{k+1}(t_1, G'_N, 2\varepsilon)$. Conversely, pick any $a_1 \in R_1^{k+1}(t_1, G'_N, \varepsilon)$ and let $\sigma'_2 : \Theta \times \mathcal{T}_2 \to \Delta(A'_{2,N})$ be a corresponding k-order ε-rationalizable conjecture. Define $\sigma_2 : \Theta \times \mathcal{T}_2 \to \Delta(A_{2,N})$ as

$$\sigma_2(\theta, t_2) = \mathsf{marg}_{A_{2,N}}\, \sigma'_2(\theta, t_2) \qquad \forall\, \theta \in \Theta,\ \forall\, t_2 \in \mathcal{T}_2.$$

Since (22) holds for k, this is a k-order 2ε-rationalizable conjecture for t_1 in G_N. Moreover, the difference in expected payoff for t_1 between any $a'_1 \in A_{1,N}$ and a_1 under σ_2 in game G_N is

$$\int\limits_{\Theta \times \mathcal{T}_2^k} \Big[g_{1,N}(a'_1, \sigma_2(\theta, t_2), \theta) - g_{1,N}(a_1, \sigma_2(\theta, t_2), \theta)\Big] t_1^{k+1}(d\theta \times dt_2^k) =$$

$$= 2 \int\limits_{\Theta \times \mathcal{T}_2^k} \Big[g'_{1,N}(a'_1, \sigma'_2(\theta, t_2), \theta) - g'_{1,N}(a_1, \sigma'_2(\theta, t_2), \theta)\Big] t_1^{k+1}(d\theta \times dt_2^k) \leq 2\varepsilon,$$

hence $a_1 \in R_1^{k+1}(t_1, G_N, 2\varepsilon)$. This shows that $R_1^{k+1}(t_1, G'_N, 2\varepsilon) \subseteq R_1^{k+1}(t_1, G_N, 2\varepsilon)$, so the proof of (21) for $k+1$ is complete.

Now we show that (22) also remains true for $k+1$, thus concluding the proof. Fix $t_2 \in \mathcal{T}_2$ and let $a_2 \in R_2^{k+1}(t_2, G_N, 2\varepsilon)$ and let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,N})$ be a corresponding k-order 2ε-rationalizable conjecture. Choose any

$$x^* \in \operatorname*{arg\,max}_{x \in \{0,1\}} \int\limits_{\Theta \times \mathcal{T}_1^k} g'_{2,N}(\sigma_1(\theta, t_1), a_2, x, \theta) t_2^{k+1}(d\theta \times dt_1^k).$$

Then the difference in expected payoff for t_2 between any $(a'_2, x) \in A'_{2,N}$ and (a_2, x^*) under σ_1 in game G'_N is

$$\int\limits_{\Theta \times \mathcal{T}_1^k} \Big[g'_{2,N}(\sigma_1(\theta, t_1), a'_2, x, \theta) - g'_{2,N}(\sigma_1(\theta, t_1), a_2, x^*, \theta)\Big] t_2^{k+1}(d\theta \times dt_1^k) \leq$$

$$\leq \int\limits_{\Theta \times \mathcal{T}_1^k} \Big[g'_{2,N}(\sigma_1(\theta, t_1), a'_2, x, \theta) - g'_{2,N}(\sigma_1(\theta, t_1), a_2, x, \theta)\Big] t_2^{k+1}(d\theta \times dt_1^k) =$$

$$= \frac{1}{2} \int\limits_{\Theta \times \mathcal{T}_1^k} \Big[g_{2,N}(\sigma_1(\theta, t_1), a'_2, \theta) - g_{2,N}(\sigma_1(\theta, t_1), a_2, \theta)\Big] t_2^{k+1}(d\theta \times dt_1^k) \leq \frac{1}{2} 2\varepsilon = \varepsilon,$$

hence $(a_2, x^*) \in R_2^{k+1}(t_2, G'_N, \varepsilon)$. This proves $R_2^{k+1}(t_2, G_N, 2\varepsilon) \subseteq \mathsf{proj}_{A_{2,N}} R_2^k(t_2, G'_N, \varepsilon)$. Conversely, let $(a_2, x) \in R_2^{k+1}(t_2, G'_N, \varepsilon)$ and let $\sigma'_1 : \Theta \times \mathcal{T}_1 \to \Delta(A'_{1,N})$ be a corresponding k-order ε-rationalizable conjecture. Then the difference in expected payoff for t_2 between any $a_2' \in A_{2,N}$ and a_2 under σ_1' in game G_N is

$$\int_{\Theta\times\mathcal{T}_1^k} \Big[g_{2,N}(\sigma_1'(\theta,t_1),a_2',\theta) - g_{2,N}(\sigma_1'(\theta,t_1),a_2,\theta)\Big] t_2^{k+1}(d\theta\times dt_1^k) =$$

$$= 2 \int_{\Theta\times\mathcal{T}_1^k} \Big[g_{2,N}'(\sigma_1'(\theta,t_1),a_2',x,\theta) - g_{2,N}'(\sigma_1'(\theta,t_1),a_2,x,\theta)\Big] t_2^{k+1}(d\theta\times dt_1^k) \le 2\varepsilon,$$

hence $a_2 \in R_2^{k+1}(t_2, G_N, 2\varepsilon)$. This proves $\mathsf{proj}_{A_{2,N}} R_2^k(t_2, G_N', \varepsilon) \subseteq R_2^{k+1}(t_2, G_N, 2\varepsilon)$, so the proof of (22) for $k+1$ is complete.

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