# Sensitivity of Pareto Solutions in Multiobjective Optimization

A. Balbás,[1] E. Galperin,[2] and P. Jiménez Guerra[3]

**Abstract.** The paper presents a sensitivity analysis of Pareto solutions on the basis of the Karush-Kuhn-Tucker (KKT) necessary conditions applied to nonlinear multiobjective programs (MOP) continuously depending on a parameter. Since the KKT conditions are of the first order, the sensitivity properties are considered in the first approximation. An analogue of the shadow prices, well known for scalar linear programs, is obtained for nonlinear MOPs. Two types of sensitivity are investigated: sensitivity in the state space (on the Pareto set) and sensitivity in the cost function space (on the balance set) for a vector cost function. The results obtained can be used in applications for sensitivity computation under small variations of parameters. Illustrative examples are presented.

**Key Words.** Sensitivity analysis, nonscalarized multiobjective programming, Pareto set, balance set.

## 1. Introduction

Since the notions of multiobjectives and of ideal point (the vector of global partial optima) were introduced in mathematical programming and control theory (Refs. 1–4), their importance to economy and engineering was recognized immediately in the research and literature that followed. Also, it became clear that, in reality, the multiobjectives were not

[1]Professor, Departamento de Economía de la Empresa, Universidad Carlos III, Madrid, Spain. Research of this author was partially supported by Grant BEC2003-09067-C04-03.
[2]Professor, Département de Mathématiques, Université du Québec à Montréal, Montréal, Québec, Canada. Research of this author was partially supported by NSERC Grant RGPIN-3492-00.
[3]Professor, Departamento de Matemáticas Fundamentales, Facultad de Ciencias, Universidad Nacional de Educación a Distancia, Madrid, Spain. Research of this author was partially supported by Grant BEC2003-09067-C04-02.

some fixed rigidly defined criteria, but rather they might contain uncertainties or depend on parameters (Refs. 5, 6). This necessitated studies on the sensitivity of multiobjective programs (Refs. 6–11), with respect to perturbations of objective functions, or utility functions, or the ordering cone (Ref. 8) to analyze the stability of solutions under certain assumptions like convexity (Ref. 10) or homogeneity (Ref. 11) or under some particular quantitative assumptions (Ref. 6).

Recently, the concepts of balance number, balance point, and balance set were introduced in Refs. 12–13 for vector optimization problems; several authors have analyzed their significant properties and have developed algorithms to compute them in practice (see Refs. 14–19 for further details).

Mainly, this approach yields a very general alternative method in vector optimization because multiobjective problems can be deeply analyzed by means of their ideal points rather than as scalarized problems. It is not necessary to seek appropriate weights to compute a balance point. Instead, one may choose a direction of preferential deviations from the ideal point in order to reach an optimal point.

Consequently, an interesting economic interpretation is possible, since the ideal point may be considered an adequate reference for the decision marker. Given an arbitrary balance point $b=(b_1, b_2, \dots, b_k), b_i$ is the difference between the final level attained in the $i$th objective and its ideal level, $i=1,2,\dots,k$; thus, a decision maker can choose another balance point when these differences are not acceptable. Furthermore, each quotient $b_i/b_j$ provides the number of units lost in the $i$th objective per unit lost in the $j$th one. When the problem is scalarized, the meaning of the weights is not so clear.

This nonscalarized procedure provides new algorithms which are very general. When we are choosing a concrete direction to detect a balance point, we are also choosing the ratios of gains and losses among the conflicting objectives.

As pointed out in Ref. 14, the set of Pareto solutions and the balance set are equivalent from a theoretical viewpoint, in the sense that there exists a simple relationship between these sets. Thus, balance set techniques apply also to study the Pareto solutions.

Both advantages, new nonscalarized algorithms and their economic interpretation, justify the interest of extending the discussion in order to address another important issue of vector optimization. So, this paper focuses on sensitivity, since it still presents many open questions when dealing with vector problems.

Regarding sensitivity, scalar problems have many deep properties whose extension to vector problems is not straightforward. This fact is pointed out clearly in Refs. 20–21. In order to overcome these difficulties, the recent literature (see for instance Refs. 3–11, 22–26) has developed new ideas and methods; we try to show here that the balance space approach may be a useful alternative and can broaden possible techniques. Since the adjoint Pareto set equals the ideal point $J$ plus the balance set [$P^* = f(P) = J + B$ (see Refs. 14–15)], the sensitivity of the balance set allows us to obtain the $P^*$ sensitivity by adding two terms. Moreover, this opens a way to study sensitivity of Pareto solutions $x \in P$ in the state space.

## 2. Statement of the Problem, Basic Concepts, and Hypotheses

Consider a global multiobjective program,

$$\inf \; f(x, p), \quad x \in X \subseteq \mathbb{R}^n, \quad f = (f_1, \ldots, f_k), \tag{1}$$

$$X = \left\{x \in R^n, \; g(x, p) \leq q, \; x \geq 0, \; q \in \mathbb{R}^m\right\}, \tag{2}$$

where $p$ is a vector of technological parameters and $q$ is a resource vector that may contain financial resources of different kinds, including deficits. We make the following assumption:

(A1) Each scalar problem

$$\inf\{f_i(x, p) : x \in X\} \tag{3}$$

attains its global optimal value at a single point $x(i, p) \in X, i = 1, 2, \ldots, k$; therefore, the ideal point or the set of global partial minima

$$J(p) = [f_1(x(1, p), p), f_2(x(2, p), p), \ldots, f_k(x(k, p), p)] \in \mathbb{R}^k$$

does exist.

To simplify notations, we denote

$$J_i(p) = f_i(x(i, p), p), \quad i = 1, 2, \ldots, k.$$

Following the approach of Ref. 15 or Ref. 16, an element $b \in \mathbb{R}^k, b \geq 0$, is said to be a balance point of (1) if

$$\{f(x) : x \in X\} \cap [J(p), J(p) + b] \neq \emptyset$$

and

$$\{f(x) : x \in X\} \cap [J(p), J(p) + b^*] = \emptyset,$$

for every $b^* \in \mathbb{R}^k$ such that[4]

$$0 \leq b^* \leq b, \quad b^* \neq b.$$

As pointed out in Ref. 15, $b \in \mathbb{R}^k$ is a balance point of (1) if and only if $J(p)+b$ is the value $f(x_p, p)$ at a Pareto point $x_p$.

Consider a direction vector $l=(l_1,\ldots,l_k) \in \mathbb{R}^k$, $l_i \geq 0$, $i=1,\ldots,k$, $l \neq 0$. With this $l$, which is supposed to sweep the whole $\mathbb{R}^k_+$, the original problem (1)–(2) can be represented by the following equivalent problem whose decision variables are $\tau \in \mathbb{R}_+$ and $x \in X \subseteq \mathbb{R}^n$:

$$\min \ \tau, \tag{4a}$$

$$\text{s.t.} \quad x \in X = \{x \in R^n, g(x,p) \leq q, \ x \geq 0, \ q \in \mathbb{R}^m\}, \tag{4b}$$

$$f(x,p) - J(p) \leq \tau l, \quad \tau \geq 0. \tag{4c}$$

Indeed, if $x_0$ is a solution of (1)–(2) in some sense, then fixing $l^0 > 0$, we can find $\min \tau \geq 0$ for which at least one of the relations in (4c) is an equality. Alternatively, fixing $\tau_0 > 0$, we can find $l \in \mathbb{R}^k_+$ for which at least one of relations in (4c) is an equality; thus, $\min \tau = \tau_0$ is attained. Viceversa, for sufficiently large $\tau$ and any fixed $l^0 > 0$, the feasible set (4b)–(4c) is nonempty, so that decreasing $\tau$ to some $\tau_0 = \min \tau$ will yield the solution $(\tau_0, x_0)$ with one or many points $x_0$ if, for $\tau < \tau_0$, the feasible set (4b)–(4c) is empty. In this equivalence, the possible nonuniqueness of corresponding solutions is a positive phenomenon inherent to multiobjective programs.

Suppose that $\tau = 0$ is a solution of (4). Since

$$J_i(p) = \inf f_i(x,p), \quad x \in X,$$

due to (A1), so

$$f_i(x,p) < J_i(p)$$

is impossible; thus, with $\tau = 0$, instead of (4c), we have the equality

$$f(x,p) = J(p).$$

It means that one and the same $x(p) \in X$ renders the global minimum value $J_i(p)$ for every $i = 1,\ldots,k$; thus, problem (1)–(2) is in fact a scalar program with different $f_i(x,p)$ attaining $\inf f_i(x,p)$ at the same point $x(p) \in X$. Such MOP is called balanced (Refs. 12, 13).

For unbalanced problems, the set (4b)–(4c) is empty for $\tau = 0$. To obtain a solution of (4), hence the corresponding solution of (1)–(2) in

[4]If $u, v \in \mathbb{R}^k$, with $u \leq v$, then $[u,v]$ denotes the set $\{x \in \mathbb{R}^k : u \leq x \leq v\}$ coordinatewise.

the direction $l$, we have to increase gradually $\tau$ until first time when the set (4b)–(4c) becomes nonempty. Clearly, for different directions $l$, the first nonempty set will contain different points. For some of those directions, this set will contain Pareto points [note that Pareto solutions always exist for any MOP; for example, the points $x(i, p)$ in (A1) are Pareto solutions]. We make the following assumption.

(A2) If the first nonempty set of (4b)–(4c) produced with increasing $\tau$ in the direction $l$ contains a Pareto point, then this set is a singleton.

This is the hypothesis of general position for nonlinear MOPs; see Hypothesis 3.1 in Ref. 17, p. 120.

In this research, we consider only those directions $l \in \mathbb{R}^k_+$ that produce with increasing $\tau$ a single Pareto point as the first nonempty set appears in (4b)–(4c). All such points form the Pareto set for (1)–(2) or (4), which (by definition) is deemed to be the optimal solution for (1)–(2) or (4); see Definition 3.1 in Ref. 17, p. 121, which is wider and admits also non-Pareto optimal solutions. Of course, different Pareto solutions may be of different quality with respect to other criteria, but this is another problem.

When the direction vector $l \in \mathbb{R}^k_+$ sweeps the whole space $\mathbb{R}^k_+$, the program (4) renders all feasible solutions of the initial program (1)–(2), By construction, the optimal solutions of (4) are the closest possible to the ideal vector $J(p)$. Therefore, it is reasonable to take the globally optimal solutions of (4) as optimal, by definition, for the initial problem (1)–(2).

Since problem (4) is a scalar program for each fixed $l \in \mathbb{R}^k_+$, we can apply to it the KKT necessary conditions for a local minimum. In this way, the $k$-parameter family of the KKT necessary conditions is obtained for the $k$-parameter family of programs (4), hence also for the single original MOP (1)–(2). Without loss of generality, the leading $k$-vector parameter $l \in \mathbb{R}^k_+$ can be normalized conveniently; e.g. $\|l\| = 1$.

Among all the minima of (4), there is the global one inf $\tau$. We make the following assumption.

(A3) For each fixed triplet $(p, q, l)$, the Karush-Kuhn-Tucker (KKT) necessary conditions are applicable to problem (4). The global solution $(\tau^0, x^0)$ of (4) is unique and is contained among the stationary points $(\tau^*, x^*)$ defined by the KKT conditions.

This assumption holds in many practical problems. The exact conditions (e.g., convexity) under which it is valid will be investigated elsewhere.

For nonlinear programs, the sensitivity investigations are based on partial derivatives (see e.g. Refs. 7 and 9–11); so, we make the following blanket assumption.

(A4) The functions in (1)–(4), including the stationary points $\tau^*(p,q,l)$, are continuous and have continuous partial derivatives.

In the sequel, the following basic lemma will be useful.

**Lemma 2.1.** If a global optimal solution $(\tau^0, x^0)$ is Pareto, then the relations (4c) are all equalities at this point.

**Proof.** By a theorem in Ref. 14, we have

$$f(P(p),p) = J(p) + B(p), \quad \forall p, \tag{5}$$

or in coordinate form,

$$f_i(x_p,p) = J_i(p) + b_i(p), \quad x_p \in P, \ i = 1, \ldots, k, \tag{6}$$

where $P(p)$ is the Pareto set corresponding to a fixed vector-parameter $p$, $B(p)$ is the balance set, and $J(p)$ is the ideal point [vector of global partial minima in (3)].

By construction, with increasing $\tau$, the first nonempty set (4b)–(4c) will have equality in at least one of relations (4c). Let it be the first one,

$$f_1(x_p,p) - J_1(p) = \tau^0(p) l_1, \quad x_p = x^0. \tag{7}$$

Since $x_p \in P$, we have by (6)

$$f_1(x_p,p) = J_1(p) + b_1(p); \tag{8}$$

thus,

$$\tau^0(p) l_1 = b_1(p),$$

where $b_1$ is the first coordinate of the balance point $b = (b_1, \ldots, b_k) \in B$.

By construction and since we consider only the rays $\tau l$ intersecting the balance set $B$, we have $\tau^0(p) l = b(p)$; thus, the relations (4c) at the Pareto points can be written as

$$f(x_p,p) - J(p) \le \tau^0(p) l = b(p). \tag{9}$$

Comparing (6) and (9) proves the lemma. □

## 3. Lagrangian Formulation and KKT Conditions

For the $k$-parameter family of problems (4), we can write the corresponding $k$-parameter family of Lagrangian functions with $l_i$, $(i = 1, \ldots, k)$, as independent parameters,

$$L(\tau, x, p, q, \mu, \lambda, v, l) = \tau + \sum_{j=1}^{m} \mu_j [g_j(x, p) - q_j] - \sum_{s=1}^{n} \lambda_s x_s + \sum_{i=1}^{k} \nu_i [f_i(x, p) - \tau l_i - J_i(p)]. \quad (10)$$

Note that, if $x_p \in P$ is a Pareto solution (thus, globally optimal), then due to the complementarity conditions (see below) and the equalities in (4c) (Lemma 2.1), we have

$$L^0(\cdot) = \tau^0(p, q, l) = \min \tau, \quad (11)$$

$$b = \tau^0(p, q, l) l = f(x_p, p) - J_p, \quad b \in B, \quad (12)$$

where $B$ is the balance set for (1)–(2).

Denote by $x^*(p), \tau^*(p) = \min \tau$ an optimal solution of (4) corresponding to a fixed value of the parameter $p$. Then, by the Karush-Kuhn-Tucker theorem, the following conditions must hold at the point $(x^*, \tau^*)$:

$$\partial L / \partial \tau = 1 - \sum_{i=1}^{k} \nu_i^* l_i = 0, \quad (13)$$

$$\partial L / \partial x_s = \sum_{j=1}^{m} \mu_j^* \, \partial g_j / \partial x_s + \sum_{i=1}^{k} \nu_i^* \, \partial f_i / \partial x_s - \lambda_s^* = 0, \quad s = 1, \ldots, n, \quad (14)$$

$$\partial L / \partial \mu_j = g_j(x^*, p) - q_j \leq 0, \quad j = 1, \ldots, m, \quad (15)$$

$$\partial L / \partial \lambda_s = -x_s^* \leq 0, \quad s = 1, \ldots, n, \quad (16)$$

$$\partial L / \partial \nu_i = f_i(x^*, p) - \tau^*(p) l_i - J_i(p) \leq 0, \quad i = 1, \ldots, k, \quad (17)$$

$$\mu_j^* [g_j(x^*, p) - q_j] = 0, \quad j = 1, \ldots, m, \quad (18)$$

$$\lambda_s^* x_s^* = 0, \quad s = 1, \ldots, n, \quad (19)$$

$$\nu_i^* [f_i(x^*, p) - \tau^*(p) l_i - J_i(p)] = 0, \quad i = 1, \ldots, k, \quad (20)$$

$$\mu_j^* \geq 0, \ \lambda_s^* \geq 0, \ \nu_i^* \geq 0, \quad j = 1, \ldots, m, \ s = 1, \ldots, n, \ i = 1, \ldots, k. \quad (21)$$

The existence of Lagrange multipliers $\mu_j^*, \lambda_s^*, \nu_i^*$ is included in Assumption (A3). The relations (13), (14), (18), (19), (20) include $1 + 2n + m + k$ equations for the same number of unknowns $\tau^*, x_s^*, \mu_j^*, \lambda_s^*, \nu_i^*$. By

Assumption (A3), these equations have one or more solutions satisfying (15)–(17), (21) and called stationary points. Some of these deliver local minima for problem (4); by Assumption (A3), one and only one of them yields the unique global $\min \tau = \tau^0(p)$. This solution $(x^0, \tau^0)$ defines a Pareto point $x^0(p)$, the balance point $b = \tau^0(p)l$ in the direction $l \in \mathbb{R}^k$, and the actual values of the cost functions [cf. (12)]

$$f_i(x^0(p), p) = \tau^0(p)l_i + J_i(p), \quad i = 1, \dots, k, \tag{22}$$

globally optimal in the direction $l = (l_1, l_2, \dots, l_k)$. To this solution, there correspond certain values $\mu_j^0, \lambda_s^0, v_i^0$ of the Lagrange multipliers.

Taking the differential of the Lagrangian in (10), we have

$$\begin{aligned} dL = d\tau + d\sum_{j=1}^{m} \mu_j[g_j(x, p) - q_j] - d\sum_{s=1}^{n} \lambda_s x_s \\ + d\sum_{i=1}^{k} v_i[f_i(x, p) - \tau l_i - J_i(p)]. \end{aligned} \tag{23}$$

**Theorem 3.1.** If $x^0 \in P$ is a Pareto solution, then

$$dL^0 = d\tau^0(p, q, l) = v^0 db, \quad b \in B, \tag{24}$$

$$db = l\, dL^0, \quad \forall p, \ \forall q. \tag{25}$$

**Proof.** On a Pareto set, the entries in (23) become composite functions, since $x = x^0(p, q, l) \in P$ and also $\tau = \tau^0(p, q, l)$ for a stationary point produced by the KKT conditions. By the theorem of invariance of the form of the first differential, the formula (23) remains valid if a stationary point (e.g., global optimal solution) travels on a subset of the feasible set (on Pareto set). Now, using the complementarity conditions (18)–(20) for the global optimal solutions $x^0(p, q, l)$, we get the first equality in (24).

The second equality of (24) is obtained as follows. Moving along the subset (18)–(19), cancel out the second and third terms on the right in (23) by the complementarity conditions (18)–(19). Then, taking the differential from the last term in (23) yields

$$dL^0 = d\tau^0 + dv^0[f(x^0, p) - \tau^0 l - J(p)] + v^0[df^0 - d\tau^0 l - dJ]. \tag{26}$$

The first bracket in (26) vanishes by Lemma 2.1. Opening the second bracket, we have

$$dL^0 = d\tau^0 + v^0[df^0 - dJ] - d\tau^0(v^0 l). \tag{27}$$

Since $v^0 l = 1$ by (13), cancelling $d\tau^0$ from (27), and noting that $df^0 - dJ = db$, due to (12), we get $dL^0 = v^0 db$ in (24). Now, multiplying this by $l$ on the left and again using (13), we get $l dL^0 = db$ of (25). □

Here and in the sequel, the star marks stationary points and multipliers satisfying the KKT conditions. The superscript $^0$ denotes the global optimal solution (Pareto) and the corresponding multipliers, which all necessarily satisfy the KKT conditions. For clarity, we indicate only those parameters which are of interest; e.g. $x^0, x^0(p), x^0(q), x^0(p, q, l)$ represent one and the same Pareto solution.

## 4. Optimal Lagrangian and Sensitivity of the Balance Set

The Lagrangian (10) can be considered as a $(2 + 2n + N_p + 2m + 2k)$-dimensional surface, where $N_p$ is the number of different technological parameters $(p_1, \ldots, p_{N_p}) = p$. The resource parameters $q_j$ are singled out because of their different economical sense; however, in some problems, $q_j$ can be conveniently included in the vector $p$ as some of its coordinates. Clearly, equality constraints are also included in (10), (15)–(17) as e.g. pairs of opposite inequality constraints.

By Assumption (A4), the surface (10) is continuous and admits continuous partial derivatives of all orders that we may need, with respect to any entries considered as independent variables. When this surface passes through stationary points satisfying the KKT conditions (13)–(21), it shrinks continuously into a less-dimensional surface, due to the complementarity relations (18)–(20). For the Pareto points, the surface (10) becomes a $(1 + N_p + m + k)$-dimensional surface given by (11), with directions $l$ restricted to intersect the balance set. We call this surface (11) the optimal Lagrangian $L^0(\cdot)$. Following classical traditions, we consider first the sensitivity with respect to the resource vector $q$. For the nonoptimal surface (10), we have the partial differential ($\nabla$ denotes a gradient)

$$d_q L = \nabla_q L dq = -\mu dq = -\sum_{j=1}^{m} \mu_j dq_j, \tag{28}$$

with sensitivities

$$\partial L / \partial q_j = -\mu_j, \tag{29}$$

that represent the shadow prices for the resources $q_j$.

Due to the complementarity conditions (18)–(20), all sums in (10) tend to zero as $x(p,q,l)$ tends to $x^*(p,q,l)$. Therefore, (10) reduces to (11) with the differential

$$d_q L^* = \nabla_q \tau^* dq = \sum_{j=1}^{m} (\partial \tau^* / \partial q_j) dq_j. \tag{30}$$

By the continuity of the partial derivatives, we have

$$\lim_{x \to x^*} d_q L = d_q L^*, \tag{31}$$

$$\lim_{x \to x^*} \partial L / \partial q_j = \partial L^* / \partial q_j = \partial \tau^* / \partial q_j = -\mu_j^*. \tag{32}$$

Since instead of $q_j, q$ we can consider any parameter or group of parameters in (10), this continuity argument proves the following fact.

**Theorem 4.1.** Sensitivity of Stationary Values. For a stationary value $\tau^*(p,q,l)$, we have

$$\partial \tau^* / \partial \beta = \partial L / \partial \beta|_{x=x^*}, \quad d_\beta \tau^* = d_\beta L|_{x=x^*}, \tag{33}$$

where $\beta$ is a parameter or a group of parameters [for the differential in (33)] from the $(N_p + m + k)$-vector $[p,q,l]$.

Note that $\tau^*(p,q,l)$ does not have to be the global optimal value. It can be a local optimal value or a nonoptimal value associated with a stationary point $x^*$ of the KKT conditons.

Theorem 4.1 can be used as follows. The computation of $\partial \tau^* / \partial \beta$ may be difficult if $\tau^* = \tau^*(p,q,l) = \tilde{\tau}^*(\beta)$ is not representable as a formula, so that $\partial \tau^* / \partial \beta$ must be approximated by neighboring solutions of problem (4). Instead of such tedious computations, one can calculate $\partial L / \partial \beta$ using the nonoptimal Lagrangian (10), then solve (4) just once and put approximate values of $x^*, \mu^*, v^*$ (as needed) into $\partial L / \partial \beta$. For example,

$$\partial \tau^* / \partial q_j = -\mu_j^*, \quad d_q \tau^* = -\mu^* dq.$$

**Application to Scalar Nonlinear Programming.** In this case, $k=1$; thus, $l_1 = 1$ and by (13) we have $v_1^* = 1$. Dropping the subscript, we get from (20)

$$f(x^*, p) - \tau^*(p,q) - J(p,q) = 0, \tag{34}$$

and since the global minimum value $J(p)$ is attained by Assumption (A1), so $\tau^* = 0$ and the problem (1)–(2) is in fact scalar with the Lagrangian

$$L(x, p, q, \mu, \lambda) = f(x, p) + \mu[g(x, p) - q] - \lambda x. \tag{35}$$

The first relation in (33) takes the form

$$\partial J(p, q)/\partial\beta = \partial f(x^*(p, q), p)/\partial\beta = \partial L/\partial\beta|_{x=x^*(p,q)}. \tag{36}$$

In particular,

$$\partial J/\partial q_j = -\mu_j^*.$$

Formula (36) coincides with the so-called envelope theorem in scalar programming; see e.g. Ref. 26.

**Example 4.1.** Consider the problem

$$\min\ f = x_2 - x_1^p, \quad p \geq 0, \tag{37}$$

$$\text{s.t.}\quad x_1 + x_2 \leq q, \quad x_1 \geq 0,\ \ x_2 \geq 0,\ \ q > 0. \tag{38}$$

The optimal solution is obvious; $x_2 = 0, x_1 = q$ if $p > 0$ and $x_1 \in [0, q]$ if $p = 0$, which is the bifurcation point with respect to the solution set.

The solution by the KKT theorem is as follows. The Lagrangian is

$$L = x_2 - x_1^p + \mu(x_1 + x_2 - q) - \lambda_1 x_1 - \lambda_2 x_2. \tag{39}$$

The KKT conditions are, for $p > 0$,

$$\partial L/\partial x_1 = -p x_1^{p-1} + \mu - \lambda_1 = 0, \quad \mu \geq 0,\ \ \lambda_1 \geq 0, \tag{40}$$

$$\partial L/\partial x_2 = 1 + \mu - \lambda_2 = 0, \quad \lambda_2 \geq 0, \tag{41}$$

$$\partial L/\partial\mu = x_1 + x_2 - q \leq 0, \quad x_1 \geq 0,\ \ x_2 \geq 0, \tag{42}$$

while the complementarity conditions are

$$\mu(x_1 + x_2 - q) = 0, \quad \lambda_1 x_1 = 0, \quad \lambda_2 x_2 = 0. \tag{43}$$

From (40)–(41), we have

$$\mu = \lambda_1 + p x_1^{p-1} = \lambda_2 - 1. \tag{44}$$

Case 1. $\mu = 0, \lambda_2 = 1 > 0, x_2 = 0, \lambda_1 = -p x_1^{p-1}, \lambda_1 x_1 = -p x_1^p = 0, x_1 = 0.$ Thus,

$$x^* = (0, 0), \quad f(x^*, p) = f(0, 0) = 0.$$

Case 2. $\mu > 0, \lambda_2 = 1 + \mu > 0, x_2 = 0, x_1 = q$; thus, $x^* = (q, 0); \lambda_1 x_1 = (\mu - pq^{p-1})q = 0, \mu = pq^{p-1}, f(q, 0) = -q^p < f(0, 0)$; thus, $x^0 = (q, 0)$, the global minimizer.

Let us verify (36). For $x^0 = (q, 0)$, we have

$$\partial f/\partial q = -pq^{p-1}, \qquad \partial L/\partial q|_{x^0} = -\mu = -pq^{p-1}, \tag{45}$$

$$\partial f/\partial p = -pq^p \log q, \quad \partial L/\partial p|_{x^0} = -p x_1^{0p} \log x_1^0 = -pq^p \log q. \tag{46}$$

For Case 1, where $x^* = (0, 0)$ is a stationary point, we have $f(x^*) = 0$, and

$$\partial f/\partial q = \partial f/\partial p = 0, \quad \partial L/\partial q_{|x^*} = -\mu = 0,$$
$$\partial L/\partial p|_{x \to x^*} = -p \lim_{x_1 \to 0} x_1^p \log x_1 = 0.$$

For $p = 0$, we have min $f = x_2 - 1$ in (37), the Lagrangian of (39) does not contain the term $-x_1^p$, and the solution of (40)–(43) with $q > 0$ is as follows:

$$\mu = \lambda_1 = 0, \ \ \lambda_2 = 1, \ \ x_2^* = 0, \ \ x_1^* \in [0, q], \ \ \min f = -1.$$

We have

$$\partial f/\partial q = 0, \quad \partial L/\partial q_{x^*} = -\mu = 0.$$

Note that this result follows from (45) as $p \to 0$, since the cost function in (37) is continuous at $p = 0$.

In order to show how the theory applies for multiobjective problems, let us add the second objective

$$\min f^* = x_1$$

to Problem (37)–(38). Then, the ideal point is $(-q^p, 0)$ and (4a)–(4c) become

$$\begin{aligned}
&\min\ \tau,\\
&\text{s.t.}\ \ x_1 + x_2 \le q,\\
&\qquad x_1 \ge 0,\\
&\qquad x_2 \ge 0,\\
&\qquad x_2 - x_1^p - \tau \le -q^p,\\
&\qquad x_1 - \tau \le 0.
\end{aligned}$$

The Lagrangian (10) becomes

$$L = \tau + \mu(x_1 + x_2 - q) - \lambda_1 x_1 - \lambda_2 x_2 + \nu_1(x_2 - x_1^p - \tau + q^p) + \nu_2(x_1 - \tau).$$

The KKT conditions (13)–(21) become

$$\nu_1 + \nu_2 = 1,$$
$$\mu - \lambda_1 - \nu_1 p x_1^{p-1} + \nu_2 = 0,$$
$$\mu - \lambda_2 + \nu_1 = 0,$$
$$\mu(x_1 + x_2 - q) = 0,$$
$$\lambda_1 x_1 = \lambda_2 x_2 = 0,$$
$$\nu_1(x_2 - x_1^p - \tau + q^p) = 0,$$
$$\nu_2(x_1 - \tau) = 0,$$

with the solution

$$\nu_1 = 1, \ \ \nu_2 = 0, \ \ \mu = 0, \ \ \lambda_1 = 0, \ \ \lambda_2 = 1, \ \ x_1 = 0, \ \ x_2 = 0, \ \ \tau = q^p.$$

Clearly,

$$\partial\tau/\partial p = \partial q^p/\partial p = q^p \log q,$$

that coincides with $\partial L/\partial p$ if one bears in mind the expression above of $L$ and the values $x_1 = x_2 = 0$. In a similar way, one can check that Theorem 4.1 also holds when considering the objective $f^* = x_1$; consequently, (6) shows that Theorem. 4.1 holds for the multiobjective problem too.

**Remark 4.1.** It is worth noticing that Theorem 4.1 follows from the continuity of the Lagrangian and its derivatives and from the complementarity conditions. It does not require even the stationarity implied by the KKT conditions, from which it is usually derived in the literature. It means that Theorem 4.1 applies to a much larger class of problems than the Karush-Kuhn-Tucker theorem.

Combining Theorems 3.1 and 4.1, we obtain the sensitivity of the balance set and an estimate for finite variations of the global optimal value $\tau^0$ with respect to variations of parameters and to displacements of $x^0$ within Pareto set.

**Theorem 4.2.** If $x^0 \in P$, a Pareto solution, then to the first order,

$$db = l \lim_{x \to x^0} dL, \quad b \in B \tag{47}$$

$$d\tau^0 = v^0 l \lim_{x \to x^0} dL, \tag{48}$$

where $v^0 l$ is the scalar product and the variations db, $d\tau^0$ can be considered with respect to small perturbations of any collection of parameters.

**Proof.** It follows from Theorem 3.1 by continuity when moving within the Pareto set □

Suppose that all the parameters are fixed and only one (say $p_1$) varies.

**Theorem 4.3.** If $x^0(p_1)$ is a Pareto solution and $\tau^0 > 0$, then the total derivative is

$$df(x^0(p_1), p_1)/dp_1 = dJ/dp_1 + l \lim_{x \to x^0(p_1)} \partial L/\partial p_1. \tag{49}$$

**Proof.** By Lemma 2.1, all the relations (4c) are equalities for a Pareto point. Differentiating these equalities with respect to $p_1$ [total derivative for $f(x^0(p_1), p_1)$], we get

$$df^0/dp_1 = dJ/dp_1 + l(d\tau^0/dp_1), \quad \tau^0 > 0. \tag{50}$$

Setting $\beta = p_1$ in (33), and noting that $\tau$ is not a composite function of $p_1$, we get

$$\partial\tau^0/\partial p_1 = d\tau^0/dp_1$$

and (49) follows from (50) and (33). □

## 5. Sensitivity of the Global Minimizers (Pareto Solutions)

For simplicity, we assume that all the parameters are fixed and only one (say $p_1$) in (49) varies. Dropping the index, denote the coordinates of the vector on the right-hand side of (49) as follows:

$$w_i^0 = dJ_i/dp + l_i \lim_{x \to x^0(p)} \partial L/\partial p, \quad i = 1, \dots, k. \tag{51}$$

Taking the total derivative at the left of (49), we have, due to (51):

$$\sum_{s=1}^{n} \partial f_i/\partial x_s \Big|_{x=x^0(p)} dx_s/dp = w_i^0 - \partial f_i/\partial p\Big|_{x=x^0(p)}, \quad i = 1, \dots, k. \tag{52}$$

Consider the set $W \in \{1, \dots, m\}$ of indices $\{j\}$ for which the constraint $g_j(x, p) \le q_j$ is saturated at $x^0(p)$, that is,

$$g_j(x^0(p), p) = q_j, \quad \text{for } j \in W, \tag{53}$$

and suppose that a variation of $p$ of size $dp$ does not disturb the equality (53). Then, we can differentiate (53) to obtain

$$\sum_{s=1}^{n} \partial g_j/\partial x_s \Big|_{x=x^0(p)} dx_s/dp = -\partial g_i/\partial p\Big|_{x=x^0(p)}, \quad j \in W. \tag{54}$$

If the number of equations in (52), (54) with linearly independent left-hand sides is less than $n$, then the equalities (14) in the KKT conditions explicitly containing $x_s$ can be differentiated to obtain additional linearly independent equations for $dx_s/dp$. Let us denote by $S$ the obtained linear system.

By Assumptions (A2), (A3), the Pareto point $x^0(p)$, a global minimizer, is unique, and this uniqueness is conserved with the variations of $p$. Since

$$x^0(p+dp) - x^0(p) = (dx/dp)dp, \tag{55}$$

where $x^0(p)$, $dp$, and $x^0(p+dp)$ are unique, we have that $dx = (dx_1, \ldots, dx_n)$ is also unique. Since the partial derivatives in $S$ and also $w_i$ of (51) are continuous, by Assumption (A4), this means that the system $S$ [Containing (52), (54), and additional equations from the KKT conditions, if any] must have a unique solution for $dx_s/dp$, $s = 1, \ldots, n$. Hence, the following result is obtained.

**Theorem 5.1.** Assume that $x^0 \in P$ is a Pareto solution such that, with small variation $dp$ of the parameter $p$,

(a) $x^0(p)$ remains Pareto;
(b) the constraints $g_j(x^0(p), p) = q_j$ remain saturated with that variation of $p$. Then, under Assumptions (Al)–(A4), the sensitivities $dx_s/dp$, $s = 1, \ldots, n$, are obtained as the unique solution of the linear system $S$.

**Remark 5.1.** There are MOPs for which the Pareto set coincides with the feasible set (see e.g. Refs. 12, 13, 15) and $x^0(p)$ may be in the interior of the feasible set. In this case, $W$ is empty, so there are no equations of type (54). However, this situation may happen only if the number of independent cost functions $k \geq n$ (see Corollary 5.2 in Ref. 19, p. 328) and in this case the Jacobian $\partial f_i/\partial x_s$ at $x = x^0(p)$ has rank $n$, so that the system (52) alone may provide the solution mentioned in Theorem 5.1.

**Example 5.1.** A good illustration to Theorem 5.1 is provided by Example 4.1, since a scalar problem is the special case of a MOP with

$k=1$, $\tau=0$. Thus, the second terms on the right of (49)–(51) vanish. We have the optimal point at $x^0=(q,0)$. Thus,

$$dx^0/dq=(1,0), \quad dx^0/dp=(0,0). \tag{56}$$

With respect to $q$, by (50), (52), (54), we have for (37)–(38)

$$\begin{aligned}&(\partial f/\partial x_1)(dx_1/dq)+(\partial f/\partial x_2)(dx_2/dq)\\&=(-px_1^{p-1})(dx_1/dq)+dx_2/dq\\&=-pq^{p-1},\\&dx_1/dq+dx_2/dq=1,\end{aligned}$$

whence

$$-pq^{p-1}(dx_1/dq)+1-dx_1/dq=-pq^{p-1},$$

and the solution is

$$dx_1/dq=1, \quad dx_2/dq=0,$$

as in (56).

With respect to $p$, we have.

$$\begin{aligned}&(dx_2/dp)-[x_1(p)]^p\log x_1(p)\,(dx_1/dp)\\&=w^0-\partial f/\partial p=-q^p\log q+q^p\log q\\&=0,\end{aligned}$$

whence

$$\begin{aligned}&dx_2/dp-q^p\log q(dx_1/dp)=0,\\&dx_1/dp+dx_2/dp=0;\end{aligned}$$

thus, $dx^0/dp=(0,0)$ as in (56).

## 6. Conclusions

This paper shows how the theory of global optimization and the balance space approach may apply in order to develop a general theory of sensitivity for vector optimization problems. This general theory points out that the balance space approach is an interesting alternative and complements the classical Pareto analysis. For instance, it yields a general envelope theorem (Theorem 4.1) that applies easily in practical situations and measures the sensitivity with respect to any parameter of the problem.

Two kinds of sensitivity are considered: sensitivity of optimal values (on the balance set) and that of a global minimizer (sensitivity of the Pareto set). The results can be applied to nonlinear multiobjective problems as is illustrated by examples.

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