# Essays on Dynamic Games

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# Contents

# CHAPTER 1

# Introduction

Dynamic game theory brings together four features that are key to many situations in economics, ecology, and elsewhere: optimizing behavior, presence of multiple agents/players, enduring consequences of decisions and lack of complete information for the agents about the system. To deal with problems bearing these four features the dynamic game theory methodology splits the modeling of the problem into three parts.

The first part deals with modeling the *environment* in which the agents act. Usually, a set of ordinary differential equations is specified to model the agents' interaction with the environment. However, there do exist many other mathematical models to describe systems which change over time, for instance, differential algebraic equations, differential equations with discontinuous right hand side, partial differential equations, time delay equations and stochastic differential equations. All of these give rise to different classes of dynamic games which have their own specific features. These equations are assumed to capture the primary (dynamical) features of the environment. A characteristic property of this specification is that these dynamic equations mostly contain a set of so called input functions. These input functions model the effect of the actions taken by the agents on the environment during the course of the game.

The next part deals with modeling the agents' *objectives*. Usually the agents' objectives are formalized as cost/utility functionals which have to be minimized (maximized). This minimization (maximization) is commonly performed subject to the specified dynamic model of the environment. Techniques developed in optimal control theory play a central role in solving dynamic games. In fact, from a historical perspective, the theory of dynamic games arose from a merge of static game theory and optimal control theory.

The information agents have about the game is crucial for the outcome of the decision making. A characteristic for a static game is that it takes place in one moment of time: all players make their choice once and simultaneously and, dependent on the choices made, each player receives his payoff. In such a formulation important issues like the *order of play* in the decision process, *information* available to the players at the time of their decisions, and the evolution of the game are suppressed. In case the agent's act in a dynamic environment these issues are, however, crucial and need to be properly specified before one can infer the outcome of the game. This specification is the third modeling part that characterizes the dynamic game theory methodology.

This thesis is based on four self contained independent chapters in the field of dynamic games. There are some differences in notation between chapters. The central theme of each chapter is related to one or more of the, above discussed, modeling components.

Chapter 2 studies the problem of finding Pareto solutions in a dynamic game. Here, the agents' interaction environment is modeled by an ordinary differential equation. Assuming that the game is played indefinitely, often called as infinite horizon, and players use the so called open loop strategies, the necessary and sufficient conditions for the existence of non improvable (Pareto) solutions are formulated. Further, these results are used to analyze a specific class of dynamic games called the linear quadratic differential game. The obtained results can also be used to analyze multi-objective optimal control problems.

Chapter 3 analyzes a noncooperative dynamic game when the agents' interaction environment is modeled by a linear constant coefficient differential algebraic equation. A method to decouple the dynamic and algebraic parts of the environment is discussed. Necessary and sufficient conditions for the existence of feedback Nash equilibria are obtained. Further, a geometric interpretation for the multiplicity of Nash equilibria is provided. Merits and drawbacks of the proposed approach are illustrated with examples.

Chapter 4 makes an attempt to introduce dynamics in coalitional games which are generally static in nature. In particular, this chapter investigates some allocation rules in certain dynamic transferable utility games. Here, the term dynamic refers to a situation where the average value of each coalition in the long run is known but its instantaneous value is unknown (interval uncertainty). These allocation rules ensure convergence of average allocations to the average core, a coalitionally stable set [77], of the average game and are designed based on complete and incomplete information of the extra reward.

Chapter 5 first analyzes the optimal control problem when the agents' interaction environment displays a switching behavior, which is captured by differential equations with discontinuous right hand side. The shallow lake model, a widely studied non linear dynamic game in environmental economics, is studied to highlight the key difference from the smooth case. The obtained results are used to compute the symmetric open loop Nash equilibrium.

This thesis is an outcome of research collaboration. Chapters 2 and 3 are co-authored with J. C. Engwerda; cf. [85, 84]. Chapter 4 is a joint work with D. Bauso; cf. [12, 11]. Finally, chapter 5 is co-authored with J. M. Schumacher and J. C. Engwerda.

# CHAPTER 2

# Necessary and Sufficient Conditions for Pareto Optimality in Infinite Horizon Cooperative Differential Games

## 2.1 Introduction

In this chapter we address the problem of finding the set of Pareto optimal solutions in the situation where a single player has multiple objectives or multiple players, $N$ players here, decide to coordinate their actions with an intent to minimize their costs. The system or the dynamic environment where the players interact is modeled by a (set of) differential equation(s), and we assume an open-loop information structure. Every player $i$ may choose his action/control trajectory, $u_i(.)$, arbitrarily from the set $\mathscr{U}^i$ of piecewise continuous functions[1]. Formally, the players are assumed to minimize the performance criteria:

$$J_i(x_0, u_1, u_2 \cdots, u_N) = \int_0^{\infty} g_i(t, x(t), u_1(t), u_2(t), \cdots, u_N(t))dt, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the solution of the differential equation (dynamic environment)

$$\dot{x}(t) = f(t, x(t), u_1(t), u_2(t), \cdots, u_N(t)),\ x(0) = x_0 \in \mathbb{R}^n. \tag{2.2}$$

Here, $u_i(t) \in \mathbb{R}^{m_i}$ with $u_i(.) \in \mathscr{U}^i$ and we denote $u = (u_1, u_2, \cdots, u_N) \in \mathscr{U}^i \times \mathscr{U}^i \times \cdots \times \mathscr{U}^N = \mathscr{U}$ with $\mathscr{U}$ being the set of admissible controls. Let $m = m_1 + m_2 + \cdots + m_N$. For the above problem to be well-defined we assume that $f(t,x,u) : \mathbb{R} \times \mathbb{R}^n \times \mathscr{U} \to \mathbb{R}^n$ and $g_i(t,x,u) : \mathbb{R} \times \mathbb{R}^n \times \mathscr{U} \to \mathbb{R}$, $i = 1, 2, \cdots N$, are continuous and all the partial derivatives of $f$ and $g_i$ w.r.t. $x$ and $u$ exist and are continuous. Further, we assume that the integrals involved in the player's objectives converge[2].

Pareto optimality plays a central role in analyzing these problems. Since we are interested in the joint minimization of the objectives of the players, the cost incurred by a single player cannot be minimized without increasing the cost incurred by other players. So, we consider solutions which cannot be improved upon by all the players simultaneously; the so called Pareto optimal solutions. Formally, the set of controls $u^* \in \mathscr{U}$ is Pareto optimal

[1] $u_i \in \mathscr{U}_i$ could be any measurable function such that $u_i(t) \in U_i \subset \mathbb{R}^{m_i}$, see e.g., [63] or [35].

[2] If the integrals do not converge there exist other notions of optimality, see [95], [26], and the analysis becomes involved.

if the set of inequalities $J_i(x_0,u) \leq J_i(x_0,u^*)$, $i = 1,2,\cdots,N$, with at least one of the inequalities being strict, does not allow for any solution in $u \in \mathscr{U}$. The corresponding point $(J_1(x_0,u^*),J_2(x_0,u^*),\cdots,J_N(x_0,u^*)) \in \mathbb{R}^N$ is called a Pareto solution. The set of all Pareto solutions is called the Pareto frontier.

In this chapter we are interested in finding Pareto optimal solutions of the infinite horizon cooperative game problem (1,2). Here, we do not consider formation of subcoalitions and the possibility of utility transfers during the course of the game. We assume players make binding agreements towards cooperation at the start of the game and continue for ever, which requires the use pre-commitment strategies by the players. This aspect motivates the use of open loop information structure in the chapter.

By varying the controls/actions in $\mathscr{U}$ one obtains a set of feasible points in $\mathbb{R}^N$ and the Pareto frontier constitutes the set of non improvable points. Further, we do not consider the aspect of selecting a particular point on the Pareto frontier, i.e., bargaining, and one may consult, e.g., chapter 6 of [35] for these issues. So, in cooperative game theory terminology the problem $(1,2)$ relates to the issue of finding costs incurred by the grand coalition in a non transferable utility game described in strategic form, see [77] and [69].

A well known way to find Pareto optimal controls is to solve a parametrized optimal control problem [118, 65]. However, it is unclear whether all Pareto optimal solutions are obtained using this procedure, see example 4.2 [38]. The closest references we could track, towards finding Pareto solutions in differential games, are [29], [19] and [107]. The necessary conditions for Pareto solutions where cost functions are just functions of the terminal state were given in [107] and the affiliated papers [102] and [66] . In [107] geometric properties of Pareto surfaces were used to derive necessary conditions which are in the spirit of maximum principle. Some difference with our work are: they assume that the admissible controls are of the feedback type and the terminal state should belong to some $n-1$ dimensional surface. Recently, [38] gives necessary and sufficient conditions for Pareto optimality for finite horizon cooperative differential games.

Almost all of the earlier works address the problem of finding Pareto solutions in the finite horizon case. In this chapter we focus on infinite horizon cooperative differential games. In section 2.2 we present a necessary and sufficient characterization of Pareto optimality which entails to reformulate the Pareto optimality problem as $N$ constrained infinite horizon optimal control problems. As a consequence, our results are in the spirit of the maximum principle, and this is also one of the reason why open loop strategies are used in this chapter.

We stress that this reformulation should not be confused with decentralization problems[3] where each player $i$ by choosing actions, without coordination, from the strategy set

[3]Marschak [71] introduced a class of cooperative decision making problems called team problems. Here, the objectives of the players are identical and as a result team optimality and person by person optimality are analogous to Pareto optimality and Nash equilibrium respectively. Team decision problems play a crucial role in understanding decentralization control problems, see [54].

$\mathscr{U}^i$ to achieve a Pareto optimum. Instead, in our approach, the constraints depend upon a Pareto solution due to the above reformulation, see lemma 2.2.2.

Due to the above reformulation our results are closely related to necessary and sufficient conditions for optimality of infinite horizon optimal control problems. Infinite horizon optimal control problems arise when no natural bound can be placed on the time horizon, for example while modeling capital accumulation processes (economic growth) and in biological sciences. In his seminal work on the theory of saving, Ramsey [83] used a dynamic optimization model defined on an unbounded time horizon, see [2] for more details. As the objectives can grow unbounded different notions of optimality have been introduced, see [26, 95] more details on the analysis. The necessary conditions for optimality given by the maximum principle are incomplete as transversality conditions are not clearly specified. As a result one obtains a large number of extremal trajectories. A natural extension of finite horizon transversality conditions, in general, is not possible, see [51]. Only by imposing certain restrictions on the system such an extension can be made, see [75], [95], [93] and more recently [5] and [109].

In section 2.3 we show, by making a particular assumption on the Lagrange multipliers, that the necessary conditions for Pareto optimality are same as the necessary conditions for optimality of a weighted sum optimal control problem. Further, we observe that an extension of finite horizon transversality conditions is a weak sufficient condition to satisfy this assumption. For discounted autonomous systems, [75] derives necessary conditions for free endpoint optimal control problems. We extend these results for the constrained problems (due to the above reformulation) and derive weak sufficient conditions for this assumption to hold true. In section 2.4 we derive sufficient conditions for Pareto optimality in the spirit of Arrow's sufficiency results in optimal control. In section 2.5 we consider regular indefinite infinite planning horizon linear quadratic differential games where the cost involved for the state variable has an arbitrary sign and the use of every control is quadratically penalized. We observe that if the dynamic system is controllable then this assumption holds true naturally. The linear quadratic case was recently solved for both a finite and infinite planning horizon in [37] assuming that the problem is a convex function of the control variables and the initial state is arbitrary. In this chapter we concentrate on the general case and where the initial state is fixed and the planning horizon is infinite. We provide some examples to illustrate subtleties and open issues. For the scalar case, by imposing a restriction on the control space, we show that all Pareto optimal solutions can be obtained using the weighting method and provide an algorithm to compute all the Pareto solutions.

**Notation:** We use the following notation. Let $\overline{N} = \{1, 2, \cdots, N\}$ denote the grand coalition and let $\overline{N}\backslash\{i\}$ denote the coalition of all players excluding player $i$. Let $\mathscr{P}_N$ denote the $N$ dimensional unit simplex. $\mathbb{R}^N_+$ denotes a cone consisting of $N$ dimensional vectors with non negative entries. $\mathbf{1}_N$ denotes a vector in $\mathbb{R}^N$ with all its entries equal to 1. $y'$

represents the transpose of the vector $y \in \mathbb{R}^N$. $|x|$ represents the absolute value of $x \in \mathbb{R}$. $||y||$ represents the Euclidean norm of the vector $y \in \mathbb{R}^N$. $|y|_i$ represents the absolute value of the $i^{th}$ entry of the vector $y$. $|A|_{(m,n)}$ represents the absolute value of entry $(m,n)$ of the matrix $A$. $A > 0$ denotes matrix $A$ is strictly positive definite. $f_x(.)$ represents the partial derivative of the function $f(.)$ w.r.t $x$. $\Phi_{f_x}(t,0) = e^{\int_0^t f_x(x,u)dt}$ represents the state transition matrix associated with the linear autonomous linear ordinary differential equation $\dot{x} = f_x(x,u)x,\ x(0) = x_0$. $\overrightarrow{\omega} \in \mathbb{R}^N_+$ denotes the vector whose entries $\omega_i$ are the weights assigned to the cost function of each player. We define the weighted sum function $G(.)$ as $G(\overrightarrow{\omega},t,x(t),u(t)) = \sum_{i\in\overline{N}} \omega_i g_i(t,x(t),u(t))$. $M_\omega$ represents the weighted matrix $\sum_{i\in\overline{N}} \omega_i M_i$ where $\overrightarrow{\omega}$ represents the weight vector, and $M_i$ is the weighting matrix that appears in player $i$'s objective. $\mathrm{sp}\{\mathtt{v}_1, \mathtt{v}_2, \cdots, \mathtt{v}_k\}$ represents the subspace spanned by the vectors $\mathtt{v}_1, \mathtt{v}_2, \cdots, \mathtt{v}_k$.

## 2.2 Pareto optimality

In this section we state conditions to characterize Pareto optimal controls. Lemma 2.2.1, given below, states that every control minimizing a weighted sum of the cost function of all players (where all weights are strictly positive) is Pareto optimal. So, varying the positive weights over the unit simplex one obtains, in principle, different Pareto optimal controls. A proof of the lemma can be found in [38, 65].

**Lemma 2.2.1.** *Let $\alpha_i \in (0,1)$, with $\sum_{i=0}^N \alpha_i = 1$. Assume $u^* \in \mathscr{U}$ is such that*

$$u^* \in \arg\min_{u\in\mathscr{U}} \left\{ \sum_{i=1}^N \alpha_i J_i(x_0,u) \right\}. \tag{2.3}$$

*Then $u^*$ is Pareto optimal.*

The above lemma implies that minimizing the weighted sum is an easy way to find Pareto optimal controls. Being a sufficient condition it is, however, unclear whether we obtain all Pareto optimal controls in this way. In fact the above procedure may yield no Pareto efficient controls, while an infinite number of Pareto solutions exist. The following example illustrates this point.

*Example* 2.2.1. Consider

$$\dot{x}(t) = u_1(t) - u_2(t),\ x(0) = 0,$$

together with the cost functions

$$J_1(x_0,u_1,u_2) = \int_0^\infty (u_1(t) - u_2(t))dt \ \text{ and } \ J_2(x_0,u_1,u_2) = \int_0^\infty x^2(t)(u_2(t) - u_1(t))dt.$$

The control spaces $\mathscr{U}_i, i = 1,2$ are defined as

$$\mathscr{U}_i = \left\{ u_i(.) \text{ is piecewise continuous} \middle| J_j(x_0,u_1,u_2) \text{ exists}, j = 1,2, \text{ and } \lim_{t\to\infty} x(t) \text{ exists} \right\}.$$

Then, by simple calculations we have $J_2(x_0,u_1,u_2) = -\frac{1}{3}\left(\lim_{t\to\infty} x(t)\right)^3 = -\frac{1}{3}J_1^3(x_0,u_1,u_2)$ for all $(u_1,u_2)$. Clearly, by choosing different values for control functions $u_i(.)$ every point in the $(J_1,J_2)$ plane satisfying $J_2 = -\frac{1}{3}J_1^3$ can be obtained. Furthermore, it is clear that every point on this curve is Pareto optimal. Now, consider the minimization of $J_\alpha(x_0,u_1,u_2) = \alpha J_1(x_0,u_1,u_2) + (1-\alpha)J_2(x_0,u_1,u_2)$ subject to the dynamics given above and with $\alpha \in (0,1)$. We choose $u_1(.) = 0$ and $u_2(.) = -\mathbb{I}_{[0,c]}$, where $\mathbb{I}_{[0,c]}$ represents the indicator function for the interval $[0,\ c]$. Then, straight forward calculations yield $J_\alpha(x_0,u_1,u_2) = \alpha c - \frac{1-\alpha}{3}c^3$. By choosing $c$ arbitrarily large, $J_\alpha(x_0,u_1,u_2)$ can be made arbitrarily small, i.e., $J_\alpha(x_0,u_1,u_2)$ does not have a minimum.

Lemma 2.2.2 mentioned below gives both a necessary and sufficient characterization of Pareto solutions. It states that every player's Pareto optimal solutions can be obtained as the solution of a constrained optimization problem. The proof is along the lines of the finite dimensional case considered in chapter 22 of [99].

**Lemma 2.2.2.** *$u^* \in \mathscr{U}$ is Pareto optimal if and only if for all $i$, $u^*(.)$ minimizes $J_i(x_0,u)$ on the constrained set*

$$\mathscr{U}_i \triangleq \left\{u | J_j(x_0,u) \le J_j(x_0,u^*),\ j = 1,\cdots,N,\ j \ne i\right\},\ \textit{for}\ i = 1,\cdots,N. \tag{2.4}$$

*Proof.* $\Rightarrow$ Suppose $u^*$ is Pareto optimal. Then $u^* \in \mathscr{U}_k$[4], $\forall k$, so $\mathscr{U}_k \ne \emptyset$. Now, if $u^*$ does not minimize $J_k(x_0,u)$ on the constraint set $\mathscr{U}_k$ for some $k$, then there exists a $u$ such that $J_j(x_0,u) \le J_j(x_0,u^*)$ for all $j \ne k$ and $J_k(x_0,u) < J_k(x_0,u^*)$. This contradicts the Pareto optimality of $u^*$.
$\Leftarrow$ Suppose $u^*$ minimizes each $J_k(x_0,u)$ on $\mathscr{U}_k$. If $\hat{u}$ does not provide a Pareto optimum, then there exists a $u(.) \in \mathscr{U}$ and an index $k$ such that $J_i(x_0,u) \ne J_i(x_0,u^*)$ for all $i$ and $J_k(x_0,u) < J_k(x_0,u^*)$. This contradicts the minimality of $u^*$ for $J_k(x_0,u)$ on $\mathscr{U}_k$. ∎

We observe that for a fixed player the constraint set $\mathscr{U}_i$ defined in (2.4) depends on the entries of the Pareto optimal solution that represents the loss of the other players. Therefore this result mainly serves theoretical purposes, as we will see, e.g., in the proof of theorem 2.3.1 and theorem 2.3.3. Using the above lemma, we next argue that Pareto optimal controls satisfy the dynamic programming principle.

**Corollary 2.2.1.** *If $u^* \in \mathscr{U}$ is a Pareto optimal control for $x(0) = x_0$ in (1,2), then for any $\tau > 0$, $u^*([\tau,\infty))$ is a Pareto optimal control for $x(\tau) = x^*(\tau)$ in (1,2). Here, $x^*(\tau) = x(t,0,u^*([0,\tau]))$ is the value of the state at $\tau$ generated by $u^*([0,\tau])$.*

*Proof.* Let $\mathscr{U}_i(\tau)$, with $x(\tau) = x^*(\tau)$, be the constrained set defined as:

$$\mathscr{U}_i(\tau) = \left\{u | J_j(x(\tau),u) \le J_j(x(\tau),u^*([\tau,\infty))),\ j = 1,\cdots,N,\ j \ne i\right\}.$$

[4] The admissible control space $\mathscr{U}_i$ defined in (2.4) is not rectangular, in general.

Consider a control $u \in \mathscr{U}_i(\tau)$ and let $u^e([0,\infty))$ be a control defined on $[0,\infty)$ such that $u^e([0,\tau)) = u^*([0,\tau))$ and $u^e([\tau,\infty)) = u$, then $x(\tau, 0, u^e([0,\tau)) = x^*(\tau)$. Further,

$$\begin{aligned}
J_j(x_0, u^e) &= \int_0^\infty g_j(t, x(t), u^e(t))dt \\
&= \int_0^\tau g_j(t, x^*(t), u^*(t))\, dt + \int_\tau^\infty g_j(t, x(t), u(t))\, dt \\
&\quad \text{(as } u \in \mathscr{U}_i(\tau)\text{, we have)} \\
&\leq \int_0^\tau g_j(t, x^*(t), u^*(t))\, dt + \int_\tau^\infty g_j(t, x^*(t), u^*(t))\, dt \\
&= \int_0^\infty g_j(t, x^*(t), u^*(t))\, dt = J_j(x_0, u^*).
\end{aligned}$$

The above inequality holds for all $j = 1, \cdots, N$, $j \neq i$. Clearly, $u^e([0,\infty)) \in \mathscr{U}_i(0)$ i.e., every element $u \in \mathscr{U}_i(\tau)$ can be viewed as an element $u^e \in \mathscr{U}_i(0)$ restricted to the time interval $[\tau,\infty)$. From the dynamic programming principle it follows directly that $u^*([\tau,\infty))$ has to minimize $J_i(x^*(\tau), u)$ for $u \in \mathscr{U}_i(\tau)$. ∎

We used lemma 2.2.2 in the proof of corollary 2.2.1. Further, being an optimal control problem, it can be shown that strategies that satisfy lemma 2.2.1 also satisfy the statement of corollary 2.2.1. Another result that follows directly from lemma 2.2.2 is that if the argument at which some player's cost is minimized is unique, then this control is Pareto optimal too (see corollary 2.5 in [38] for the proof).

**Corollary 2.2.2.** *Assume $J_1(x_0, u)$ has a minimum which is uniquely attained at $u^*$. Then $\big(J_1(x_0, u^*), J_2(x_0, u^*), \cdots, J_N(x_0, u^*)\big)$ is a Pareto solution.*

We give the following result from lemma 2.2.2. Since, Pareto optimality is preserved for every strictly monotonic transformation of the cost functions, if the player's costs are modified as $\tilde{J}_i(x_0, u) = J_i(x_0, u) - c,\ c \in \mathbb{R},\ \forall i \in \overline{N}$, then we have the following corollary.

**Corollary 2.2.3.** *The set of Pareto optimal strategies for the games with player's objectives as $J_i(x_0, u)$ and $\tilde{J}_i(x_0, u)$, $i \in \overline{N}$, is the same.*

## 2.3 Necessary conditions for the general case

In this section, using lemma 2.2.2 we derive necessary conditions of Pareto optimality for the problem (2.2,2.1) in a general setting. Before proceeding in this direction we give the following notation for the $N$ person infinite horizon cooperative differential game:

$$\begin{aligned}
(P) \qquad & \text{for each } i \in \overline{N} \ \min_{u \in \mathscr{U}} \int_0^\infty g_i(t, x(t), u(t))dt \\
& \text{subject to } \ \dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0.
\end{aligned}$$

Let $u^*$ be a Pareto optimal strategy for the problem $(P)$ and $x^*$ be the trajectory generated by $u^*$. Using lemma 2.2.2, $(P)$ is equivalently written as $N$ constrained optimal control problems, denoted by $(P_i)$ for each player $i \in \overline{N}$, as follows:

$$(P_i) \qquad \min_{u \in \mathscr{U}_i} \int_0^\infty g_i(t,x(t),u(t))dt$$
$$\text{subject to } \dot{x}(t) = f(t,x(t),u(t)), \quad x(0) = x_0.$$

The control space $\mathscr{U}_i$ in $(P_i)$ is constrained and depends on the Pareto optimal solution of players $j \in \overline{N} \backslash \{i\}$. Introducing the auxiliary states $\tilde{x}^i_j(t),\ j \in \overline{N} \backslash \{i\}$ as

$$\tilde{x}^i_j(t) = \int_0^t g_i(t,x(t),u(t))dt,\ \tilde{x}^i_j(0) = 0,$$

the constraint set $\mathscr{U}_i$ can be represented as

$$\mathscr{U}_i \triangleq \Big\{ u \in \mathscr{U} \,\big|\, \dot{\tilde{x}}^i_j(t) = g_j(t,x(t),u(t)), \quad \tilde{x}^i_j(0) = 0,$$
$$\lim_{t \to \infty} \tilde{x}^i_j(t) \le \tilde{x}^{i^*}_j = \int_0^\infty g_i(t,x^*(t),u^*(t))dt,\ \forall j \in \overline{N} \backslash \{i\} \Big\}.$$

The unconstrained representation, w.r.t to control space, of $(P_i)$ is then given as:

$$(P_i) \qquad \min_{u \in \mathscr{U}} \int_0^\infty g_i(t,x(t),u(t))dt$$
$$\text{subject to } \dot{x}(t) = f(t,x(t),u(t)),\ x(0) = x_0$$
$$\dot{\tilde{x}}^i_j(t) = g_j(t,x(t),u(t)),\ \tilde{x}^i_j(0) = 0,\ \lim_{t \to \infty} \tilde{x}^i_j(t) \le \tilde{x}^{i^*}_j,\ \forall j \in \overline{N} \backslash \{i\}.$$

Collecting the above and from lemma 2.2.2, we have the following proposition.

**Proposition 2.3.1.** *$u^*$ is a Pareto optimal control for the cooperative game problem $(P)$ $\Leftrightarrow$ $u^*$ is an optimal control for the problems $(P_i)$, $i \in \overline{N}$.*

The optimal control problems $(P_i)$ have mixed end point constraints, i.e., $\lim_{t \to \infty} x(t)$ is free and $\lim_{t \to \infty} \tilde{x}^i_j(t),\ j \in \overline{N} \backslash \{i\}$ are constrained. Let $H_i$ denote the Hamiltonian associated with the problem $(P_i)$ and be defined as (with abuse of notation) $H_i = \lambda^0_i g_i + \lambda'_i f + \sum_{j \in \overline{N} \backslash \{i\}} \mu^i_j g_j$. From proposition 2.3.1, by applying Pontryagin maximum principle for $(P_i)$ one can obtain necessary conditions for Pareto optimality for $(P)$. These necessary conditions give a set of extremal trajectories and the associated transversality conditions allow one to single out the optimal one. If the game problem $(P)$ is finite horizon type, then the transversality conditions associated with the problem $(P_i)$ are $\lambda_i(T) = 0,\ 0 < T < \infty$ and $\mu^i_j \ge 0$ for $j \in \overline{N} \backslash \{i\},\ i \in \overline{N}$ and the maximum principle holds in normal form i.e., $\lambda^0_i = 1,\ i \in \overline{N}$ (refer to proposition 3.16, [48]). Using these ideas it was shown in [38] that necessary conditions for Pareto optimality for $(P)$ are same as the necessary conditions for optimality of a weighted sum optimal control problem. The main result there hinges upon the transversality conditions and normality of the problems $(P_i)$. Unfortunately, for the infinite horizon case the necessary conditions for optimality of the problems

$(P_i)$ are incomplete (see pg. 234, theorem 12 of [95]). The above finite horizon transversality conditions generally do not naturally carry over to the infinite horizon case. Refer to [51, 48, 75, 95, 5] for counterexamples to illustrate this behavior.

We will see in the following discussion that $\mu_j^i(t) = \mu_j^i$ (constants) due to the special structure of $\tilde{x}_j^i(t)$. Let $\overrightarrow{\lambda}_i = \left(\mu_1^i, \cdots, \mu_{i-1}^i, \lambda_i^0, \mu_{i+1}^i, \cdots, \mu_N^i\right)'$, $i \in \overline{N}$. In the theorem 2.3.1 below, by making an assumption on $\overrightarrow{\lambda}_i$ we show, using proposition 2.3.1, that necessary conditions of Pareto optimality of $(P)$ are the same as the necessary conditions for optimality of a weighted sum optimal control problem.

*Assumption* 2.3.1. For each problem $(P_i)$, the Lagrange multipliers associated with the objective function and the states $(\tilde{x}_j^i(t))$ are non negative with at least one of them strictly positive, i.e., $\overrightarrow{\lambda}_i \in \mathbb{R}_+^N \backslash \{0\}$.

**Theorem 2.3.1.** *If $(J_1(x_0,u^*), J_2(x_0,u^*), \cdots, J_N(x_0,u^*))$ is a Pareto candidate for problem (P) and assumption 2.3.1 holds, then there exists an $\overrightarrow{\alpha} \in \mathscr{P}_N$, a co-state function $\lambda(t) : [0,\infty) \to \mathbb{R}^n$ such that, with Hamiltonian defined as $H\left(\overrightarrow{\alpha}, t, x(t), u(t), \lambda(t)\right) = \lambda'(t) f(t,x(t),u(t)) + G(\overrightarrow{\alpha}, t, x(t), u(t))$, the following conditions are satisfied.*

$$\begin{aligned}
H(\overrightarrow{\alpha},t,x^*(t),u^*(t),\lambda(t)) \leq& H(\overrightarrow{\alpha},t,x^*(t),u(t),\lambda(t)) && (2.5a)\\
H^0(\overrightarrow{\alpha},t,x^*(t),\lambda(t)) =& \min_{u(t)} H(\overrightarrow{\alpha},t,x^*(t),u(t),\lambda(t)) && \\
\dot{\lambda}(t) =& -H_x^0(\overrightarrow{\alpha},t,x^*(t),\lambda(t)) && (2.5b)\\
\dot{x^*}(t) =& H_\lambda^0(\overrightarrow{\alpha},t,x^*(t),\lambda(t)) \quad s.t\ x^*(0) = x_0 && (2.5c)\\
\left(\overrightarrow{\alpha},\lambda(t)\right) \neq& 0,\ \forall t \in [0,\infty),\ \overrightarrow{\alpha} \in \mathscr{P}_N. && (2.5d)
\end{aligned}$$

*Proof.* From proposition 2.3.1, if $u^*$ is Pareto optimal for $(P)$ then the pair $(x^*,u^*)$ is optimal for the problem $(P_i)$. We define the Hamiltonian associated with $(P_i)$ as:

$$H_i(t,x(t),u(t),\lambda(t)) \triangleq \lambda_i'(t) f(t,x(t),u(t)) + \lambda_i^0 g_i(t,x(t),u(t)) + \sum_{j\in\overline{N}\backslash\{i\}} \mu_j^i(t) g_j(t,x(t),u(t)). \tag{2.6}$$

So, from Pontryagin's maximum principle there exist a constant $\lambda_i^0$ and co-state functions (continuous and piecewise continuously differentiable) $\lambda_i(t) \in \mathbb{R}^n$ and $\mu_j^i(t) \in \mathbb{R},\ j \in S_N^i$ such that:

$$\begin{aligned}
(\lambda_i^0, \lambda_i(t), \mu_j^i(t)) &\neq (0,0,0),\ j \in \overline{N}\backslash\{i\},\ t \in [0,\infty) && (2.7a)\\
H_i(t,x^*(t),u^*(t),\lambda(t)) &\leq H_i(t,x^*(t),u(t),\lambda(t)) && (2.7b)\\
H_i^0(t,x(t),\lambda(t)) &= \min_{u(t)} H_i(t,x(t),u(t),\lambda(t)) && \\
\dot{\lambda}_i(t) &= -H_{i\,x}^0(t,x^*(t),\lambda(t)) && (2.7c)\\
\dot{\mu}_j^i(t) &= -H_{i\,\tilde{x}_j^i}^0(t,x^*(t),\lambda(t)). && (2.7d)
\end{aligned}$$

Since $(H_i^0)_{\tilde{x}_j^i} = 0$, multipliers associated with the auxiliary variables $\mu_j^i(t) = \mu_j^i$ (constants) and the Hamiltonian can be written as $H_i(t,x(t),u(t),\lambda(t)) = \lambda_i'(t)f(t,x(t),u(t)) + G(\overrightarrow{\lambda}_i,t,x(t),u(t))$. The first order conditions are:

$$\lambda_i'(t)f(t,x^*(t),u^*(t))+G(\overrightarrow{\lambda}_i,t,x^*(t),u^*(t)) \leq \lambda_i'(t)f(t,x^*(t),u(t)) + G(\overrightarrow{\lambda}_i,t,x^*(t),u(t)) \tag{2.8}$$

$$\dot{\lambda}_i(t) = -f_x'(t,x^*(t),u^*(t))\lambda_i(t) - G_x(\overrightarrow{\lambda}_i,t,x^*(t),u^*(t)). \tag{2.9}$$

Taking a sum over $i \in \overline{N}$ for (2.8) and (2.9) yields

$$\sum_{i\in\overline{N}} \Big(\lambda_i'(t)f(t,x^*(t),u^*(t)) + G(\overrightarrow{\lambda}_i,t,x^*(t),u^*(t))\Big) \leq$$
$$\sum_{i\in\overline{N}} \big(\lambda_i'(t)f(t,x^*(t),u(t)) + G(\overrightarrow{\lambda}_i,t,x^*(t),u(t))\big), \tag{2.10}$$

$$\sum_{i\in\overline{N}} \dot{\lambda}_i(t) = -f_x'(t,x^*(t),u^*(t)) \sum_{i\in\overline{N}} \lambda_i(t) - \sum_{i\in\overline{N}} G_x(\overrightarrow{\lambda}_i,t,x^*(t),u^*(t)). \tag{2.11}$$

Let us introduce $d = \sum_{i\in\overline{N}} \Big(\lambda_i^0 + \sum_{j\in\overline{N}\setminus\{i\}} \mu_j^i\Big)$. By assumption 2.3.1 we have $d > 0$. We define $\lambda(t) = \frac{1}{d}\sum_{i\in\overline{N}} \lambda_i(t)$, $\alpha_i = \frac{1}{d}\Big(\lambda_i^0 + \sum_{j\in\overline{N}\setminus\{i\}} \mu_i^j\Big)$, $i \in \overline{N}$ and a vector $\overrightarrow{\alpha} = \big(\alpha_1, \cdots, \alpha_N\big)'$. Notice that $\overrightarrow{\alpha} \in \mathscr{P}_N$ by assumption (1). Dividing the equation (2.10) by $d$ we have:

$$\lambda'(t)f(t,x^*(t),u^*(t))+G(\overrightarrow{\alpha},t,x^*(t),u^*(t)) \leq \lambda'(t)f(t,x^*(t),u(t)) + G(\overrightarrow{\alpha},t,x^*(t),u(t)) \tag{2.12}$$

$$\dot{\lambda}(t) = -f_x'(t,x^*(t),u^*(t))\lambda(t) - G_x(\overrightarrow{\alpha},t,x^*(t),u^*(t)). \tag{2.13}$$

Next we define the modified Hamiltonian as

$$H(\overrightarrow{\alpha},t,x(t),u(t),\lambda(t)) \triangleq \lambda'(t)f(t,x(t),u(t)) + G(\overrightarrow{\alpha},t,x(t),u(t)).$$

Then necessary conditions for $u^*$ to be Pareto optimal control can be rewritten as (2.5). ■

*Remark* 2.3.1. The necessary conditions given by (2.5) are closely related to the minimization of $\sum_{i\in\overline{N}} \alpha_i J_i$ subject to (2.2), i.e., the weighted sum optimal control problem. There are, however, some subtle differences. When the weighted sum optimal control problem admits maximum principle in normal form then one obtains necessary conditions as (2.5).

A natural extension of finite horizon transversality conditions to the infinite horizon case for the problem $(P_i)$, $i \in \overline{N}$ leads to $\lambda_i^0 = 1$ and $\mu_j^i \geq 0$ for $i \in \overline{N}$, and as result guarantees assumption 2.3.1. For the analysis that follows from now onwards we focus on weak sufficient conditions that allow such an extension. Towards that end, we first consider non-autonomous systems. In general, such an extension is achieved by imposing restrictions on the system parameters, also called as growth conditions. Specializing theorem 3.16 [95] or example 10.3 [93] to the problem $(P_i)$, we have the following corollary:

**Corollary 2.3.1.** *If there exist non-negative numbers a, b and c with $c > Nb$ such that the following conditions are satisfied for $t \geq 0$ and all $x(t)$:*

$$(|g_{ix}(t,x(t),u(t)|)_m \leq ae^{-ct}, \; m = 1,\cdots,N, \; \forall i \in \overline{N} \tag{2.14a}$$

$$(|f_x(t,x(t),u(t)|)_{(l,m)} \leq b, \; l = 1,\cdots,N, \; m = 1,\cdots,N. \tag{2.14b}$$

*Then assumption 2.3.1 is satisfied. Consequently, for every Pareto solution the necessary conditions given by (2.5) hold true and in addition* $\lim_{t\to\infty}\lambda(t) = 0$ *is satisfied.*

*Proof.* If conditions (2.14) hold true, then by theorem 3.16 of [95], the finite horizon transversality conditions do extend to the infinite horizon case. As a result, $\lambda_i^0 = 1$, $\mu_i^j \geq 0, \; \forall j \in \overline{N}\backslash\{i\}$ and $\lim_{t\to\infty}\lambda_i(t) = 0$ are satisfied for the constrained optimal control problem $(P_i)$ we have $\overrightarrow{\lambda}_i \in \mathbb{R}_+^N\backslash\{0\}$. Clearly assumption 2.3.1 is satisfied. So, the necessary conditions given by (2.5) hold true and in addition $\lim_{t\to\infty}\lambda(t) = 0$. ■

The following example demonstrates the application of theorem 2.3.1 and corollary 2.3.1.

*Example* 2.3.1. Consider the following game problem:

$$
\begin{aligned}
(P) \quad J_1(x_0,u_1,u_2) &= \int_0^\infty e^{-\rho t/2}(u_1(t)-u_2(t))dt \quad \text{and} \\
J_2(x_0,u_1,u_2) &= \int_0^\infty e^{-3\rho t/2}x^2(t)(u_2(t)-u_1(t))dt \\
\text{subject to } \dot{x}(t) &= \frac{\rho}{2}x(t)+u_1(t)-u_2(t), \; x(0)=0, \; t\in[0,\infty) \\
u \in \mathscr{U}, \text{ s.t. } \mathscr{U} &= \left\{u(.)\middle|\, \forall t\geq 0, \; u(t)\in E\subset\mathbb{R}^2, \; E \text{ is a closed and bounded set}\right\}.
\end{aligned}
$$

Taking the transformations $\tilde{x}(t) = e^{-\rho t/2}x(t)$ and $\tilde{u}_i(t) = e^{-\rho t/2}u_i(t)$, we transform the game $(P)$ as a new game $(\tilde{P})$ given by:

$$
\begin{aligned}
(\tilde{P}) \qquad J_1(x_0,\tilde{u}_1,\tilde{u}_2) &= \int_0^\infty (\tilde{u}_1(t)-\tilde{u}_2(t))dt \qquad \text{and} \\
J_2(x_0,\tilde{u}_1,\tilde{u}_2) &= \int_0^\infty \tilde{x}^2(t)(\tilde{u}_2(t)-\tilde{u}_1(t))dt \\
\text{subject to} \qquad \dot{\tilde{x}}(t) &= \tilde{u}_1(t)-\tilde{u}_2(t), \; \tilde{x}(0)=0, \; t\in[0,\infty).
\end{aligned} \tag{2.15}
$$

We have $J_i(x_0,u_1,u_2) = J_i(x_0,\tilde{u}_1,\tilde{u}_2), \; i=1,2$. The player's objectives are simplified as:

$$
\begin{aligned}
J_1(x_0,\tilde{u}_1,\tilde{u}_2) &= \int_0^\infty (\tilde{u}_1(t)-\tilde{u}_2(t))\,dt = \lim_{t\to\infty}\tilde{x}(t) \\
J_2(x_0,\tilde{u}_1,\tilde{u}_2) &= \int_0^\infty \tilde{x}^2(t)\,(\tilde{u}_2(t)-\tilde{u}_1(t))\,dt = -\frac{1}{3}\left(\lim_{t\to\infty}\tilde{x}(t)\right)^3.
\end{aligned}
$$

By construction, $|u_i(t)| < c$ for some $c > 0$, $i = 1,2$, $\forall\, t \geq 0$ and $\lim_{t\to\infty}\tilde{x}(t) \leq 2c/\rho$. We notice that $J_2 = -\frac{1}{3}J_1^3$ for all $(u_1,u_2)$ and choosing different values for the control functions

$u_i(.)$, every point in the $(J_1, J_2)$ plane satisfying $J_2 = -\frac{1}{3}J_1^3$ can be attained. Moreover, every point on this curve is Pareto optimal. This conclusion can be derived from the application of theorem 2.3.1 and corollary 2.3.1 too. With straightforward calculations we can show that for $(\tilde{P})$, $|f_{\tilde{x}}(.)| = 0$, $|g_{1\tilde{x}}(.)| = 0$, $|g_{2\tilde{x}}(.)| \leq 2|\tilde{x}(t)||\tilde{u}_1(t) - \tilde{u}_2(t)| \leq 4c^2/\rho$. The growth conditions mentioned in corollary 2.3.1 hold true for the game problem $(\tilde{P})$. Then from theorem 2.3.1 there exists a co-state function $\tilde{\lambda}(t)$, with Hamiltonian defined as

$$H(.) = \tilde{\lambda}(t)\,(\tilde{u}_1(t) - \tilde{u}_2(t)) + \big(\alpha - (1-\alpha)\tilde{x}^2(t)\big)\,(\tilde{u}_2(t) - \tilde{u}_1(t)).$$

Further, $H(.)$ attains a minimum w.r.t $\tilde{u}_i(.),\ i = 1,2$ only if

$$\tilde{\lambda}(t) + \left(\alpha - (1-\alpha)\tilde{x}^{*^2}(t)\right) = 0 \text{ for all } t \in [0,\infty). \tag{2.16}$$

As the growth conditions are satisfied we have $\lim_{t\to\infty}\tilde{\lambda}(t) = 0$. The adjoint variable $\tilde{\lambda}(t)$ satisfies (by differentiating (2.16))

$$\dot{\tilde{\lambda}}(t) = 2(1-\alpha)\tilde{x}^*(t)(\tilde{u}_2(t) - \tilde{u}_1(t)),\ \forall t \in [0,\infty)\ \ \lim_{t\to\infty}\tilde{\lambda}(t) = 0.$$

We see that the necessary condition (7b) also results in the same differential equation for $\tilde{\lambda}(t)$, $\forall\, t \in [0,\infty)$. Using (2.16) we can conclude that for arbitrary choices of $u_1(.)$ and $u_2(.)$, theorem 2.3.1 holds true by choosing $\alpha$ such that $\lim_{t\to\infty}\tilde{x}^{*^2}(t) = \frac{\alpha}{1-\alpha}$. So, all the controls $u_1(.)$ and $u_2(.)$ are candidates for Pareto solutions (as they satisfy the necessary conditions). To show that the candidates are indeed Pareto optimal we have to show that the necessary conditions are sufficient too. This aspect is treated in example 2.4.1.

### 2.3.1 Discounted autonomous systems

The growth conditions given in corollary 2.3.1 ensure that assumption 2.3.1 is satisfied. However, the conditions (2.14) are quite strict. In this subsection we analyze games defined by autonomous systems with exponentially discounted player's costs and for this class of problems assumption 2.3.1 is guaranteed under mild conditions. The discount factor $\rho$ is assumed to be strictly positive. We represent the game problem as $(P^\rho)$ and the related optimal control problem as $(P_i^\rho)$. For discounted autonomous systems, Michel [75] gives necessary conditions for optimality for free endpoint infinite horizon optimal control problems. However, in the present case the problems $(P_i^\rho)$ are constrained with constraints taking a special structure. In the following discussion, owing to this special structure, we show that the conditions given by Michel [75] are sufficient to guarantee assumption 2.3.1. As a result, the necessary conditions for Pareto optimality of $(P^\rho)$ are the same as the necessary conditions for optimality of a weighted sum optimal control problem.

As a first step, we derive the necessary conditions for optimality for the mixed endpoint constrained optimal control problem $(P_i^\rho)$ (in similar lines of [75][5]). If $u^*$ is a Pareto

[5]Notice, the necessary conditions given by Michel [75] considers only the free end point optimal control problem.

optimal strategy for the game problem $(P^\rho)$, then from proposition 2.3.1 $u^*$ is optimal for the constrained optimal control problem $(P_i^\rho)$, $i \in \overline{N}$ given by:

$$
\begin{aligned}
(P_i^\rho) \qquad & \min_{u \in \mathscr{U}} \int_0^\infty e^{-\rho t} g_i(x(t), u(t)) dt \\
\text{subject to} \quad & \dot{x}(t) = f(x(t), u(t)),\ x(0) = x_0,\ u \in \mathscr{U} \\
& \dot{\tilde{x}}_j(t) = e^{-\rho t} g_j(x(t), u(t)),\ \tilde{x}_j^i(0) = 0,\ \lim_{t \to \infty} \tilde{x}_j^i(t) \le \tilde{x}_j^{i^*},\ \forall j \in \overline{N} \backslash \{i\}.
\end{aligned}
$$

Let $(x^*(t), u^*(t)),\ 0 \le t \le \infty$ be the optimal admissible pair for problem $(P_i^\rho)$, we fix $T > 0$ and define $h_i(z),\ i \in \overline{N}$ as:

$$
h_i(z) = \int_z^\infty e^{-\rho t} g_i\left(x^*(t - z + T), u^*(t - z + T)\right) dt. \tag{2.17}
$$

To derive the necessary conditions for optimality of $u^*$, we first consider the following truncated and augmented problem $(P_{i_T}^\rho)$ (associated with the problem $(P_i^\rho)$) defined as follows:

$$
\begin{aligned}
(P_{i_T}^\rho) \quad & \min_{u \in \mathscr{U}}\ h_i(z(T) - T) + \int_0^T v(t) e^{-\rho z(t)} g_i(Y(t), U(t)) dt \\
\text{subject to} \quad & \dot{Y}(t) = v(t) f(Y(t), U(t)),\ Y(0) = Y_0,\ Y(T) = x^*(T),\ U \in \mathscr{U} \\
& \dot{\tilde{Y}}_j^i(t) = v(t) e^{-\rho z(t)} g_j(Y(t), U(t)), \\
& \qquad \tilde{Y}_j^i(0) = 0,\ \tilde{Y}_j^i(T) + h_j(z(T) - T) \le \tilde{x}_j^{i^*},\ \forall j \in \overline{N} \backslash \{i\} \qquad (2.18) \\
& \dot{z}(t) = v(t),\ z(0) = 0,\ v(t) \in \big[1/2, \infty\big).
\end{aligned}
$$

*Remark* 2.3.2. Notice that the above problem is a mixed end point constrained finite horizon problem, i.e., $X(T)$ is fixed and $Y_j^i(T)$ is constrained. Further, (2.18) captures the constraint set $\mathscr{U}_i$ defined in lemma 2.2.2.

The following lemma, which is useful in theorem 2.3.2, relates the optimal solution of $(P_i^\rho)$ to the optimal solution $(P_{i_T}^\rho)$. The proof of lemma 2.3.1 is given in the appendix. We notice that the special structure of constraints given by (2.18) plays a role in arriving at this conclusion.

**Lemma 2.3.1.** *If $(x^*(t), u^*(t))$ is an optimal admissible pair for the problem $(P_i^\rho)$ then $\big(x^*(t), t, u^*(t), 1\big)$, $t \in [0,\ T]$, is an optimal admissible pair for the problem $(P_{i_T}^\rho)$.*

Using the above lemma, we give necessary conditions for optimality of problem $(P_i^\rho)$ in the following theorem (see appendix for the proof).

**Theorem 2.3.2.** *If $(x^*(t), u^*(t)),\ t \in \big[0,\ \infty\big)$ is an optimal pair for the problem $(P_i^\rho)$ then there exist $\overrightarrow{\lambda}_i \in \mathbb{R}_+^N$, $l_i^0 \in \mathbb{R}^n$ and continuous functions $\lambda_i(t) \in \mathbb{R}^n$ and $\gamma_i(t) \in \mathbb{R}$ respectively* *such that*

$$\left(\overrightarrow{\lambda}_i, \lambda_i(t), \gamma_i(t)\right) \neq (0,0,0),\ \forall t \geq 0,\ \left\| \left(\overrightarrow{\lambda}_i, \lambda_i(0)\right) \right\| = 1 \tag{2.19a}$$

$$\dot{\lambda}_i(t) = -e^{-\rho t} G_x(\overrightarrow{\lambda}_i, x^*(t), u^*(t)) - f_x'(x^*(t), u^*(t))\lambda_i(t),\ \lambda_i(0) = l_i^0 \tag{2.19b}$$

$$\dot{\gamma}_i(t) = \rho\, e^{-\rho t} G(\overrightarrow{\lambda}_i, x^*(t), u^*(t)),\ \lim_{t\to\infty} \gamma_i(t) = 0 \tag{2.19c}$$

$$H(\overrightarrow{\lambda}_i, t, x^*(t), u(t), \lambda(t)) \triangleq \lambda_i'(t) f(x^*(t), u^*(t)) + e^{-\rho t} G(\overrightarrow{\lambda}_i, x^*(t), u^*(t))$$

$$H(\overrightarrow{\lambda}_i, t, x^*(t), u^*(t), \lambda(t)) \leq H(\overrightarrow{\lambda}_i, t, x^*(t), u(t), \lambda(t)) \quad \forall u(t) \tag{2.19d}$$

$$H(\overrightarrow{\lambda}_i, t, x^*(t), u^*(t), \lambda(t)) = -\gamma_i(t). \tag{2.19e}$$

*Remark* 2.3.3. Though the approach in lemma 2.3.1 and theorem 2.3.2 is similar to the one given in [75], the main differences lie in the problem formulation. In [75], the necessary conditions are obtained for the free endpoint unconstrained infinite horizon optimal control problem. However, the game problem ($P^\rho$), due to proposition 2.3.1, leads to $N$ mixed endpoint constrained optimal control problems ($P_i^\rho$).

From theorem 2.3.2, if $u^*$ is optimal for the problem ($P_i$) then there exists a $\overrightarrow{\lambda}_i \in \mathbb{R}_+^N$ such that the conditions (2.19) hold true. These necessary conditions are closely related to the minimization of a weighted sum optimal control problem with the weight vector $\overrightarrow{\lambda}_i$. This observation is evident in the non-autonomous case as well, see (2.8) and (2.9). Due to the special structure of the constraint set $\mathscr{U}_i$, the term $G(\overrightarrow{\lambda}_i, x^*(t), u^*(t))$, weighted instantaneous undiscounted cost of the players, appears in the necessary conditions for optimality of all the problems ($P_i^\rho$), $i \in \overline{N}$. Now, from (2.19a) and (2.19c) if $\lim_{t\to\infty} \lambda_i(t) = 0$ then assumption 2.3.1 holds true. As the scrap value associated with problem ($P_i^\rho$) is zero, $\lim_{t\to\infty} \lambda_i(t) = 0$ is the natural transversality condition. In the discussion that follows we give two possible ways, in corollary 2.3.2 and corollary 2.3.3, to ensure $\lim_{t\to\infty} \lambda_i(t) = 0$. For autonomous systems [75] gives assumptions under which the natural transversality condition holds for the free endpoint case with maximization criterion. We show in the corollary 2.3.2 that these conditions, formulated as assumption 2.3.2 below, also suffice to conclude that this transversality condition holds for the problem ($P_i^\rho$). The proof, given in the appendix, is along the same lines of [75], and requires the following assumption.

*Assumption* 2.3.2. $g_i(x(t), u(t))$, $\forall i \in \overline{N}$ is non positive and there exists a neighborhood $V$ of $0 \in \mathbb{R}^n$ which is contained in the set of possible velocities $f(x^*(t), u(t))$ for all $u \in \mathscr{U}$ if $t \to \infty$.

**Corollary 2.3.2.** *Let assumption 2.3.2 hold true. Then, an optimal solution for the problem ($P_i^\rho$) satisfies in addition to the conditions (2.19), the following transversality condition:* $\lim_{t\to\infty} \lambda_i(t) = 0$.

*Remark* 2.3.4. a) The nonpositivity assumption 2.3.2 can be relaxed in the following way. If the instantaneous undiscounted costs of players $g_i(x(t), u(t))$, $i \in \overline{N}$ are bounded above for all pairs $(x(t),u(t)), t \in [0,\infty)$ then by assigning a new cost $\tilde{g}_i(x(t),u(t)) = g_i(x(t),u(t)) - M$ with $M = \max_{i\in\overline{N}} \sup_{t\in[0,\infty)} g_i(x(t),u(t))$ leaves $\tilde{g}(x(t),u(t))$ nonpositive. Now, by defining a new game $(\tilde{P}^\rho)$ with $\tilde{g}(.)$ as the instantaneous undiscounted costs for player $i$ we observe that Pareto optimal controls (if they exist) of $(P^\rho)$ and $(\tilde{P}^\rho)$ coincide. We will use this idea in example 2.3.2 to find Pareto optimal controls.

b) Notice that the second condition in assumption 2.3.2 is identical to the notion of state reachability when the state dynamics is described by a linear constant coefficient differential equation.

The conditions given in assumption 2.3.2 are mild but they are difficult to verify except for special cases, see remark 2.3.4(b) and example 2.3.2. Another possibility is to seek for (growth) conditions so as to obtain a bound on $||\lambda_i(t)||$. Recently, [95, 5] discuss such conditions for a class of free end point optimal control problems. In the following corollary 2.3.3, which is in the same spirit as the above works, we give (growth) conditions for the problem $(P_i^\rho)$. Towards that end, we make the following assumption. The proof of the corollary is provided in the appendix.

*Assumption* 2.3.3. a) There exist a $s \geq 0$ and an $r \geq 0$ such that

$$||g_{i_x}(x(t),u(t))|| \leq s\,(1+||x(t)||^r) \text{ for all } x(t) \in \mathbb{R}^n, u \in \mathscr{U} \text{ and } i \in \overline{N}.$$

b) There exist nonnegative constants $c_1, c_2, c_3$ and $\lambda \in \mathbb{R}$, such that for every admissible pair $(x(t),u(t))$, one has

$$\begin{aligned} ||x(t)|| &\leq c_1 + c_2 e^{\lambda t} \text{ for all } t \geq 0 \\ ||\Phi_{f_x}(t,0)|| &\leq c_3 e^{\lambda t} \text{for all } t \geq 0. \end{aligned}$$

c) For every admissible pair $(x(t),u(t))$ the eigenvalues of $f_x(x(t),u(t))$ are strictly positive.

**Corollary 2.3.3.** *Let assumption 2.3.3 hold true. Then, an optimal solution for problem* $(P_i^\rho)$ *satisfies in addition to the conditions (2.19), the following transversality condition* $\lim_{t\to\infty} \lambda_i(t) = 0$ *if* $\rho > (1+r)\lambda$.

*Remark* 2.3.5. a) In the above corollary, by selecting a high discount factor the natural transversality condition is obtained. When the state evolution dynamics is linear and player's objectives are convex in the control variable, the growth conditions (a) and (b) given in assumption 2.3.3 are similar to those obtained in [5] and [6].

b) In [5], the free endpoint infinite horizon optimal control problem is approximated with a series of free endpoint finite horizon problems whereas in the current approach $(P_i^\rho)$ is approximated with fixed endpoint problems. As a result, an additional condition (c) appears in the assumption 2.3.3, see the proof in appendix for more details.

To summarize, Corollaries 2.3.2 and 2.3.3 are sufficient conditions to ensure that assumption 2.3.1 holds true. Now, collecting the above results we proceed to the main result of the subsection.

**Theorem 2.3.3.** *Let assumption 2.3.2 or 2.3.3 hold true. If* $\big(J_1(x_0,u^*),J_2(x_0,u^*),\cdots,$ $J_N(x_0,u^*)\big)$ *is a Pareto optimal solution for problem (P) then there exists an* $\overrightarrow{\alpha} \in \mathscr{P}_N$ *and a co-state function* $\lambda(t):[0,\infty)\to\mathbb{R}^n$ *such that the following conditions are satisfied.*

$$H(\overrightarrow{\alpha},t,x(t),u(t),\lambda(t)) \triangleq \lambda'(t)f(t,x(t),u(t))+e^{-\rho t}G(\overrightarrow{\alpha},x(t),u(t)) \tag{2.20a}$$
$$H(\overrightarrow{\alpha},t,x^*(t),u^*(t),\lambda(t)) \leq H(\overrightarrow{\alpha},t,x^*(t),u(t),\lambda(t)),\ \forall u(t) \tag{2.20b}$$
$$H^0(\overrightarrow{\alpha},t,x^*(t),\lambda(t)) = \min_{u(t)} H(\overrightarrow{\alpha},t,x^*(t),u(t),\lambda(t))$$
$$\dot{\lambda}(t) = -H^0_x(\overrightarrow{\alpha},t,x^*(t),\lambda(t)),\ \lambda(0)=l_0\in\mathbb{R}^n,\ \lim_{t\to\infty}\lambda(t)=0 \tag{2.20c}$$
$$\dot{x}^*(t) = H^0_\lambda(\overrightarrow{\alpha},t,x^*(t),\lambda(t)),\ x^*(0)=x_0 \tag{2.20d}$$
$$\dot{\gamma}(t) = \rho\ G(\overrightarrow{\alpha},x^*(t),u^*(t)),\ \lim_{t\to\infty}\gamma(t)=0 \tag{2.20e}$$
$$H^0(\overrightarrow{\alpha},t,x^*(t),\lambda(t)) = -\gamma(t) \tag{2.20f}$$
$$\big(\overrightarrow{\alpha},\gamma(t),\lambda(t)\big)\neq 0,\ \forall t\in[0,\infty),\ \overrightarrow{\alpha}\in\mathscr{P}_N. \tag{2.20g}$$

*Proof.* If assumptions 2.3.2 or 2.3.3 hold true then for each problem $(P_i^\rho)$ $\overrightarrow{\lambda}_i\in\mathbb{R}^N_+\backslash\{0\}$ and $\lim_{t\to\infty}\lambda_i(t)=0$. We define $d=\sum_{i\in\overline{N}}\overrightarrow{\lambda}'_i\mathbf{1}_N$, $\alpha_i=\frac{1}{d}\left(\lambda_i^0+\sum_{j\in\overline{N}\backslash\{i\}}\mu_i^j\right)$, $i\in\overline{N}$ and a vector $\overrightarrow{\alpha}=(\alpha_1,\cdots,\alpha_N)'$. We notice that $\overrightarrow{\alpha}\in\mathscr{P}_N$. Taking the summation of equation (2.19b) for all $i\in\overline{N}$ and defining $\lambda(t)=\frac{1}{d}\sum_{i\in\overline{N}}\lambda_i(t)$ we observe that conditions (2.20b) and (2.20c) are satisfied. Taking the summation of equation (2.19c) for all $i\in\overline{N}$ and defining $\gamma(t)=\frac{1}{d}\sum_{i\in\overline{N}}\gamma_i(t)$, the conditions (2.20e) and (2.20f) are satisfied. Since, $\overrightarrow{\alpha}\in\mathscr{P}_N$ and $\lim_{t\to\infty}\lambda(t)=0$ we observe that (2.20g) is satisfied. ■

Next, we consider an example from [56] to illustrate usage of assumption 2.3.2 and theorem 2.3.3.

*Example* 2.3.2. Consider a fishery game with two players. The evolution of the stock of fish, in a particular area, is governed by the differential equation

$$\dot{x}(t)=ax(t)-bx(t)\ln x(t)-u_1(t)-u_2(t),\ x(0)=x_0\geq 2, \tag{2.21}$$

where $x(t)$ refers to the stock of fish, and $a>0$, $b>0$. It is assumed that $x(t)\geq 2, t\in[0,\infty)$. In (2.21), the stock of fish $x(t)$ depends upon $ax(t)$ births, $bx(t)\ln x(t)$ deaths and the fishing efforts of player $i$, $u_i(t)=w_i(t)x(t)$, at each point in time $t$. Each fisherman tries to maximize his utility $J_i(.)$, given by:

$$J_i(x_0,u_1,u_2)=\int_0^\infty e^{-\rho t}\ln u_i(t)\ dt.$$

We assume $0 < \varepsilon \leq w_i(t) < \infty$ for the utility to be well defined. By taking the transformation $y(t) = \ln x(t)$ the system (2.21) is modified as:

$$\dot{y}(t) = a - by(t) - w_1(t) - w_2(t),\ y(0) = \ln x(0), \tag{2.22}$$

and the player's utility is transformed as:

$$J_i(x_0, u_1, u_2) = \int_0^{\infty} e^{-\rho t} \left(y(t) + \ln w_i(t)\right)\, dt. \tag{2.23}$$

We notice that the instantaneous undiscounted reward is bounded below. As the system (2.22) is controllable, from remark 2.3.4.a) we notice that assumption 2.3.2 is satisfied. So all Pareto candidates can be obtained by solving the necessary conditions associated with the weighted sum optimal control problem :

$$\min_{w_1, w_2} \left\{ -\int_0^{\infty} e^{-\rho t} \left(y(t) + \alpha \ln w_1(t) + (1-\alpha) \ln w_2(t)\right) dt \right\},$$

subject to (2.22). We define the Hamiltonian as

$$\begin{aligned} H(\alpha, t, y, w_1, w_2, \lambda) \triangleq\ & \lambda(t)\left(a - by(t) - w_1(t) - w_2(t)\right) \\ & - e^{-\rho t}\left(y(t) + \alpha \ln w_1(t) + (1-\alpha)\ln w_2(t)\right). \end{aligned}$$

Taking $H_{w_i} = 0,\ i = 1,2$ gives $w_1(t) = -\frac{\alpha}{\lambda(t)} e^{-\rho t}$ and $w_2(t) = -\frac{1-\alpha}{\lambda(t)} e^{-\rho t}$. The adjoint variable is governed by

$$\dot{\lambda}(t) = b\lambda(t) + e^{-\rho t},\ \lim_{t \to \infty} \lambda(t) = 0,$$

and the solution is given as $\lambda(t) = -\frac{e^{-\rho t}}{\rho + b}$. The candidates for Pareto optimal strategies in open loop form are given by:

$$\begin{aligned} u_1^*(t) &= \alpha(\rho + b) e^{m(t, x_0)} \\ u_2^*(t) &= (1-\alpha)(\rho + b) e^{m(t, x_0)}, \end{aligned}$$

where $m(t, x_0) = e^{-bt} \ln x_0 + \frac{a - (b+\rho)}{b}\left(1 - e^{-bt}\right)$. The candidates for Pareto solutions are given as

$$\begin{aligned} J_1^*(x_0, u_1, u_2) &= \frac{\rho \ln x_0 + a - (\rho + b)}{\rho(\rho + b)} + \frac{\ln(\alpha(\rho + b))}{\rho} \\ J_2^*(x_0, u_1, u_2) &= \frac{\rho \ln x_0 + a - (\rho + b)}{\rho(\rho + b)} + \frac{\ln((1-\alpha)(\rho + b))}{\rho}. \end{aligned} \tag{2.24}$$

## 2.4 Sufficient conditions for Pareto optimality

It is well known [42] that if the action spaces as well as the players objective functions are convex then minimization of the weighted sum of the objectives results in all Pareto solutions. We give the following theorem from [35].

**Theorem 2.4.1.** *If $\mathscr{U}$ is convex and $J_i(x_0,u)$ is convex in $u$ for all $i=1,2,\cdots,N$ then for all Pareto optimal $u^*$ there exist $\overrightarrow{\alpha} \in \mathscr{P}_N$, such that $u^* \in \arg\min_{u\in\mathscr{U}} \sum_{i=1}^N \alpha_i J_i(x_0,u)$.*

Recently in [37], this property was used to obtain both necessary and sufficient conditions for the existence of Pareto optimal solutions for regular convex linear quadratic differential games. In general it is a difficult task to check if the players objectives are convex functions of controls. However, under some conditions the solutions of (2.20) result in Pareto optimal strategies. In this section we derive sufficient conditions for a strategy to be Pareto optimal. The sufficient conditions given in the theorem below are inspired by Arrow's sufficient conditions [95] in optimal control. Further, these sufficient conditions are given for non-autonomous systems and they hold true for discounted autonomous systems as well.

**Theorem 2.4.2.** *Assume that there exists $\overrightarrow{\alpha} \in \mathscr{P}_N$ and a co-state function $\lambda(t) : [0,\infty) \to \mathbb{R}^n$ satisfying (2.20c). Introduce the Hamiltonian $H(t,\overrightarrow{\alpha},x(t),u(t),\lambda(t)) = f(t,x(t),u(t))$ $+G(\overrightarrow{\alpha},t,x(t),u(t))$. Assume that the Hamiltonian has a minimum w.r.t $u(t)$ for all $x(t)$, denoted by $H^0(\overrightarrow{\alpha},t,x(t),\lambda(t)) = \min_{u(t)} H(\overrightarrow{\alpha},t,x(t),u(t),\lambda(t))$. If $H^0\big(\overrightarrow{\alpha},t,x(t),\lambda(t)\big)$ is convex in $x(t)$ and $\liminf_{t\to\infty} \lambda'(t)\,(x^*(t)-x(t)) \geq 0$, then $u^*(t)$ is Pareto optimal.*

*Proof.* From the convexity of $H^0(\overrightarrow{\alpha},t,x(t),\lambda(t))$ we have:

$$H^0(\overrightarrow{\alpha},t,x(t),\lambda(t)) - H^0(\overrightarrow{\alpha},t,x^*(t),\lambda(t)) \geq H_x^0(\overrightarrow{\alpha},t,x^*(t),\lambda(t))\,(x(t)-x^*(t))$$

Since, $H(\overrightarrow{\alpha},t,x(t),u(t),\lambda(t)) \geq H^0(\overrightarrow{\alpha},t,x(t),\lambda(t)) = H(\overrightarrow{\alpha},t,x^*(t),u^*(t),\lambda(t))$ we have:

$$\begin{aligned} H(\overrightarrow{\alpha},t,x(t),u(t),\lambda(t)) - H(\overrightarrow{\alpha},t,x^*(t),u^*(t),\lambda(t)) &\geq H_x^{0'}(\overrightarrow{\alpha},t,x^*(t),\lambda(t))\,(x(t)-x^*(t)) \\ &= -\dot{\lambda}'(t)\,(x(t)-x^*(t)) \ \text{(by (2.20c))}. \end{aligned}$$

Using the definition of Hamiltonian, the above inequality can be written as:

$$\begin{aligned} \lambda'(t)\,(f(t,x(t),u(t)-f(t,x^*(t),u^*(t))) + G(\overrightarrow{\alpha},t,x(t),u(t)) - G(\overrightarrow{\alpha},t,x^*(t),u^*(t)) \\ \geq -\dot{\lambda}'(t)\,(x(t)-x^*(t)) \end{aligned}$$

$$\begin{aligned} \big(G(\overrightarrow{\alpha},t,x(t),u(t)) - G(\overrightarrow{\alpha},t,x^*(t),u^*(t))\big) &\geq \dot{\lambda}'(t)\,(x^*(t)-x(t)) + \lambda'(t)\,(\dot{x}^*(t)-\dot{x}(t)) \\ &= \frac{d}{dt}\big(\lambda'(t)\,(x^*(t)-x(t))\big). \end{aligned}$$

Taking the integrals on both sides we have

$$\int_0^T \big(G(\overrightarrow{\alpha},t,x(t),u(t)) - G(\overrightarrow{\alpha},t,x^*(t),u^*(t))\big)\,dt \geq \big(\lambda'(t)\,(x^*(t)-x(t))\big)\Big|_0^T.$$

As $x^*(0) = x(0) = x_0$ and $\lambda(0)$ is bounded the above inequality is given as:

$$\int_0^T \left(G(\overrightarrow{\alpha},t,x(t),u(t)) - G(\overrightarrow{\alpha},t,x^*(t),u^*(t))\right) dt \geq \left(\lambda'(T)\left(x^*(T) - x(T)\right)\right).$$

Taking $T \to \infty$

$$J(u) - J(u^*) \geq \lim_{t\to\infty} \lambda'(t)\left(x^*(t) - x(t)\right) \geq \liminf_{t\to\infty} \lambda'(t)\left(x^*(t) - x(t)\right) \geq 0.$$

Clearly, by lemma 2.2.1 $u^*$ is Pareto optimal. ■

*Example* 2.4.1. (**sufficient conditions**): We illustrate theorem 2.4.2 by considering example 2.3.1 again. First, we notice that $H^0(t,\tilde{x}^*,\tilde{\lambda}) = 0$ so $H^0(t,\tilde{x}^*,\tilde{\lambda})$ is convex in $\tilde{x}(t)$. Next, $\lim_{t\to\infty}\tilde{x}(t)$ exists and is finite and $\lim_{t\to\infty}\tilde{\lambda}(t) = 0$, so $\liminf_{t\to\infty}\tilde{\lambda}(t)\ (\tilde{x}^*(t) - \tilde{x}(t)) = 0$. So, by theorem 2.4.2 every control $(u_1,u_2)$ is Pareto optimal.

*Example* 2.4.2. (**sufficient conditions**): For example 2.3.2 the candidates for Pareto solutions are given by (2.24). If the model in example 2.3.2 satisfies the sufficient conditions mentioned in theorem 2.4.2, then all Pareto solutions are indeed given by (2.24). The minimized Hamiltonian is given by:

$$H^0(t,y(t),\lambda(t)) = -\frac{\rho\, y(t)}{\rho+b}\, e^{-\rho t} + e^{-\rho t}\left(1 - \frac{a}{\rho+b} - \ln\left(\alpha^\alpha(1-\alpha)^{1-\alpha}(\rho+b)\right)\right).$$

Clearly, $H^0(t,y(t),\lambda(t))$ is convex (linear here) in $y(t)$. Since $w_i(t),\ i = 1,2$, is bounded we have $|y(t)| \leq (c_1 + c_2e^{-bt})$. Further, $\lambda(t) = -\frac{e^{-\rho t}}{\rho+b}$, thus we have $\liminf_{t\to\infty}\lambda(t)(y^*(t) - y(t)) = 0$. The fishery model satisfies the sufficient conditions as given by theorem 2.4.2. So, all the candidates given by (2.24) are Pareto solutions.

## 2.5 Linear quadratic case

In this section we consider the discounted infinite planning horizon linear quadratic cooperative differential game (denoted as ($P^{LQ}$)). Player $i \in \overline{N}$ may choose his control trajectory, $u_i(.)$ from the set of admissible controls $\mathscr{U}$ where the specific choice of control space will be clarified below. The problem is to determine the set of Pareto solutions for the cooperative game defined by

$$(P^{LQ}) \qquad \min_{u\in\mathscr{U}} J_i(x_0,u) \tag{2.25}$$

$$J_i(x_0,u) = \int_0^\infty e^{-\rho t}(x'(t)\ u'(t))M_i(x'(t)\ u'(t))'dt$$

$$\text{subject to } \dot{x}(t) = Ax(t) + \sum_{i\in\overline{N}} B_iu_i(t),\ x(0) = x_0. \tag{2.26}$$

Where $M_i = \begin{bmatrix} Q_i & S_i' \\ S_i & R_i \end{bmatrix}$ is symmetric, $R_i \geq 0,\ i = 1,\cdots,N$. We define the following spaces:

a) $L^N_{2,\text{loc}} \triangleq \{u \mid \int_0^\infty u'(t)u(t)dt < \infty\}$, i.e., the set of locally square-integrable functions.

b) $L^+_{2,\text{s}}(x_0;x,A) \triangleq \left\{u \in L^N_{2,\text{loc}} \mid \text{s.t} \lim_{t\to\infty} x(t) = 0,\ \dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0\right\}$.

It can be proved, for instance see lemma 2.1 [38], that the control spaces mentioned above are convex. We take $L^N_{2,\text{loc}}$ as the choice of control space $\mathscr{U}$ unless otherwise specified. In the following corollary we give conditions under which assumption 2.3.1 is satisfied.

**Lemma 2.5.1.** *For ($P^{LQ}$), if the pair $(A,B)$ is controllable then assumption 2.3.1 holds true.*

*Proof.* If $u^*$ is Pareto optimal then by proposition 2.3.1 $u^*$ is optimal for the $P_i^{LQ}$, $i \in \overline{N}$. Here, $P_i^{LQ}$ represents the linear quadratic analog of the constrained subproblem $P_i$ defined in section 2.3. The necessary conditions for optimality of $u^*$ are given by equations (2.19). Assuming an interior solution $u^*(t)$, the first order condition translates to $e^{-\rho t}\left(2R_{\lambda_i}u^*(t) + 2S_{\lambda_i}x^*(t)\right) + B'\lambda(t) = 0$. If $\overrightarrow{\lambda}_i = 0$, then conditions lead to $\dot{\lambda}_i(t) = -A'\lambda_i(t)$, $\dot{\gamma}_i(t) = 0$ and the above first order condition would be $B'\lambda_i(t) = 0$. This implies $B'\dot{\lambda}_i(t) = -B'A'\lambda_i(t) = 0$. Repeating the same $n-1$ times we see that $\lambda'(t)\begin{bmatrix}B & AB & A^2B & \cdots & A^{n-1}B\end{bmatrix} = 0$. As, $(A,B)$ is controllable we necessarily have $\lambda_i(t) = 0$ for all $t$. Further as $\dot{\gamma}_i(t) = 0$ and $\lim_{t\to\infty}\gamma_i(t) = 0$ we have $\gamma_i(t) = 0$ for all $t$. But this violates the necessary condition (2.19a). So, $\overrightarrow{\lambda}_i \in \mathbb{R}^N_+\backslash\{0\}$ and assumption 2.3.1 holds true. ∎

*Remark* 2.5.1. Specializing corollary 2.3.2 to the linear quadratic case to guarantee assumption 2.3.1 may require restrictions on the system parameters and control space $\mathscr{U}$[6].

In the next theorem we specialize theorem 2.3.3 for the linear quadratic case. Towards that end, we define $z(t) = e^{-\rho t/2}x(t)$, $v(t) = e^{-\rho t/2}u(t)$, $p(t) = e^{\rho t/2}\lambda(t)$ and $\tilde{A} = A - \frac{\rho}{2}I$.

**Theorem 2.5.1.** *Let $(A,B)$ is controllable. If $(J_1(x_0,u^*),\cdots,J_N(x_0,u^*))$ is a Pareto solution for the problem (2.25,2.26) then there exists an $\overrightarrow{\alpha} \in \mathscr{P}_N$ such that*

$$\begin{aligned} &e^{-\rho t}\left[x^{*\prime}(t) \quad u^{*\prime}(t)\right]M_\alpha\left[x^{*\prime}(t) \quad u^{*\prime}(t)\right]' + \lambda'(t)\left(Ax^*(t) + Bu^*(t)\right) \leq \\ &\qquad e^{-\rho t}\left[x^{*\prime}(t) \quad u^{*\prime}(t)\right]M_\alpha\left[x^{*\prime}(t) \quad u'(t)\right]' + \lambda'(t)\left(Ax^*(t) + Bu(t)\right) \end{aligned} \tag{2.27a}$$

$$\dot{x}^*(t) = Ax^*(t) + Bu^*(t),\ x^*(0) = x_0, \tag{2.27b}$$

$$\dot{\lambda}(t) = -A'\lambda(t) - e^{-\rho t}\left(2Q_\alpha x^*(t) + 2S_\alpha u^*(t)\right)\ \lambda(0) = l_0 \in \mathbb{R}^n. \tag{2.27c}$$

*In case $\alpha$ is such that $R_\alpha > 0$, the above equations can be equivalently rephrased as that every Pareto optimal control satisfies $u^*(t) = e^{\rho t/2}v^*(t)$ where $v^*(t) = -R_\alpha^{-1}\big(S_\alpha z^*(t)+$*

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[6] A sufficient condition to satisfy assumption 2.3.1 can be shown as $(A,\ B)$ controllable, $A$ is stable and $u(t) \in E \subset \mathbb{R}^m$, with $E$ being a bounded set.

$B'p(t)\big)$. $(z^*(t),p(t))$ *is the solution of the linear autonomous differential equation given by:*

$$\begin{bmatrix}\dot{z}^*(t)\\ \dot{p}(t)\end{bmatrix} = G_\alpha \begin{bmatrix} z^*(t)\\ p(t)\end{bmatrix},\ z(0)=x_0 \text{ (given)},\ p(0)=l_0\in\mathbb{R}^n. \tag{2.28}$$

*Proof.* $(P^{LQ})$ is a special case of $(P^\rho)$. Again using proposition 2.3.1 we have $u^*$ is optimal for $P_i^{LQ}$, $i\in\overline{N}$. Since $(A,B)$ is controllable from lemma 2.5.1 assumption 2.3.1 holds true, i.e., $\overrightarrow{\lambda}_0^i \in \mathbb{R}_+^N\backslash\{0\}, i\in\overline{N}$. So, the necessary conditions (2.27a-2.27c) follow directly from theorem 2.3.3. ■

*Remark* 2.5.2. Here, $G_\alpha$ is a Hamiltonian matrix given by

$$G_\alpha \triangleq \begin{bmatrix} \tilde{A}-BR_\alpha^{-1}S_\alpha & -BR_\alpha^{-1}B' \\ -\left(Q_\alpha - S_\alpha R_\alpha^{-1}S_\alpha\right) & -\left(\tilde{A}-BR_\alpha^{-1}S_\alpha\right)'\end{bmatrix}.$$

The extremal trajectories generated by the Hamiltonian flow (2.28) depend on $x_0$ and $\alpha$. The additional information that we have is $p(0)=l_0\in\mathbb{R}^n$ is bounded. The eigenvalues of the Hamiltonian matrix $G_\alpha$ are symmetric w.r.t real and imaginary axis. So, $G_\alpha$ has at most $n$ eigenvalues with negative real part. Bounded trajectories of (2.28) evolve on the stable manifolds and converge towards the equilibrium points of (2.28). The state of the Hamiltonian system, $\begin{bmatrix} z'(t) & p'(t)\end{bmatrix}'$, has $2n$ variables and out of which only $n$, related to $x_0$, are free. The co-state variable $p(t)$ can be obtained as a result of the above boundedness restriction and as a result depends on the initial state $x_0$. In nonlinear models, it is very common to have multiple co-state trajectories, converging to the equilibrium point, resulting in the same optimal cost, see [48, chapter 5].

To restrict the number of possible extremal trajectories we make the following assumption on admissible controls.

*Assumption* 2.5.1. The admissible controls $v$ satisfy the property: $v\in L_{2,s}^+(x_0,z,\tilde{A})$.

Notice, assumption 2.5.1 requires $\lim_{t\to\infty} z(t)=0$ whereas $x(t)=e^{\rho t/2}z(t)$ can grow unbounded. Strong restrictions on the system parameters ensure $\lim_{t\to\infty} x(t)=0$ , see section 5 of [26] for more details. Theorem 2.5.1 only gives necessary conditions and solving these equations we obtain Pareto candidates. Further, we notice that these necessary conditions are similar, with controllability assumption, to necessary conditions for optimality of a weighted sum optimal control problem. The following theorem relates Pareto optimality with weighted sum minimization. We first define the weighted sum objective as:

$$J_\beta(x_0,u)=\int_0^\infty e^{-\rho t}\begin{bmatrix} x'(t) & u'(t)\end{bmatrix} M_\beta \begin{bmatrix} x'(t) & u'(t)\end{bmatrix}' dt,$$

where $\overrightarrow{\beta}\in\mathscr{P}_N$, $R_i>0$, $i\in\overline{N}$ and $x(t)$ solves

$$\dot{x}(t)=Ax(t)+Bu(t),\ x(0)=x_0. \tag{2.29}$$

**Theorem 2.5.2.** *Let (A,B) be controllable. If $u^*$ is Pareto optimal then there exists a $\overrightarrow{\beta} \in \mathscr{P}_N$ such that the following condition holds true*

$$J_\beta(x_0,u) - J_\beta(x_0,u^*) + \lim_{t\to\infty} \lambda'(t)\,(x(t)-x^*(t)) = J_\beta(0,u-u^*). \tag{2.30}$$

*Proof.* First we notice,

$$\begin{aligned}
J_\beta(x_0,u) - J_\beta(x_0,u^*) &= \int_0^\infty e^{-\rho t}\begin{bmatrix} x(t) \\ u(t)\end{bmatrix}' M_\beta \begin{bmatrix} x(t) \\ u(t)\end{bmatrix} dt - \int_0^\infty e^{-\rho t}\begin{bmatrix} x^*(t) \\ u^*(t)\end{bmatrix}' M_\beta \begin{bmatrix} x^*(t) \\ u^*(t)\end{bmatrix} dt \\
&= \int_0^\infty e^{-\rho t}\begin{bmatrix} x(t)-x^*(t) \\ u(t)-u^*(t)\end{bmatrix}' M_\beta \begin{bmatrix} x(t)-x^*(t) \\ u(t)-u^*(t)\end{bmatrix} dt \\
&\quad + 2\int_0^\infty e^{-\rho t}\begin{bmatrix} x^*(t) \\ u^*(t)\end{bmatrix}' M_\beta \begin{bmatrix} x(t)-x^*(t) \\ u(t)-u^*(t)\end{bmatrix} dt.
\end{aligned}$$

We recognize the first part of the sum on the righthand side of the above equations as $J_\beta(0,u-u^*)$. Since $u^*$ is Pareto optimal from theorem 2.5.1 there exists a $\overrightarrow{\beta} \in \mathscr{P}_N$ such that (2.27a-2.27c) hold true. Now, we observe using conditions (2.27a) and (2.27c) that

$$\begin{aligned}
\lim_{t\to\infty} \lambda'(t)\,(x(t)-x^*(t)) - \lambda'(0)\,(x(0)-x^*(0)) &= \int_0^\infty \frac{d}{dt}\left(\lambda'(t)\,(x(t)-x^*(t))\right) dt \\
&= -2\int_0^\infty e^{-\rho t}\begin{bmatrix} x^*(t) \\ u^*(t)\end{bmatrix}' M_\beta \begin{bmatrix} x(t)-x^*(t) \\ u(t)-u^*(t)\end{bmatrix} dt.
\end{aligned}$$

Since $\lambda(0)$ is bounded, the second term in sum on the lefthand side of the above equation vanishes and results in equation (2.30). ■

Theorem 2.5.3 given below states that, under controllability condition, for a fixed initial state a weighted sum (single player) linear quadratic optimal control problem has a solution if and only if the cost function is convex in $u$, the necessary conditions resulting from the maximum principle and a transversality condition are satisfied.

**Theorem 2.5.3.** *We have the following assertions for $J_\beta(x_0,v)$.*

*a) (Convexity) For any $\alpha \in [0,1]$, $u_i \in \mathscr{U}$, $i = 1,2$ and $\overrightarrow{\beta} \in \mathscr{P}_N$ we have*

$$\begin{aligned}
\alpha J_\beta(x_0,u_1) + (1-\alpha)J_\beta(x_0,u_2) - J_\beta(z_0,\alpha u_1 + (1-\alpha)u_2)& \\
= \alpha(1-\alpha)J_\beta(0,u_1-u_2).&
\end{aligned} \tag{2.31}$$

*b) Let $(A,B)$ be controllable, then $u^* = \arg\min_{u\in\mathscr{U}} J_\beta(x_0,u)$ exists if*

*i) $\min_{u\in\mathscr{U}} J_\beta(0,u)$ exists (and equals zero).*

*ii) there exist $u^*$, $x^*$ and $\lambda$ that satisfy, for all $t \geq 0$*

$$e^{-\rho t}\left(2Q_\beta x^*(t) + 2R_\beta u^*(t)\right) + B'\lambda(t) = 0, \text{ where} \tag{2.32a}$$

$$\dot{x^*}(t) = Ax^*(t) + Bu^*(t)\ x^*(0) = x_0 \tag{2.32b}$$

$$\dot{\lambda}(t) = -A'\lambda(t) - e^{-\rho t}\left(2Q_\beta x^*(t) + 2S_\beta u^*(t)\right)\ \lambda(0) = l_0 \in \mathbb{R}^n. \tag{2.32c}$$

*Conversely, if (i), (ii) and in addition* $\liminf_{t\to\infty} \lambda'(t)(x^*(t) - x(t)) \geq 0$ *holds true then* $u^* = \arg\min_{u\in\mathscr{U}} J_\beta(x_0, u)$.

*Proof.* a) By linearity of the system (2.29) if $x_i(t)$ is generated by $u_i(t)$ with $x_i(0) = x_0$ for $i = 1, 2$. Then for $\alpha \in [0, 1]$, $\alpha u_1(t) + (1-\alpha)u_2(t)$ generates $\alpha x_1(t) + (1-\alpha)x_2(t)$ with initial state as $x_0$.

$$\begin{aligned}
&J_\beta(x_0, \alpha u_1(t) + (1-\alpha)u_2(t)) \\
&\qquad = \int_0^\infty e^{-\rho t}\begin{bmatrix}\alpha x_1(t) + (1-\alpha)x_2(t) \\ \alpha u_1(t) + (1-\alpha)u_2(t)\end{bmatrix}' M_\beta \begin{bmatrix}\alpha x_1(t) + (1-\alpha)x_2(t) \\ \alpha u_1(t) + (1-\alpha)u_2(t)\end{bmatrix} dt \\
&\qquad = \alpha^2 J_\beta(x_0, u_1) + (1-\alpha)^2 J_\beta(x_0, u_2) \\
&\qquad\quad + 2\alpha(1-\alpha)\int_0^\infty e^{-\rho t}\begin{bmatrix}x_1(t) \\ u_1(t)\end{bmatrix}' M_\beta \begin{bmatrix}x_2(t) \\ u_2(t)\end{bmatrix} dt
\end{aligned}$$

$$\begin{aligned}
&\alpha J_\beta(x_0, u_1) + (1-\alpha) J_\beta(x_0, u_2) - J_\beta(x_0, \alpha u_1(t) + (1-\alpha)u_2(t)) \\
&\qquad = \alpha(1-\alpha)\int_0^\infty e^{-\rho t}\begin{bmatrix}x_1(t) - x_2(t) \\ u_1(t) - u_2(t)\end{bmatrix}' M_\beta \begin{bmatrix}x_1(t) - x_2(t) \\ u_1(t) - u_2(t)\end{bmatrix} dt.
\end{aligned}$$

Using the linearity property we identify the integral on the right hand side as $J_\beta(0, u_1 - u_2)$.

b) $\Rightarrow$ First, we have $x(t, x_0, u) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds$. So, introducing $p(t) = e^{At}x_0$ and $w(t) = \int_0^t e^{A(t-s)}Bu(s)ds$, we have $x(t) = p(t) + w(t)$. Some elementary calculations then show that

$$J(x_0, u) = J(0, u) + \int_0^\infty e^{-\rho s}p'(s)Q_0 p(s)ds + 2\int_0^\infty e^{-\rho s}\left(p'(s)Q_0 w(s) + p'(s)V_0 u(s)\right)ds.$$

Therefore for any $\tau \in \mathbb{R}$ we have

$$J(x_0, \tau u) = \tau^2 J(0, u) + \int_0^\infty e^{-\rho s}p'(s)Q_0 p(s)ds + 2\tau\int_0^\infty e^{-\rho s}\left(p'(s)Q_0 w(s) + p'(s)V_0 u(s)\right)ds.$$

So, if $J(0, u) < 0$ for some $u \in \mathscr{U}$, $J(z_0, \tau u)$ can be made arbitrarily small by choosing $\tau$ large enough. Therefore we conclude that if $\min_{u\in\mathscr{U}} J(z_0, u)$ exists, necessarily $J(0, u) \geq 0$ for every $u \in \mathscr{U}$. Since $J(0, 0) = 0$ it is obvious that condition (i) holds. Since $u^*$ is

a minimizer the necessary conditions for optimality hold in normal form following the reasoning given by lemma 2.5.1, and (ii) holds true.
$\Leftarrow$ If there exists a $u^*$ satisfying (2.32) then following the proof of theorem 2.5.2 we have $J_\beta(x_0,u) - J_\beta(x_0,u^*) + \lim_{t\to\infty}\lambda'(t)\,(x(t)-x^*(t)) = J_\beta(0,u-u^*)$ for all $u \in \mathscr{U}$. From (i) we have $J_\beta(x_0,u) - J_\beta(x_0,u^*) \geq \lim_{t\to\infty}\lambda'(t)\,(x^*(t)-x(t))$. By assumption, we have $\liminf_{t\to\infty}\lambda'(t)\,(x^*(t)-x(t)) \geq 0$. So, $J_\beta(x_0,u) \geq J_\beta(x_0,u^*)$ for all $u \in \mathscr{U}$. ■

### 2.5.1 Fixed initial state

In this section we give additional properties of Pareto solutions that arise due to linear quadratic nature of the game $(P^{LQ})$. First, the following properties hold true due to the linearity of the game $(P^{LQ})$, refer lemma 3.2 [38] for a detailed proof.

**Lemma 2.5.2.** *Assume $u^*$ is a Pareto optimal control for (2.25, 2.26). Then $\mu u^*$ is a Pareto optimal control for (2.25, 2.26) with $x(0) = \mu x_0$.*

As a result of the above lemma, if for the initial state $x_0 = 0$ there exists a Pareto solution different from zero, then all points on the half-line connecting this point and zero are also Pareto solutions. We state this observation formally in the next corollary.

**Corollary 2.5.1.** *Consider the cooperative game (2.25, 2.26) with $x_0 = 0$. Assume $u^*$ is a Pareto optimal control for this game yielding the Pareto solution $J^*$. Then for all $\mu \in \mathbb{R}$, $\mu u^*$ yields the Pareto solution $\mu^2 J^*$.*

For the two player case in a finite horizon setting theorem 3.6 [38] shows that all Pareto solutions can be obtained using the weighting method using a technical lemma 3.5 [38]. The following theorem is an infinite horizon counter part of theorem 3.6 [38].

**Theorem 2.5.4.** *Let (A,B) be controllable. Consider the two-player case of the problem with $R_i > 0$.*

1. *If $(J_1(x_0,u^*), J_2(x_0,u^*))$ is a Pareto solution for problem $(P^{LQ})$, then there exists an $\alpha \in [0,1]$ such that*
   - *(i) (2.32) holds (with $u^*$ defined correspondingly) and*
   - *(ii) for this $\alpha$, $\inf_u J_\alpha(0,u) = \inf_u(\alpha J_1(0,u) + (1-\alpha)J_2(0,u))$ exists.*
2. *Conversely, if there exists an $\alpha \in (0,1)$ such that (i) and (ii) above hold true, then $\big(J_1(x_0,u^*),\, J_2(x_0,u^*)\big)$ is a Pareto solution.*

*Proof.* 1) From lemma 3.5 and item (1) of theorem 3.6 in [38], it follows that if $(J_1(x_0,u^*), J_2(x_0,u^*))$ is a Pareto solution then $u^*$ minimizes the weighted sum $J_\alpha(x_0,u)$. The remaining part follows from theorem item (b) of 2.5.3. 2) Follows from item (b) of theorem 2.5.3 and a direct application of lemma 2.2.1. ■

*Remark* 2.5.3. Note that theorem 2.5.4 does not assume that the cost functions $J_i$ are convex, and almost all Pareto solutions can be obtained using the weighting method. Here 'almost all' refers to Pareto solutions which are obtained by using the weights in the interval $(0,\ 1)$. The only additional Pareto optimal solutions that may exist are obtained by considering strategies $\tilde{u} = \arg\min_{u\in\mathscr{U}} J_i(x_0,u)$ for some $i \in \overline{N}$, i.e., giving a weight equal to 1 for a single player. It is still unclear if the same conclusion can be derived in a $N(>2)$ player setting.

### 2.5.2 Arbitrary initial state

In this section we consider conditions under which $(P^{LQ})$ has a Pareto solution for an arbitrary initial state. It is well known [110, 119] that the solution of (2.28) is closely related to the existence of the solution to the following algebraic Riccati equation (ARE):

$$\tilde{A}'X + X\tilde{A} - (XB + S_\alpha)R_\alpha^{-1}\left(B'X + S_\alpha\right) + Q_\alpha = 0. \qquad \text{(ARE)}$$

Using theorem 5 [110] and theorem 13.9 [119], we state the following proposition:

**Proposition 2.5.1.** *Let $(\tilde{A},B)$ be controllable. Then the following are equivalent.*

*a) The frequency domain inequality (FDI) satisfies*

$$\Psi_\alpha(j\omega) = \begin{bmatrix} B'(-j\omega-\tilde{A}')^{-1} & I \end{bmatrix} M_\alpha \begin{bmatrix} (j\omega-\tilde{A})^{-1}B \\ I \end{bmatrix} \geq \varepsilon B'(-j\omega-\tilde{A}')^{-1}(j\omega-\tilde{A})^{-1}B,$$

*for some $\varepsilon > 0$ and $0 \leq \omega \leq \infty$.*

*b) The (ARE) has a unique real symmetric stabilizing solution $X_\alpha$ such that $\sigma(\tilde{A} - BR_\alpha^{-1}(B'X_\alpha + S_\alpha)) \in \mathbb{C}_-$.*

*c) The Hamiltonian matrix $G_\alpha$ has no $j\omega -$ axis eigenvalues and there exists an $n$ dimensional stable graph subspace[7].*

**Lemma 2.5.3.** *Let $(\tilde{A},B)$ be controllable. If the (ARE) has a real symmetric stabilizing solution for $\overrightarrow{\alpha}_1,\cdots,\overrightarrow{\alpha}_k \in \mathscr{P}_N$ then (ARE) has a solution for all $\overrightarrow{\alpha}$ in the cone $\mathscr{K}(\overrightarrow{\alpha}_1,\overrightarrow{\alpha}_2,\cdots,\overrightarrow{\alpha}_k)$, where*

$$\mathscr{K}(\overrightarrow{\alpha}_1,\overrightarrow{\alpha}_2,\cdots,\overrightarrow{\alpha}_k) = \left\{ \overrightarrow{\alpha} \in \mathscr{P}_N \middle| \overrightarrow{\alpha} = \sum_{i=1}^{k} \kappa_i \overrightarrow{\alpha}_i,\ \kappa_i > 0,\ i = 1,2,\cdots,k \right\}.$$

---

[7] Let $S$ represent the subspace spanned by eigenvectors, denoted by $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_{2n\times n}$, associated with stable eigenvalues. If $X_{1_{n\times n}}$ is invertible then we call $S$ a stable graph subspace.

*Proof.* From proposition 2.5.1(b), if the (ARE) has a solution for $\overrightarrow{\alpha}_i \in \mathscr{P}_N$ then $\Psi_{\alpha_i}(j\omega)$ satisfies the inequality given in proposition 2.5.1(a) for $i = 1,2,\cdots,k$. So, for any $\overrightarrow{\alpha} \in \mathscr{K}(\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \cdots, \overrightarrow{\alpha}_k)$, $\Psi_\alpha(j\omega) = \sum_{i=1}^k \kappa_i \Psi_{\alpha_i}(j\omega) \geq \varepsilon B'(-j\omega - \tilde{A}')^{-1}(j\omega - \tilde{A})^{-1}B$ for some $\varepsilon > 0$ and $0 \leq w \leq \infty$. Again from proposition 2.5.1 we have that (ARE) with $\overrightarrow{\alpha} \in \mathscr{K}(\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \cdots, \overrightarrow{\alpha}_k)$ has a real symmetric stabilizing solution. ■

In the theorem 2.5.5 given below we consider the special case when (ARE) has a stabilizing solution for the vertices of the simplex $\mathscr{P}_N$.

**Theorem 2.5.5.** *Let $(\tilde{A}, B)$ be stabilizable and $v \in L^+_{2,s}(x_0, z, \tilde{A})$. Assume (ARE) has a solution for $\overrightarrow{\alpha} = \overrightarrow{e}_i$, $i = 1,2,\cdots,N$, where $\overrightarrow{e}_i$ is the $i^{th}$ standard unit vector in $\mathbb{R}^N$. Then for all initial states a Pareto solution exists. For a fixed initial state the set of all Pareto solutions is given by $\{(J_1(x_0, u^*_\alpha), \cdots, J_N(x_0, u^*_\alpha))\}$. Here, for a fixed $\overrightarrow{\alpha} \in \mathscr{P}_N$,*

$$u^*_\alpha(t) = -e^{-\rho t/2} R_\alpha^{-1} \left(B'X_\alpha + V_\alpha\right) z^*(t),$$

*where $z^*(t)$ satisfies $\dot{z}^*(t) = \left(\tilde{A} - BR_\alpha^{-1}\left(B'X_\alpha + V_\alpha\right)\right) z^*(t)$, $z^*(0) = x_0$.*

*Proof.* If $\overrightarrow{\alpha} = \overrightarrow{e}_i$ then the game problem reduces to a single player optimal control problem. Recalling theorem 2.7 [38], if $(\tilde{A}, B)$ is stabilizable then $\min_{u \in \mathscr{U}} J_i(x_0, u)$ exists if and only if (ARE) has a unique stabilizing real symmetric solution $X$. Under this condition $\min_{u \in \mathscr{U}} J_i(x_0, u) = x_0' X_i x_0$ is attained uniquely by $u^*(t) = -e^{\rho t/2} R_i^{-1}\left(B'X_i + V_i\right) z^*(t)$, where $z^*(.)$ solves $\dot{z}^*(t) = \left(\tilde{A} - BR_i^{-1}\left(B'X_i + V_i\right)\right) z^*(t)$. Clearly, we have $J_i(0,u) \geq \min_u J_i(0,u) = 0$. So, from theorem 2.5.3, $J_i(x_0, u)$ is strictly convex in $u$. As a result if (ARE) has a real symmetric stabilizing solution for $\overrightarrow{\alpha} = \overrightarrow{e}_i$, $i = 1,2,\cdots,N$ then players' objectives $J_i(x_0, u)$ are strictly convex in $u$. Since the choice of control space is convex, from theorem 2.4.1 it follows that for all Pareto optimal $u^*$ there exist $\overrightarrow{\alpha} \in \mathscr{P}_N$ such that $u^* = \arg\min_{u \in \mathscr{U}} \sum_i^N \alpha_i J_i(x_0, u)$ for all $x_0$. Notice, we only require that $(\tilde{A}, B)$ to be stabilizable for this conclusion. Further, there exists a one-one correspondence between Pareto surface and $\mathscr{P}_N$. ■

In the following lemma we show under certain conditions that a Pareto optimal control minimizes a weighted sum optimal control problem.

**Lemma 2.5.4.** *Let $(\tilde{A}, B)$ be controllable and $v \in L^+_{2,s}(x_0, z, \tilde{A})$. If $u^*$ is Pareto optimal then there exists a $\overrightarrow{\beta} \in \mathscr{P}_N$ such that conditions (2.30) holds true. Further, if (ARE) has a unique real symmetric stabilizing solution for this $\overrightarrow{\beta}$ then $u^*$ minimizes $J_\beta(x_0, u)$.*

*Proof.* If $u^*$ is Pareto optimal then from theorem 2.5.2 there exists a $\overrightarrow{\beta} \in \mathscr{P}_N$ such that (2.30) holds true. Suppose if (ARE) has a unique real symmetric stabilizing solution $X_\beta$, then $J_\beta(x_0, u)$ is strictly convex. So, there exists a unique $\tilde{u} = \arg\min_{u \in \mathscr{U}} J_\beta(x_0, u)$ such that $J(x_0, \tilde{u}) = x_0' X_\beta x_0$, in particular $\min_{u \in \mathscr{U}} J_\beta(0, u) = 0$. To show $u^* = \tilde{u}$ we proceed as follows. From (2.30) and above arguments we have $J_\beta(x_0, u) - J_\beta(x_0, u^*) + \lim_{t \to \infty} \lambda'(t)$

$(x(t) - x^*(t)) \geq 0$. With straightforward calculations we can show that $\lim_{\to\infty} p'(t)(z^*(t) - z(t)) = \lim_{\to\infty} \lambda'(t)(x^*(t) - x(t))$. From, proposition 2.5.1 we have that $G_\beta$ has an $n$ dimensional stable graph subspace. So, the optimal co-state rule is given uniquely by $p(t) = X_\beta z^*(t)$. As $X_\beta$ is stabilizing and $v \in L^+_{2,s}(x_0, z, \tilde{A})$, we have that $\lim_{t\to\infty}(z(t) - z^*(t)) = 0$, and as a result $\lim_{t\to\infty} p'(t)(z(t) - z^*(t)) = 0$ . Clearly, $J_\beta(x_0, u) - J_\beta(x_0, u^*) \geq 0$ for all $x_0$. So, $u^*$ also minimizes $J_\beta(x_0, u)$. From the uniqueness of the minimizer we have $u^* = \tilde{u}$. ■

A question which next naturally arises is whether we can characterize all Pareto solutions in a way similar to theorem 2.5.5 if there exists some player who can obtain arbitrarily low costs if he is allowed to manipulate all control instruments that affect the system, that is, if not all cost functions are convex. Such situations occur if, say player 1 could, by choosing the actions of player 2, achieve arbitrarily low costs (i.e., gains). This occurs at the expense of player 2, whose costs increase using the corresponding control scheme. We use lemma 2.5.3 and lemma 2.5.4 to address this issue in the following corollary.

**Corollary 2.5.2.** *Let $(A, B)$ be controllable and $v \in L^+_{2,s}(x_0, z, \tilde{A})$. Consider the problem with $R_i > 0$. Assume (ARE) has a solution for $\overrightarrow{\alpha}_i \in \mathscr{P}_N$, $i = 1, 2, \cdots, k$. Then for all initial states a Pareto solution exists. For a fixed initial state and for all $\overrightarrow{\alpha} \in \mathscr{K}(\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \cdots, \overrightarrow{\alpha}_k)$, $\{(J_1(x_0, u^*_\alpha), \cdots, J_N(x_0, u^*_\alpha))\}$ yield Pareto solutions. Here $u^*_\alpha(t) = e^{-\rho t/2} v^*(t)$, $v^*(t) = -R_\alpha^{-1}(S_\alpha + B'X_\alpha) z^*(t)$, $z^*(t)$ solves the differential equation $\dot{z}^*(t) = \big(\tilde{A} - BR_\alpha^{-1}\big(S_\alpha + B'X_\alpha\big)\big) z^*(t)$, $z^*(0) = x_0$ and $X_\alpha$ solves (ARE).*

The number of extremal trajectories are considerably reduced by assumption 2.5.1. The co-state rule, see section 5.3 of [48], which defines the co-state trajectory, in general, depends on the initial state. However, if (ARE) has a unique stabilizing solution it is defined by $p(t) = X_\alpha z^*(t)$ and does not depend on the choice of the initial state. The following example demonstrates these subtleties.

*Example* 2.5.1. Consider the cooperative game with

$$J_i(x_0, u) = \int_0^\infty e^{-\rho t} \sum_{j=1}^{2} \big(q_{ij}^2 x_j^2(t) + r_{ij}^2 u_j^2(t)\big)\, dt,\ i = 1, 2$$

$$\text{subject to} \qquad \dot{x}(t) = (A + \frac{\rho}{2} I) x(t) + B_1 u_1(t) + B_2 u_2(t),\ x(0) = x_0 \neq 0,\ \rho > 0$$

$$v(t) = e^{\rho t/2} v(t),\ v \in L^+_{2,s}(x_0, z, \tilde{A}).$$

Here we take $A = \text{diag}(a_1, a_2)$, $[B_1\ B_2] = I$. Choosing $r_{ij} > 0$, $q_{i2} > 0$, $i, j = 1, 2$, $q_{11} = -a_1^2 r_{11}$ and $q_{21} = -a_1^2 r_{21}$ we can show that the eigenvalues of the Hamiltonian matrix $G_\alpha$ as $\{-s,\ 0,\ 0,\ s\}$, where $s = \sqrt{a_2^2 + \frac{\alpha q_{12} + (1-\alpha) q_{22}}{\alpha r_{12} + (1-\alpha) r_{22}}} > 0$. The eigenvector and generalized eigenvector corresponding to eigenvalue zero are $\mathrm{v}_0^1 = \begin{bmatrix} 1 & 0 & a_1(\alpha r_{11} + (1-\alpha) r_{12}) & 0 \end{bmatrix}'$ and $\mathrm{v}_0^2 = \begin{bmatrix} 1/a_1 & 0 & 0 & 0 \end{bmatrix}'$. The eigenvector corresponding to the stable eigenvalue $-s$ is calculated as $\mathrm{v}_{-s}^1 = \begin{bmatrix} 0 & \frac{s - a_2}{a q_{12} + (1-a) q_{22}} & 0 & 1 \end{bmatrix}'$. The admissible extremal trajectories could belong to one of the subspaces, namely $\mathrm{sp}\{\mathrm{v}_0^1\}$, $\mathrm{sp}\{\mathrm{v}_0^1,\ \mathrm{v}_0^2\}$, $\mathrm{sp}\{\mathrm{v}_{-s}^1\}$ and $\mathrm{sp}\{\mathrm{v}_{-s}^1,\ \mathrm{v}_0^1\}$. Taking assumption 2.5.1 some trajectories can be ruled out.

$\underline{(z^*(t),p(t)) \in \mathrm{sp}\{\mathrm{v}_0^1\}}$: In this case the we obtain $v_1^*(t) = -a_1 z_1^*(t)$, $v_2^*(t) = 0$, $z_1^*(t) = x_1(0)$, $z_2^*(t) = e^{a_2 t} x_2(0) = 0$, $p_1(t) = -a_1(\alpha r_{11} + (1-\alpha) r_{12}) z_1^*(t)$ and $p_2(t) = 0$ for all $t$.

$\underline{(z^*(t),p(t)) \in \mathrm{sp}\{\mathrm{v}_0^1,\ \mathrm{v}_0^2\}}$: In this case the we obtain, in addition to the above extremal, $v_1^*(t) = 0$, $v_2^*(t) = 0$, $z_1^*(t) = e^{a_1 t} x_1(0)$, $z_2^*(t) = e^{a_2 t} x_2(0) = 0$, $p_1(t) = 0$ and $p_2(t) = 0$ for all $t$.

$\underline{(z^*(t),p(t)) \in \mathrm{sp}\{\mathrm{v}_0^1,\ \mathrm{v}_{-s}^1\}}$: In this case the we obtain $v_1^*(t) = -a_1 z_1^*(t)$, $v_2^*(t) = -\frac{(s-a_2)(\alpha r_{12}+(1-\alpha) r_{22})}{\alpha q_{12}+(1-\alpha) q_{22}} z_2^*(t)$, $z_1^*(t) = x(0)$, $z_2^*(t) = e^{-st} x_2(0)$, $p_1(t) = a_1(\alpha r_{11} + (1-\alpha) r_{12}) z_1^*(t)$ and $p_2(t) = \frac{(s-a_2)}{\alpha q_{12}+(1-\alpha) q_{22}} z_2^*(t)$ for all $t$.

$\underline{(z^*(t),p(t)) \in \mathrm{sp}\{\mathrm{v}_{-s}^1\}}$: In this case the we obtain $v_1^*(t) = 0$, $v_2^*(t) = -\frac{(s-a_2)(\alpha r_{12}+(1-\alpha) r_{22})}{\alpha q_{12}+(1-\alpha) q_{22}} z_2^*(t)$, $z_1^*(t) = e^{a_1 t} x(0) = 0$, $z_2^*(t) = e^{-st} x_2(0)$, $p_1(t) = 0$ and $p_2(t) = \frac{(s-a_2)}{\alpha q_{12}+(1-\alpha) q_{22}} z_2^*(t)$ for all $t$.

Since $v \in L_{2,s}^+(x_0, z, \tilde{A})$, admissible extremals satisfy $\lim_{t\to\infty} z^*(t) = 0$. In the first two cases the obtained extremal trajectories are possible only with $x_0 = 0$. In the latter two cases we see that the obtained extremals are possible with $x_0 = (0, x_2(0))$ where $x_2(0)$ is arbitrary. Further, these extremals are same. Now we check if this extremal is Pareto optimal using the sufficient conditions given by theorem 2.4.2. The minimized Hamiltonian is given by $H^0(.) = 0$. Since admissibility requires $\lim_{t\to\infty} z(t) = 0$ we see that $\lim_{t\to\infty} p'(t)(z^*(t) - z(t)) = 0$. So, the obtained controls are indeed Pareto optimal. Further, we see that not all initial states, in particular $x_1(0) \neq 0$, result in a Pareto solution.

### 2.5.3 The scalar case

In this subsection we discuss the scalar case in more detail. Notice, assuming $b_i \neq 0$, $i = 1, 2, \cdots, N$, controllability is trivially satisfied. Using lemma 2.5.2 some interesting observations can be derived for the scalar case. The choice of control space is taken as $L_{2,s}^+(x_0, z, \tilde{A})$. So, all the admissible trajectories satisfy $\lim_{t\to\infty} z(t) = 0$. The Hamiltonian matrix takes the form $G_\alpha = \begin{bmatrix} e & -f \\ -g & -e \end{bmatrix}$, with $f > 0$. Now, we have the following 3 possible cases.

a) $G_\alpha$ has eigenvalue zero with geometric multiplicity 1. Straightforward calculations show that $z(t) = x_0 + (ex_0 - fl_0)t$ and $p(t) = l_0 - (gx_0 - el_0)t$. For admissibility of extremal trajectories we require $\lim_{t\to\infty} z(t) = 0$, and this is possible only if $x_0 = 0$ and $l_0 = 0$ as $f \neq 0$. The possibility of eigenvalue zero with geometric multiplicity

2 is ruled out as $f \neq 0$.

b) $G_\alpha$ has complex eigenvalues. Again, straight forward calculations show that $z(t) = x_0 \cos(wt) + (ex_0 - fl_0)\sin(wt)$ and $p(t) = l_0 \cos(wt) - (gx_0 + el_0)\sin(wt)$. Following the same reasoning as above we have that $x_0 = 0$ and $l_0 = 0$.

c) The eigenvector corresponding to the stable eigenvalue is always a graph subspace. If $\sigma > 0$ is an eigenvalue of $G_\alpha$ then $-\sigma$ is also an eigenvalue of $G_\alpha$. As $f > 0$, the eigenvector corresponding to $-\sigma$ can always be taken as $\begin{bmatrix} 1 & (e+\sigma)/f \end{bmatrix}'$.

Theorem 2.5.6 below states that if $(P^{LQ})$ has a Pareto solution for a non zero initial state, then it can be found using the weighting method.

**Lemma 2.5.5.** *Consider the scalar system with $R_i > 0,\ i \in \overline{N}$ and $v \in L^+_{2,s}(x_0, z, \tilde{A})$. Let $x_0 \neq 0$. If $(P^{LQ})$ has a Pareto optimal control $\tilde{u}^*(x_0)$ then there exists an $\overrightarrow{\alpha} \in \mathscr{P}_N$ such that* $\min \sum_{i\in\overline{N}} \alpha_i J_i$ *subject to (2.26) has a solution for all initial states (including $x_0 = 0$).*

*Proof.* Let $(J_1(x_0,u^*), J_2(x_0,u^*), \cdots, J_N(x_0,u^*))$ is a Pareto solution for some $x_0 \neq 0$. Since $x_0$ is a scalar, from lemma 2.5.2 the scalar game $(P^{LQ})$ has a Pareto solution for every initial state. Let $x_0 \neq 0$ be fixed. Let $\alpha$, $z^*(t)$ and $p(t)$, with a corresponding solution $v^*(x_0)$, solve (2.28). Due to linearity we see that for the same choice of $\alpha$, $\mu x_0$, $\mu z^*(t)$ and $\mu p(t)$ also solves (2.28). This means (2.28) has a solution for all $x_0$. Since, $v \in L^+_{2,s}(x_0, z, \tilde{A})$ we have $\lim_{t\to\infty} z(t) = 0$ for all $x_0$. Following the discussion above $G_\alpha$ must have an eigenvalue with negative real part. Which means $G_\alpha$ has no eigenvalues on the imaginary axis and there exists a stable graph subspace. So, the (ARE) has a real stabilizing solution from proposition 2.5.1. Again from lemma 2.5.4 this means that $u^*$ minimizes the weighted sum objective with $\alpha$ as the weight vector for all initial states $x_0$. ∎

*Remark* 2.5.4. a) As already noticed in the proof of lemma 2.5.5 it follows directly from lemma 2.5.2 that, in case the scalar game has a Pareto solution for some initial state different from zero, the game (2.25, 2.26) has a Pareto solution for every initial state.

b) From the proof of lemma 2.5.5 we can in fact conclude the following result. If for $x_0 \neq 0$ there exists a Pareto solution and (2.28) has a solution with the choice of $\overrightarrow{\alpha} = (\bar{\alpha}_1, \cdots, \bar{\alpha}_N) \in \mathscr{P}_N$, then the scalar optimization problem $\min \sum_{i\in\overline{N}} \bar{\alpha}_i J_i$ subject to (2.26) has a solution for all initial states $x_0 \in \mathbb{R}$. In other words all candidates obtained from theorem 2.5.1 are indeed Pareto solutions.

**Theorem 2.5.6.** *Consider the scalar system with $R_i > 0, \quad i \in \overline{N}$ and $v \in L^+_{2,s}(x_0, z, \tilde{A})$. Then for some $x_0 \neq 0$ (2.25, 2.26) has a Pareto optimal control $\tilde{u}^*(x_0)$ if and only if there exists $\overrightarrow{\alpha} \in \mathscr{P}_N$ such that for every $x_0$,* $\min \sum_{i\in\overline{N}} \alpha_i J_i$ *subject to (2.26) has a solution.*

*Proof.* $\Rightarrow$ In particular, it follows that if (2.25, 2.26) has a Pareto optimal solution for some $x_0 \neq 0$ then lemma 2.5.5 yields the advertized result.

$\Leftarrow$ Since $R_\alpha > 0$, if there exists an $\overrightarrow{\alpha} \in \mathscr{P}_N$ such that for every $x_0$ $\min \sum_{i \in \overline{N}} \alpha_i J_i$ has a solution then the associated (ARE) has a stabilizing solution. The control strategy thus obtained is unique. Further, this control is indeed Pareto optimal from theorem 2.5.1. ∎

In other words, theorem 2.5.6 shows that to find all Pareto solutions of the game (2.25, 2.26), with arbitrary initial state (or for some initial state different from zero) one has to determine all $\overrightarrow{\alpha} \in \mathscr{P}_N$ for which (ARE) has a stabilizing solution. From remark 2.5.4.b and theorem 2.5.6 we have the following algorithm to find all the Pareto solutions for a scalar game.

**Algorithm 2.5.1.** *With assumption 2.5.1 holding true, consider the scalar system with $R_i > 0$, $i \in \overline{N}$. Let us define the index set $I(\alpha)$ as*

$$I(\alpha) = \left\{ \overrightarrow{\alpha} \in \mathscr{P}_N \middle| \text{(ARE) has a stabilizing solution } X_\alpha \right\}.$$

1. *To find all Pareto solutions of the game (2.25, 2.26) with arbitrary initial state one has to determine the set $I(\alpha)$. Then, the set of Pareto optimal controls is given by*

$$\tilde{u}^*(t) = -R_\alpha^{-1}\left(B'X_\alpha + S_\alpha\right) e^{\left(\tilde{A} - BR_\alpha^{-1}(B'X_\alpha + S_\alpha)\right)t} x_0 \text{ with } \overrightarrow{\alpha} \in I(\alpha), \tag{2.33}$$

   *where $K_\alpha$ solves the corresponding (ARE).*

2. *To find all Pareto solutions for which the game (2.25, 2.26) has a Pareto solution for a fixed initial state $x_0 \neq 0$ use step (1).*

3. *If $x_0 = 0$ and $I(\alpha)$ is non empty, $u^* \equiv 0$ is a Pareto optimal control. We know from corollary 2.5.1 that if we find a Pareto solution different from zero, then all points on the half-line through this solution and zero are Pareto solutions too.*

We consider example 6.2 from [35] to illustrate the usage of algorithm 2.5.1.

*Example* 2.5.2. Consider the situation in which there are two individuals who invest in a public stock of knowledge. Let $x(t)$ be the stock of knowledge at time $t$ and $u_i(t)$ the investment of player $i$ in public knowledge at time $t$. Assume that the stock of knowledge evolves according to the accumulation equation

$$\dot{x}(t) = -\beta x(t) + u_1(t) + u_2(t),\ x(0) = x_0,$$

where $\beta$ is the depreciation rate. Assume that each player derives quadratic utility from the consumption of the stock of knowledge and that the cost of investment increases quadratically with the investment effort. That is, the cost function of both players is given by

$$J_i = \int_0^\infty e^{-\theta t}\left\{-q_i x^2(t) + r_i u_i^2(t)\right\} dt.$$

Since the investment efforts are bounded and the system is controllable all Pareto solutions can be obtained using the weighting method. It can be easily verified that the associated (ARE) has a stabilizing solution if

$$\left(\beta+\frac{\theta}{2}\right)^2-(\alpha q_1+(1-\alpha)q_2)\left(\frac{1}{\alpha r_1}+\frac{1}{(1-\alpha)r_2}\right)>0.$$

For the choice of model parameters $\beta=2,\ \theta=0.05,\ r_i=q_i=1,\ i=1,2$ the above condition is written as $\alpha(1-\alpha)>0.2380$, $0<\alpha<1$. Applying the algorithm 2.5.1 we find $I(\alpha)=(0.3902,\ 0.6098)$ and all Pareto efficient controls are given by (2.33).

Next, we consider an example to illustrate that in general the set of Pareto optimal control actions is not convex.

*Example* 2.5.3. Consider the cooperative game with

$$J_1=\int_0^\infty e^{-\rho t}\left(x^2(t)+\frac{9}{10}u_1^2(t)+\frac{1}{10}u_2^2(t)\right)dt \text{ and}$$
$$J_2=\int_0^\infty e^{-\rho t}\left(x^2(t)+\frac{1}{10}u_1^2(t)+\frac{9}{10}u_2^2(t)\right)dt$$
$$\text{subject to the system } \dot{x}=\frac{\rho}{2}x+\frac{4}{10}(u_1+u_2),\quad x(0)=x_0\neq 0,\ \rho>0.$$

Here player $i$ controls $u_i$. Using the algorithm 2.5.1 the game has a Pareto optimal solution for those $\alpha\in[0,\ 1]$ for which the (ARE) $-sx_\alpha^2+1=0$, where $s=\frac{16}{(9\alpha-8)(8\alpha+1)}$. Taking $x_\alpha=\sqrt{s}$ here, we see that for all $\alpha\in[0,1]$ there is a Pareto solution. The set of all Pareto optimal controls is given by

$$u_\alpha^*(t)=e^{\rho t/2}v_\alpha^*(t),\ v_\alpha^*(t)=-\frac{4x_\alpha}{10}\begin{bmatrix}\frac{10}{1+8\alpha}\\ \frac{10}{9\alpha-1}\end{bmatrix}z^*(t),\ \dot{z}^*(t)=\frac{4}{10}\begin{bmatrix}1 & 1\end{bmatrix}v_\alpha^*(t),\ z^*(0)=x_0\neq 0$$

For $\alpha=\frac{1}{4}$ and $\alpha=\frac{3}{4}$ the Pareto controls are given by $v_{\frac{1}{4}}^*(t)=\begin{bmatrix}-1.5275\\-0.6547\end{bmatrix}z^*(t)$ and $v_{\frac{3}{4}}^*(t)=\begin{bmatrix}-0.6547\\-1.5275\end{bmatrix}z^*(t)$ respectively. Now consider the convex combination $\tilde{u}(t)=\frac{1}{2}u_{\frac{1}{4}}^*(t)+\frac{1}{2}u_{\frac{3}{4}}^*(t)$. With straightforward calculations it can be verified that this choice of control yields the same cost, $\tilde{J}=1.2548, i=1,2$, for both players. On the other hand choosing $\alpha=\frac{1}{2}$ we observe that the Pareto control $u_{\frac{1}{2}}^*$ yields a lower cost, same for both the players, $J_{\frac{1}{2}}^*=1.25<1.2548=\tilde{J}$. So, $\tilde{u}$ is not Pareto optimal, and this example demonstrates that in general the set of Pareto optimal controls is not convex.

## 2.6 Conclusions

In this chapter we derive necessary conditions for the existence of Pareto solutions in an infinite horizon cooperative differential game with open loop information structure. We consider non-autonomous and discounted autonomous systems for the analysis. These conditions are in the spirit of the maximum principle. For autonomous systems we derive some necessary conditions for optimality by exploiting the special constraint structure (due to reformulation of Pareto optimality). We gave some weak conditions, related to the extension of finite horizon transversality conditions, under which the necessary conditions for Pareto optimality are same as those of a weighted sum optimal control problem. Furthermore, we derive conditions under which the necessary conditions are also sufficient.

Later, the obtained results are used to analyze the regular indefinite infinite horizon linear quadratic differential game. We show that if the dynamic system is controllable then all Pareto candidates can be obtained by solving the necessary conditions for optimality of a weighted sum optimal control problem. For the two player case we show that almost all Pareto solutions can be obtained by using the weighting method even if player's cost functions are not convex. For the $N$ player scalar case we present an algorithm to calculate all the Pareto solutions if the initial state differs from zero. This algorithm proceeds by determining the elements in the unit simplex for which the associated weighted algebraic Riccati equation has a solution. We illustrate the subtleties with relevant examples.

The condition that players cooperate indefinitely could be too restrictive in some real world problems, for instance, joint ventures. A method based on moving horizons seems to be a logical and flexible alternative to the infinite horizon approach. Further, it would be interesting to see how the necessary and sufficient conditions can be formulated if the feedback information pattern is assumed.

## 2.A Appendix

*Proof of Lemma 2.3.1.* We prove the lemma using a contradiction argument. Suppose $\big(x^*(t),\ t, u^*(t), 1\big)$, $t \in [0,\ T]$ is not optimal for the problem $(P^{\rho}_{i_T})$ then there exists an admissible pair $(\hat{Y}(t), \hat{z}(t), \hat{U}(t), \hat{v}(t))$, $t \in [0,\ T]$ such that

$$h_i(\hat{z}(T) - T) + \int_0^T \hat{v}(t) e^{-\rho \hat{z}(t)} g_i(\hat{Y}(t), \hat{U}(t))dt < \int_0^\infty e^{-\rho t} g_i(x^*(t), u^*(t))dt$$
$$h_j(\hat{z}(T) - T) + \int_0^T \hat{v}(t) e^{-\rho \hat{z}(t)} g_j(\hat{Y}(t), \hat{U}(t))dt \le \int_0^\infty e^{-\rho t} g_j(x^*(t), u^*(t))dt,\ \forall j \in \overline{N}\backslash\{i\}.$$

Since $\hat{v}(t) \in [1/2,\ \infty)$, $\hat{z}(t)$ is an increasing function defined on $[0,\ T]$ so by the inverse function theorem $\hat{z}(t)$ is invertible on $[0,\ \hat{z}(T)]$. We define $\hat{x}(s) = \hat{Y}(\hat{z}^{-1}(s))$ and $\hat{u}(s) = \hat{U}(\hat{z}^{-1}(s))$ for $s \in [0, \hat{z}(T)]$ and observe that $\hat{x}(0) = \hat{Y}(\hat{z}^{-1}(0)) = x_0$ and $\hat{x}(z(T)) = \hat{Y}(\hat{z}^{-1}(\hat{z}(T))) = \hat{Y}(T) = x^*(T)$. Further, we have $\hat{x}(s)$ defined on $s \in [0,\ \hat{z}(T)]$ as:

$$\hat{x}(z(t)) = x_0 + \int_0^t \dot{\hat{Y}}(t)dt = x_0 + \int_0^T \hat{v}(t) f(Y(t), U(t))dt.$$

Taking $s = z(t)$ we have

$$\hat{x}(s) = x_0 + \int_0^s f(Y(z^{-1}(s)), U(z^{-1}(s)))ds = x_0 + \int_0^s f(\hat{x}(s), \hat{u}(s))ds,$$

for $s \in [0,\ \hat{z}(T)]$. Since $\hat{x}(s)$ satisfies the above integral equation we have $\dot{\hat{x}}(s) = f(\hat{x}(s), \hat{u}(s))$, $\hat{x}(0) = x_0$, $s \in [0, \hat{z}(T)]$. Next, for $s > \hat{z}(T)$, we define $\hat{x}(s) = x^*(s - \hat{z}(T) + T)$ and $\hat{u}(s) = u^*(s - \hat{z}(T) + T)$. Then we observe that $\dot{x}(s) = f(x(s), u(s))$ with $x(\hat{z}(T)) = x^*(T)$. Clearly the pair $(\hat{x}(s), \hat{u}(s))$, $s \in \big[0,\ \infty\big)$ is admissible for problem $(P_i^\rho)$ and satisfies the following conditions

$$\int_0^\infty e^{-\rho t} g_i(\hat{x}(t), \hat{u}(t))dt < \int_0^\infty e^{-\rho t} g_i(x^*(t), u^*(t))dt$$
$$\int_0^\infty e^{-\rho t} g_j(\hat{x}(t), \hat{u}(t))dt \leq \int_0^\infty e^{-\rho t} g_j(x^*(t), u^*(t))dt, \ \ \forall j \in \overline{N} \backslash \{i\},$$

which clearly violates the optimality of $(x^*(t), u^*(t))$ for the problem $(P_i^\rho)$. ■

<u>*Proof of Theorem 2.3.2.*</u> $(P_{i_T}^\rho)$ is a mixed endpoint constrained finite horizon optimal control problem. We first define the Hamiltonian $H_{iT}\left(\overrightarrow{\lambda}_{i_T}, z(t), Y(t), v(t), U(t)\right)$ as:

$$H_{i_T}(.) \triangleq v(t)e^{-\rho z(t)} \left(\lambda_{i_T}^0 g_i(Y(t), U(t)) + \sum_{j \in \overline{N} \backslash \{i\}} \mu_{j_T}^i(t) g_j(Y(t), U(t))\right) + v(t)\lambda_i'(t) f(Y(t), U(t)) + v(t)\gamma_i(t). \quad (2.34)$$

The necessary conditions are: there exist $\lambda_{i_T}^0 \in \mathbb{R}_+$, $\mu_{j_T}^i(t) \in \mathbb{R}$, $\lambda_{i_T}(t) \in \mathbb{R}^n$, $\gamma_{i_T}(t) \in \mathbb{R}$ such that for almost every $t \in [0, T]$ (the partial derivatives of the Hamiltonian given below are evaluated at the optimal pair $((x^*(t), t), (u^*(t), 1)))$:

$$\dot{\lambda}_{i_T}(t) = -\left(H_{i_T}\right)_Y = -f_x'(x^*(t), u^*(t))\lambda_{i_T}(t) - \lambda_{i_T}^0 e^{-\rho t} g_{ix}(x^*(t), u^*(t)) - \sum_{j \in \overline{N} \backslash \{i\}} \mu_{j_T}^i(t) e^{-\rho t} g_{j_x}(x^*(t), u^*(t)), \quad \lambda_{i_T}(T) \text{ free}$$
$$\dot{\mu}_{j_T}^i(t) = -\left(H_{i_T}\right)_{\tilde{Y}_j^i},\ \mu_{j_T}^i(T) \geq 0, \quad \mu_{j_T}^i(T)\left(\tilde{x}_j(T) + h_j(0) - \tilde{x}_j^*\right) = 0, \quad \forall j \in \overline{N} \backslash \{i\}$$
$$\dot{\gamma}_{i_T}(t) = -\left(H_{i_T}\right)_z, \quad \gamma_{i_T}(T) = \lambda_{i_T}^0 h_{ix}(0) + {\mu^i}_{j_T}(T) h_{j_x}(0)$$
$$\left(\lambda_{i_T}^0,\ \{\mu_{j_T}^i(t),\ j \in \overline{N} \backslash \{i\}\}, \lambda_{i_T}(t), \gamma_{i_T}(t)\right) \neq (0, \cdots, 0).$$

Since $\left(H_{i_T}\right)_{\tilde{x}_j^i} = 0$, we have $\mu_{j_T}^i(t) = \mu_{j_T}^i \geq 0$, $\forall j \in \overline{N} \backslash \{i\}$. Let $\overrightarrow{\lambda}_{i_T} = \big(\mu_{1_T}^i, \cdots, \mu_{i-1_T}^i, \lambda_{i_T}^0, \mu_{i+1_T}^i, \cdots, \mu_{N_T}^i\big)$. Then $\overrightarrow{\lambda}_{iT} \in \mathbb{R}_+^N$. Next we show by contradiction that also from the necessary conditions $\left(\overrightarrow{\lambda}_{i_T}, \lambda_{i_T}(0)\right) \neq (0,0)$. For, if $\left(\overrightarrow{\lambda}_{i_T}, \lambda_{i_T}(0)\right) = (0,0)$ then the necessary conditions give that

$$\dot{\lambda}_{i_T}(t) = -f_x'(x^*(t), u^*(t))\lambda_{i_T}(t),$$

which results in $\lambda_{i_T}(t) = 0$ for $t \in [0, T]$. Further, $\overrightarrow{\lambda}_{i_T} = 0$ leads to $\gamma_{i_T}(t) = 0$ for $t \in [0, T]$ which violates the necessary condition $\left(\overrightarrow{\lambda}_{i_T}, \lambda_{i_T}(t), \gamma_{i_T}(t)\right) \neq (0,0,0)$ for all $0 \leq$

$t \leq T$. Since $\lambda_{i_T}(T)$ is free, we choose (without loss of generality) $\lambda_{i_T}(0)$ such that $\left\|\left(\overrightarrow{\lambda}_{i_T}, \lambda_{i_T}(0)\right)\right\| = 1$. The adjoint variable $\lambda_{i_T}(t)$ satisfies:

$$\dot{\lambda}_{i_T}(t) = -f'_x(x^*(t), u^*(t))\lambda_{i_T}(t) - e^{-\rho t} G_x(\overrightarrow{\lambda}_{i_T}, x^*(t), u(t)), \;\; \lambda_{i_T}(0) = l^0_{i_T}, \tag{2.35}$$

whereas $\gamma_{i_T}(t)$ satisfies

$$\dot{\gamma}_{i_T}(t) = -\left(H_{i_T}\right)_z = \rho\, G(\overrightarrow{\lambda}_{i_T}, x^*(t), u^*(t)). \tag{2.36}$$

From the definition of $h(.)$ we have:

$$\begin{aligned} \gamma_{i_T}(T) &= \rho\lambda^0_{i_T} \int_T^\infty e^{-\rho t} g_i(x^*(t), u^*(t))dt + \rho \sum_{j \in \overline{N}\setminus\{i\}} \mu^i_{j_T} \int_T^\infty e^{-\rho t} g_j(x^*(t), u^*(t))dt \\ &= \rho \int_T^\infty e^{-\rho t} G(\overrightarrow{\lambda}_{i_T}, x^*(t), u^*(t))dt. \end{aligned} \tag{2.37}$$

The Hamiltonian is linear in $v(t)$ and the minimum w.r.t $(U(t), v(t))$ on the set $\mathscr{U} \times [1/2, \infty)$ is attained at $(u^*(t), 1)$. The minimum of the Hamiltonian w.r.t $v(t)$ for $U(t) = u^*(t)$ is achieved at an interior point of $[1/2, \infty)$, so we have:

$$e^{-\rho t} G(\overrightarrow{\lambda}_{i_T}, x^*(t), u^*(t)) + \lambda'_{i_T}(t) f(x^*(t), u^*(t)) + \gamma_{i_T}(t) = 0, \;\; \forall t \in [0, T]. \tag{2.38}$$

The minimum of the Hamiltonian w.r.t $U(t)$ is independent of $v(t)$ (positive scaling) and does not depend on the term $\gamma_i(t)v(t)$. So, the minimum of $H_{i_T}$ w.r.t $U(t)$ at $v(t) = 1$ is achieved at

$$\begin{aligned} u^*(t) &= \arg\min_{U(t)} H_{i_T}(\overrightarrow{\lambda}_{i_T}, t, x^*(t), z(t) = t, U(t), v(t) = 1, \lambda_{i_T}(t), \gamma_{i_T}(t)) \\ &= \arg\min_{u(t)} \left(e^{-\rho t} G(\overrightarrow{\lambda}_{i_T}, x^*(t), u(t)) + \lambda'_{i_T}(t) f(x^*(t), u(t))\right), \;\; \forall t \in [0, T]. \end{aligned} \tag{2.39}$$

Now, consider an increasing sequence $\{T_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} T_k = \infty$. We can associate an optimal control problem $(P_{i_{T_k}})$ with each $T_k$ such that the necessary conditions as discussed above hold true. Then there exists a sequence $\left\{\left(\overrightarrow{\lambda}_{i_{T_k}}, l^0_{i_{T_k}}\right)\right\}$ such that $\overrightarrow{\lambda}_{i_{T_k}} \in \mathbb{R}^N_+$ and $\left\|\left(\overrightarrow{\lambda}_{i_{T_k}}, l^0_{i_{T_k}}\right)\right\| = 1$. We know from the Bolzano-Weierstrass theorem that every bounded sequence has a convergent subsequence. Using the same indices for such a subsequence we infer that there exists $\overrightarrow{\lambda}_i \in \mathbb{R}^N_+$ and $l^0_i$ such that

$$\lim_{k\to\infty} \overrightarrow{\lambda}_{i_{T_k}} = \overrightarrow{\lambda}_i \in \mathbb{R}^N_+ \;\; \text{and} \;\; \lim_{k\to\infty} l^0_{i_{T_k}} = l^0_i \;\; \text{such that} \; \left\|\left(\overrightarrow{\lambda}_i, l^0_i\right)\right\| = 1. \tag{2.40}$$

We observe that (2.35) is a linear ODE. So we can write $\lambda_{i_{T_k}}(t)$ as:

$$\lambda_{i_{T_k}}(t) = \Phi_{-f'_x}(t, 0) l^0_{i_{T_k}} - \int_0^t \Phi_{-f'_x}(t, s) e^{-\rho s} G_x(\overrightarrow{\lambda}_{i_T}, x^*(s), u^*(s))ds, \tag{2.41}$$

where $\Phi_{-f_x'}(t,s)$ is the state transition matrix associated with $\dot{z}(t) = -f_x'(x^*(t),u^*(t))\, z(t)$. Since the weights of $\overrightarrow{\lambda}_{i_{T_k}}$ appear linearly in $G_x(.)$ as $k \to \infty$, $\lambda_i(t)$ satisfies the differential equation (2.19b). A similar argument holds for $\gamma_{i_{T_k}}(t)$, condition (2.38) and (2.39) resulting in (2.19c), (2.19e) and (2.19d) respectively. ■

*Proof of Corollary 2.3.2.* From the necessary conditions (2.19d) and (2.19e) of theorem 2.3.2 we have

$$\begin{aligned} & e^{-\rho t} G(\overrightarrow{\lambda}_i, x^*(t), u(t)) + \lambda_i'(t) f(x^*(t), u(t)) \\ & \qquad \geq e^{-\rho t} G(\overrightarrow{\lambda}_i, x^*(t), u^*(t)) + \lambda_i'(t) f(x^*(t), u^*(t)) = -\gamma_i(t). \end{aligned}$$

By assumption 2 we have $\lambda_i'(t) f(x^*(t),u(t)) \geq -\gamma_i(t)$. Next define $q(t)$ as follows:

$$q(t) = \frac{\lambda_i(t)}{\max\{1, ||\lambda_i(t)||\}}. \quad \text{Since } ||q(t)|| \leq 1 \text{ we have } \limsup_{t\to\infty} ||q(t)|| = l.$$

If $l = 0$ there is nothing to prove. So assume $l > 0$ and consider a sequence $\{t_n\}$ converging to infinity such that $||q(t_n)|| > l/2$. Since there exists $u(t)$ such that $\mathscr{B}_{\delta>0}$[8] $\subset f(x^*(t),u(t))$, there exists an $\varepsilon > 0$ such that $\frac{2\varepsilon}{l} < \delta$. So, there exists $u_n(t_n)$ such that $f(x^*(t_n),u_n(t_n)) = -(2\varepsilon/l)\, q(t_n)$. Since, $\lim_{t\to\infty} \gamma_i(t) = 0$ we take the above sequence $\{t_n\}$ such that $-l\varepsilon/2 \leq -\gamma(t_n) \leq l\varepsilon/2$. Collecting all the above we have:

$$\lambda_i'(t_n) f(x^*(t_n),u_n(t_n)) = -\max\{1, ||\lambda_i(t_n)||\}\,(2\varepsilon/l)\,||q(t_n)||^2 \geq -\gamma_i(t_n)$$

$$\gamma_i(t_n) \geq \max\{1, ||\lambda_i(t_n)||\}\,(2\varepsilon/l)\,||q(t_n)||^2 > l\varepsilon/2.$$

Clearly, this is a contradiction and thus $\lim_{t\to\infty} \lambda_i(t) = 0$. ■

*Proof of Corollary 2.3.3.* From the assumption 2.3.3 there exist constants $c_4 \geq 0$, $c_5 \geq 0$, $c_6 \geq 0$ and $c_7 \geq 0$ such that:

$$e^{-\rho t} |g_i(x(t),u(t))| \leq e^{-\rho t}\left(c_4 + c_5 ||x(t)||^{r+1}\right) \leq c_6 e^{-\rho t} + c_7 e^{-(\rho-(r+1)\lambda)t}.$$

Since $\rho > (1+r)\lambda$, the player's costs $J_i(x_0,u)$ converge for every admissible pair $(x(t),u(t))$. For the problem $(P_i)$ we rewrite $\lambda_i(t)$ (from (2.19b)) as follows:

$$\begin{aligned} \lambda_i(t) &= \Phi_{-f_x'}(t,0)\, l_i^0 - \int_0^t e^{-\rho s} \Phi_{-f_x'}(t,s)\, G_x(x^*(s),u^*(s))ds \\ &= \Phi_{-f_x'}(t,0) \left( l_i^0 - \int_0^t e^{-\rho s} \Phi_{-f_x'}(0,s)\, G_x(x^*(s),u^*(s))ds \right) \\ & \qquad \left( \text{we know } \Phi_{-f_x'}(t,s) = \left(\Phi_{f_x}^{-1}(t,s)\right)' = \Phi_{f_x}'(s,t) \right) \\ &= \Phi_{-f_x'}(t,0) \left( l_i^0 - \int_0^t e^{-\rho s} \Phi_{f_x}'(s,0)\, G_x(x^*(s),u^*(s))ds \right). \end{aligned}$$

[8] Unit ball in $\mathbb{R}^n$ of radius $\delta > 0$.

The norm of $\lambda_i(t)$ is bounded as:

$$\begin{aligned}
||\lambda_i(t)|| &\leq \left\|\Phi_{-f'_x}(t,0)\right\| \left(||l_i^0|| + \int_0^t e^{-\rho s} \left\|\Phi_{f_x}(s,0)\right\| \|G_x(x^*(s),u^*(s))\| ds\right) \\
&\quad \text{(from assumption 2.3.3b there exist a } c_8 \geq 0\text{, a } c_9 \geq 0 \text{ such that)} \\
&\leq \left\|\Phi_{-f'_x}(t,0)\right\| \left(||l_i^0|| + \int_0^t e^{-\rho s} \left(c_8 e^{\lambda s} + c_9 e^{(1+r)\lambda s}\right) ds\right) \\
&\quad (l_i^0 \text{ is bounded, so there exist a } c_{10} \geq 0\text{, a } c_{11} \geq 0 \text{ and } c_{12} \geq 0) \text{ such that} \\
&\leq \left\|\Phi_{-f'_x}(t,0)\right\| \left(c_{10} + c_{11} e^{-(\rho-\lambda)t} + c_{12} e^{-(\rho-(1+r)\lambda)t}\right).
\end{aligned}$$

Let $\phi^0(t)$ and $\phi_0(t)$ denote the largest and smallest eigenvalues of the Hermitian part[9] of $-f'_x(x^*(t),u^*(t))$. By assumption $-f'_x(x^*(t),u^*(t))$ is bounded and has strictly negative eigenvalues, so we have $-\infty < \phi_0(t) \leq \phi^0(t) < 0$, $\forall t \geq 0$. From [52, lemma 4.2] we have:

$$\exp\left(\int_0^T \mu_0(s) ds\right) \leq \left\|\Phi_{-f'_x}(t,0)\right\| \leq \exp\left(\int_0^T \mu^0(s) ds\right).$$

Since $\mu^0(s) < 0$ for all $s \geq 0$ we have $\lim_{t\to\infty} \left\|\Phi_{-f'_x}(t,0)\right\| = 0$. By assumption $\rho > (1+r)\lambda$, so $\lim_{t\to\infty} \lambda_i(t) = 0$ follows directly. ■

[9]The Hermitian part of matrix $A$ is defined here as $A^H = \frac{1}{2}(A + A')$.

## CHAPTER 3

# Feedback Nash Equilibria for Descriptor Differential Games using Matrix Projectors

## 3.1 Introduction

Dynamic game theory captures multi person decision making processes that occur in time. These problems arise in several disciplines such as monetary policy coordination, ecology and several others, refer to [34] for a detailed overview. The dynamic environment where the players interact is often modeled by a set of ordinary differential equations and the related theory for such games is well established [7]. Complex systems, however, include modeling with both differential and algebraic equations, i.e., differential algebraic equations. Problems of this kind appear in studying systems which are constrained and which operate under different timescales, for example in environmental economics where global warming is assumed to be a system which has slow dynamics that is affected by various processes that have fast dynamics. Descriptor systems, linear differential algebraic equations, approximate such multiple time scale systems. Differential games for descriptor systems were e.g., already studied by Xu and Mizukami in [115]. Further, these authors considered the leader follower information structure in [116]. Glizer [46] considers the asymptotic behavior of the zero-sum game solution from a cheap control perspective. In [68] the necessary and sufficient conditions for optimal $H_\infty$ control, also a differential game, for linear constant coefficient descriptor systems with arbitrary index have been formulated. More recently, Engwerda et al. [40] give solvability conditions, with open loop information pattern, for the index[1] 1 linear quadratic differential game and in [41] for higher order indices. [39] considers the index 1 case with feedback information pattern. The main tool used to analyze the descriptor system in most of the works uses Kronecker canonical form (KCF). The differential and algebraic parts of a descriptor system can be decoupled using KCF. It is known, however, that computing KCF for a descriptor system is still a challenge, see [55] for some recent developments.

In this chapter, we consider the linear quadratic differential game on a descriptor system with feedback information pattern. Feedback strategies are generally preferred due to

[1] Index roughly translates to the number of differentiations required to represent a differential algebraic equation as a differential equation, see section 3.2.

attractive properties such as strong time consistency. We consider a geometric approach towards decoupling the descriptor system instead of KCF. We lean heavily on the matrix projector techniques pioneered by März et al. [49, 73]. We review some important results on matrix projectors in appendix A. Further, we propose an algorithm to compute matrix projectors that completely decouple a descriptor system into algebraic and differential parts. In section 3.2 we introduce the game problem. We analyze effect of feedback on a decoupled system and analyze the informational non uniqueness property of feedback strategies. In section 3.3 we give solvability conditions for the game problem for the index 1 case. Later, for descriptor systems with higher order indices, we give an approach to recast the problem as an index 1 case. We demonstrate the limitation of this method with an example. Finally section 3.4 concludes.

**Notation:** We use the following notation. $X'$ represents the transpose of $X \in \mathbb{R}^{n\times m}$. $X^\dagger$ represents Moore-Penrose pseudo inverse of $X$. For $n > m$, $X^\dagger$ gives a left inverse of $X$, when $X$ has full column rank. Im$X$ and ker$X$ represents the column space and null space of $X$ respectively. The symbol $\oplus$ denotes direct sum.

## 3.2 Preliminaries

In this section we assume that players $i = 1,2$ like to minimize $J_i(x_0,u_1,u_2)$,

$$\text{where} \quad J_i(x_0,u_1,u_2) = \int_0^\infty \left[x'(t)\ u_1'(t)\ u_2'(t)\right] M_i \left[x'(t)\ u_1'(t)\ u_2'(t)\right]' dt, \tag{3.1}$$

$$M_i = \begin{bmatrix} D_i & V_i & W_i \\ V_i' & R_{1i} & N_i \\ W_i' & N_i' & R_{2i} \end{bmatrix},\ D_i = D_i^T,\ R_{ii} > 0,\ i = 1,2, \text{ and } x(t) \text{ satisfies}$$

$$E\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t),\ x(0) = x_0, \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^{m_i}$, $i = 1,2$, $A \in \mathbb{R}^{n\times n}$, $E \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$, $i = 1,2$ and dimensions of other matrices are defined appropriately. Games with discounted objectives can be reformulated as (3.1-3.2) by a simple change of variables as $A \to A - \frac{\theta}{2}E$, where $\theta > 0$ is the discount factor, see example 3.3.2 for an illustration. The information pattern considered in this chapter is of the feedback type and we assume that players use linear feedback strategies of the form $u_i(t) = F_ix(t)$, $i = 1,2$. $u^* = (u_1^*, u_2^*)$ is called a feedback Nash equilibrium (FBNE) if the usual inequalities apply, i.e., no player can improve his performance by a unilateral deviation from this set of equilibrium actions. We introduce the notation $F_{-i}^*$ which corresponds to the strategies, $u_j(t) = F_jx(t), j = 1,2, j \neq i$, used by all the players excluding the player $i$. Now, the formal definition of FBNE reads as follows:

**Definition 3.2.1.** *$F^* = (F_1^*, F_2^*)$ is called a feedback Nash equilibrium if for $i = 1,2$, $J_i(x_0,F_i^*,F_{-i}^*) \leq J_i(x_0,F_i,F_{-i}^*)$ for every input $x_0$ and $F_i$.*

The following lemma follows directly from the definition of FBNE.

**Lemma 3.2.1.** *Let $H \in \mathbb{R}^{n\times d}$. $(F_1^*,\ F_2^*)$ is a FBNE for the game defined by the cost $J_i(x_0, F_1H, F_2H)$ and the system $\dot{x}(t) = (A + B_1F_1H + B_2F_2H)x(t),\ x(0) = x_0$ if and only if $(G_1^*,\ G_2^*)$ is a FBNE for the game defined by the cost $J_i(x_0, G_1, G_2)$ and the system $\dot{x}(t) = (A + B_1G_1 + B_2G_2)x(t),\ x(0) = x_0$ and $(F_1^*, F_2^*)$ solve the set of equations $F_1^*H = G_1^*$ and $F_2^*H = G_2^*$*[2].

For $E = I$, we recall the following result from theorem 8.5 of [35].

**Theorem 3.2.1.** *Assume that matrix $G = \begin{bmatrix} R_{11} & N_1 \\ N_2' & R_{22} \end{bmatrix}$ is invertible. Then the differential game (3.1,3.2) with $E = I$, has a feedback Nash equilibrium $(F_1^*, F_2^*)$ for every initial state if and only if*

$$\begin{bmatrix} F_1^* \\ F_2^* \end{bmatrix} = -G^{-1} \begin{bmatrix} B_1'X_1 + V_1' \\ B_2'X_2 + W_2' \end{bmatrix}. \tag{3.3}$$

*Here $(X_1, X_2)$ are a symmetric stabilizing solution of the coupled algebraic Riccati equations*

$$\begin{aligned} D_1 + W_1F_2^* + F_2^{*\prime}W_1' + F_2'R_{21}F_2^* - F_1^{*\prime}R_{11}F_1^* + X_1(A + B_2F_2^*) + (A + B_2F_2^*)'X_1 &= 0 \\ D_2 + V_2F_1^* + F_1^{*\prime}V_2' + F_1^{*\prime}R_{12}F_1^* - F_2^{*\prime}R_{22}F_2^* + X_2(A + B_1F_1^*) + (A + B_1F_1^*)'X_2 &= 0. \end{aligned}$$

The above coupled algebraic Riccati equations are solvable only for special cases, see [36] for a discussion on algorithms. The system (3.2) is solvable, under assumptions on smoothness of $u(t)$ and consistent initial states $x_0$, if the pencil $\lambda E - A$ is regular i.e., $\det(\lambda E - A) \neq 0$ for at least one $\lambda$. We call $(E,\ A)$ a regular pair if the pencil $\lambda E - A$ is regular. The index, denoted by a whole number $\mu$, of the descriptor system is the number of differentiations required to solve the descriptor system as an ordinary differential equation, see section 2 of [22]. The solution of the descriptor system depends on the derivatives of the input up to an order of $\mu - 1$. There are several approaches used to analyze the descriptor system and the most notable of such methods is the KCF. Though it serves the purpose of analyzing a descriptor system, computation of the KCF for a regular pencil is still a challenge, see [55] for details. There are algorithms such as GUPTRI, see [33], which reveal the Jordan structure of the pencil without actually computing the canonical form. However, a complete knowledge of the canonical form, i.e., the matrices that represent the left and right deflating subspaces of the matrix pencil, is required to compute the Nash strategies. We summarize the main problem addressed in this section as follows:

**Problem 3.2.1.** *Consider the performance criterion (3.1) and system (3.2), where rank$(E)$ $= r < n = \dim(x)$. Assume $(E, A)$ is regular with index $\mu$. Find conditions under which (3.1,3.2) has a feedback Nash solution $u_i = F_ix$.*

[2] Solvable if and only if $\mathrm{Ker}H \subset \mathrm{Ker}G_i^*,\ i = 1, 2$.

If $E$ is non-singular, the FBNE are obtained using theorem 3.2.1. The result given in theorem 3.2.1 is based on the dynamic programming principle, see [7, 35]. The optimal control problem of descriptor systems has been recently solved by Mehrmann et al. [61] using ideas from behavioral systems theory [82]. They derive necessary conditions for optimality which result in a differential algebraic equation. Further, it was shown that an optimal solution is obtained by solving the resulting optimal system instead of a Riccati equation.

We solve the problem 3.2.1 by first decoupling algebraic and dynamic components of the descriptor system and later eliminate the algebraic components. The problem considered above is posed in a very general setting. We did not make assumptions with respect to stabilizing properties of the feedback strategies, which is generally the case when $E$ is non-singular.

In his seminal paper Smale [101] puts forward the idea of interpreting differential algebraic equations as vector fields on manifolds. In the later works by Rheinboldt, in [87], and Reich, in [86], the authors discuss the classes of DAEs which can be seen as vector fields on constraint manifolds (due to algebraic constraints). For linear DAEs, März et al., [49] develop a matrix chain approach to decouple a descriptor system into differential and algebraic parts. If the pencil $\lambda E - A$ is regular, then it was shown in theorem 3.1 of [73] that the existence of so called *canonical projectors* is guaranteed. Using canonical projectors a descriptor system can be decoupled canonically as a vector field and as a constraint manifold. Recently, Wong [113, 114] demonstrates the computational advantage of the matrix projector methods in engineering applications. In appendix A, we review important details of matrix projectors and discuss an algorithm to generate the canonical projectors for a regular matrix pencil. A regular descriptor system can be decoupled completely, using canonical projectors $(Q_i, P_i)$, $i = 0, 1, \cdots, \mu - 1$, as follows:

$$\dot{m}(t) = P_0 \cdots P_{\mu-1} E_\mu^{-1} A m(t) + P_0 \cdots P_{\mu-1} E_\mu^{-1} B u(t),\ m(0) = P_0 \cdots P_{\mu-1} x_0 \tag{3.4a}$$

$$n(t) = -\sum_{i=0}^{\mu-1} N_i E_\mu^{-1} B u^i(t),\ n(0) = (I - P_0 \cdots P_{\mu-1}) x_0 \tag{3.4b}$$

$$x(t) = m(t) + n(t),\ m(t) = P_0 \cdots P_{\mu-1} x(t),\ n(t) = \big(I - P_0 \cdots P_{\mu-1}\big) x(t), \tag{3.4c}$$

where $m(t)$ and $n(t)$ are projections of the state $x(t)$. Here $P_i$, $Q_i = I - P_i, i = 0, 1, \cdots, \mu - 1$ are canonical projectors generated by a regular pair $(E, A)$, refer to appendix A for details. This decomposition is unique for a regular pair $(E, A)$.

*Remark* 3.2.1. Since $m(0) \in \mathrm{Im}(P_0 P_1 \cdots P_{\mu-1})$ and $\dot{m}(t) \in \mathrm{Im}(P_0 P_1 \cdots P_{\mu-1})$, from lemma 3.2.1 of [9], we have $m(t) \in \mathrm{Im}(P_0 P_1 \cdots P_{\mu-1})$, $\forall t \geq 0$. In other words, equation (3.4a) represents an invariant flow and $m(t)$ belongs to the invariant subspace $\mathrm{Im}(P_0 P_1 \cdots P_{\mu-1}) \subset \mathbb{R}^n$ for all $t \geq 0$.

Using the above remark we have the following lemma:

**Lemma 3.2.2.** *$m(t) \in \mathbb{R}^n$ is isomorphic to a vector $m_l(t) \in \mathbb{R}^d$, where $d = \dim\big(Im(P_0P_1 \cdots P_{\mu-1})\big) < n$. Further, there exist matrices $Y_{d\times n}$ and $Z_{n\times d}$ such that $m_l(t) = Ym(t)$, $m(t) = Zm_l(t)$ and $ZY = P_0P_1 \cdots P_{\mu-1}$.*

*Proof.* The first part of the lemma is obvious from remark 3.2.1. For the second part we know from [49] that $P_0P_1 \cdots P_{\mu-1}$ is a projector. So, we have $m(t) = P_0P_1 \cdots P_{\mu-1}m(t)$, taking the SVD of $P_0P_1 \cdots P_{\mu-1}$ as $U_{n\times d}\Sigma_{d\times d}V'_{d\times n}$ we have $m(t) = U\Sigma V'm(t)$. Identifying $Z = U\Sigma$ and $Y = V'$ we have that $m_l(t) = Ym(t)$ and $m(t) = Zm_l(t)$. ■

From (3.4a-3.4b) it is clear that $x(t)$ is continuous if $u(t)$ is at least $\mu - 1$ times differentiable. We denote $\mathscr{U}$ to be the set of admissible controls, i.e., $u(.) \in \mathscr{U}$ is atleast $\mu - 1$ times differentiable. Notice, that at $t = 0$ the initial state, $x_0$, and higher order derivatives of the input satisfy the equation (3.4b). The initial states that satisfy this constraint are referred to as consistent. Let $\mathscr{X}_0$ denote the consistent initial state manifold, then it is characterized in the following lemma:

**Lemma 3.2.3.** *For every $u(.) \in \mathscr{U}$, the descriptor system* (3.2) *yields a unique continuous state trajectory if $x_0 \in \mathscr{X}_0$.*

*Proof.* Violation of the algebraic constraint (3.4b) at $t = 0$ results in a jump, i.e., $x(0^+) = \lim_{t\downarrow 0} x(t) \neq \lim_{t\uparrow 0} x(t) = x(0^-) = x(0)$. Let us define

$$\mathscr{X}_0 \triangleq \left\{ x_0 \in \mathbb{R}^n \,\middle|\, \big(I - P_0P_1 \cdots P_{\mu-1}\big)\, x_0 = -\sum_{i=0}^{\mu-1} N_iE_\mu^{-1}Bu^i(0) \right\}. \tag{3.5}$$

For an admissible $u(t)$, such jumps can be avoided if $x_0 \in \mathscr{X}_0$. ■

From lemma 3.2.2 the low dimensional representation of the inherent ODE (3.4a) is given by:

$$\dot{m}_l(t) = YP_0P_1 \cdots P_{\mu-1}E_\mu^{-1}\left(AZm_l(t) + B_1u_1(t) + B_2u_2(t)\right),\; m_l(0) = YP_0P_1 \cdots P_{\mu-1}x_0. \tag{3.6}$$

We define the following classes of feedback strategies:

a) Class of partial state information feedbacks $\mathscr{P}$, where players use $u_i(t) = K_im_l(t)$.

b) Class of full state information feedbacks $\mathscr{F}$, where players use $u_i(t) = F_ix(t)$.

An application of a particular feedback strategy can alter the behavior of the descriptor system, for instance, change in the index and regularity may fail to hold. However, if players use strategies from class $\mathscr{P}$ we show, in the theorem 3.2.2 below, that both regularity and index are preserved. In the discussion that follows , taking $m = m_1 + m_2$, we denote $u(t) = [u_1'(t)\;\; u_2'(t)]' \in \mathbb{R}^m$, $B = [B_1\;\; B_2] \in \mathbb{R}^{n\times m}$, $F = [F_1'\;\; F_2']' \in \mathbb{R}^{m\times n}$ and $K = [K_1'\;\; K_2']' \in \mathbb{R}^{m\times d}$. We denote by $K \in \mathscr{P}$ if players use strategies $u_i(t) = K_im_l(t),\; i = 1,2$ and by $F \in \mathscr{F}$ if players use strategies $u_i(t) = F_ix(t),\; i = 1,2$. We need the following auxiliary result to prove theorem 3.2.2.

**Lemma 3.2.4.** *For the admissible projectors $Q_i$ and $P_i$, $i = 0, 1, \cdots, \mu - 1$ we have the following:*

*a)* $P_0 P_1 \cdots P_{\mu-1} Q_i = 0,\ i \in \{0, 1, \mu - 2\}$

*b)* $P_0 P_1 \cdots P_{\mu-1} P_i = P_0 P_1 \cdots P_{\mu-1},\ i \in \{0, 1, \mu - 2\}$.

*Proof.* a) For $i \in \{0, 1, \mu - 2\}$, $P_0 P_1 \cdots P_{\mu-1} Q_i = P_0 P_1 \cdots P_{\mu-2}(I - Q_{\mu-1}) Q_i$, from admissibility of $Q_i$'s (see appendix A) we know $Q_j Q_i = 0$ for $j > i$. So, we have $P_0 P_1 \cdots P_{\mu-1} Q_i = P_0 P_1 \cdots P_{\mu-2} Q_i$, repeating this $\mu - i - 1$ times we have $P_0 P_1 \cdots P_{\mu-1} Q_i = P_0 P_1 \cdots P_i Q_i = 0$.

b) For $i \in \{0, 1, \mu - 2\}$, $P_0 P_1 \cdots P_{\mu-1} P_i = P_0 P_1 \cdots P_{\mu-1}(I - Q_i)$, from part (a), derived above, we have $P_0 P_1 \cdots P_{\mu-1} P_i = P_0 P_1 \cdots P_{\mu-1}$. ■

**Theorem 3.2.2.** *The class $\mathscr{P}$ is index preserving, i.e., $(E,\ A)$ and $\big(E,\ A + BKYP_0 P_1 \cdots P_{\mu-1}\big)$ have the same index for any $K \in \mathscr{P}$.*

*Proof.* Consider the regular index-$\mu$ descriptor system. From theorem 3.A.1 we know $E_\mu$ is non singular. The descriptor system with partial information feedback is given by:

$$E\dot{x} = Ax + BKm_l = \big(A + BKYP_0 P_1 \cdots P_{\mu-1}\big)x.$$

Let $(\tilde{E}_i,\ \tilde{A}_i)$ be the matrix chain obtained by taking $\tilde{E}_0 = E$ and $\tilde{A}_0 = A + BKYP_0 P_1 \cdots P_{\mu-1}$. Then we need to show that the matrix chain stops after $\mu$ steps i.e., $\tilde{E}_\mu$ is nonsingular. We prove the result using an induction argument. For $i = 0$, clearly $\tilde{E}_i = E_i = E$ and $\tilde{A}_i = A_i + BKYP_0 P_1 \cdots P_{\mu-1}$. For $i = 1$ as $\ker \tilde{E}_0 = \ker E_0 = \ker E$, we take $\tilde{Q}_0 = Q_0$

$$\begin{aligned}
\tilde{E}_1 &= E + \big(A + BKYP_0 P_1 \cdots P_{\mu-1}\big) Q_0 \\
&= E + AQ_0 + BKYP_0 P_1 \cdots P_{\mu-1} Q_0 \\
&= E_1 + BKYP_0 P_1 \cdots P_{\mu-1} Q_0 = E_1 \quad \text{(follows from lemma 3.2.4.a)},
\end{aligned}$$

$$\begin{aligned}
\tilde{A}_1 &= \big(A + BKYP_0 P_1 \cdots P_{\mu-1}\big) P_0 \\
&= AP_0 + BKYP_0 P_1 \cdots P_{\mu-1} P_0 \\
&= A_1 + BKYP_0 P_1 \cdots P_{\mu-1} \quad \text{(follows from lemma 3.2.4.b)}.
\end{aligned}$$

We assume $\tilde{E}_k = E_k$ and $\tilde{A}_k = A_k + BKYP_0 P_1 \cdots P_{\mu-1}$ for $0 \le k \le i$ and we show that $\tilde{E}_k = E_k$ and $\tilde{A}_k = A_k + BKYP_0 P_1 \cdots P_{\mu-1}$ for $0 \le k \le i + 1$. For $i = 1$ the above assumption holds true since we already showed $\tilde{E}_1 = E_1$ and $\tilde{A}_1 = A_1 + BKYP_0 P_1 \cdots P_{\mu-1}$. Now, since $\tilde{E}_k = E_k$ we have $\tilde{Q}_k = Q_k$ and $\tilde{P}_k = P_k$ for $0 \le k \le i$ as only the image of $Q_k$ is fixed (as

$\ker E_k$).

$$\begin{aligned}
\tilde{E}_{i+1} &= \tilde{E}_i + \tilde{A}_i\tilde{Q}_i \\
&= E_i + \left(A_i + BKYP_0P_1\cdots P_{\mu-1}\right)Q_i \\
&= E_i + A_iQ_i + BKYP_0P_1\cdots P_{\mu-1}Q_i = E_{i+1} \quad \text{(follows from lemma 3.2.4.a),}
\end{aligned}$$

$$\begin{aligned}
\tilde{A}_{i+1} &= \tilde{A}_i\tilde{P}_i \\
&= \left(A_i + BKYP_0P_1\cdots P_{\mu-1}\right)P_i \\
&= A_{i+1} + BKYP_0P_1\cdots P_{\mu-1}P_i \quad \text{(follows from lemma 3.2.4.b).}
\end{aligned}$$

So, by induction we have $\tilde{E}_k = E_k$ and $\tilde{A}_k = A_k + BKYP_0P_1\cdots P_{\mu-1}$ for $0 \le k \le \mu$. Further, $E_\mu$ is nonsingular, since the pencil $(A,B)$ is regular with index-$\mu$ is nonsingular, we have $\tilde{E}_\mu$ is nonsingular as well. So, by theorem 3.A.1 the pencil $(E,\ A + BKYP_0P_1\cdots P_{\mu-1})$ is regular with index $\mu$. ■

*Remark* 3.2.2. We cannot guarantee such index preserving property, in general, for any $F \in \mathscr{F}$. Let us denote the index preserving subclass by $\mathscr{F}_\mu$, of $\mathscr{F}$. We notice for any $K \in \mathscr{P}$ the players use $u(t) = Km_l(t) = KYm(t) = \left(KYP_0P_1\cdots P_{\mu-1}\right)x(t)$. Thus for every strategy $K \in \mathscr{P}$ there exists an index preserving strategy $\left(KYP_0P_1\cdots P_{\mu-1}\right) \in \mathscr{F}_\mu \subset \mathscr{F}$.

*Example* 3.2.1. We demonstrate the subtleties in remark 3.2.2 with an example. Consider the descriptor system with $E = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $A = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$. Taking $Q_0 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ as the projector to Ker$E$ we see that $E_1 = E + AQ_0 = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$. So, from theorem 3.A.1 the pencil $(E,A)$ has an index equal to 1. We define $\mathscr{F} = \left\{\begin{bmatrix}f_1 & f_2\\ f_3 & f_4\end{bmatrix} \in \mathbb{R}^{2\times 2}\right\}$. Now, the index of the pencil $(E, A+BF)$ for any $F \in \mathscr{F}$ is determined by the rank of the matrix $E_1 = E + (A+BF)Q_0 = \begin{bmatrix}1 & 1+f_2+f_4\\ 0 & 1+f_2+f_4\end{bmatrix}$. So, the index preserving class is given by $\mathscr{F}_1 = \left\{\begin{bmatrix}f_1 & f_2\\ f_3 & f_4\end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, 1+f_2+f_4 \ne 0\right\}$. For any $F \notin \mathscr{F}_1$ the index of the pencil $(E, A+BF)$ is greater than 1.

### 3.2.1 Informational non uniqueness

When players use a strategy $F \in \mathscr{F}$, let the index of the resulting autonomous descriptor system (3.2) be $\tilde{\mu}$. Here, feedback alters the static and dynamic spectral properties of the pencil $(A,B)$ and as a result $\tilde{\mu}$ is different from $\mu$ unless $\mathscr{F} = \mathscr{F}_\mu$. A canonical decomposition of the descriptor system, for $t > 0$, once the players use an $F \in \mathscr{F}$ is given by:

$$\begin{aligned}
\dot{\tilde{m}}(t) &= \tilde{P}_0\cdots\tilde{P}_{\tilde{\mu}-1}E_{\tilde{\mu}}^{-1}A\tilde{m}(t),\ \tilde{m}(0) = \tilde{P}_0\cdots\tilde{P}_{\tilde{\mu}-1}x_0 \qquad (3.7)\\
\tilde{n}(t) &= 0.
\end{aligned}$$

So, for $t > 0$ the players actually use $\tilde{m}(t)$, as $x(t) = \tilde{m}(t) + \tilde{n}(t) = \tilde{m}(t)$, and $\tilde{m}(t) \in \mathrm{Im}(\tilde{P}_0 \cdots \tilde{P}_{\tilde{\mu}-1}) \subset \mathbb{R}^n$. From lemma 3.2.2 we have that $\tilde{m}(t)$ is isomorphic to $\tilde{m}_l(t)$. As a result, for $t > 0$, we see that $x(t) \in \mathbb{R}^n$ is isomorphic to $\tilde{m}_l(t) \in \mathbb{R}^d$ with $d < n$. The algebraic constraints in the descriptor system render some of the state variables in $x(t)$ redundant. This feature is captured in the canonical decomposition (3.7) by an invariant flow. However, we notice that a canonical decomposition (3.7) could be given only after applying the full state feedback $F \in \mathscr{F}$. Further, the canonical projectors and the players' costs vary with $F$ in a way that is not easily tractable for applications like optimal control. This limitation motivates an investigation for the existence of index preserving feedback strategies. In theorem 3.2.2 we showed that the class $\mathscr{P}$ is index preserving. So, a question arises if players restrict their strategies $\mathscr{F}_\mu$, see remark 3.2.2, can we infer that it is sufficient for players to use a strategy in $\mathscr{P}$? We show in the theorem 3.2.3 below that this observation is true for $\mu = 1$. Consider an index 1 descriptor system (3.2), then with a canonical decomposition we have:

$$\begin{aligned} \dot{m}_l(t) &= YP_0E_1^{-1}Am(t) + YP_0E_1^{-1}Bu(t),\ m_0(0) = YP_0x_0 && (3.8a)\\ n(t) &= -Q_0E_1^{-1}Bu(t),\ n(0) = Q_0x_0. && (3.8b) \end{aligned}$$

**Lemma 3.2.5.** *Assume $(E,\ A)$ regular and index 1, then for any $F \in \mathscr{F}_1$ we have that $(I + Q_0E_1^{-1}BF)$ is non-singular.*

*Proof.* Since $(E,\ A)$ is index 1, for any projector $Q_0$ such that $\mathrm{Im}Q_0 = \mathrm{Ker}E$ we have by theorem 3.A.1, $E + AQ_0$ is non-singular. For any $F \in \mathscr{F}_1$ we have $(E, A + BF)$ is index 1. As a result, for any $Q_0$ with $\mathrm{Im}Q_0 = \mathrm{Ker}E$, we have $E + (A + BF)Q_0$ is non-singular.

$$E + (A + BF)Q_0 = E + AQ_0 + BFQ_0 = E_1 + BFQ_0 = E_1\left(I + E_1^{-1}BFQ_0\right).$$

Clearly, $\left(I + E_1^{-1}BFQ_0\right)$ is non-singular. We know,

$$\det\left(I + E_1^{-1}BFQ_0\right) = \det\left(\begin{bmatrix} I & -E_1^{-1}BF \\ Q_0 & I \end{bmatrix}\right) = \det\left(I + Q_0E_1^{-1}BF\right),$$

so, $\left(I + Q_0E_1^{-1}BF\right)$ is also non-singular. ■

**Theorem 3.2.3.** *Every full state information feedback $F \in \mathscr{F}_1$ can be realized as a partial state feedback $K = \Omega(F) = F\left(I + Q_0E_1^{-1}BF\right)^{-1}Z$.*

*Proof.* Using an $F \in \mathscr{F}_1$, i.e., with $u(t) = Fx(t)$, in the system (3.8a-3.8b) we have:

$$\begin{aligned} \dot{m}_l(t) &= YP_0E_1^{-1}AZm_l(t) + YP_0E_1^{-1}BFx(t),\ m_l(0) = YP_0x_0\\ n(t) &= -Q_0E_1^{-1}BFx(t). \end{aligned}$$

From the second equation we have $n(t) = -Q_0E_1^{-1}BFZm_l(t) - Q_0E_1^{-1}BFn(t)$. Consequently, using lemma 3.2.5, we have $n(t) = -\left(I + Q_0E_1^{-1}BF\right)^{-1}Q_0E_1^{-1}BFZm_l(t)$. Thus,

$x(t)$ is given as:

$$\begin{aligned} x(t) &= m(t)+n(t) \\ &= \left(I-\left(I+Q_0E_1^{-1}BF\right)^{-1}Q_0E_1^{-1}BF\right)Zm_l(t) \\ &= \left(I+Q_0E_1^{-1}BF\right)^{-1}Zm_l(t). \end{aligned} \tag{3.9}$$

Therefore the inherent ODE is given by:

$$\dot{m}_l(t) = YP_0E_1^{-1}\left(A+BF\left(I+Q_0E_1^{-1}BF\right)^{-1}\right)Zm_l(t).$$

Now, let us define a mapping $\Omega : \mathscr{F}_1 \to \mathscr{P}$, i.e., $K = \Omega(F)$, as

$$K = \Omega(F) = F\left(I+Q_0E_1^{-1}BF\right)^{-1}Z. \tag{3.10}$$

Using the identity $Q_0E_1^{-1}BF\left(I+Q_0E_1^{-1}BF\right)^{-1} = \left(I+Q_0E_1^{-1}BF\right)^{-1}Q_0E_1^{-1}BF$, we have:

$$\begin{aligned} \dot{m}_l(t) &= YP_0E_1^{-1}AZm_l(t)+YP_0E_1^{-1}BKm_l(t) \\ n(t) &= -\left(I+Q_0E_1^{-1}BF\right)^{-1}Q_0E_1^{-1}BFZm_l(t) \\ &= -Q_0E_1^{-1}BKm_l(t). \end{aligned}$$

■

*Remark* 3.2.3. The autonomous regular descriptor system, after $F$ is applied, can be seen canonically as a vector field on a manifold (a proper linear subspace $\mathrm{Im}P_0$ here). What theorem 3.2.3 says is that applying an $F \in \mathscr{F}_1$ to (3.2) is same as applying $\Omega(F) \in \mathscr{P}$ to the vector field (3.4a). Since $\Omega(.)$ is a many to one mapping we expect to have more than one $F \in \mathscr{F}_1$ resulting in the same closed loop behavior. Further, for applications like optimal control or dynamic games it will suffice to regulate (3.4a) using a partial state feedback, say $K \in \mathscr{P}$, then reconstruct the full state feedback as $\Omega^{-1}(K) \in \mathscr{F}_1$.

We analyze some properties of $\Omega^{-1}(.)$ in the following theorem.

**Theorem 3.2.4.** *For any $K \in \mathscr{P}$, the inverse map $\Omega^{-1}(.)$ is given by*

$$\Omega^{-1}(K) = KS^{\dagger}+T(I-SS^{\dagger}), \tag{3.11}$$

*where $S = Z-Q_0E_1^{-1}BK$ and $T \in \mathbb{R}^{m\times n}$ is arbitrary. Further, $\Omega^{-1}(K)$ is non empty.*

*Proof.* From (3.10) we have

$$Z-Q_0E_1^{-1}BK = \left(I-Q_0E_1^{-1}BF\left(I+Q_0E_1^{-1}BF\right)^{-1}\right)Z = \left(I+Q_0E_1^{-1}BF\right)^{-1}Z.$$

Since $Z$ has full column rank and from the above we notice that $Z-Q_0E_1^{-1}BK$ has full column rank. Now, taking $S = Z-Q_0E_1^{-1}BK$, (3.10) is given by $K = FS$. Now, for given a $K \in \mathscr{P}$, all solutions $F \in \mathscr{F}_1$ are characterized, see pg. 295 [1], by

$$F = \Omega^{-1}(K) = KS^{\dagger}+T\left(I-SS^{\dagger}\right), \tag{3.12}$$

where $T \in \mathbb{R}^{m\times n}$ is arbitrary. For the non emptiness part, applying a feedback $K \in \mathscr{P}$ means $u(t) = Km_l(t) = KYm(t) = KYZm_l(t)$. So, $K$ and $KYZ$ are same strategies resulting in same closed-loop behavior. Replacing $K$ with $KYZ$ and choosing $T = KY = KYZY$ in (3.12) we have $F = KYZS^\dagger + KYZY\left(I - SS^\dagger\right) = KYZY + KYZS^\dagger \; - KYZYSS^\dagger \; = KY + KYZS^\dagger - KYZYZS^\dagger - KYP_0Q_0E_1^{-1}BKS^\dagger \; = KY$. So, for every $K \in \mathscr{P}$, there is a trivial solution $\Omega^{-1}(K) = F = KY \in \mathscr{F}_1$, and this observation coincides with remark 3.2.2. ∎

*Remark* 3.2.4. The game (3.1,3.2) with strategy set $\mathscr{P}$ is informationally inferior (see section 6.3 [7]) to $\mathscr{F}_1$, since $\mathscr{P} \subseteq \mathscr{F}_1$. However, we showed that for every $F \in \mathscr{F}_1$ there exists a $\Omega(F) \in \mathscr{P}$. Here, $K = \Omega(F)$ encapsulates entire state information. In other words, being a deterministic optimization problem providing more information about the state does not improve the optimal solution. We recall from proposition 6.3.2 [7], that players cannot realize a cost lesser than what is achieved with an equilibrium strategy from $\mathscr{P}$, by searching in $\mathscr{F}_1$. We show later that there exist many informationally non unique equilibrium strategies in $\mathscr{F}_1$ which correspond to a single solution of the game. However, the situation can be quite different in a stochastic setting, see section 6.7 of [7].

*Remark* 3.2.5. Notice that the analysis in this section is restricted to index 1. For higher order indices the algebraic constraints cannot be eliminated easily as they include derivatives of inputs. However, we address this case with a different approach in section 3.3.2.

# 3.3 Feedback Nash equilibria

## 3.3.1 Index 1 case

In this section we collect all the results discussed in the previous section to derive FBNE for the differential game (3.1, 3.2). We consider the index 1 case. The game (3.1, 3.2), when players use strategies $u_i = F_i x,\ i = 1,2$, with $F = [F_1\ F_2] \in \mathscr{F}_1$, can be written as:

$$J_i(x_0,F_1,F_2) = \int_0^\infty x'(t) \begin{bmatrix} I \\ F_1 \\ F_2 \end{bmatrix}' \begin{bmatrix} D_i & V_i & W_i \\ V_i' & R_{i1} & N_i \\ W_i' & N_i' & R_{2i} \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ F_2 \end{bmatrix} x(t)dt$$

$$E\dot{x}(t) = (A + BF)x(t),\ x(0) = x_0.$$

Using theorem 3.2.3 the game (3.1, 3.2) is transformed as follows:

$$J_i(m_l(0),K_1,K_2) = \int_0^\infty m_l'(t) \begin{bmatrix} I \\ K_1 \\ K_2 \end{bmatrix}' \begin{bmatrix} \tilde{D}_i & \tilde{V}_i & \tilde{W}_i \\ \tilde{V}_i & \tilde{R}_{1i} & \tilde{N}_i \\ \tilde{W}_i & \tilde{N}_i & \tilde{R}_{2i} \end{bmatrix} \begin{bmatrix} I \\ K_1 \\ K_2 \end{bmatrix} m_l(t)\ dt \tag{3.14a}$$

$$\dot{m}_l(t) = \left(\tilde{A} + \tilde{B}K\right) m_l(t),\ m_l(0) = YP_0x_0, \tag{3.14b}$$

where $F=\begin{bmatrix}F_1' & F_2'\end{bmatrix}'$, $B=\begin{bmatrix}B_1 & B_2\end{bmatrix}$, $K=\begin{bmatrix}K_1' & K_2'\end{bmatrix}'$, $\tilde{A}=YP_0E_1^{-1}AX$, $\tilde{B}=YP_0E_1^{-1}B$ and

$$\begin{bmatrix}\tilde{D}_i & \tilde{V}_i & \tilde{W}_i\\ \tilde{V}_i & \tilde{R}_{1i} & \tilde{N}_i\\ \tilde{W}_i & \tilde{N}_i & \tilde{R}_{2i}\end{bmatrix}=\mathscr{X}'\begin{bmatrix}D_i & V_i & W_i\\ V_i' & R_{i1} & N_i\\ W_i' & N_i' & R_{2i}\end{bmatrix}\mathscr{X},\quad \mathscr{X}=\begin{bmatrix}Z & -Q_0E_1^{-1}B_1 & -Q_0E_1^{-1}B_2\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}.$$

Notice that the game (3.1,3.2) is transformed into a game where the players' objectives and system dynamics are given by (3.14a) and (3.14b) respectively. We are interested in first finding FBNE of the resulting lower order game (3.14a,3.14b) and later use theorem 3.2.4 to obtain FBNE of the game (3.1,3.2). We seek for stabilizing strategies to keep the players' objectives bounded. To this end, we make an assumption that $K\in\mathscr{P}$ is stabilizing i.e., $\tilde{A}+\tilde{B}K$ is stable [3]. Further, if $m_l(t)\to 0$ then from (3.9) we have that $x(t)\to 0$. Moreover, $K\in\mathscr{P}$ and $\Omega^{-1}(K)\in\mathscr{F}_1$ have the same closed-loop behavior and as a result the FBNE of the game (3.1,3.2) are stabilizing too. Now, The main result is given as follows:

**Theorem 3.3.1.** *Let $\mu=1$ and assume that matrix $\tilde{G}=\begin{bmatrix}\tilde{R}_{11} & \tilde{N}_1\\ \tilde{N}_2' & \tilde{R}_{22}\end{bmatrix}$ is invertible and the matrices $\tilde{R}_{ii}>0,\ i=1,2$. Then $(F_1^*,\ F_2^*)$ is a stabilizing FBNE for (3.1, 3.2) for every consistent initial state if and only if $F^*\in\Omega^{-1}(K)$ where $K$ is given by*

$$K=\begin{bmatrix}K_1\\ K_2\end{bmatrix}=-\tilde{G}^{-1}\begin{bmatrix}\tilde{B}_1'X_1+\tilde{V}_1'\\ \tilde{B}_2'X_2+\tilde{W}_2'\end{bmatrix},\tag{3.15}$$

*and $(X_1,\ X_2)$ are a symmetric stabilizing solution of the coupled algebraic Riccati equations*

$$\tilde{D}_1+\tilde{W}_1K_2+K_2'\tilde{W}_1'+K_2'\tilde{R}_{21}K_2-K_1'\tilde{R}_{11}K_1+X_1\left(\tilde{A}+\tilde{B}_2K_2\right)+\left(\tilde{A}+\tilde{B}_2K_2\right)'X_1=0\tag{3.16a}$$

$$\tilde{D}_2+\tilde{V}_2K_1+K_1'\tilde{V}_2'+K_1'\tilde{R}_{12}K_1-K_2'\tilde{R}_{22}K_2+X_2\left(\tilde{A}+\tilde{B}_1K_1\right)+\left(\tilde{A}+\tilde{B}_1K_1\right)'X_2=0.\tag{3.16b}$$

*Moreover, $J_i=x_0'P_0'Y'X_iYP_0x_0$.*

*Proof.* Let us define $H=\left(I+Q_0E_1^{-1}BF\right)^{-1}Z$ and we notice that $H$ has full column rank. From the above reformulation it follows directly from lemma 3.2.1 that $(F_1^*,F_2^*)$ is a FBNE for (3.1,3.2) for every consistent initial state if and only if $(K_1,K_2)$ is a FBNE for the game (3.14a,3.14b) and $(F_1^*,F_2^*)$ solve $F^*\in\Omega^{-1}(K)$. Further, by theorem 3.2.1 we have that $(K_1,\ K_2)$ is a FBNE for every initial state for the game (3.14a,3.14b) if and only if (3.15) holds where $(X_1,X_2)$ are a stabilizing solution of (3.16a,3.16b). ■

[3] It is not clear as to which player can influence stability of the closed loop system. However, the stabilization constraint can be justified from the supposition that both players have a first priority in stabilizing the system. Whether this coordination actually takes place, depends on the outcome of the game. Only if the players have the impression that their actions are such that the system becomes unstable, will they coordinate their actions in order to realize this meta-objective and adapt their actions accordingly. See section 8.3 of [35] for a detailed discussion.

*Remark* 3.3.1. The algebraic constraint is given by $n(t) = -Q_0 E_1^{-1} BKm_l(t) = -Q_0 E_1^{-1} B KYm(t)$. Then, the consistent initial state manifold is characterized as:

$$\mathscr{X}_0 = \left\{ x_0 \in \mathbb{R}^n \middle| \; Q_0(I + E_1^{-1} BKYP_0)x_0 = 0 \right\},$$

where $K$ is the lower order FBNE given by (3.15).

### 3.3.2 Index $\mu > 1$ case

Lemma 3.2.5 and theorem 3.2.3 were obtained by restricting the class of feedbacks to $\mathscr{F}_1$ and as a result the projectors $P_0$ and $Q_0$ were retained even for the modified pencil $(E,\ A+BF)$. However, for $\mu > 1$ the projector chains change with an exception for the class $\mathscr{P}$, as demonstrated in theorem 3.2.2. For the index 1 case, the map $\Omega(.)$, as shown in (3.9), removes all the redundant state information. However, for $\mu > 1$, due to presence of derivatives in the algebraic constraints, even if the strategies are restricted to $\mathscr{F}_\mu$ it is not clear if a mapping $\Omega : \mathscr{F}_\mu \to \mathscr{P}$, similar to (3.10), exists. However, we have the following sufficient condition.

**Proposition 3.3.1.** *If $D_i\big(I - P_0 \cdots P_{\mu-1}\big) = 0$, $V_i'\big(I - P_0 \cdots P_{\mu-1}\big) = 0$ and $W_i'\big(I - P_0 \cdots P_{\mu-1}\big) = 0$ for $i = 1,2$, then players' objectives do not include the algebraic constraints. Further, the game (3.1, 3.2) can be solved using a partial state feedback.*

*Proof.* Using (3.4c), the integrand in (3.1) is rewritten as:

$$\begin{aligned}(m(t)+n(t))' D_i (m(t)+n(t)) + (m(t)+n(t))' V_i u_1(t) + (m(t)+n(t))' W_i u_2(t) \\ + u_1'(t) V_i' (m(t)+n(t)) + u_1'(t) W_i' (m(t)+n(t)) + \cdots . \end{aligned} \tag{3.17}$$

Again from (3.4c), we have $n(t) = \big(I - P_0 \cdots P_{\mu-1}\big)x(t)$. Now, if the conditions given in the statement of the lemma hold true then (3.17) is given by

$$m'(t)D_i m(t) + m'(t)V_i u_1(t) + m'(t)W_i u_2(t) + u_1'(t)V_i' m(t) + u_1'(t)W_i' m(t) + \cdots .$$

As $m(t) = Zm_l(t)$, the game (3.1, 3.2) is same as the game obtained by replacing (3.1) with

$$\int_0^{t_f} \big[m_l'(t) \;\; u_1'(t) \;\; u_2'(t)\big] \tilde{M}_i \big[m_l'(t) \;\; u_1'(t) \;\; u_2'(t)\big]' dt, \text{ where } \tilde{M}_i = \begin{bmatrix} Z'D_iZ & Z'V_i & Z'W_i \\ V_i'Z & R_{1i} & N_i \\ W_i'Z & N_i' & R_{2i} \end{bmatrix},$$

and (3.2) with (3.6) respectively. Clearly, we see that the algebraic part of the descriptor system does not influence player's objectives and the resulting game is solved using theorem 3.2.1 with partial state feedback strategies of the type $u_i(t) = K_i m_l(t),\ i = 1,2$. ■

The conditions given in the above proposition can be too restrictive and we suggest a different approach so as to recast the problem to the index 1 case. The response of a descriptor system is characterized by the eigenstructure of the pencil $(E,A)$. In the classical pole placement problem for systems where $E$ is non-singular a desired eigenstructure can be achieved, under a controllability assumption of the system, by using a derivative or proportional feedback. For descriptor systems there are several notions of controllability, for example, see [30, 117]. A descriptor system (3.2) is impulse controllable if $\operatorname{rank}([E \;\; A\mathcal{N}_\infty \;\; B]) = n$, where the columns of $\mathcal{N}_\infty$ span $\operatorname{Ker}E$. If the descriptor system is impulse controllable then it was shown, in corollary 7 [24], that there exists a matrix $G \in \mathbb{R}^{m\times n}$ such that the pencil $(E,\, A+BG)$ is regular and has index at most 1.

If the dynamic environment where the players interact is modeled by a descriptor system with index $\mu$ then the strategies of players should be sufficiently smooth, and as a result players cannot adapt their strategies quickly. We make an assumption that players are obliged first, with an incentive that they can adapt quickly later, so as to regularize the system using a proportional state feedback. The players use strategies of type $u_i = G_i x + v_i$ such that the resulting descriptor system, given by $E\dot{x} = (A + B_1G_1 + B_2G_2)x + B_1v_1 + B_2v_2 = (A+BG)x + Bv$, $x(0) = x_0$ has index 1. This involves finding matrices $G$ such that the projector chain stops after one step, i.e., $E_1 = E + (A+BG)Q_0$ is non singular, where $Q_0$ is a projector such that $\operatorname{Im}Q_0 = \operatorname{Ker}E$. An SVD based algorithm is presented in [24] to construct such regularizing feedbacks.

Now, once the higher index descriptor system is regularized we can apply the theory developed in section 3.3.1 to compute the FBNE. This involves replacing the matrix $A$ with $A+BG$ in the analysis of sections 3.2.1 and 3.3.1. Notice that there exists more than one regularizing matrix $G$ and not all of them give stabilizing solutions to the Riccati equations. We demonstrate this drawback with an example in the next section.

### 3.3.3 Examples

*Example* 3.3.1. We consider [45] the issue of convergence of feedback Nash solution of the singularly perturbed system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) + 2x_2(t) + u_1(t) + u_2(t),\; x_1(0) = 1 \\ \varepsilon\dot{x}_2(t) &= -x_1(t) - 2x_2(t) + 2u_1(t) + 2u_2(t),\; x_2(0) = 2, \end{aligned}$$

with performance criteria

$$J_i = \int_0^\infty \left\{ [x_1'(t)x_1'(t)]Q_i[x_1'(t)x_1'(t)]' + u_i'(t)R_{ii}u_i(t) + u_j'(t)R_{ij}u_j(t) \right\} dt,\;\; i \neq j,\; i,j = 1,2,$$

where $D_i = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $R_{ii} = 1$ and $R_{ij} = 1$. In this example the converged feedback Nash equilibrium strategies are $F_i^* = \lim_{\varepsilon\downarrow 0} F_i(\varepsilon) = [-1.50908,\; -0.7321], i = 1,2$. The reduced order system, obtained by taking $\varepsilon = 0$, is a descriptor system. With $Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ as the initial projector and $E_1 = E + AQ_0 = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$, the canonical projector is obtained as $Q_0 = \begin{bmatrix} 0 & 0 \\ 1/2 & 1 \end{bmatrix}$. Using the canonical projector we have $E_1 = \begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$, $Z = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$

and $Y = \begin{bmatrix} -1 & 0 \end{bmatrix}$. The transformed system parameters (cf. (3.14a, 3.14b)) are $\tilde{A} = 0$, $\tilde{B} = \begin{bmatrix} -3 & -3 \end{bmatrix}$, $\tilde{Q}_i = 3/2$, $\tilde{V}_i = 0$, $\tilde{W}_i = 0$, $\tilde{R}_{ii} = 3$ , $\tilde{R}_{ij} = 4$, $j \neq i$ and $\tilde{N}_i = 2$, $i = 1,2$. Straightforward calculations show that the set of FBNE for this game, defined by $F_i = [f_{i1}\ f_{i2}]$, i=1,2, satisfy:

$$f_{i2} = -\frac{2}{2\sqrt{2}+1} f_{i1} - \frac{\sqrt{2}}{2\sqrt{2}+1},\ i = 1,2,\ f_{i1} \text{ is arbitrary.}$$

As observed in [45], we see that the FBNE of the original game, i.e., $F_i^* = \lim_{\varepsilon \downarrow 0} F_i(\varepsilon)$, does not belong to the set of the FBNE obtained from the lower order game as shown in the figure 3.1. If $F_i^* = \lim_{\varepsilon \downarrow 0} F_i(\varepsilon) \in \Omega^{-1}(K)$, then the game is considered to be well posed and this property is desired as the lower order FBNE, i.e., $\Omega^{-1}(K)$, are robust against uncertainties of the parameter $\varepsilon$. See [57] for a detailed discussion.

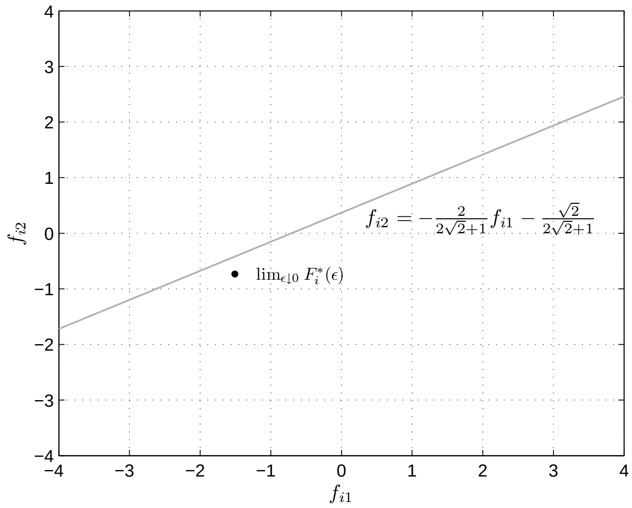



**Figure 3.1** – The FBNE of the singularly perturbed system, i.e., ($\varepsilon \neq 0$), is represented as the black dot and the set of FBNE of the lower order game is represented by the gray line.

*Example* 3.3.2. In this example we show that the suggested regularization approach, as given in section 3.3.2, for higher order index cases gives different solutions based on the choice of $G$ used. The example, from [41], is a macro-economic stabilization problem. Assume that a monetary and fiscal authority like to stabilize some key macro-economic variables, i.e., the real interest rate, $r$, inflation, $\dot{p}$, and the output gap, $y$, after a shock has occurred. The system is described by the following equations:

$$r(t) = i(t) - \dot{p}(t) \tag{3.18a}$$

$$\dot{y}(t) = -\alpha i(t) + \alpha \dot{p}(t) + \beta f(t) \tag{3.18b}$$

$$m(t) - p(t) = \gamma y(t) - \delta i(t). \tag{3.18c}$$

Here $p(t)$ is the price level, $i(t)$ denotes the nominal interest rate, $m(t)$ is the money supply and $f(t)$ the fiscal policy. The first two instruments, the nominal interest rate and money supply, are determined by the monetary authority of the country, whereas the level of the third instrument, the fiscal policy, is set by the government. Here, Equation (3.18a) models the real interest rate, (3.18b) is a simple growth equation of the output gap and (3.18c) models asset market equilibrium. Assume that an initial shock in the real interest rate, price level and output gap has occurred, all equal to one (in the respective units). Introducing as the state variable $x(t) = [r(t)\ p(t)\ y(t)]'$, $u_1(t) = [i(t)\ m(t)]'$ and $u_2(t) = f(t)$ the model can be rewritten as (3.2), where

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha & 1 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -\gamma \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ -\alpha & 0 \\ \delta & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} \text{ and } x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The performance criteria of the players are given as:

$$J_i = \int_0^\infty e^{-\theta t} \left[x'(t)\ u_1'(t)\ u_2'(t)\right] M_i \left[x'(t)\ u_1'(t)\ u_2'(t)\right]' dt,\ i = 1,2.$$

Since the performance indices are discounted, we make the following change of variables $A \to A - \frac{\theta}{2}E$. It can be easily verified that the system is regular if $1 + \alpha\gamma \neq 0$. We observe that the pencil $(E,A)$ has index 2. Taking $\alpha = 1/2$, $\beta = 3/4$, $\gamma = 1$, $\delta = 1/2$ and $\theta = 0.15$, the initial choice of $Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $G = (g_{ij})$ we see that with $g_{11} + 2g_{21} \neq 0$, the system can be regularized. The matrices entering the players costs are $D_1 = \text{diag}\{2,2,1\}$, $D_2 = \text{diag}\{1,1,2\}$, $R_{11} = \text{diag}\{2,2\}$, $R_{21} = 1$, $R_{12} = \text{diag}\{1,1\}$, $R_{22} = 2$, $V_i = 0$, $W_i = 0$, $N_i = 0$, $i = 1,2$. For the following choices of regularizing matrices $G = \begin{bmatrix} 2 & -1 & -1 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ the eigenvalues of $\tilde{A} + \tilde{B}K$ are found to be $(-1.2292, -0.58473)$ and $(-1.7905, -0.57729)$ respectively. Further, if the choice of the regularizing matrix is $G = \begin{bmatrix} 2 & 1 & 1 \\ 1/2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}$, we observe that the coupled Riccati equations do not have a solution. So, we see that the choice of regularizing matrix $G$ does affect the solution of the game. However, if players agree, before switching to noncooperative mode of play, upon a particular choice of the regularizing matrix then this method can be used to find FBNE for games defined with higher order descriptor systems.

## 3.4 Conclusions

In this chapter we consider the regular indefinite infinite planning horizon linear quadratic differential game for descriptor systems. Firstly, we develop an algorithm to generate canonical projectors for a regular matrix pair, and using these projectors it is possible to canonically decouple a descriptor system into differential and algebraic parts. Later, we analyze the effect of feedback on a regular descriptor system. We show, for the index 1 case, that there exists a many to one mapping from full state index preserving feedbacks to the projected state feedbacks leading to informational non-uniqueness. Further, we discuss the properties of the inverse mapping and derived necessary and sufficient conditions for the existence of FBNE. These conditions were stated in terms of a projected system. A complete parametrization was derived for the set of FBNE. For the higher order index case, under an impulse controllability assumption, we suggest a regularization based approach to recast the problem to an index 1 case. However, it is unclear as to how the Riccati equations depend upon the class of regularizing matrices. We demonstrated the drawback of the approach with an example. The obtained theoretical results can be generalized straightforwardly to the $N$ player case. We observed that the closed-loop system evolves invariantly on a proper linear subspace of $\mathbb{R}^n$, the configuration space. So, the FBNE is an inverse projection of the FBNE obtained from the lower order (projected) system. Further, in case there exists a FBNE, usually, there exists an infinite number of feedback Nash equilibria which all give rise to the same closed-loop behavior of the system. For future work, it would be interesting, for instance, to investigate the question whether in a singularly perturbed game the full order FBNE belongs to the set of inverse projections of the lower order FBNE, this aspect was studied in [45, 57]. Further, it would be interesting to search for equilibria within this set that satisfy some additional properties, like e.g. robustness.

## 3.A Appendix

For a regular index-$\mu$ pencil $\lambda E - A$, we consider the following sequences of matrices, subspaces and projectors

$$E_0 = E,\ A_0 = A \tag{3.19a}$$

$$\text{for } i \geq 0 \qquad E_{i+1} = E_i + A_i Q_i,\ A_{i+1} = A_i P_i \tag{3.19b}$$

$$Q_i^2 = Q_i,\ \mathrm{Im} Q_i = \mathrm{Ker} E_i,\ P_i = I - Q_i. \tag{3.19c}$$

For the matrix chain (3.19a-3.19c) an important result, following [49], is given as:

**Theorem 3.A.1.** *If $(E,A)$ is a regular pencil with the index $\mu$, then the matrices $E_0,\ E_1,\ \cdots,\ E_{\mu-1}$ are singular, whereas $E_\mu$ is non-singular. Conversely, if $E_\mu$ is non-singular and $E_0,\ E_1,\ \cdots,\ E_{\mu-1}$ are singular, then $(E,A)$ is a regular pencil with index $\mu$.*

The projectors $Q_i,\ i = 0, 1, \cdots, \mu - 1$ are not unique since the range of $Q_i$ is fixed as $\mathrm{Ker} E_i$ and $\mathrm{Ker} Q_i$ is arbitrary. Later, we see that this arbitrariness is helpful to construct an algorithm. We review some important properties of matrix projectors below. Refer [73, 113] for details.

1. $(\text{Ker}E_i \cap \text{Ker}A_i) = (\text{Ker}E_i \cap \text{Ker}E_{i+1}) \subseteq (\text{Ker}E_{i+1} \cap \text{Ker}E_{i+2})$.

2. Following theorem 2.1 of [73], for a regular pencil with index $\mu$, the projectors $Q_0,\ Q_1, \cdots,\ Q_{\mu-1}$ can be constructed such that $Q_jQ_i = 0$ for $j > i$. Projectors satisfying this property are called *admissible* projectors.

The above properties hold true irrespective of the choice of projectors. We elaborate more on properties 1 and 2 here. For a regular pencil with index $\mu$, $E_\mu$ is non-singular so $\text{Ker}E_\mu = \{0\}$. From property (1), given above, we have $(\text{Ker}E_0 \cap \text{Ker}E_1) \subseteq \cdots \subseteq \big(\text{Ker}E_{\mu-2} \cap \text{Ker}E_{\mu-1}\big) \subseteq \big(\text{Ker}E_{\mu-1} \cap \text{Ker}E_\mu\big) = \{0\}$. This implies $\big(\text{Ker}E_0 \oplus \cdots \oplus \text{Ker}E_{i-1}\big) \cap \text{Ker}E_i = \{0\}$ or $(\text{Im}Q_0 \oplus \text{Im}Q_1 \oplus \cdots \oplus \text{Im}Q_{i-1}) \cap \text{Im}Q_i = \{0\}$, see lemma 2.6 [72] for details. We recall the following lemma:

**Lemma 3.A.1** (lemma 2.5, [73])**.** *For two subspaces of* $\mathbb{R}^m$ $L = span\{l_1, l_2, \cdots, l_s\}$, $N = span\{n_1, n_2, \cdots, n_t\}$, $L \cap N = 0$, *there is a projector* $U$ *such that* $ImU = L$, $KerU \supseteq N$.

*Proof.* Denote by $R$ the $m \times (s+t)$ matrix consisting of the columns $l_1, \cdots, l_s, n_1, \cdots, n_t$. Since $N \cap L = 0$, $s+t \leq n$, and $l_1, \cdots, l_s, n_1, \cdots, n_t$ are linearly independent. Then, $R$ is full column rank and the desired projector $U$ is constructed as

$$U = R \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \left(R'R\right)^{-1} R', \text{ where } R = \begin{bmatrix} l_1, \cdots, l_s & n_1 \cdots, n_t \end{bmatrix}. \tag{3.20}$$

■

Using lemma 3.A.1 we can choose $Q_i$ such that $(\text{Im}Q_0 \oplus \text{Im}Q_1 \oplus \cdots \oplus \text{Im}Q_{i-1}) \subseteq \text{Ker}Q_i$ and $\text{Im}Q_i = \text{Ker}E_i$. The constructed projectors, called as *admissible projectors*, satisfy $Q_jQ_i = 0,\ j > i$. Later we propose an algorithm to generate these *admissible* projectors given a regular pencil with index $\mu$.

### 3.A.1 Illustration for index 1

In this section we demonstrate the application of matrix projectors towards decoupling a descriptor system. Consider the following regular index 1 descriptor system:

$$E\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0. \tag{3.21}$$

The matrix chain is given by $E_0 = E$, $A_0 = A$, $Q_0 = Q_0^2$, $\text{Im}Q_0 = \text{Ker}E_0$, $P_0 = I - Q_0$, $E_1 = E_0 + A_0Q_0 = E + AQ_0$ and $E_1$ is non singular. We verify that $E_1P_0 = EP_0 + AQ_0P_0 = EP_0 \to P_0 = E_1^{-1}EP_0$ and $E_1Q_0 = EQ_0 + AQ_0Q_0 = AQ_0 \to Q_0 = E_1^{-1}AQ_0$. Further, $x(t)$ can be decomposed as $x(t) = Ix(t) = (P_0 + Q_0)x(t) = P_0x(t) + Q_0x(t)$ and we represent

$m(t) = P_0x(t)$ and $n(t) = Q_0x(t)$. Now, pre-multiplying the above descriptor system with $E_1^{-1}$ leads to

$$\begin{aligned} E_1^{-1}E\dot{x}(t) &= E_1^{-1}Ax(t) + E_1^{-1}Bu(t) \\ E_1^{-1}E(P_0\dot{x}(t) + Q_0\dot{x}(t)) &= E_1^{-1}AP_0x(t) + E_1^{-1}AQ_0x(t) + E_1^{-1}Bu(t) \\ P_0\dot{x}(t) &= E_1^{-1}AP_0x(t) + Q_0x(t) + E_1^{-1}Bu(t). \end{aligned}$$

Multiplying the above equation with $P_0$ gives the inherent ODE

$$\dot{m}(t) = P_0E_1^{-1}Am(t) + P_0E_1^{-1}Bu(t), \tag{3.22}$$

and with $Q_0$ gives the algebraic constraint

$$0 = Q_0E_1^{-1}Am(t) + n(t) + Q_0E_1^{-1}Bu(t). \tag{3.23}$$

The above decoupling is not complete as $m(t)$ appears in (3.23). Let us define $\tilde{Q}_0 = Q_0E_1^{-1}A$. To see $\tilde{Q}_0$ as a valid projector onto Ker$E_0$, we note that $\tilde{Q}_0Q_0 = Q_0E_1^{-1}AQ_0 = Q_0$ and $Q_0\tilde{Q}_0 = Q_0Q_0E_1^{-1}A = \tilde{Q}_0$ which implies $\tilde{Q}_0$ is a projector onto the same range as $Q_0$. Next, we discuss the canonicity [4] of $\tilde{Q}_0$. Let $Q_{0_1}$ and $Q_{0_2}$ be two projectors with range constrained to Ker$E$, then $Q_{0_1}Q_{0_2} = Q_{0_2}$ and $Q_{0_2}Q_{0_1} = Q_{0_1}$. Using this we can show $Q_{0_1}P_{0_2} = -Q_{0_2}P_{0_1}$. We write $E_{1_2} = E + AQ_{0_2} = E + AQ_{0_1}Q_{0_2} = E + AQ_{0_1} - AQ_{0_1}P_{0_2} = E + AQ_{0_1} + AQ_{0_2}P_{0_1} = (E + AQ_{0_1})(I + Q_{0_2}P_{0_1}) = E_{1_1}(I + Q_{0_2}P_{0_1})$. Thus we have $E_{1_2}^{-1} = (I - Q_{0_2}P_{0_1})E_{1_1}^{-1}$. Now, $\tilde{Q}_{0_2} = Q_{0_2}E_{1_2}^{-1}A = Q_{0_2}(I - Q_{0_2}P_{0_1})E_{1_1}^{-1}A = Q_{0_1}E_{1_1}^{-1}A = \tilde{Q}_{0_1}$. $\tilde{Q}_0$ is called a *canonical projector* and it is unique, so we have that $\tilde{Q}_0 = \tilde{Q}_0\tilde{E}_1^{-1}A$, where $\tilde{E}_1 = E + A\tilde{Q}_0$. Repeating the above decoupling procedure by replacing $Q_0$ with $\tilde{Q}_0$ we arrive at the same decoupled equations (3.22, 3.23) as above, but the cross term disappears in (3.23), i.e., $n(t) = -Q_0E_1^{-1}Bu(t)$, leading to a complete decoupling.

### 3.A.2 Canonical projectors

In the previous section we showed that an index 1 system can be decoupled completely with the existence of a unique canonical projector $\tilde{Q}_0$. For index $\mu > 1$, by theorem 3.1 [73], the existence of canonical projectors is guaranteed for a regular descriptor system. Furthermore, the canonical projectors are also admissible and satisfy:

$$Q_i = Q_iP_{i+1}\cdots P_{\mu-1}E_\mu^{-1}A_i,\ i = 0, 1, \cdots, \mu - 2 \tag{3.24}$$

$$Q_{\mu-1} = Q_{\mu-1}E_\mu^{-1}A_{\mu-1}, \tag{3.25}$$

where $A_i$, $E_\mu$ are defined in (3.19a-3.19c). Collecting ideas discussed above we present an algorithm to generate canonical projectors for an index $\mu$ regular matrix pair $(E, A)$ as follows:

[4] This discussion follows from [114] and we repeat it for the sake of completeness.

**Algorithm 3.A.1.**

1. *Start with* $E_0 = E$, $A_0 = A$, $Q_0 = I - E_0^{\dagger}E_0$, $P_0 = I - Q_0$ *and* $V = KerE_0$.
2. ***(Admissible projectors)*** *for* $i \in \{1, 2, \ \mu - 1\}$
   (a) $E_i = E_{i-1} + A_{i-1}Q_{i-1}$, $A_i = A_{i-1}P_{i-1}$, $U = KerE_i$
   (b) *Define* $R = \begin{bmatrix} U & V \end{bmatrix}$
   (c) $Q_i = R\begin{bmatrix} I & \\ & 0 \end{bmatrix}(R'R)^{-1}R'$, $P_i = I - Q_i$
   (d) $V = \begin{bmatrix} V & U \end{bmatrix}$
3. ***(Canonical projectors)***
   (a) *Set for* $i = 0, \cdots, \mu - 1$, $Q_i^{(0)} = Q_i$, $E_i^{(0)} = E_i$, $A_i^{(0)} = A_i$
   (b) *Make* $Q_{\mu-1}^{(0)}$ *canonical by* $Q_{\mu-1}^{(0)} = Q_{\mu-1}^{(0)}\left(E_\mu^{(0)}\right)^{-1}A_{\mu-1}^{(0)}$
   (c) *for* $j = 0$ *to* $\mu - 1$
      i. $Q_{\mu-1}^{(j)} = Q_{\mu-1}^{(j-1)}\left(E_\mu^{(j-1)}\right)^{-1}A$
      ii. $Q_i^{(j)} = Q_i^{(j-1)}P_{i+1}^{(j-1)}\cdots P_{\mu-1}^{(j-1)}\left(E_\mu^{(j-1)}\right)^{-1}A, \quad i = 0, 1, \cdots, \mu - 2$
      iii. $\left(E_\mu^{(j)}\right)^{-1} = \left(I - Q_0^jP_0^{j-1} - Q_1^jP_1^{j-1} - \cdots - Q_{\mu-1}^jP_{\mu-1}^{j-1}\right)\left(E_\mu^{(j-1)}\right)^{-1}$

After obtaining the set of admissible projectors from step (2), canonical projectors are obtained from step (3). For a discussion on the additional properties, such as admissibility of the intermediate projector chains $\left(Q_i^{(j)}, P_i^{(j)}\right)$, $i = 0, 1, \cdots, \mu - 1$ and canonicity of the resulting projectors $Q_i^{(\mu-1)}$, $i = 0, 1, \cdots, \mu - 1$ refer to [113].

### 3.A.3 Illustration for index $\mu > 1$

Similar to index 1 case, the canonical projectors completely decouple a regular descriptor system with index $\mu > 1$. For more details on the procedure refer [73]. We, however, give few important steps. Firstly, canonical projectors satisfy the following identity:

$$\begin{aligned} x =& P_0x + Q_0x = (P_0P_1)x + (P_0Q_1 + Q_0)x = \cdots \\ & \cdots = \left(P_0 \cdots P_{\mu-1}\right)x + \left(P_0 \cdots P_{\mu-2}Q_{\mu-1} + \cdots + P_0Q_1 + Q_0\right)x. \end{aligned} \tag{3.26}$$

We see $x(t)$ can be projected into $\mu + 1$ subspaces. The first projection constitutes a vector field that evolves invariantly in the subspace $\mathrm{Im}\left(P_0 \cdots P_{\mu-1}\right)$ and the remaining $\mu$ projections constitute algebraic constraints. The descriptor system (3.21) is decoupled completely as follows:

$$\left(P_0 \cdots P_{\mu-1}\right)\dot{x} = \left(P_0 \cdots P_{\mu-1}\right)E_\mu^{-1}A\left(P_0 \cdots P_{\mu-1}\right)x + \left(P_0 \cdots P_{\mu-1}\right)E_\mu^{-1}Bu$$

$$
\begin{bmatrix}
Q_0 & Q_0P_1 & Q_0P_1P_2 & \cdots & Q_0P_1\cdots P_{\mu-2} \\
 & Q_1 & Q_1P_2 & \cdots & Q_1P_2\cdots P_{\mu-2} \\
 & & \ddots & & \vdots \\
 & & & Q_{\mu-3} & Q_{\mu-3}P_{\mu-2} \\
 & & & & Q_{\mu-2}
\end{bmatrix}
\begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{\mu-2} \\ Q_{\mu-1} \end{bmatrix} \dot{x} = -
\begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{\mu-3} \\ Q_{\mu-2} \end{bmatrix} x
- \begin{bmatrix} Q_0P_1\cdots P_{\mu-1} \\ Q_1P_2\cdots P_{\mu-1} \\ \vdots \\ Q_{\mu-3}P_{\mu-2}P_{\mu-1} \\ Q_{\mu-2}P_{\mu-1} \end{bmatrix} E_\mu^{-1}Bu
$$

$$
0 = Q_{\mu-1}x + Q_{\mu-1}E_\mu^{-1}Bu.
$$

The last two equations are written compactly as follows:

$$
\begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{\mu-3} \\ Q_{\mu-2} \\ Q_{\mu-1} \end{bmatrix} x = -\mathscr{M}
\left( \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{\mu-2} \\ Q_{\mu-1} \\ 0 \end{bmatrix} \dot{x} -
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ I \end{bmatrix} E_\mu^{-1}Bu \right),
$$

$$
\mathscr{M} = \begin{bmatrix}
Q_0 & Q_0P_1 & P_1P_2 & \cdots & Q_0P_1\cdots P_{\mu-1} \\
 & Q_1 & Q_1P_2 & \cdots & Q_1P_2\cdots P_{\mu-1} \\
 & & \ddots & \vdots & \vdots \\
 & & & Q_{\mu-3}P_{\mu-2} & Q_{\mu-3}P_{\mu-2}P_{\mu-1} \\
 & & & Q_{\mu-2} & Q_{\mu-2}P_{\mu-1} \\
 & & & & Q_{\mu-1}
\end{bmatrix}.
$$

By careful elimination of derivatives, of terms $Q_jx,\ 1 \leq j \leq \mu-1$, on the right hand side, we have:

$$
\begin{bmatrix} Q_0 \\ P_0Q_1 \\ \vdots \\ P_0\cdots P_{\mu-3}Q_{\mu-2} \\ P_0P_1\cdots P_{\mu-2}Q_{\mu-1} \end{bmatrix} x = -
\begin{bmatrix}
\text{X} & \text{X} & \text{X} & \cdots & \text{X} & -Q_0P_1\cdots P_{\mu-1} \\
 & \text{X} & \text{X} & \cdots & \text{X} & -P_0Q_1P_2\cdots P_{\mu-1} \\
 & & \ddots & & \vdots & \vdots \\
 & & & & \text{X} & -P_0\cdots P_{\mu-3}Q_{\mu-2}P_{\mu-1} \\
 & & & & & -P_0\cdots P_{\mu-2}Q_{\mu-1}
\end{bmatrix}
\begin{bmatrix} E_\mu^{-1}Bu^{\mu-1} \\ E_\mu^{-1}Bu^{\mu-2} \\ \vdots \\ E_\mu^{-1}Bu^{1} \\ E_\mu^{-1}Bu \end{bmatrix}.
\tag{3.27}
$$

From property (3.26), representing $m = \big(P_0\cdots P_{\mu-1}\big)x$ and $n = \big(P_0\cdots P_{\mu-2}Q_{\mu-1} + \cdots$

$+P_0Q_1+Q_0\big)x=(I-P_0\cdots P_{\mu-1})x$, the descriptor system can be decoupled as:

$$\dot{m}(t)=P_0\cdots P_{\mu-1}E_\mu^{-1}Am(t)+P_0\cdots P_{\mu-1}E_\mu^{-1}Bu(t),\ m(0)=P_0\cdots P_{\mu-1}x_0 \tag{3.28a}$$

$$n(t)=-\sum_{i=0}^{\mu-1}N_iE_\mu^{-1}Bu^i(t), \tag{3.28b}$$

where $N_i$ in (3.28b) are obtained by manipulation of terms in (3.27).

# CHAPTER 4

# Lyapunov stochastic stability and control of robust dynamic coalitional games with transferable utilities

## 4.1 Introduction

This chapter deals with robustness and dynamics in coalitional games with transferable utility (TU), along the line of [13, 14]. Coalitional TU games, introduced first by von Neumann and Morgenstern [80], have recently sparked much interest in the control and communication engineering community [90]. In essence, coalitional TU games are constituted by a set of players who can form coalitions and by a characteristic function associating a real number with every coalition. The real number represents the value of the coalition and can be thought of as a monetary value that can be divided among the members of the coalition according to some appropriate fairness allocation rule. The value of a coalition also reflects the monetary benefit demanded by a coalition to be a part of the grand coalition.

In the context of coalitional TU games, *robustness* and *dynamics* naturally arise in all the situations where the coalitions values are uncertain and time-varying. We next discuss each of the two aspects and try to connect them to the existing literature.

Robustness has to do with modeling coalitions' values as unknown entities and this is in spirit with some literature on stochastic coalitional games [103, 105]. However, we deviate from these last works since we model coalitions values as Unknown But Bounded (UBB) variables within an a priori known polytopic set [15]. It is worth to mention that this formulation is in line with the recent literature on interval valued games [3], where the authors use intervals to describe coalitions values similarly to what is done in this chapter. The interval nature of coalitions' values arises generally due to the optimistic and pessimistic expectations of the coalitions [27] when cooperation is achieved from a strategic form game. We also recognize some differences in that we focus more on the time-varying nature of the coalitions values. In doing this, we also link the approach to the set invariance theory [18] and stochastic stability theory [67] which provides us some nice tools for stability analysis (see, e.g., the resort to a Lyapunov function in the proof of theorem 4.4.1).

Dynamics enters in the form of a system state evolution. The state accounts for the accumulated discrepancy between coalitions' values and allocations up to the current time with the assumption that the game is played continuously over time. So, the state accounts for the extra reward or excess that a coalition has received up to the current time. However, this excess is different from the coalitional excess that appears commonly, e.g., in the definition of nucleolus [92]. At each time, different coalitions' values realize and these values are undisclosed to the Game Designer (GD) who then adjusts allocations based on the available information on the system state. Bringing dynamical aspects into the framework of coalitional TU games is an element in common with other papers [43, 53, 60]. The main difference with those works is that the values of coalitions are realized exogenously[1] and no relation exists between consecutive samples.

This chapter is in spirit with a few other recent attempts to bring robustness and dynamics within the framework of coalitional TU games [11, 12, 13, 14]. In [13, 14] the authors dealt with robust stabilizability of the excesses in a discrete time setting. Here, we are more concerned with convergence almost surely of a function of the excesses and we show that this translates into the long-run almost sure convergence of the average allocation to the core[2] of the average game. We assume that the average value of the coalitons' converges to a balanced game which the GD has access to. Convergence conditions together with the idea that allocation rules use a measure of the extra reward that a coalition has received up to the current time by re-distributing the budget among the players are a main issue in a number of other papers [4, 28, 50, 64, 96] as well. However, this chapter departs from the aforementioned contributions mainly in that dynamics in those works is captured by a bargaining mechanism with fixed coalitions' values while we let the values be time-varying and uncertain. This last element adds some robustness in our allocation rule which has not been dealt with before.

The main contribution of this chapter is summarized by the following three results. Firstly, we design an allocation rule based on full information on the extra reward so that the average allocation can be driven to a specific point in the core of the average game. Secondly, we design a new allocation rule based on partial information on the extra reward so that convergence to the core can be still guaranteed but not to a specific point of the core. Convergence of both allocation rules is proved via Lyapunov stochastic stability theory. The third result highlights connections of the Lyapunov stochastic stability theory to the approachability theory [17, 64].

A few minor contributions are present in this chapter as, for instance, the definition of integral and average game whose role becomes fundamental when the coalitions' values variations are known with delay by the GD. Beside this, also the reformulation of the problem as a network flow control problem[3] where the allocation rule turns into a robust

[1] For endogenous formulations of coalitions' values see section 9 of [77].

[2] A set of allocations is in the core when it is coalitionally rational. That is, the core consists of those allocations for which no coalition would be better off if it would separate itself and get its coalitional worth. Or, stated differently, a set of allocations belongs to the core if there is no incentive to any coalition to break off from the grand coalition. See [77] for more details.

[3] See [16], for connections between network flow problems and bankruptcy games in a static context.

control policy is a novel aspect. The importance of such a reformulation lies in the fact that we can prove the convergence of the allocations using the strong tools of the Lyapunov stochastic stability theory. We conclude this introduction by remarking the novelty of the idea of turning a coalitional TU game set up into a control theoretic problem, which represents, by far, the main characteristics of this work.

This chapter is organized as follows. In section 4.2, we formulate the problem. In section 4.3, we present the basic idea of our solution approach. In section 4.4 we state the three main results of this work and postpone the derivation of such results to section 4.5. In section 4.6, we provide some numerical illustrations. Finally, in section 4.7, we draw some concluding remarks.

## 4.2 Problem formulation

In this section, we formulate the problem in its generic form and elaborate on the role of information. Consider a set of players $N = \{1, \ldots, n\}$ and all possible nonempty coalitions $S \subseteq N$ arising among these players. Introduce a time-varying characteristic function $v_S(t)$ which assigns a real value to each coalition $S$ at time $t \geq 0$:

$$v_S(t) : S \times \mathbb{R}_+ \to \mathbb{R},$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers. We also denote by $m = 2^n - 1$, the number of possible coalitions except the empty coalition (indicated by $\emptyset$) and view the characteristic function $v_S(t)$ as returning a vector in the $m$-dimensional space:

$$v(t) = [v_S(t)]_{S \subseteq N} \in \mathbb{R}^m, \quad \forall t \geq 0.$$

We make use of the above time-varying characteristic function to define the following dynamic coalitional games.

**Definition 4.2.1.** ***(dynamic TU game)*** *For each time $t \geq 0$, the instantaneous, integral and average dynamic games are defined as:*

- ***(instantaneous game)*** $< N, v(t) >$, *with* $v(t) \in \mathbb{R}^m$
- ***(integral game)*** $< N, \tilde{v}(t) >$, *with* $\tilde{v}(t) = \int_0^t v(\tau) d\tau$
- ***(average game)*** $< N, \bar{v}(t) >$, *with* $\bar{v}(t) = \frac{\tilde{v}(t)}{t}$.

The motivation of formalizing the above dynamic TU games is in that such games are suitable to model scenarios where the coalitions' values vary with time. In this perspective, while the instantaneous game accounts for the instantaneous variations of the values, the integral game describes the accumulated values over time whereas the average game describes the mean of the coalitions values over time. The importance of the average game increases in those situations where $v(t)$ is known with a certain delay. In these cases, because of its smooth variations, an average game based on past values represents a better approximation than any past instantaneous game. We consider the following example, taken from [13], to motivate the problem formulation.

*Example* 4.2.1. Consider a single-period one-warehouse multi-retailer inventory system. A warehouse $W$ serving three retailers $R_1$, $R_2$ and $R_3$. Each retailer faces a demand bounded by a minimum and a maximum value. For instance $R_i$ faces a demand $d_i$ in the interval $[d_i^-, d_i^+]$. After demands $d_i$ are realized, each retailer $R_i$ must choose whether to fulfill the demand or not. The retailers do not hold any private inventory. Therefore, if they wish to fulfill their demands, they must reorder goods at the central warehouse. The retailers may share the total transportation cost $K$. Before demands are realized, the warehouse holder decides how to allocate the transportation costs among the retailers. This decision is only based on the knowledge of the minimum demand $d_i^-$ and maximum demand $d_i^+$. This allocation problem can be reformulated as a TU game. However, the value of a coalition is interval valued due to interval uncertainty. To see the dynamic aspect of the application, consider a situation where the discussed scenario occurs repeatedly in time, i.e., at each time (day, week), and the warehouse manager allocates the costs and demands are realized. Now, the warehouse manager can design allocation rules based on past allocations and excesses of coalitions so that the retailers can reorder jointly.

*Remark* 4.2.1. Since a TU game represented in characteristic function form suppresses the strategic aspect, i.e., players influencing the evolution of the game using strategies, this chapter deviates from the other chapters. Introducing time aspects to a static TU game opens the possibility for modeling aspects such as intertemporal transfers, patience and expectations of players/coalitions. A generic dynamic coalitional game description should capture these features. This chapter addresses modeling a class of problems of the type given in example 4.2.1.

In line with the above preamble, the underlying assumption throughout this chapter is that $v(t)$ is unknown to the game designer (GD) or central planner but confined within a convex set $\mathscr{V}$[4] at any time. We also assume that $v(t)$ is a mean ergodic stochastic process. The limitations due to our choice of modeling the coalitions values as a stationary stochastic process are justified by an increased simplicity in the analysis and efficacy of the obtained results.

*Assumption* 4.2.1. **(UBB and mean ergodic)** Signal $v(t)$ is UBB within a given convex set $\mathscr{V}$, i.e., $v(t) \in \mathscr{V} \in \mathbb{R}^m$. Furthermore, the expected value of $v(t)$ coincides with the long term average $\lim_{t\to\infty} \bar{v}(t)$.

From the Bondareva-Shapley theorem, see [98], it follows that the core of a TU game is non empty if and only if the game is balanced. One interpretation of balancedness in a TU game is that the players can distribute one unit of working time to any coalition and in doing so cannot generate more value than the grand coalition. We do not make assumptions regarding the balancedness of the instantaneous games which is the case with [13]. Under the above assumption, the core of the instantaneous game can be empty at some time $t$, i.e., the instantaneous games are, in general, not balanced. We, however, assume that the

[4]The interval nature of the uncertainty naturally leads to convexity of $\mathscr{V}$.

core of the average game is nonempty in the long run in order to make the problem more tractable. Before introducing the assumption, denote $v_{\text{nom}} = \lim_{t\to\infty} \bar{v}(t)$ and let $C(v_{\text{nom}})$ be the core of the average game. Hereafter, we will also call $v_{\text{nom}}$ the (long run) average coalitions' values. This is a reasonable assumption since only in principle one could rely on infinite budgets.

*Assumption* 4.2.2. **(balancedness)** The core of the average game is nonempty in the limit: $C(v_{\text{nom}}) \neq \emptyset$.

We can view the above assumption as introducing some steady-state (average) conditions on a game scenario subject to instantaneous fluctuations. Now, assume that the GD can take actions in terms of instantaneous allocations denoted by $a(t) \in \mathbb{R}^n$ with the following budget constraints.

*Assumption* 4.2.3. **(bounded allocation)** The instantaneous allocation is bounded within a hyperbox in $\mathbb{R}^n$

$$a(t) \in \mathscr{A} = \{a \in \mathbb{R}^n : a_{\min} \leq a \leq a_{\max}\},$$

with a priori given lower and upper bounds $a_{\min}, a_{\max} \in \mathbb{R}^n$.

A first assumption about the information available to the GD is that he knows the long run average coalitions' values. Under this assumption, as we see in the following sections, the GD can design allocation rules that can converge to the core of the average game.

*Assumption* 4.2.4. **(on available information)** The GD knows $v_{\text{nom}}$.

In spirit with a number of other papers [4, 28, 50, 64, 96], we aim at finding allocation rules that use a measure of the extra reward that a coalition has received up to the current time by re-distributing the budget among the players. To do this, a first step is to define excesses for the coalitions. For any coalition $S \subseteq N$, we define *excess (extra reward) at time* $t \geq 0$ as the *excess at time* $t = 0$ plus the difference between the total integral reward, given to it, and the integral value of the coalition itself, i.e.,

$$\varepsilon_S(t) = \sum_{i\in S} \tilde{a}_i(t) - \tilde{v}_S(t) + \varepsilon_S(0).$$

Furthermore, assume without loss of generality $\varepsilon_S(0) = 0$, we say that *$S$ is in excess at time $t \geq 0$* if the excess is nonnegative, i.e., $\sum_{i\in S} \tilde{a}_i(t) \geq \tilde{v}_S(t)$. To summarize, coalitions in excess are those with respect to which the grand coalition of the integral game is stable against deviations. Let $\varepsilon(t)$ represent the vector of accumulated coalitions' excesses formally given as:

$$\varepsilon(t) = \{\varepsilon_S(t)\}_{N \supseteq S \neq \emptyset}.$$

We are interested in answering three questions for this class of games.

- **Question 1:** Are there allocation rules such that the average allocation converges? If yes, let us denote by $\mathscr{A}_0$ the set where the average of allocations converge to.

- **Question 2:** If the average allocation converges, can we make it converge to the core of the average game $\mathscr{A}_0 \subseteq C(v_{\mathrm{nom}})$?
- **Question 3:** Furthermore, can we guarantee the convergence to a specific point of the core that we have a priori selected?

To motivate the above questions let us think of a situation where the objective of the GD is to maintain stability of the grand coalition in an average sense, in which case one would expect $\mathscr{A}_0 \subseteq C(v_{\mathrm{nom}})$. We are now in the position to formally state the problem in its the generic form. Henceforth, we use the symbol w.p.1 to mean "with probability one".

**Problem 4.2.1.** *Find an allocation rule* $f : \mathbb{R}^m \to \mathscr{A} \in \mathbb{R}^n$*, such that if* $a(t) = f(\varepsilon(t))$ *then* $\lim_{t\to\infty} \bar{a}(t) \in \mathscr{A}_0 \subseteq C(v_{nom})$ *w.p.1.*

Observe that because of the random nature of the coalitions' values $v(t)$, both the excesses $\varepsilon(t)$ and the allocations $a(t)$ are random and as such we look at the convergence of $\bar{a}(t)$ w.p.1. Essentially, we require that the probability of $\bar{a}(t)$ converging in the limit to $\mathscr{A}_0 \subseteq C(v_{\mathrm{nom}})$ is 1. This type of convergence is also known in the literature as *almost sure* convergence [67]. To include the above questions 2 and 3 in the problem statement (problem 4.2.1), we need to make different assumptions on $\mathscr{A}_0$.

i) $\mathscr{A}_0$ is a specific point in the core a priori selected, call it nominal allocation and denote it as $a_{\mathrm{nom}}$, i.e.,
$$\mathscr{A}_0 = a_{\mathrm{nom}} \in C(v_{\mathrm{nom}}),$$
ii) $\mathscr{A}_0$ is the whole average core $C(v_{\mathrm{nom}})$, namely
$$\mathscr{A}_0 = C(v_{\mathrm{nom}}).$$

We will show that solving the first case requires the exact knowledge of the excesses $\varepsilon(t)$ at time $t$ on the part of the GD. Conversely, in the second case it suffices that the GD has a partial knowledge about $\varepsilon(t)$. This is in line with our intuition about the fact that case i) requiring convergence to a specific point of the core appears as a more constrained version of problem ii) where convergence is required to any point of the core.

## 4.3 Flow transformation based dynamics

The basic idea of our solution approach is to recast the problem into a flow control one. To do this, consider the hyper-graph $\mathscr{H}$ with vertex set $V$ and edge set $E$ as:

$$\mathscr{H} = \{V, E\}, \quad V = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}, \quad E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}.$$

The vertex set $V$ has one vertex per each coalition whereas the edge set $E$ has one edge per each player. A generic edge $i$ is incident to a vertex $\mathbf{v}_j$ if the player $i$ is in the coalition associated to $\mathbf{v}_j$. So, incidence relations are described by matrix $B_{\mathscr{H}}$whose rows are the characteristic vectors $c^S \in \mathbb{R}^n$. We recall that the components of a characteristic vector $c_i^S = 1$ if $i \in S$ and $c_i^S = 0$ if $i \notin S$. So, for a 3 player situation we have coalitions $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\{1,2,3\}$ associated with vertices $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, $\mathbf{v_4}$, $\mathbf{v_5}$, $\mathbf{v_6}$ and $\mathbf{v_7}$ respectively. The matrix incidence $B_{\mathscr{H}}$ for a 3 player case is given as:

$$B_{\mathscr{H}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The edge $\mathbf{e_i}$ corresponds to the player $i$. The first column of $B_{\mathscr{H}}$ says that player 1 is present in the coalitions $\{1\}$, $\{1,2\}$, $\{1,3\}$ and $\{1,2,3\}$. The flow control reformulation arises naturally if we view allocation $a_i(t)$ as the flow on edge $\mathbf{e}_i$ and the coalition value $v_S(t)$ of a generic coalition $S$ as the demand $d_j(t)$ in the corresponding vertex $\mathbf{v}_j$, namely $v_S(t) = d_j(t)$. In view of this, allocation in the core translates into over-satisfying the demand at the vertices. Specifically,

$$a(t) \in C(v(t)) \quad \Leftrightarrow \quad B_{\mathscr{H}} a(t) \geq d(t), \tag{4.1}$$

with the last inequality satisfied with the equal sign due to the efficiency condition of the core, i.e, $\sum_{i=1} a_i(t) = d_m(t)$, where $d_m(t)$ denotes the $m^{\text{th}}$ component of $d(t)$ and is equal to the grand coalition value $v_N(t)$. Now, since $d(t)$ is unknown at time $t$, we need to introduce some error dynamics which accounts for the derivatives of the excesses. Since, $\varepsilon(t)$ represents the accumulated coalition excess we have:

$$\dot{\varepsilon}(t) = B_{\mathscr{H}} a(t) - d(t), \quad d(t) \in \mathscr{V}. \tag{4.2}$$

From (4.1) and averaging and taking the limit in (4.2), we can reformulate problem 4.2.1 as a flow control problem where a controller wishes to drive the quantity $\lim_{t\to\infty} \frac{\varepsilon(t)-\varepsilon(0)}{t}$ to the target set below w.p.1 :

$$\mathscr{T} = \{\tau \in \mathbb{R}^m : \tau_m = 0, \tau_j \geq 0, \forall j = 1, \ldots, m-1\}.$$

Note, $\tau_m = 0$ due to efficiency of allocations.

*Remark* 4.3.1. Driving the average allocations to a particular point $a_{\text{nom}} \in \mathscr{A}_0 \subseteq C(v_{\text{nom}})$ results in reaching a specific point in the target set $\mathscr{T}$. To see this, note that when $\lim_{t\to\infty} \bar{a}(t) = a_{\text{nom}}$ we have $\mathscr{T} \ni B_{\mathscr{H}} a_{\text{nom}} - v_{\text{nom}} \geq 0$ due to the property of the core. Thus, we also have that $\lim_{t\to\infty} \frac{\varepsilon(t)-\varepsilon(0)}{t}$ is driven to the point $B_{\mathscr{H}} a_{\text{nom}} - v_{\text{nom}} \in \mathscr{T}$.

The inequality condition in (4.1) is transformed into equality type by introducing, from standard LP techniques, $m-1$ slack variables (one per each coalition other than the grand coalition). This increases the control space of the GD from $m$ to $n+m-1$ and the dynamics (4.2) can be rewritten as follows:

$$\dot{x}(t) = Bu(t) - d(t),\ d(t) \in \mathscr{V}, \tag{4.3}$$

where $B = \left[ B_{\mathscr{H}} \;\middle|\; \begin{matrix} -I_{m-1} \\ 0 \end{matrix} \right] \in \mathbb{R}^{m \times n+m-1}$. Variable $x(t)$ represents the state of the system and captures deviation from the balanced system, i.e., the system characterized by $a_{\text{nom}}$ and $v_{\text{nom}}$. We introduce the set of feasible controls as:

$$U = \left\{ u(t) \in \mathbb{R}^{n+m-1} : u(t) = [a^T(t)\ \ s^T(t)]^T,\ a(t) \in \mathscr{A},\ s(t) \geq 0 \right\}. \tag{4.4}$$

In preparation to the reformulation of the problem as a stochastic stabilizability one, we introduce the following preliminary result.

**Lemma 4.3.1.** *The following two statements are equivalent:*

*i) the average allocations converge to the core of the average game,*

$$\lim_{t\to\infty} \bar{a}(t) \in C(v_{nom}),\ \text{w.p.1}, \tag{4.5}$$

*ii) the new variable $\frac{x(t)-x(0)}{t}$ is asymptotically stable almost surely, i.e.,*

$$\lim_{t\to\infty} \frac{x(t)-x(0)}{t} = 0, \text{w.p.1.}$$

*Proof.* Since by assumption the average game is balanced the core is nonempty. To see why i) implies ii) first observe that

$$\bar{a}(t) \in C(\bar{v}(t)) \iff B\bar{u}(t) = \bar{v}(t). \tag{4.6}$$

To see why (4.6) holds true, note that $\bar{a}(t) \in C(v(t))$ means that there exists a feasible allocation $\bar{a}(t)$ such that $B_{\mathscr{H}}\bar{a}(t) - \bar{v}(t) \geq 0$. Now, since the surplus variables are under the control of the GD, it is always possible to set $\bar{s}(t) = B_{\mathscr{H}}\bar{a}(t) - \bar{v}(t)$ so that we can obtain $B\bar{u}(t) = \bar{v}(t)$ w.p.1 and this proves condition (4.6). On the other hand, by integrating (2), we also have that

$$\frac{x(t)-x(0)}{t} = B\bar{u}(t) - \bar{v}(t).$$

Now, from (4.5), (4.6) and Assumption 2 we infer that $\lim_{t\to\infty} B\bar{u}(t) - \bar{v}(t) = 0$ w.p.1, which in turn means $\lim_{t\to\infty} \frac{x(t)-x(0)}{t} = 0$ w.p.1. So, we can conclude ii).
To see why ii) implies i), note that, $\lim_{t\to\infty} \frac{x(t)-x(0)}{t} = 0$ w.p.1 implies $\lim_{t\to\infty} B\bar{u}(t) - \bar{v}(t) = 0$ w.p.1. Since from (3) we also have that $\lim_{t\to\infty} \bar{s}(t) > 0$, then we can rewrite $\lim_{t\to\infty} B\bar{u}(t) = v_{\text{nom}}$ w.p.1, as $\bar{s}(t) = B_{\mathscr{H}}\bar{a}(t) - \bar{v}(t)$ and $v_{\text{nom}}$ is balanced by Assumption 2. Thus we have $\lim_{t\to\infty} \bar{a}(t) \in C(v_{\text{nom}})$ w.p.1. ∎

It is worth noting that condition (4.5) is part of problem 4.2.1. In other words when solving problem 4.2.1 we always guarantee (4.5). If this is clear then, we can use the above lemma to rephrase problem 4.2.1. In doing this we need to make a partial distinction between case i) and ii). More specifically, the case ii) where $\mathscr{A}_0 = C(v_{\text{nom}})$ can be restated as follows:

$$\text{Find } u(t) = \phi(x(t)) \in U \text{ s.t } \lim_{t\to\infty} \frac{x(t) - x(0)}{t} = 0 \text{ w.p.1.} \tag{4.7}$$

Note that we can alternatively look at condition (4.5) as a constraint that requires $x(t)$ to remain bounded for all $t$. Actually any control $\phi(x(t))$ so that $x(t)$ is bounded w.p.1 implies that the numerator $x(t) - x(0)$ is bounded as well and taking the limit for $t \to \infty$ then the fraction goes to zero. With this in mind, let us anticipate here that the problem that we will solve is a stricter one where $x(t)$ not only remains bounded but also approaches zero w.p.1. A strict reformulation of the above problem is given as:

$$\text{Find } u(t) = \phi(x(t)) \in U \text{ s.t } \lim_{t\to\infty} x(t) = 0 \text{ w.p.1.} \tag{4.8}$$

*Remark* 4.3.2. If $\mathscr{V} \subseteq \text{int}\{BU\}$, then from [10] we can drive $x(t)$ to zero asymptotically in a deterministic sense, i.e., $\lim_{t\to\infty} x(t) = 0$, and as a result also condition (5) is satisfied. This approach was carried out in [13] to analyze the convergence of allocations to a specific point in the core, in a discrete time setting. In this chapter, we relax the assumption $\mathscr{V} \in \text{int}\{BU\}$ in favor of the weaker condition $v_{\text{nom}} \in \text{int}\{\mathscr{V} \cap BU\}$. We will still be able to prove that $\lim_{t\to\infty} x(t) = 0$ w.p.1 (in stochastic sense).

As regards problem 4.2.1 case i), we need to make an additional comment. Actually, it turns out that if we wish to reach a specific point $a_{\text{nom}}$ then the condition (4.5) is only necessary. So, the problem we will solve is a stricter version of (4.8).

## 4.4 Main results

In this section we present the three main results of this work. The first one relates to the case where the GD has full information on $x(t)$ in which case the average allocation can be driven to a specific point in the core of the average game. The second result applies to the case where the GD has partial information on $x(t)$, and convergence to the core can be still guaranteed but not to a specific point of the core. The third result highlights connections of the implemented solution approach to the approachability principle [17, 64].

### 4.4.1 Full information case

In this section we solve problem 4.2.1 with $\mathscr{A}_0 = a_{\text{nom}}$ and under the assumption that the GD has complete information regarding excesses $\varepsilon(t)$ and therefore $x(t)$ as well. We recall that inferring $x(t)$ from $\varepsilon(t)$ is possible as the surplus $s(t)$ are selected by the GD. As we have said before, the problem that we solve is a stricter version of (4.8). This version derives from augmenting the state of dynamics (4.3) as explained in the rest of this section. Before introducing the augmentation technique let us start by assuming that the fluctuations of the coalitions' values around the mean $v_{\text{nom}}$ are independent of the state $x(t)$. We formalize this in the next assumption where we denote by $\Delta d(t) = d(t) - v_{\text{nom}}$ the above fluctuations.

*Assumption* 4.4.1. The state $x(t)$ and the coalitions' values fluctuations $\Delta d(t)$ are uncorrelated.

Introducing the fluctuations $\Delta d(t)$ allows us to rewrite dynamics (4.3) in a more convenient way. To do this, note first that, as $u = [a^T \ s^T]^T$, if $a_{\text{nom}}$ is fixed then $u_{\text{nom}}$ is also fixed as $Bu_{\text{nom}} = v_{\text{nom}}$. Let us denote $\Delta u(t) = u(t) - u_{\text{nom}}$. The dynamics (4.3) can be rewritten as follows:

$$\begin{aligned}\dot{x}(t) = Bu(t) - d(t) &= Bu(t) - (v_{\text{nom}} + (d(t) - v_{\text{nom}})) = Bu(t) - v_{\text{nom}} - \Delta d(t) \\ &= B\left(u(t) - u_{\text{nom}}\right) - \Delta d(t) = B\Delta u(t) - \Delta d(t).\end{aligned}$$

We said before that we will focus on a stricter version of (4.8). We do this by augmenting the state as shown next. First, denote by $B^\dagger$ a generic pseudo inverse matrix of $B$ and complete matrices $B$ and $B^\dagger$ with matrices $C$ and $F$ such that

$$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} B^\dagger & F \end{bmatrix} = I. \tag{4.9}$$

Then, building upon the new square matrix $\begin{bmatrix} B \\ C \end{bmatrix}$, let us consider the augmented system

$$\begin{array}{rcl} \dot{x}(t) & = & B\Delta u(t) - \Delta d(t) \\ \dot{y}(t) & = & C\Delta u(t). \end{array} \tag{4.10}$$

Here $y(t)$ plays the role of a compensator[5] and we assume that $d(t)$ is uncorrelated to $y(t)$ as well. After integrating the above system (see (4.11), right) we define a new variable $z(t)$ as follows:

$$z(t) = \begin{bmatrix} B^\dagger & F \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix} z(t). \tag{4.11}$$

It turns out that to drive $x(t)$ to zero w.p.1, and obtain $u_{\text{nom}}$ as average allocation on the long run, we can rely on a simple function $\hat{\phi}(.)$, which depends on $z(t)$. Before introducing this function, for future purposes observe that the dynamics for $z(t)$ satisfies the first-order

[5]The additional dynamic variable $y(t)$ has the goal of keeping track of the load unbalancing with respect to the desired average 0.

differential equation:

$$\begin{aligned}
\dot{z}(t) &= \begin{bmatrix} B^{\dagger} & F \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} \\
&= \begin{bmatrix} B^{\dagger} & F \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} \Delta u(t) - \begin{bmatrix} B^{\dagger} & F \end{bmatrix} \begin{bmatrix} \Delta d(t) \\ 0 \end{bmatrix} \\
&= \Delta u(t) - B^{\dagger} \Delta d(t).
\end{aligned} \tag{4.12}$$

Let $\Delta u^{\min}$ and $\Delta u^{\max}$ be the minimal and maximal values of $\Delta u(t)$ for the following constraints to hold true: $u(t) = u_{\text{nom}} + \Delta u(t) \in U$. Then, let us formally define $\hat{\phi}(z(t))$ as:

$$\hat{\phi}(z(t)) = u_{\text{nom}} + \Delta u(t) \in U, \quad \Delta u(t) = sat_{[\Delta u^{\min},\, \Delta u^{\max}]}(-z(t)), \tag{4.13}$$

where with $sat_{[a,b]}(\xi)$ we denote the saturated function that, given a generic vector $\xi$ and lower and upper bounds $a$ and $b$ of same dimensions as $\xi$, returns

$$sat_{[a,b]}(\xi) = \begin{cases} b_i & \text{for all } i \quad \xi_i > b_i \\ a_i & \text{for all } i \quad \xi_i < a_i \\ \xi_i & \text{for all } i \quad a_i \leq \xi_i \leq b_i. \end{cases}$$

Now, taking the control $u(t) = \hat{\phi}(z(t))$, we obtain the dynamic system $\dot{z}(t) = B\hat{\phi}(z(t)) - d(t)$. With the above preamble in mind, we are ready to state the following convergence property.

**Theorem 4.4.1.** *Using the controller $\hat{\phi}(z(t))$, as in* (4.13)*, we have* $\lim_{t\to\infty} z(t) = 0$ *w.p.1 and therefore* $\lim_{t\to\infty} \bar{u}(t) = u_{nom}$.

In the next corollary, we use the previous result to provide an answer to problem 4.2.1.

**Corollary 4.4.1.** *The state $x(t)$ is driven to zero w.p.1 as expressed in* (4.8) *and the average allocation converges to the nominal allocation i.e.,* $\lim_{t\to\infty} \bar{a}(t) = a_{nom}$.

*Proof.* This is a direct consequence of the result proved in the previous theorem. From (4.11), and $[B^{\dagger}\ F]$ being a non singular matrix, we have $\lim_{t\to\infty} x(t) = 0$ w.p.1. From the previous theorem we also have $\lim_{t\to\infty} \bar{u}(t) = u_{\text{nom}}$. Since $u(t) = [a^T(t)\ s^T(t)]^T$, we have that $\lim_{t\to\infty} \bar{a}(t) = a_{\text{nom}}$. ∎

To summarize, in the full information case, the controller $u(t)$ defined by (4.13) induces an allocation sequence $a(t)$ such that the average $\bar{a}(t)$ converges to $\mathscr{A}_0 = a_{\text{nom}}$ and the average excess of coalitions are driven to $s_{nom} \in \mathscr{T}$.

### 4.4.2 Partial information case

In the previous section we observed that if the GD has complete knowledge of the excesses and therefore of $x(t)$ then he can design an allocation rule so that the average allocations are driven to $a_{\text{nom}}$. In this section we solve problem 4.2.1 with $\mathscr{A}_0 = C(v_{\text{nom}})$ and under the assumption that the GD has partial information about $x(t)$. In particular, we assume that the GD knows the sign of $x(t)$. An information structure based on the sign of $x(t)$ has an oracle-based interpretation which we discuss in detail in Subsection 4.4.2.1.

Similarly to the previous section, suppose that we know a particular allocation $a_{\text{nom}}$ in the core $C(v_{\text{nom}})$, and let us study the convergence properties of the average allocations. In particular, using an allocation rule $u(t) = \phi(x(t))$, we require that $x(t)$ satisfying the dynamics $\dot{x}(t) = B\phi(x(t)) - d(t)$, converge to zero in probability. In this section, we state the second main result of this work which proposes a solution to problem 4.2.1 with partial information structure. To do this, let us denote again by $B^\dagger$ a generic pseudo inverse matrix of $B$ and take a feasible allocation $u_{\text{nom}}$ such that

$$Bu_{\text{nom}} = v_{\text{nom}} = \lim_{t\to\infty} \bar{v}(t), \quad u_{\text{nom}} \in U.$$

Also, for future purposes, define a function $\hat{\phi}(.)$, which depends only on the sign of $x(t)$, as follows:

$$\hat{\phi}(sign(x(t))) = u_{\text{nom}} + \Delta u(t) \in U, \quad \Delta u(t) = -\delta B^\dagger sign(x(t)). \tag{4.14}$$

Now, taking the control $u(t) = \hat{\phi}(sign(x(t)))$, we obtain the dynamic system $\dot{x}(t) = B\,\hat{\phi}(sign(x(t))) - d(t)$. Now, we state the following convergence property.

**Theorem 4.4.2.** *Using the controller* $u(t) = \hat{\phi}(sign(x(t)))$ *as in* (4.14) *we have* $\lim_{t\to\infty} x(t) = 0$ *w.p.1.*

**Corollary 4.4.2.** *The average allocation converges to the core of the average game:*

$$\lim_{t\to\infty} \bar{a}(t) \in C(v_{nom}),\ w.p.1.$$

*Proof.* From theorem 4.4.2, we know that $\lim_{t\to\infty} x(t) = 0$ w.p.1. This last result in turn implies that $\lim_{t\to\infty} \frac{x(t)-x(0)}{t} = 0$ w.p.1 and thus, invoking lemma 4.3.1, we also have that $\lim_{t\to\infty} \bar{a}(t) \in C(v_{\text{nom}})$, w.p.1. ■

*Remark* 4.4.1. In both cases, i.e., with full and partial information on $x(t)$ the GD could drive $\lim_{t\to\infty} x(t)$ to zero w.p.1. As a result we have $\lim_{t\to\infty} \frac{x(t)-x(0)}{t} = 0$ and by lemma 4.3.1 convergence of allocations to the core $C(v_{\text{nom}})$ is guaranteed. With full information on $x(t)$ a particular allocation $a_{\text{nom}}$ can be achieved i.e., $\mathscr{A}_0 = a_{\text{nom}}$ on average whereas in the partial information case, the average of the allocations converge to the core of the average game, i.e., $\mathscr{A}_0 = \lim_{t\to\infty} C(\bar{v}(t))$.

#### 4.4.2.1 Oracle based interpretation

In this subsection we elaborate more on the partial information structure. In particular, we highlight how the feedback on state $x(t)$ can be reviewed as the result of an oracle-based procedure. To see this, assume that the GD knows the sign of $x(t)$. Since $x(t) = (\varepsilon(t) - \tilde{s}(t)) - (\varepsilon(0) - x(0))$, $\text{sign}(x(t))$ reflects over-satisfaction of coalitions with respect to the threshold $\tilde{s}(t)$. In particular, take without loss of generality $\varepsilon(0), x(0) = 0$, then with reference to component $j$, the sign of $x_j(t)$ yields:

$$sign\left(x_j(t)\right) = \begin{cases} 1 & \varepsilon_j(t) > \tilde{s}_j(t) \\ 0 & \varepsilon_j(t) = \tilde{s}_j(t) \\ -1 & \varepsilon_j(t) < \tilde{s}_j(t). \end{cases} \tag{4.15}$$

To summarize, we can think of a situation where the GD approaches an oracle that tells him the sign of $x(t)$. Since $s(t)$ is chosen by the GD for every $t$, the accumulated surplus, $\tilde{s}(t)$, is given as an input to the oracle. The oracle returns "yes" if the actual accumulated excess is greater than $\tilde{s}(t)$ and "no" otherwise. The use of oracles is an element in common with the ellipsoid method in optimization and with a large literature [79] on cutting planes.

Recall that nonnegativeness of the threshold has its roots in the feasibility condition $u(t) \in U$ for all $t \geq 0$ with feasible set $U$ as in (4.4). Nonnegativeness of the threshold provides us with a further comment on the information available to the GD. Actually, from the first condition in (4.15), we can conclude that coalitions associated to a positive state $x(t)$ are certainly in excess. This is clear if we observe that $sign\left(x_j(t)\right) = 1$ implies $\varepsilon_j(t) > \tilde{s}_j(t) \geq 0$. We can then summarize the information content available to the GD as follows, let $S$ be the generic coalition associated to component $j$:

$$sign\left(x_j(t)\right) = \begin{cases} 1 & \text{then coalition } S \text{ in excess} \\ -1, 0 & \text{nothing can be said.} \end{cases}$$

Trivially, the development in the full information case in Section 4.4.1, which is all based on control strategy (4.13), fits remarkably well with the case where $x(t)$ is revealed completely as abundantly elaborated in [14]. In this last case, the fact that the GD knows $x(t)$ implies that he knows $\varepsilon(t)$ as well. Also, it is intuitive to infer that in this last set up, exact knowledge of $x(t)$ can only influence positively the GD in terms of speed of convergence of allocations to the core of the average game.

*Remark* 4.4.2. As the GD knows a priori the nominal game and a corresponding nominal allocation vector a natural question that arises is why one has to design an allocation rule as given by (4.13) and (4.14) instead of a stationary rule $\hat{\phi}(.) = u_{\text{nom}}$. The rules given by (4.13) and (4.14) intuitively translate to meeting the instantaneous needs of coalitions in an average sense which involves redistribution of excesses in some optimal way. This feature provides an incentive for players to stay in grand coalition.

### 4.4.3 Connections to approachability

Approachability theory was developed by Blackwell as early as 1956 [17] and culminates to the well known Blackwell's theorem. Along the lines of Section 3.2 in [64], we recall next the geometric (approachability) principle that lies behind the Blackwell's theorem. The contribution of this section is to show that such a geometric principle shares striking similarities with the solution approach used in the previous sections.

To introduce the approachability principle, let $\Phi$ be a closed and convex set in $\mathbb{R}^m$ and let $P(y)$ denote the projection of any point $y \in \mathbb{R}^m$ (closest point to $y$ in $\Phi$). Also denote by $\bar{y}_k$ the average of $y_1 \dots, y_k$, i.e., $\bar{y}_k = \frac{\sum_{t=0}^{k} y_t}{k}$ and let $dist(\bar{y}_k, \Phi)$ the Euclidean distance between point $\bar{y}_k$ and set $\Phi$.

**Lemma 4.4.1.** *(Approachability principle [64]) Suppose that a sequence of uniformly bounded vectors $y_k$ in $\mathbb{R}^m$ satisfies condition (4.16), then* $\lim_{k\to\infty} dist(\bar{y}_k, \Phi) = 0$*:*

$$[\bar{y}_k - P(\bar{y}_k)]^T [y_{k+1} - P(\bar{y}_k)] \leq 0. \tag{4.16}$$

Now, to make use of the above principle in our set up, let us consider the discrete time analog of the excess dynamics (4.3):

$$x_{k+1} = x_k + B\Delta u_k - \Delta d_k,$$

and define a new variable $y_k = x_k - x_{k-1}$ so that we can look at the sequence of $y_k$ in $\mathbb{R}^m$. Likewise, consider the discrete time version of control (4.14) as displayed below:

$$\hat{\phi}(sign(x_k)) = u_{\text{nom}} + \Delta u_k \in U, \quad \Delta u_k = -\delta B^{\dagger} sign(x_k - x_0). \tag{4.17}$$

We are now in the position of stating the main result of this section.

**Theorem 4.4.3.** *Using the controller $u_k = \hat{\phi}(sign(x_k - x_0))$ as in* (4.17) *we have equivalently that*

*i) the average allocations converge to the core of the average game,*

$$\lim_{k\to\infty} \bar{a}_k \in C(v_{nom}),\ w.p.1 \tag{4.18}$$

*ii) the vector* 0 *is approachable by the sequence $\bar{y}_k$,*

$$\lim_{k\to\infty} \bar{y}_k = 0,\ w.p.1. \tag{4.19}$$

The strength of the above result is in that it sheds light on how the convergence problem dealt with in this work has a stochastic stability interpretation as well as an approachability one.

# 4.5 Derivation of the main results

## 4.5.1 Proof of theorem 4.4.1

This proof is derived in the context of Lyapunov stochastic stability theory [67]. We start by observing that using $u(t) = \hat{\phi}(z(t))$ we have:

$$\dot{z}(t) = B\hat{\phi}(z(t)) - d(t). \tag{4.20}$$

Consider a candidate Lyapunov function $V(z(t)) = \frac{1}{2}z^T(t)z(t)$. The idea is to inspect that $E[\dot{V}(z(t))] < 0$ [6] for all $t \geq 0$. Actually, the theory establishes that if the last condition holds true, then $V(z(t))$ is a supermartingale and therefore by the martingale convergence theorem $\lim_{t\to\infty} V(z(t)) = 0$ w.p.1 (almost surely). To see that $E[\dot{V}(z(t))] < 0$ is true, observe that from (4.12) we have

$$\begin{aligned} E[\dot{V}(z(t))] &= E[z^T(t)\dot{z}(t)] \\ &= E[z^T(t)\Delta u(t)] - E[z^T(t)B^\dagger \Delta d(t)] \\ &= E[z^T(t)sat(-z(t))] < 0, \end{aligned}$$

where condition $E[z^T(t)B^\dagger \Delta d(t)] = 0$ is a direct consequence [7] of the assumption that $\Delta d(t)$ is uncorrelated with $x(t)$ and $y(t)$. But the above condition implies that $\lim_{t\to\infty} V(z(t))$ is equal to 0 w.p.1 and therefore we have $\lim_{t\to\infty} z(t) = 0$ w.p.1. So far we have proved the first part of the statement, i.e., that the dynamic system (4.20) converges to zero w.p.1. For the second part, after integrating dynamics (4.12), we have

$$\lim_{t\to\infty} \frac{\int_0^t [\Delta u(\tau) - B^\dagger \Delta d(\tau)]d\tau}{t} = \lim_{t\to\infty} \frac{z(t) - z(0)}{t} = 0.$$

This last condition together with the assumption $v_{\text{nom}} = \lim_{t\to\infty} \bar{v}(t)$ yields

$$\lim_{t\to\infty} \frac{\int_0^t B^\dagger \Delta d(\tau)d\tau}{t} = \lim_{t\to\infty} \frac{\int_0^t \Delta u(\tau)d\tau}{t} = 0,$$

from which we can conclude $\lim_{t\to\infty} \bar{u}(t) = \lim_{t\to\infty} \frac{\int_0^t u_{\text{nom}} + \Delta u(\tau)d\tau}{t} = u_{\text{nom}}$ as claimed in the statement.

---

[6] Stochastic stability involves time derivative of the expectation of $V(x(t))$. However, since $V(.)$ is non-negative and smooth, the limit (due to derivative) and expectation can be interchanged by using the dominated convergence theorem [111].

[7] If $\Delta d(t)$ is uncorrelated with $x(t)$ and $y(t)$ then $C\Delta d(t)$ is uncorrelated with $z(t) = Ax(t) + By(t)$.

### 4.5.2 Proof of theorem 4.4.2

Consider a candidate Lyapunov function $V(x(t)) = \frac{1}{2}x^T(t)x(t)$. The idea is to inspect that $E[\dot{V}(x(t))] < 0$ for all $t \geq 0$. For this to be true, it must be

$$\begin{aligned}
E[\dot{V}(x(t))] &= E[x^T(t)\dot{x}(t)] \\
&= E[x^T(t)Bu(t)] - E[x^T(t)d(t)] \\
&= E[x^T(t)Bu_{\mathrm{nom}}] + E[x^T(t)B\Delta u(t)] - E[x^T(t)v_{\mathrm{nom}}] - \underbrace{E[x^T(t)\Delta d(t)]}_{=0} \\
&= E[x^T(t)B\Delta u(t)] < 0,
\end{aligned}$$

where condition $E[x^T(t)\Delta d(t)] = 0$ is a direct consequence of Assumption 4.4.1. But the above condition $E[x^T(t)B\Delta u(t)] < 0$ is satisfied since $B\Delta u(t) = -\delta sign(x)$, which in turn implies

$$E[x^T(t)B\Delta u(t)] = E[-\delta\|x(t)\|_1] < 0.$$

Then we derive that $\lim_{t\to\infty} V(x(t)) = 0$ w.p.1 and therefore also $\lim_{t\to\infty} x(t) = 0$ w.p.1 and this concludes the proof.

### 4.5.3 Proof of theorem 4.4.3

We first prove the equivalence between (4.18) and (4.19). Invoking the discrete time reformulation of lemma 4.3.1, we can infer that condition (4.18) is equivalent to driving $\bar{y}_k = \frac{1}{k}\sum_k y_k$ to zero as $k \to \infty$ w.p.1. Actually, lemma 4.3.1 in discrete time establishes that $\lim_{k\to\infty} \bar{a}_k \in C(v_{\mathrm{nom}})$, w.p.1 is equivalent to $\lim_{k\to\infty} \frac{x_k - x_0}{k} = 0$ w.p.1. Observing that $\bar{y}_k = \frac{x_k - x_0}{k}$ then we have that $\lim_{k\to\infty} \bar{a}_k \in C(v_{\mathrm{nom}})$, w.p.1 is equivalent to $\lim_{k\to\infty} \bar{y}_k = 0$ w.p.1.

We now prove that using the controller $u_k = \hat{\phi}(sign(x_k))$ as in (4.17) then (4.19) holds true. To see this, let us invoke the approachability principle in lemma 4.4.1 and observe that a sufficient condition for approachability of $\bar{y}_k$ to 0 is $\bar{y}_k^T y_{k+1} \leq 0$ for all $k$. This is evident if we take set $\Phi$ including only the zero vector, $\Phi = \{0\}$, and thus $P(\bar{y}_k) = 0$ in (4.16). For the present case, using the definition of $y_k$, condition $\bar{y}_k^T y_{k+1} \leq 0$ would be $\frac{1}{k}(x_k - x_0)^T(x_{k+1} - x_k) \leq 0$, which implies $(x_k - x_0)^T B\Delta u_k - (x_k - x_0)^T \Delta d_k \leq 0$ for all $k$. Taking the expectation, from Assumption 4.4.1 we know that $E[(x_k - x_0)^T \Delta d_k] = 0$ and so we can write

$$\begin{aligned}
E[(x_k - x_0)^T B\Delta u_k - (x_k - x_0)^T \Delta d_k] &= E[(x_k - x_0)^T B\Delta u_k] \\
&= E[(x_k - x_0)^T B(-\delta B^{\dagger} sign(x_k - x_0))] \leq 0.
\end{aligned}$$

From the above condition we derive that $\bar{y}_k^T y_{k+1} \leq 0$ w.p.1 for all $k$ and this concludes our proof.

## 4.6 Numerical illustrations

Consider a 3 player coalitional TU game, so $m = 7$, with values of coalitions in the following intervals:

$$v(\{1\}) \in [0,4],\ v(\{2\}) \in [0,4],\ v(\{3\}) \in [0,4]$$
$$v(\{1,2\}) \in [0,4],\ v(\{1,3\}) \in [0,6]$$
$$v(\{2,3\}) \in [0,7],\ v(\{1,2,3\}) \in [0,12].$$

The convex set $\mathscr{V}$ is then a hyperbox characterized by the above intervals. From Assumption 4.2.4, the GD knows the long run average game, i.e., $\lim_{t\to\infty} \bar{v}(t) = v_{\text{nom}}$. Without loss of generality we take the balanced nominal game be as $v_{\text{nom}} = [1\ 2\ 3\ 4\ 5\ 6\ 10]^T$. In other words, during the simulations we randomize the instantaneous games $v(t) \in \mathscr{V}$ so that it satisfies the average behavior given by:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t v(\tau) d\tau = v_{\text{nom}}. \tag{4.21}$$

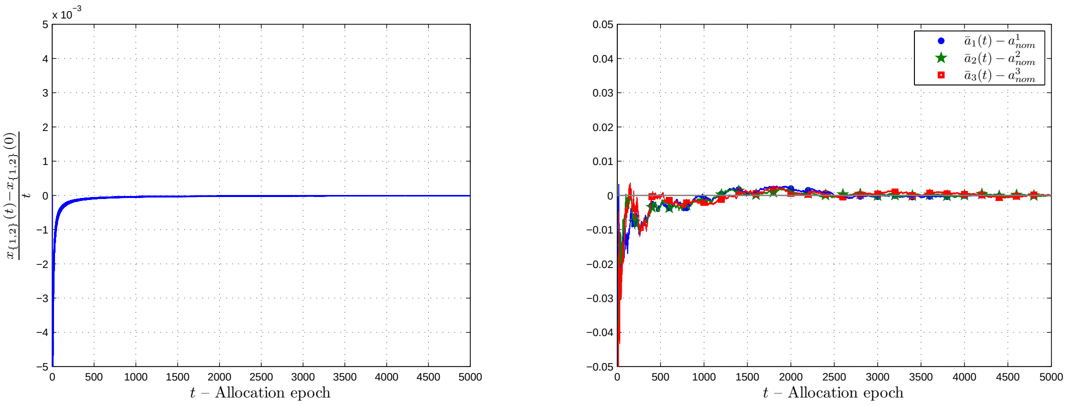
Next, we describe an algorithm to generate $v(t) \in \mathscr{V}$ such that the above condition holds true.

**Algorithm 4.6.1.**

1. *Generate m random points,* $r_i \in \mathscr{V} \subset \mathbb{R}^m,\ i = 1,2,\cdots,m$.
2. *Solve* $R.p = v_{nom}$, *with* $R = [r_1,\ r_2,\cdots\ r_m]$.
3. *If* $p \geq 0$ *and* $\mathbf{1}^T p > 0$, *then go to* 4 *else go to* 1.
4. *Rescale* $R$ *as* $R = \left(\mathbf{1}^T p\right) R$ *and* $p$ *as* $p = \frac{p}{(\mathbf{1}^T p)}$.
5. *If* $r_i \in \mathscr{V},\ i = 1,2,\cdots,m$, *then go to* 6 *else go to* 1.
6. *STOP.*

By construction, $v_{\text{nom}}$ is in the relative interior of the convex hull generated by the columns of the matrix $R$. If an instance of the game $v(t)$ is chosen as $r_i$ with probability $p_i$ from the pair $(R,p)$, Assumption 4.2.4 is satisfied. For simulations we ran the algorithm 10 times to generate 10 $(R,p)$ pairs in $\mathscr{V}$. Further, from each pair $(R,p)$ we take $100,000$ random selections (using Matlab `randsrc` function) to realize $v(t)$. The step size is set to $\Delta = 0.05$. The results are averaged over the 10 pairs. The nominal choice of allocations and surplus is taken as $u_{\text{nom}} = [2.5\ 3\ 4.5\ 1.5\ 1\ 1.5\ 1.5\ 2\ 1.5]^T$. It can be verified that $Bu_{\text{nom}} = v_{\text{nom}}$.

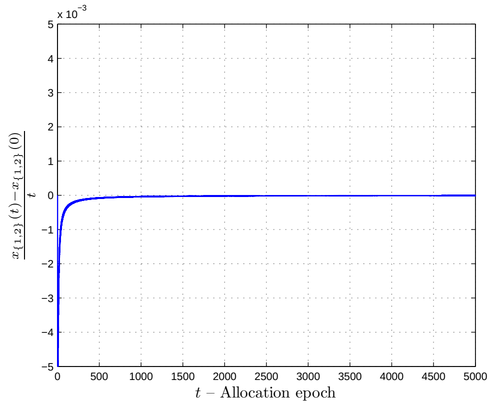
**Full information case:** The saturation thresholds $\Delta u^{\min}$ and $\Delta u^{\max}$ are chosen so as to ensure $u(t) \in U$. This condition translates into $U_{\min} \leq u_{\text{nom}} + sat_{[\Delta u^{\min},\ \Delta u^{\max}]} \leq U_{\max}$. Denote $\mathbf{1}$ as a vector with all entries equal to 1. For the instantaneous game a negative allocation/surplus is not allowed, so $U_{\min} \geq 0 \cdot \mathbf{1}$. Further, an allocation/surplus greater than the value of grand coalition is not allowed, so $U_{\max} \leq v_{\text{nom}}(N) \cdot \mathbf{1}$. For the given game parameters, we see that the lower and upper thresholds for the saturation function are $-1$ and 5.5 respectively. Next, we present the performance results of the robust control law given by equation (4.13). From theorem 4.4.1, $\lim_{t\to\infty} z(t)$ converges to zero w.p.1 and as a result $\lim_{t\to\infty} \frac{x(t)-x(0)}{t}$ converges to zero. Fig. 4.1(a) illustrates this behavior for the first component of coalition $\{1,2\}$. Further, by corollary 4.4.1, the same control law ensures that the average allocations converge to the nominal allocations in the long run, in other words $\lim_{t\to\infty} \bar{a}(t) = a_{\text{nom}}$ and Fig. 4.1(b) illustrates this behavior.



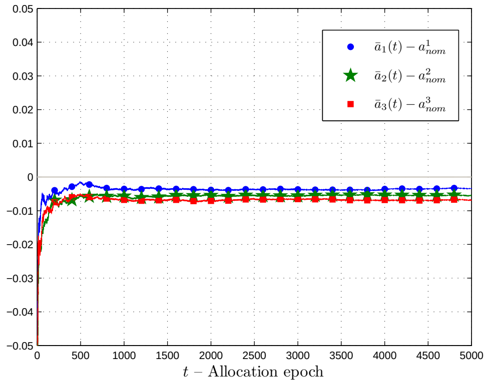

(a) Plot of $\frac{x_{\{1,2\}}(t)-x_{\{1,2\}}(0)}{t}$ (b) Plot of $\lim_{t\to\infty} \bar{a}(t) - a_{\text{nom}}$

**Figure 4.1** – Performance of the control law given by (4.13).

**Partial information case:** The choice of $\delta$ is crucial so as to ensure $u(t) \in U$. This condition translates to $U_{\min} \leq u_{\text{nom}} + \delta B^{\dagger} sign(x) \leq U_{\max}$. We observe $-\sum_j |B^{\dagger}_{ij}| \leq \left(B^{\dagger} sign(x)\right)_i \leq \sum_j |B^{\dagger}_{ij}|$. A conservative estimate of $\delta$ is obtained as $U_{\min} \leq u_{\text{nom}} \pm \delta \max_i\{\sum_j |B^{\dagger}_{ij}|\} \leq U_{\max}$. For $m = 7$, we have $\max_i\{\sum_j |B^{\dagger}_{ij}|\} = 2.11$. For the instantaneous game a negative allocation/surplus is not allowed, so $U_{\min} \geq 0.\mathbf{1}$. Furthermore, an allocation/surplus greater than the value of grand coalition is not allowed, so $U_{\max} \leq v_{\text{nom}}(N).\mathbf{1}$. We chose $\delta = 1$, which satisfies the above stated requirements. Next, we present performance results of the robust control law given by equation (4.14). From theorem 4.4.2, $x(t)$ converges to zero in probability with a specific choice of control law and as a result $\lim_{t\to\infty} \frac{x(t)-x(0)}{t}$ converges to zero. Fig. 4.2(a) illustrates this behavior for the first component of coalition $\{1,2\}$. Further, by Corollary 4.4.2, the same control law ensures that the average allocations converge to the core $C(v_{\text{nom}})$ and from equation (4.14) it is clear that the instantaneous allocations lie in a neighborhood of nominal allocations. As a result there is an uncertainty in the convergence of average allocations towards nominal allocations on the long run and Fig. 4.2(b) illustrates this behavior.




(a) Plot of $\frac{x_{\{1,2\}}(t)-x_{\{1,2\}}(0)}{t}$




(b) Plot of $\lim_{t\to\infty} \bar{a}(t) - a_{\text{nom}}$

**Figure 4.2** – Performance of the control law given by (4.14).

## 4.7 Conclusions

In this chapter we study dynamic cooperative games where at each instant of time the value of each coalition of players is unknown but varies within a bounded polyhedron. With the assumption that the average value of each coalition in the long run is known with certainty we present robust allocations schemes, which converge to the core, under two informational settings. We proved the convergence of both allocation rules using Lyapunov stochastic stability theory. Furthermore, we highlight the connections between Lyapunov theory and approachability theory in a discrete time setting. The control laws or allocation schemes are derived on the premise that the GD knows a priori, the nominal allocation vector. If this information is not available then the problem can be treated as a learning process where the GD is trying to learn the (balanced) nominal game from the instantaneous games. The allocation rules designed in this chapter assure stability of the coalitions in average, and as a result capture patience and expectations of the players in an integral sense. The modeling aspects of generic dynamic coalitional games, as mentioned in the remark 4.2.1, are open questions at this point of time.

**CHAPTER 5**

# Optimal Management and Differential Games in the Presence of Threshold Effects - The Shallow Lake Model

## 5.1 Introduction

Most of the optimal decision making problems studied in economics and ecology are complex in nature. These complexities generally arise while modeling the inherent behavior of the dynamic environment, which includes agents interacting with the system. Modeling with hybrid systems [47, 106] capture some of these complex situations. The behavior of such systems is described by the integration of continuous and discrete dynamics. An abrupt change in the discrete state of the system is called a switch. If a decision maker influences a switch then it is said to be controlled/external, whereas an internal switch generally results when the continuous state variable satisfies some constraints. Threshold effects are autonomous switches that happen when the continuous state variable hits a boundary. Some examples in this direction are, a firm going bankrupt when its equity is negative and regime shifts in ecology [91] etc. Optimal control of hybrid systems received considerable interest in control engineering, see for instance [112, 104, 21, 97, 89, 94]. These works include formulation of different variations of the necessary conditions. For computational issues related to optimal control of hybrid systems, see [62, 20, 97].

In this chapter, we study optimal management and differential games in a pollution control model called the shallow lake problem, see [70, 59, 108, 23]. The production function of this model is non linear, convex-concave in particular. As a result, the optimal vector field displays several interesting qualitative behaviors such as existence of multiple steady states, Skiba points[1] [100] and bifurcations due to parameter variations. A complete bifurcation analysis of this vector field is provided, recently, in [58]. The inflection point of this convex-concave function acts as a soft threshold. In this chapter, we approximate this nonlinear effect with simple and hysteresis switching, by using deterministic hard thresholds, and study optimal management and open loop Nash equilibrium policies. Some literature incorporating threshold effects include [78], [81] and references cited in those papers. Though in [78], the author uses necessary conditions, in the line of [94], the

[1] Starting from such a point the optimal control problem has more than one optimal solution, and as a result the decision maker is indifferent to a particular solution.

objective function is quadratic and the state variable admits jumps, whereas, the present chapter deals with discontinuous dynamics. Recently, in [81] the authors consider an optimal management problem, with probabilistic thresholds and simple dynamics, and use dynamic programming to derive optimal policies.

In this chapter we do not attempt to solve the optimal controls or equilibrium strategies for a generic class of (hybrid) differential games. Instead, we study the shallow lake problem with switching approximation and highlight the key differences with the smooth case, the classical shallow lake model. This chapter is organized as follows. In section 5.2, we review optimal control of a specific class of hybrid systems, known as switched systems. In section 5.3, we introduce a class of differential games, that arise in pollution control, with threshold effects. We study optimal management and open loop Nash equilibrium policies related to the shallow lake model in section 5.4. Finally, section 5.5 concludes.

# 5.2 Optimal control of switched systems

In this section we review necessary conditions to solve optimal control for a specific switched system. The switching system that we have in mind has the following description:

**Definition 5.2.1 (Switched System).** *A switched system is a triple $\mathscr{S} = (\mathscr{I}, \mathscr{F}, \mathscr{E})$ where*

- *$\mathscr{I}$ is a finite set, called the set of discrete states representing the vertices of a graph.*
- *$\mathscr{F} = \{f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ i \in \mathscr{I}\}$ is a collection of vector fields. We denote $\dot{x}(t) = f_i(x(t), u(t))$ to be the vector field at location $i \in \mathscr{I}$.*
- *$\mathscr{E}$ is a finite set of edges called transitions. A transition $(i, j) \in \mathscr{E}$ is triggered by events (internal or external) resulting in abrupt change in dynamics from $f_i$ to $f_j$.*

In this chapter we consider internal switchings i.e., transitions from vertex $i$ to $j$ happen when certain state constraints, say $\phi_{ij}(x(t)) = 0$, are satisfied. Let $\Phi_{ij} \triangleq \Big\{x \in \mathbb{R}^n : \phi_{ij}(x) = 0\Big\}$ be the switching surface associated with transition $i$ to $j$ and $\Phi \triangleq \cup\Phi_{ij}$. Now, we introduce a class of discounted autonomous infinite horizon optimal control problems with internal switching dynamics, described by $\mathscr{S} = (\mathscr{I}, \mathscr{F}, \Phi)$, as follows.

$$\max J, \qquad J = \int_0^\infty e^{-rt} g(x(t), u(t)) dt \tag{5.1}$$

$$\dot{x}(t) = f_i(x(t), u(t)),\ i \in \mathscr{I},\ f_i \in \mathscr{F},\ \Phi = \cup\Phi_{ij} \tag{5.2}$$

$$x(0) = x_0 \in \mathbb{R}^n, u(.) \in \mathscr{U}. \tag{5.3}$$

*Assumption* 5.2.1. The real valued functions $f_i(.), i \in \mathscr{I}$ and $g(.)$ are continuous, $\frac{\partial f_i(.)}{\partial x}$ and $\frac{\partial g(.)}{\partial x}$ exist and are continuous. The control space $\mathscr{U}$ consists of piecewise continuous functions with $u(t) \in U$, where $U$ is a bounded set included in $\mathbb{R}^m$. Further, we assume the left- and right-hand limits for $u(.)$ exist and $x(.)$ is continuous and piecewise continuously differentiable, which satisfies (5.2) for all points $t$ where $u(.)$ is continuous. The initial state satisfies $x_0 \notin \Phi$. The velocity vector satisfies $\dot{x}(\tau) \neq 0$ during a transition where $\tau$ is the switching instant.

We call a pair $(x(.), u(.))$ *admissible* for the problem (5.1-5.3) if assumption 5.2.1 is satisfied. Let $k(.)$ represent the switching sequence associated with $\mathscr{S}$, i.e., when the system is in mode $i$ at time $t$ we have $k(t) = i$. Since, the switchings happen internally we see that $u(.)$ induces a switching sequence $k(.)$. The necessary condition for a pair $(x^*(.), u^*(.))$ to be optimal for the problem (5.1-5.3) is given by the following theorem [2], see theorem 2.3 of [89] or theorem 1 of [88] or theorem 3 of [94] or theorem 2.2 and theorem 2.3 of [97] for more details.

**Theorem 5.2.1.** *If $(x^*(t),\ u^*(t))$ represent an optimal admissible pair for the problem (5.1-5.3) , there exists a piecewise absolutely continuous function $\lambda_{k^*(.)}(.)$ and a constant $\lambda_0 \geq 0$, $\left(\lambda_0, \lambda_{k(t)}\right) \neq 0$ on $[0,\infty)$ such that:*

*a) let Hamiltonian be defined as $H_{k(t)}(t, x(t), u(t), \lambda_{k(t)}(t), \lambda_0) \triangleq \lambda_0 e^{-rt} g(x(t), u(t)) + \lambda'_{k(t)} f_{k(t)}(x(t), u(t))$ then for a given $(\lambda_{k^*(t)}(t), \lambda_0, x^*(t))$ at a given time t, except at the switching instants, the following maximum condition holds*

$$H_{k^*(t)}(t, x^*(t), u^*(t), \lambda_{k^*(t)}(t), \lambda_0) \geq H_{k^*(t)}(t, x^*(t), v, \lambda_{k^*(t)}(t), \lambda_0)\ \forall\ v \in U, \quad (5.4)$$

*and if $u^*(t)$ is an interior solution then $\frac{\partial H_{k(.)}(.)}{\partial u}\Big|_{u(t)=u^*(t)} = 0$.*

*b) the adjoint process $\lambda_{k^*(t)}(t)$ satisfies $\dot{\lambda}_{k^*(t)}(t) = -\frac{\partial H_{k(.)}(.)}{\partial x}\Big|_{x(t)=x^*(t)}$ for all $t \geq 0$ except at the switching instants.*

*c) if $\tau$ is a switching instant then the following conditions hold true:*

1. *$x(\tau) \in \Phi$, $\tau \in [0,\infty)$*
2. *(adjoint jump condition)*
   *there exists a $\beta \in \mathbb{R}$ such that $\lambda_{k^*(\tau^-)} = \lambda_{k^*(\tau^+)} + \beta \left(\phi_{k^*(\tau^-)k^*(\tau^+)}(x(\tau))\right)_x$*
3. *(Hamiltonian continuity)*

$$H_{k^*(\tau^-)}(\tau^-, x^*(\tau^-), u^*(\tau^-), \lambda(\tau^-), \lambda^0) = H_{k^*(\tau^+)}(\tau^+, x^*(\tau^+), u^*(\tau^+), \lambda(\tau^+), \lambda^0).$$

The natural transversality condition $\lim_{t\to\infty} \lambda_{k^*(t)}(t) = 0$ is not guaranteed except with some additional conditions. The above necessary conditions, when solved, generally give a large number of candidates and the optimal solution is obtained by comparing the objective evaluated along the candidate trajectories. The non-switched analog of problem

[2] There exist several variations of the theorem in a more general setting, for instance refer [97, 94, 89, 104]. Here, we consider a specific system $\mathscr{S}$ where switchings happen internally.

(5.1-5.3) is the classical discounted autonomous infinite horizon optimal control problem, a well studied problem in economics literature. For this class of problems, when necessary conditions hold true in normal form, i.e., $\lambda_0 = 1$, the objective evaluated along a candidate trajectory is given by $\frac{1}{r}H(0,x^*(0),u^*(0),\lambda(0),1)$, see proposition 3.75 of [48] for a proof[3]. We show in the next lemma, for the problem (5.1-5.3), that the objective evaluated along the optimal candidates with a finite number of switchings also satisfies this property when $\lambda_0 = 1$. The proof of the lemma, given in the appendix, hinges upon the Hamiltonian continuity property.

**Lemma 5.2.1.** *Let $(x^*(.),u^*(.))$ be an optimal admissible candidate pair with a finite number of switchings for the problem (5.1-5.3), then the objective evaluated along the candidate trajectory is given by* $\frac{1}{r}H_{k^*(0)}\left(0,x^*(0),u^*(0),\lambda_{k^*(0)}(0),1\right)$.

## 5.3 A class of differential games with threshold effects

In this section we discuss briefly a differential game model in pollution control, taken from [59], and modify the dynamics to include threshold effects. Assume a situation where $N$ economic agents, sharing a natural system, take actions $a_i(t),\ i = 1,2,\cdots,N$ at time $t$, and as a result affect the state $x(t)$ of a natural system. The economic agents could be societies, dealing with eutrophication of a lake that they manage, or countries, worried about climate change. The stock of pollutant in the natural system admits a dynamics described by:

$$\dot{x}(t) = f(x(t),a_1(t),a_2(t),\cdots,a_N(t)) = \sum_{i=1}^{N} a_i(t) - bx(t) + h(x(t)),\ x(0) = x_0 \geq 0. \quad (5.5)$$

The state variable $x(t)$ could be interpreted as accumulated greenhouse gases or accumulated phosphorous in a lake. Besides the activity of economic agents, the sources that promote $x(t)$ are the nonlinear internal dynamics captured by the term $h(x(t))$. The sinks that contribute to the reduction of $x(t)$ are abstracted as the linear decay rate $b > 0$. The second part of the model deals with economic analysis of the agents. An agent $i$, with action $a_i$, generates benefits according to a strictly increasing and concave utility function $B(a_i)$. The stock of pollutants $x(t)$ causes damage to the natural system according to a strictly increasing and convex damage function $D(x)$, sometimes referred as disutility of agents. The net profit that an agent $i$ receives at a point of time $t$ is then given by $B(a_i(t)) - D(x(t))$. Each agent uses a strategy $a_i(.)$ to maximize the present value of net benefits over an infinite time horizon, i.e.,

$$\max_{a_i(.)} \int_0^\infty e^{-rt}\left(B(a_i(t)) - D(x(t))\right)dt, \quad i = 1,2,\cdots,N, \quad (5.6)$$

subject to (5.5), where $r > 0$ is a discount rate. Here, the quantities $B(.)$, $D(.)$ and $r$ are assumed to be the same for all the agents. In this chapter, $h(x)$ is assumed to be a convex-concave function; that is for lower stocks of $x(t)$ there is relatively low marginal return to

[3]This result follows from the additional necessary condition that limit of the maximized Hamiltonian, along the candidate trajectory, is zero when $t$ goes to infinity, see [75].

the system, whereas for the higher stocks this marginal return increases. When the maximal feedback rate is greater than the decay rate, i.e., $\max(h'(x)) > b$, the system (5.5) exhibits three equilibria for a certain values of inputs $a = \sum_i a_i$. There will be two stable steady states, one corresponding to low $x$ 'clear state' which is highly valued by concerned users of the natural system[4], but also a relatively high $x$ 'polluted state' which is valued by the agents due to economic interest. This nonlinear positive feedback effect is a potential source for complex qualitative behaviors in optimal solutions in the model (5.5-5.6). The region of stock near the inflection point of $h(x)$ acts as a soft threshold distinguishing the clear and polluted regions. A piecewise approximation of the non linearity in $h(x)$ results

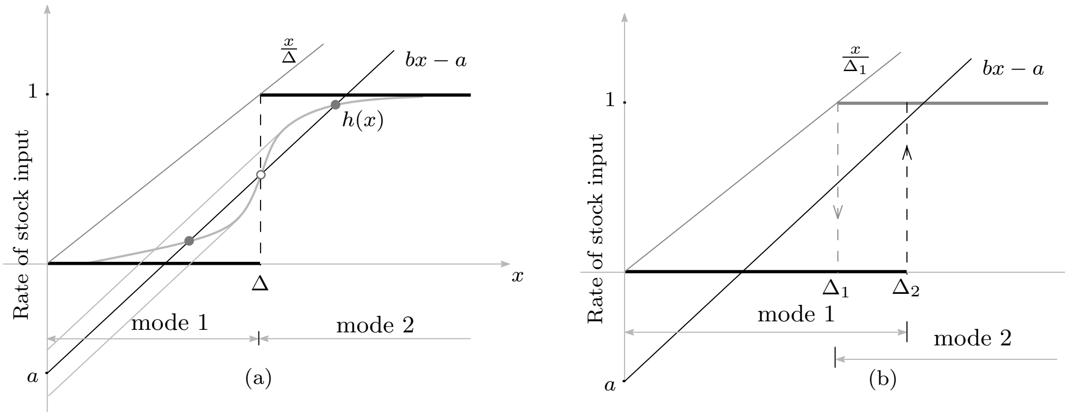



**Figure 5.1** – Approximation of $h(x)$ with hard thresholds

in a dynamics with discontinuous right hand side; that is when the amount of stock reaches a pre-specified threshold the dynamics changes abruptly. In this chapter we consider two approximations of $h(x)$, firstly with simple switching and later with ideal hysteresis switching. Hereafter, we address 'clean state' as mode 1 and 'polluted state' as mode 2. Let $a(t)$ denote total agents' activities i.e., $a(t) \triangleq \sum_i a_i(t)$.

### 5.3.1 Simple switching

Figure 5.1(a) illustrates the situation when $h(x)$ is approximated with a Heaviside step function. Thus, the dynamics of the system (5.5) is given by:

$$
\begin{aligned}
&\mathscr{S} = \{\mathscr{I}, \mathscr{F}, \mathscr{E}\} \text{ where} \\
&\mathscr{I} = \{1,2\},\ \mathscr{F} = \{f_1(x,a) = a - bx,\ f_2(a,x) = a - bx + 1\} \\
&\mathscr{E} = \{\phi_{12}(x(t)) = \phi_{21}(x(t)) = x(t) - \Delta = 0\} \\
&\dot{x}(t) = a(t) - bx(t),\ \text{for } x(t) < \Delta \\
&\dot{x}(t) = a(t) - bx(t) + 1,\ \text{for } x(t) > \Delta.
\end{aligned}
$$

For certain values of $a$ the above system exhibits two steady states, one each in mode 1 and mode 2. If the decay rate, $b$, is larger than $\frac{1}{\Delta}$ it is possible to reach the steady state in

[4]Could be people using a lake for recreation etc.

mode 1, from mode 2, by lowering the external loading $a$. So, if $b\Delta < 1$ a steady state in mode 1 cannot be reached from mode 2 even by setting $a = 0$.

### 5.3.2 Hysteresis switching

Figure 5.1(b) illustrates the situation when $h(x)$ is approximated with ideal hysteresis[5]. The dynamics of the system (5.5) is given by:

$$
\begin{aligned}
&\mathscr{S} = \{\mathscr{I}, \mathscr{F}, \mathscr{E}\} \text{ where}\\
&\mathscr{I} = \{1,2\},\ \mathscr{F} = \{f_1(x,a) = a - bx,\ f_2(a,x) = a - bx + 1\}\\
&\mathscr{E} = \{\phi_{12}(x(t)) = x(t) - \Delta_2 = 0,\ \phi_{21}(x(t)) = \Delta_1 - x(t) = 0,\ \Delta_2 > \Delta_1\}\\
&\dot{x}(t) = a(t) - bx(t),\ \text{for}\ x(t) < \Delta_2\\
&\dot{x}(t) = a(t) - bx(t) + 1,\ \text{for}\ x(t) > \Delta_1.
\end{aligned}
$$

Here, the dynamics admit history dependence. Again, if $b\Delta_1 < 1$, we see that a transition from mode 2 to mode 1 is not possible even when $a$ is set to zero.

The above models, with threshold effects, can be useful in analyzing several problems that arise in pollution management. In the next section, we consider a particular example, the shallow lake model, and analyze the optimal management and open loop Nash equilibrium policies.

## 5.4 The shallow lake model

The shallow lake model has received considerable interest over the last two decades. The essential dynamics [70] of the eutrophication process can be modeled by the differential equation

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t) + 1},\ x(0) = x_0,$$

where $x(t)$ is the amount of phosphorus in the lake at time $t$, $a(t)$ is the total input of phosphorus washed into the lake due to farming activities, $b$ is the rate of loss of phosphorus due to sedimentation, and the last term captures internal biological processes for the production of phosphorus. An agent $i$ receives benefits, by an action $a_i(t)$, as $B(a_i(t)) = \ln a_i(t)$[6] and incurs a cost, towards cleaning, as $D(x(t)) = cx^2(t)$. Here, the parameter $c$ models the relative cost of pollution. Next, we study the above canonical model with threshold effects by approximating the non-linearity with simple switching and with hysteresis. We first consider the optimal management problem with simple switching.

[5]Hysteresis effects can be modeled using smooth nonlinear functions, for instance, in [8], the author formulates dynamic programming methods for some optimal control problems with hysteresis.

[6]Notice that since $B(a_i(t)) = \ln a_i(t)$ we implicitly assume $a_i(t) > 0$ for all $t$.

### 5.4.1 Optimal management

The necessary conditions for $(x^*(t), a_1^*(t), \cdots, a_N^*(t))$ to be optimal for the optimal management problem with simple switching are given as follows:

$$
\begin{aligned}
\text{mode 1} \quad & H_1(.) = \lambda_0 e^{-rt}\left(\sum_i \ln a_i(t) - Ncx^2(t)\right) + \lambda_1(t)\left(\sum_i a_i(t) - bx(t)\right) \\
& \frac{\partial H_1(.)}{\partial a_i}|_{a_i=a_i^*} = 0 \text{ gives } \ \lambda_0 e^{-rt}\frac{1}{a_i^*(t)} + \lambda_1(t) = 0, \ i = 1,2,\cdots,N \\
& \dot{x}^*(t) = \sum_i a_i^*(t) - bx^*(t), \ x^*(0) = x_0 \\
& \dot{\lambda}_1(t) = -\frac{\partial H_1(.)}{\partial x}|_{x=x^*} = b\lambda_1(t) + 2cN\lambda_0 e^{-rt}x^*(t) \\
& (\lambda_0, \lambda_1(t)) \neq 0, \forall t \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{mode 2} \quad & H_2(.) = \lambda_0 e^{-rt}\left(\sum_i \ln a_i(t) - Ncx^2(t)\right) + \lambda_2(t)\left(\sum_i a_i(t) - bx(t) + 1\right) \\
& \frac{\partial H_2(.)}{\partial a_i}|_{a_i=a_i^*} = 0 \text{ gives } \ \lambda_0 e^{-rt}\frac{1}{a_i^*(t)} + \lambda_2(t) = 0, \ i = 1,2,\cdots,N \\
& \dot{x}^*(t) = \sum_i a_i^*(t) - bx^*(t) + 1, \ x^*(0) = x_0 \\
& \dot{\lambda}_2(t) = -\frac{\partial H_2(.)}{\partial x}|_{x=x^*} = b\lambda_2(t) + 2cN\lambda_0 e^{-rt}x^*(t) \\
& (\lambda_0, \lambda_2(t)) \neq 0, \forall t \geq 0.
\end{aligned}
$$

If $\lambda_0 = 0$ we see that $\lambda_i(t) = 0, \ i = 1,2, \ \forall t \geq 0$ in the above necessary conditions. So, necessary conditions hold in normal form, i.e., $\lambda_0 = 1$. We still cannot assume the natural transversality condition to hold true. During a switching instant $\tau > 0$ we have $x^*(\tau) = \Delta$ and the adjoint jump condition given by $\lambda_i(\tau^-) = \lambda_j(\tau^+) + \beta, \ i \neq j, \ i,j = 1,2, \ \beta \in \mathbb{R}$ holds true. Further, if $k^*(\tau^-) = i$ and $k^*(\tau^+) = j$, i.e., during a transition from mode $i$ to mode $j$, we have $H_i\big(\tau^-, x^*(\tau^-), a_1^*(\tau^-), \cdots, a_N^*(\tau^-), \lambda_i(\tau^-)\big) = H_j\big(\tau^+, x^*(\tau^+), a_1^*(\tau^+), \cdots, a_N^*(\tau^+), \lambda_j(\tau^+)\big)$. So, the Hamiltonian continuity conditions are given by:

$$H_1(\tau^-, \Delta, a_1^*(\tau^-), \cdots, a_N^*(\tau^-), \lambda_1(\tau^-)) = H_2(\tau^+, \Delta, a_1^*(\tau^+), \cdots, a_N^*(\tau^+), \lambda_2(\tau^+)) \quad (5.7)$$
$$H_2(\tau^-, \Delta, a_1^*(\tau^-), \cdots, a_N^*(\tau^-), \lambda_2(\tau^-)) = H_1(\tau^+, \Delta, a_1^*(\tau^+), \cdots, a_N^*(\tau^+), \lambda_1(\tau^+)). \quad (5.8)$$

The above optimality equations, excepting equations (5.7) and (5.8), remain unaltered even for ideal hysteresis approximation. For the latter case, the parameter $\Delta$ in (5.7) and (5.8) should be replaced with $\Delta_2$ and $\Delta_1$ respectively. The above necessary conditions can be reformulated in the current value form by defining the current value Hamiltonian as $H^c_{k^*(t)}(.) = e^{rt}H_{k^*(t)}(.)$ and the current value adjoint variable as $\lambda^c_{k^*(t)}(t) = e^{rt}\lambda_{k^*(t)}(t)$. Further, the obtained optimal dynamics and switching conditions in $(x^*(t), \lambda_c(t))$ space are transformed in $(x^*(t), a_1^*(t), \cdots, a_N^*(t))$ space as follows:

$$\dot{x}^*(t) = \sum_i a_i^*(t) - bx^*(t) + \alpha,\ x^*(0) = x_0,\ \alpha = \begin{cases} 0 & x^*(t) < \Delta \\ 1 & x^*(t) > \Delta \end{cases}$$
$$\dot{a}_i^*(t) = -(r+b)a_i^*(t) + 2Nca_i^{*2}(t)x^*(t),\ i = 1, \cdots, N \text{ (except at the switching instants).}$$

During the switching instant $t = \tau$, we have $x^*(\tau) = \Delta$ and the Hamiltonian continuity conditions given by (5.7) and (5.8) are satisfied. The above necessary conditions lead to an $N+1$ dimensional optimal vector field. In order to analyze the optimal dynamics we consider symmetric strategies, i.e., $a_i^*(t) = a^*(t)/N$. The symmetry assumption results in a 2 dimensional vector field which can be analyzed using the phase plane diagram. The necessary conditions with symmetry are now given as:

$$\dot{x}^*(t) = a^*(t) - bx^*(t) + \alpha,\ x^*(0) = x_0,\ \alpha = \begin{cases} 0 & x^*(t) < \Delta \\ 1 & x^*(t) > \Delta \end{cases} \tag{5.9}$$
$$\dot{a}^*(t) = -(r+b)a^*(t) + 2ca^{*2}(t)x^*(t) \text{ (except at the switching instants).} \tag{5.10}$$

Again, during the switching instant $t = \tau$ the Hamiltonian continuity conditions lead to the following equations.

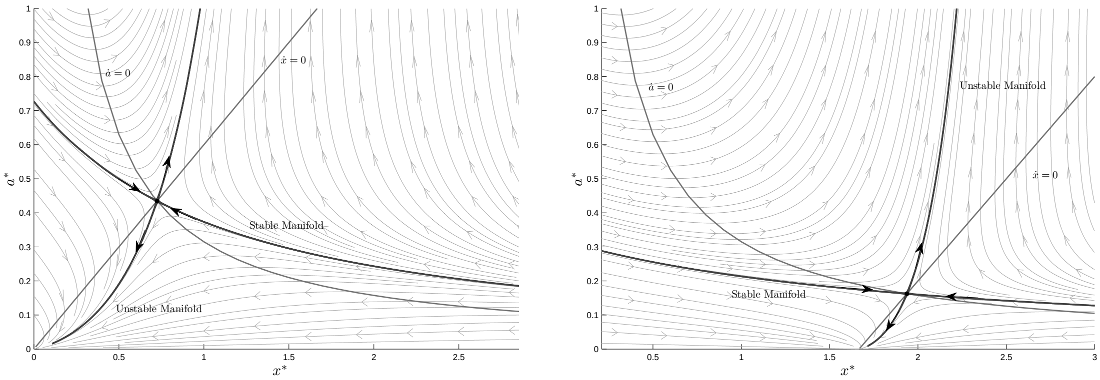
$$\text{If } k^*(\tau^-) = 1 \text{ and } k^*(\tau^+) = 2, \text{ then } \ln a^*(\tau^-) + \frac{b\Delta}{a^*(\tau^-)} = \ln a^*(\tau^+) + \frac{b\Delta - 1}{a^*(\tau^+)} \tag{5.11}$$
$$\text{If } k^*(\tau^-) = 2 \text{ and } k^*(\tau^+) = 1, \text{ then } \ln a^*(\tau^-) + \frac{b\Delta - 1}{a^*(\tau^-)} = \ln a^*(\tau^+) + \frac{b\Delta}{a^*(\tau^+)}. \tag{5.12}$$

The equations (5.9-5.12) constitute the optimal vector field of the shallow lake model with simple switching. Due to symmetry, we have the following observation (see appendix for the proof):

**Lemma 5.4.1.** *The symmetric open loop Nash equilibrium problem can be solved as a symmetric optimal management problem with $c$ replaced by $\frac{c}{N}$.*

Due to lemma 5.4.1, the symmetric open loop Nash equilibrium problem can be solved from the necessary conditions of the symmetric optimal management problem with parameter $c$ replaced by $\frac{c}{N}$. So, the symmetric open loop Nash equilibrium problem is a potential game [7]. In the following discussion we study the optimal management problem in detail.

[7] A potential game [76] facilitates to compute Nash equilibria as an optimization problem instead of a fixed point problem.




(a) Dynamics in mode 1 (b) Dynamics in mode 2

**Figure 5.2** – Phase plane analysis

#### 5.4.1.1 Phase plane analysis

The equilibrium points of the optimal vector field are:

$$\text{mode 1} \qquad \left(x_{eq}, a_{eq}\right) = \left\{(0,0), \left(\sqrt{\frac{r+b}{2cb}}, \sqrt{\frac{b(r+b)}{2c}}\right), \left(-\sqrt{\frac{r+b}{2cb}}, -\sqrt{\frac{b(r+b)}{2c}}\right)\right\}$$

$$\text{mode 2} \qquad \left(x_{eq}, a_{eq}\right) = \left\{\left(\frac{1}{b}, 0\right), \left(\frac{1}{2b} + \sqrt{\frac{1}{4b^2} + \frac{r+b}{2cb}}, \sqrt{\frac{1}{4} + \frac{b(r+b)}{2c}} - \frac{1}{2}\right), \right.$$
$$\left. \left(-\frac{1}{2b} - \sqrt{\frac{1}{4b^2} + \frac{r+b}{2cb}}, -\sqrt{\frac{1}{4} + \frac{b(r+b)}{2c}} + \frac{1}{2}\right)\right\}.$$

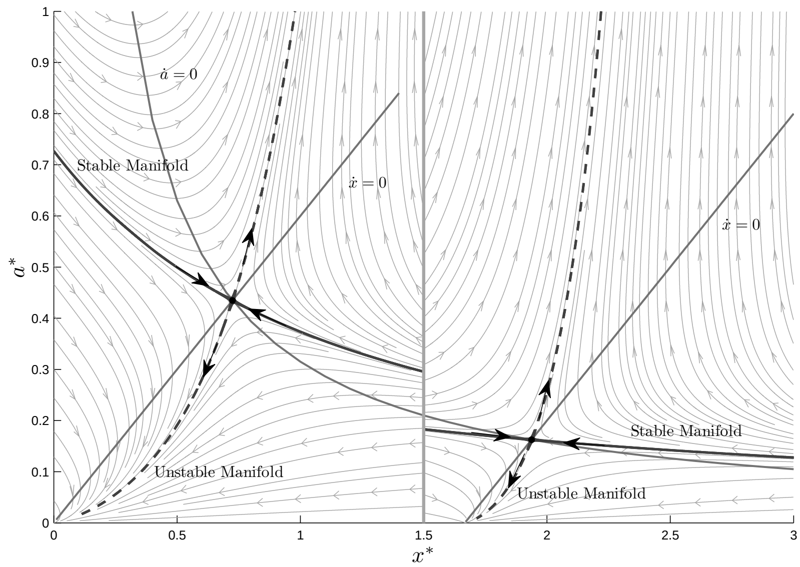
Here, it should be noted that the presence of equilibrium points in the above optimal vector field depends on the location of the threshold. We discuss these issues related to bifurcations in section 5.4.1.3.

Since the cost term involves $\ln(a)$, the nutrient loading should satisfy $a(t) > 0$. So, only the second equilibrium point is chosen for each of the modes. Let $x_{eq}^1$ and $x_{eq}^2$ denote these equilibrium points. Then we have, $0 < x_{eq}^1 < x_{eq}^2$ and $x_{eq}^2 > \frac{1}{b}$. The eigenvalues of the Jacobian matrix, for the linearized dynamics near the equilibrium points, are:

$$\text{mode 1} \qquad \frac{r}{2} \pm \frac{\sqrt{8b^2 + 8br + r^2}}{2}$$

$$\text{mode 2} \qquad \frac{r}{2} \pm \frac{\sqrt{r^2 + 8b^2 + 8br + 4c - 4\sqrt{c^2 + 2brc + 2b^2c}}}{2}.$$

By inspection, the equilibrium point in mode 1 is clearly a saddle point. Now we have:




(a) Switching dynamics with $\Delta = 1.5$

**Figure 5.3** – Phase plane analysis

$$
\begin{aligned}
8b^2+8br+4c-4\sqrt{c^2+2bcr+2b^2c} &= 4\left(c+2b(r+b)-\sqrt{c\,(c+2b(r+b))}\right)\\
&= 4\sqrt{c+2b(r+b)}\left(\sqrt{c+2b(r+b)}-\sqrt{c}\right) > 0.
\end{aligned}
$$

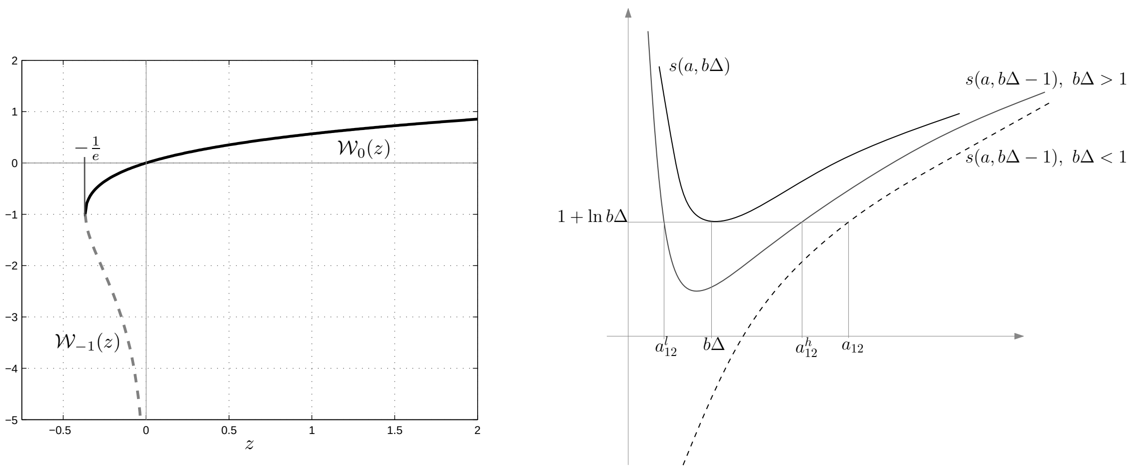
So, the equilibrium point in mode 2 is also a saddle point. Figures 5.2(a), 5.2(b) and 5.3(a) illustrate the phase portrait of the optimal dynamics in mode 1 mode 2 and with simple switching respectively. The chosen parameters are $b = 0.6$, $c = 0.5$, $r = 0.03$ and $\Delta = 1.5$. Any trajectory approaching the surface at $x = \Delta$ undergoes a switching according to the rules (5.11) and (5.12). Next, we analyze these switching rules in detail.

#### 5.4.1.2 Switching analysis

Before proceeding with the actual switching analysis we discuss solvability of the equation

$$
s(x,m) = \ln(x) + \frac{m}{x} = n, \quad m,n \in \mathbb{R}, x > 0. \tag{5.13}
$$

If $m = 0$ then $x = e^n$. We consider the case $m \neq 0$. After rearranging terms the above equation can be written as $ye^y = l,\ y = -\frac{m}{x},\ l = -me^{-n}$. Solution of the reformulated equation is given by $y = \mathscr{W}(l)$. where $\mathscr{W}(.)$ is the Lambert W function [31]. Here, $\mathscr{W}(z)$ is single valued for $\{z \geq 0\} \cup \{-\frac{1}{e}\}$, multiple valued for $-\frac{1}{e} < z < 0$, and not defined for $z < -\frac{1}{e}$. Figure 5.4(a) shows two branches of $\mathscr{W}(z)$ namely, $\mathscr{W}_0(z)$ and $\mathscr{W}_{-1}(z)$. Thus, the solution of (5.13) is given by $x = -\frac{m}{\mathscr{W}(-me^{-n})}$.




(a) Two branches of Lambert W function

(b) Analysis of jumps in control action at the switching instant $\tau$

**Figure 5.4** – Switching analysis

mode 1 to mode 2: When the optimal system switches from mode 1 to mode 2, following (5.11), the jump in the control satisfies:

$$s(a^-, b\Delta) \;=\; s(a^+, b\Delta - 1).$$

Here, $s(., b\Delta) : (0,\ \infty) \to \big[\ln b\Delta + 1,\ \infty\big)$. If $b\Delta = 1$, then $a^+ = e^{-s(a^-, b\Delta)}$. If $b\Delta \neq 1$, then $a^+ = -\frac{b\Delta - 1}{\mathscr{W}\left(-(b\Delta-1)e^{-s(a^-, b\Delta)}\right)}$. For $b\Delta < 1$, we have $-(b\Delta-1)e^{-s(a^-, b\Delta)} > 0$. So, a jump results in $a^+$ in the interval $\big[a_{12},\ \infty\big)$, $a_{12} = \frac{1-b\Delta}{\mathscr{W}_0\left(\left(\frac{1}{b\Delta}-1\right)\frac{1}{e}\right)}$. Here, $\mathscr{W}(z)$ increases for $z > 0$. For $b\Delta > 1$, we have $-(b\Delta-1)e^{-s(a^-, b\Delta)} < 0$. So, a jump results in $a^+_l$ in the interval $\big(0,\ a^l_{12}\big]$ and $a^+_h$ in the interval $\big[a^h_{12},\ \infty\big)$ $a^l_{12} = \frac{1-b\Delta}{\mathscr{W}_0\left(\left(\frac{1}{b\Delta}-1\right)\frac{1}{e}\right)}$ and $a^h_{12} = \frac{1-b\Delta}{\mathscr{W}_{-1}\left(\left(\frac{1}{b\Delta}-1\right)\frac{1}{e}\right)}$. Here, $\mathscr{W}(z)$ decreases for $z < 0$. Superscripts $l$ and $h$ denote the lower and higher values which are computed at different branches of $\mathscr{W}(z)$ for $-\frac{1}{e} < z < 0$.

mode 2 to mode 1: When the optimal system switches from mode 2 to mode 1, following (5.12), the jump in the control satisfies:

$$s(a^-, b\Delta - 1) \;=\; s(a^+, b\Delta).$$

Then, $a^+ = \frac{b\Delta}{\mathscr{W}\left(-b\Delta e^{-s(a^-, b\Delta-1)}\right)}$. We know $a^+$ is well defined only if $0 < b\Delta e^{-s(a^-, b\Delta-1)} \leq \frac{1}{e}$ which implies $s(a^-, b\Delta-1) \geq \ln b\Delta + 1$. Further, for $b\Delta e^{s(a^-, b\Delta-1)} = \frac{1}{e}$ which implies $s(a^-, b\Delta-1) = \ln b\Delta + 1$. So, $a^-$ should satisfy $s(a^-, b\Delta-1) \geq \ln b\Delta + 1$ for a jump to happen from mode 2 to mode 1. In such a case, a jump results in two points, namely $a^+_l \in (0, b\Delta]$ and $a^+_h \in [b\Delta, \infty)$.

A graphical illustration of the switchings is given in figure 5.4(b). Here, $a^-$ is called a predecessor of $a^+$, so $a^+$ is a successor of $a^-$. All switchings happen on the surface $x = \Delta$. Notice, there always exists a successor during transitions from mode 1 to mode 2, whereas some points on the switching surface may not have predecessors in mode 1. Further, in some cases there exist more than one successor or predecessor. These characteristics of the optimal vector field should be considered while analyzing the optimal candidates. We discuss these issues in the next section.

#### 5.4.1.3 Analysis of optimal control

In this subsection we use the results from subsections 5.4.1.1 and 5.4.1.2 to analyze the optimal system (5.9-5.12) and arrive at conclusions regarding the optimal candidates and control actions. First, we notice that a solution trajectory, say $\gamma(t)$, of the optimal system (5.9-5.12) starting at a point $(x_0, a_0)$, with $x_0 > 0$ and $a_0 > 0$, either

1. converge to the equilibrium points as $t \to \infty$, or
2. leads to a control $a^*(t)$ that goes to infinity, or
3. leads to a closed orbit.

*A) Solutions approaching stable equilibrium points*

First, we notice that a trajectory $\gamma(t)$ approaching any equilibrium point admits a finite number of switchings. So, the truncated candidate trajectory in the last interval satisfies necessary conditions similar to a classical problem[8]. So, the transversality condition, given by $\lim_{t\to\infty} -\frac{Ne^{-rt}}{a^*(t)} = 0$, holds true. Next, we show that the trajectory $\gamma(t)$ approaching the stable equilibrium points $(0,0)$ and $\left(\frac{1}{b},0\right)$ fails to satisfy the transversality condition. First, consider a linearization around stable equilibrium point $(0,0)$. The eigenvectors associated with the eigenvalues are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\-r\end{bmatrix}$. Thus, the trajectory $\gamma(t)$ approaching the stable equilibrium points can be approximated as $\gamma(t) = \begin{bmatrix}x^*(t)\\a^*(t)\end{bmatrix} = c_1 e^{-bt}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{-(r+b)t}\begin{bmatrix}1\\-r\end{bmatrix} + \begin{bmatrix}o(e^{-bt})\\o(e^{-(r+b)t})\end{bmatrix}$. We see that the transversality condition is violated for trajectories approaching stable equilibrium points, i.e., $\lim_{t\to\infty} -\frac{e^{-rt}}{-c_2 r e^{-(r+b)t} + o(e^{-(r+b)t})} = \lim_{t\to\infty} \frac{e^{bt}}{c_2 r - o(1)} \neq 0$. Following the same reasoning it can be shown that trajectories approaching the other stable equilibrium point also fail to satisfy the transversality condition.

*B) Solutions going to infinity*[9]

We show that it is not possible for a trajectory $\gamma(t)$ to grow unbounded while $x^*(t)$ remains bounded. If the latter condition holds, then we have $x^*(t) < M$ for $t > 0$, which

[8]We used a similar argument in the proof of lemma 5.2.1.

[9]The analysis presented here is similar to [108].

implies $\dot{x}^* = a^* - bx^* > a - bM$ for dynamics in mode 1 and $\dot{x}^* = a^* - bx^* + 1 > a^* - bM + 1$ for dynamics in mode 2. However, since $(x^*(t), a^*(t)) \to \infty$, there exists $T_0 > 0$ such that $a^*(t) > bM + 2$ for all $t > T_0$. So, for $T = T_0 + M$, $x^*(T) \geq M$, which contradicts the assumption $x^*(t) < M$ for all $t$.

The transversality condition allows for solutions that go to infinity where $a^*(t)$ grows at a rate greater than $-r$. Since, $a^*(t)$ goes to zero only at a rate $-(r+b)$, we see that only positive growth rates are possible. It is possible that these diverging paths can be more beneficial than a path that approaches an equilibrium. We discard these candidates as the control set is bounded, see assumption 5.2.1. So, we are left with candidates that approach saddle node equilibria and closed orbits. The divergence of the non-switched analogue of the optimal system, in state-adjoint coordinates, is equal to $r > 0$. So, using Poincaré - Bendixson criterion the existence of closed orbits is ruled out, see [108]. In a switched system, however, it is unclear as to how the notion of divergence should be defined. We show later, with a numerical simulation, that in this study closed orbits may not exist. So, we have the following assumption.

*Assumption* 5.4.1. We only consider optimal candidates that reach saddle node equilibria as $t \to \infty$.

*C) Optimal candidates and objective*

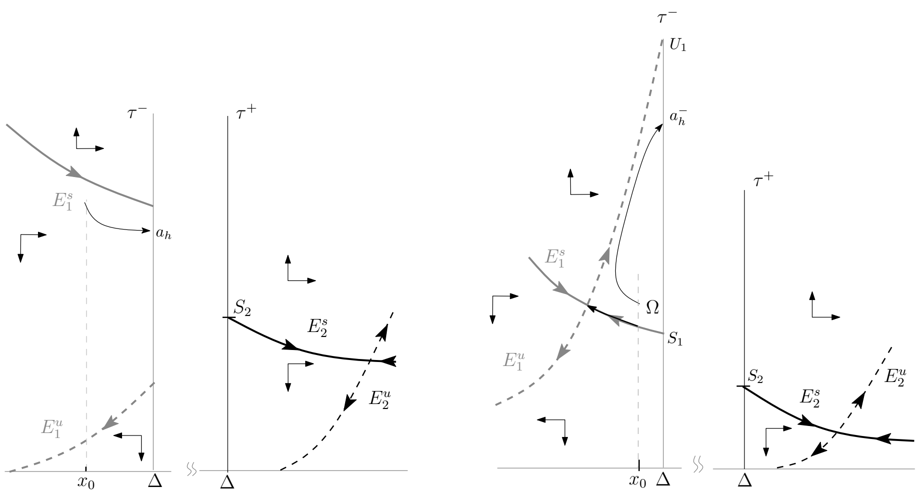
Let $E_i^u$ and $E_i^s$ denote the unstable and stable manifolds in the mode $i$. If $x_0 \in \mathbb{R}_+$ is the initial state of the lake then the optimal candidates are obtained by first tracing the trajectories backwards starting at the equilibrium points. Let $\gamma(t) \in \mathbb{R}_+^2$ be one such candidate, then the initial nutrient loading, i.e., $a(0) = a_0$, is obtained as the intersection of $\gamma(t)$ with the line $x = x_0$, and as a result multiple candidates, starting at $x_0$, are possible. Due to assumption 5.4.1, these candidates undergo only a finite number of switchings and as a result lemma 5.2.1 can be used to compare the objectives along the candidate trajectories.

The optimal vector field of the classical shallow lake problem with smooth nonlinearities admits complex qualitative behaviors such as multiple steady states, existence of indifference or Skiba points and bifurcations due to variations in the parameters $b$, $c$ and $r$, refer [58] for a complete analysis. In the present model with deterministic thresholds, we consider bifurcations due to variations in the switching surface. Further, we make the following assumption.

*Assumption* 5.4.2. The switching surface does not coincide with the equilibrium points, i.e., $\Delta \notin \left\{0, \frac{1}{b}, x_{eq}^1, x_{eq}^2\right\}$.

*D) Bifurcations due to switching surface*

The qualitative behavior of the optimal dynamics depends upon the position of the switching surface. Let $S_i$ and $U_i$ be points where the stable and unstable manifolds in mode $i$ touch the switching surface. As described in section 5.3, the critical decay rate $b = \frac{1}{\Delta}$ plays a crucial role. We consider the following situations:




(a) $\Delta < x_{eq}^1$ (b) $\Delta > x_{eq}^1$

**Figure 5.5** – Bifurcation analysis for $b\Delta < 1$

$\underline{b\Delta < 1:}$ First, we notice that if $b\Delta < 1$ any trajectory approaching the switching surface from mode 1, after entering mode 2 satisfies $\dot{x} = a - bx + 1$. Near the switching surface in mode 2 we have $\dot{x}(\tau^+) = a - (b\Delta - 1) > 0$. So, the trajectory never returns to mode 1, i.e., if the lake switches to turbid state it can never return to clear state.

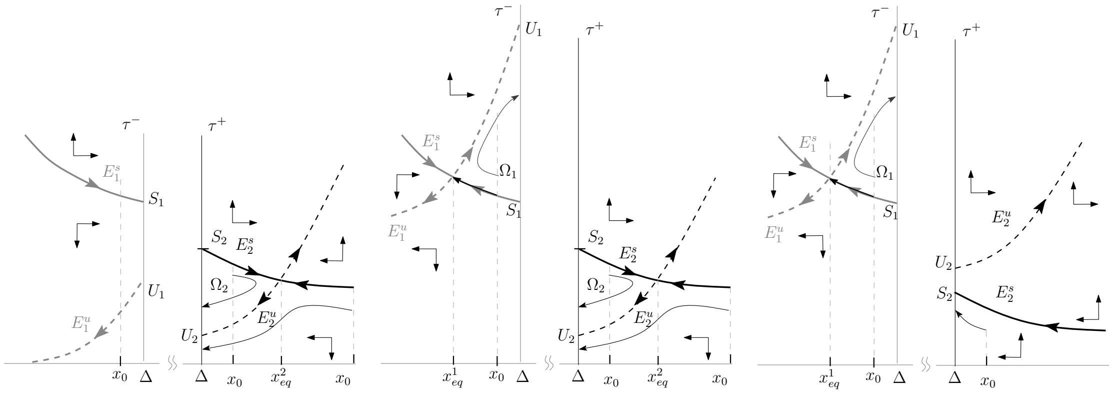
(a) Consider the case with $\Delta < x_{eq}^1$ and the related phase diagram in figure 5.5(a). For any $x_0 < \Delta$, the optimal candidate is the one that switches to the point $S_2$ from mode 1. If $S_2$ does not have a predecessor in mode 1 then there is no optimal solution [10]. If $x_0 > \Delta$, then the optimal candidate is the trajectory starting at $(x_0, E_2^s(x_0))$. So, the admissible candidates converge to the steady state in mode 2.

(b) Next, we consider the case $\Delta > x_{eq}^1$ and the related phase diagram in figure 5.5(b). Following the discussion in 5.4.1.3B there always exists a closed region $\Omega$ such that a trajectory originating in $\Omega$ leaves $\Omega$ in a finite time. A detailed analysis includes finding the predecessors of the point $S_2$. The optimal candidates can reach the steady states either in mode 1 or in mode 2. However, starting in mode 2 the steady state in mode 1 cannot be reached, whereas the steady state in mode 2 can be reached starting in mode 1.

$\underline{b\Delta > 1:}$ The following observations can be made about the optimal candidates if $b\Delta > 1$. The economic agents, by lowering the nutrient loading, can reverse the lake to mode 1 from mode 2, i.e., $\dot{x}(\tau^+) < 0$ near the switching surface. We have the following three cases.

(c) Consider the case with $\Delta < x_{eq}^1$ and the related phase diagram in figure 5.6(a). A trajectory starting in mode 1 either switches to mode 2 or approaches the origin.

[10]Transversality condition allows for trajectories with $a(t)$ going to $\infty$.




(a) $\Delta < x_{eq}^1$ (b) $x_{eq}^1 < \Delta < x_{eq}^2$ (c) $x_{eq}^2 < \Delta$

**Figure 5.6** – Bifurcation analysis for $b\Delta > 1$

Trajectories reaching the equilibrium point in mode 2 are optimal. The existence of the closed region $\Omega_2$ follows from the discussion in section 5.4.1.3A. A detailed analysis includes tracing the predecessors for the point $S_2$ on the surface $x = \Delta$ at $\tau^-$.

(d) Consider the case with $x_{eq}^1 < \Delta < x_{eq}^2$ and the related phase diagram in figure 5.6(b). The optimal candidates can reach either of the steady states in mode 1 and mode 2 by first reaching the points $S_1$ and $S_2$. So, the optimal candidates are obtained by finding the predecessors of these points using the switching rules devised in section 5.4.1.2.

(e) Consider the case with $\Delta > x_{eq}^2$ and the related phase diagram in figure 5.6(c). Optimal candidates reaching the steady state in mode 1 are optimal and these are obtained by using the switching rules for finding the predecessors.

Next, we demonstrate the subtleties in finding the optimal candidates for a specific choice of the above mentioned possibilities. We consider the case 5.4.1.3D.(e), i.e, when $b\Delta > 1$ and $\Delta > x_{eq}^2$, and explain in detail about the optimal candidates for all initial states $x_0$. Towards that end, we have the following algorithm/procedure to generate points on the switching surface which eventually reach the steady state in mode 1.

**Algorithm 5.4.1.** *Construct sequences $d_k,\ e_k,\ g_k,\ h_k,\ k = 0,1,2,\cdots$ using the following steps:*

1. *For $k = 0$, set $d_0 = S_1$ and obtain $e_0,\ g_0,\ h_0$ by solving the equation (follows from section 5.4.1.2)*

$$\begin{aligned} s(d_0, b\Delta) &= s(e_0, b\Delta) = s(g_0, b\Delta - 1) = s(h_0, b\Delta - 1) \\ s.t \quad d_0 < b\Delta < e_0, &\quad g_0 < a_{12}^l < b\Delta - 1 < a_{12}^h < h_0. \end{aligned}$$

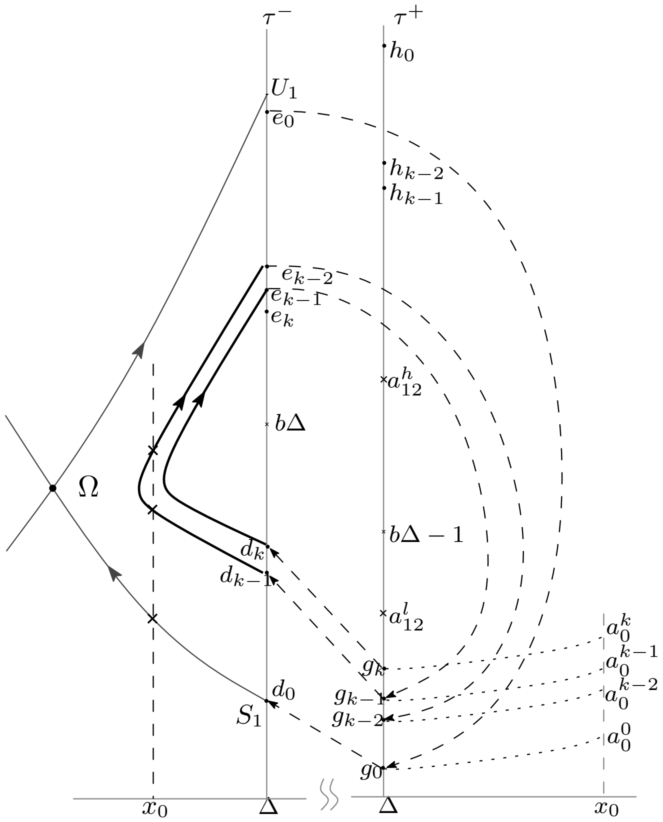



**Figure 5.7** – Graphical illustration of algorithm 5.4.1.

2. *If* $e_0 < U_1$ *go to step* 3 *else STOP.*
3. *For,* $k \geq 1$*, solve the boundary value problem to obtain* $a(0)$ *(solution exists due to the property of the region* $\Omega$*)*

$$\dot{x} = a - bx,\ \dot{a} = -(r+b)a + 2ca^2x,\ x(0) = \Delta,\ x(\tau) = \Delta,\ a(\tau) = e_{k-1}.$$

   *Set* $d_k = a(0)$
4. *Solve* $s(d_k, b\Delta) = s(e_k, b\Delta) = s(g_k, b\Delta - 1) = s(h_k, b\Delta - 1)$, $d_k < b\Delta < e_k$, $g_k < a_{12}^l < b\Delta - 1 < a_{12}^h < h_k$.
5. *Set* $k = k+1$ *and go to step 3.*

*We notice, that the above procedure generates sequences that satisfy the following condition:*

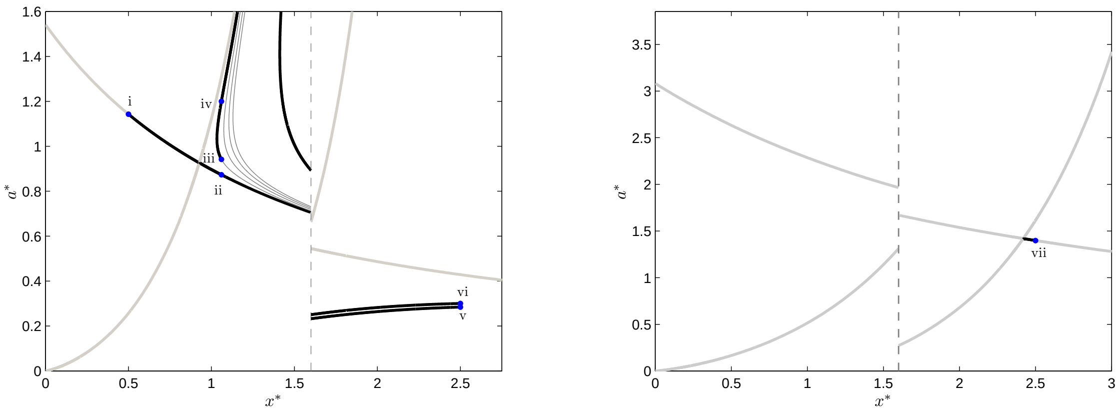
$$g_0 < g_1 < g_2 < \cdots < g_k < \cdots < a_{12}^l < b\Delta - 1 < a_{12}^h < \cdots < h_k < \cdots < h_2 < h_1 < h_0$$
$$\text{and} \quad d_0 < d_1 < d_2 < \cdots < d_k < \cdots < b\Delta < \cdots < e_k < \cdots < e_2 < e_1 < e_0.$$

A graphical illustration of the algorithm is given by figure 5.7. Any trajectory starting at $(\Delta, d_k)$, $(\Delta, e_k)$ and $(\Delta, g_k)$ will eventually reach the steady state in mode 1. Now, we consider the three situations w.r.t. choice of initial state $x_0$ and find the optimal candidates for each one of them.

- First, if $x_0 > \Delta$, i.e., starting in mode 2, let $(x_0, a_0^k)$ denote the initial state of the trajectory which reaches the point $(\Delta, g_k)$ (shown as dotted lines in figure 5.7). Then a candidate trajectory starting at $(x_0, a_0^k)$ will undergo $k$ cycles, that spiral out, before reaching the stable manifold $E_1^s$ starting at $d_0$. So, we have countably infinite candidates that satisfy the necessary conditions. The objective along each path can be calculated using lemma 5.2.1.

- $x_{eq}^1 < x_0 < \Delta$. If $e_0 > U_1$, then there may exist two candidates that approach steady state in mode 1. The first one is the stable manifold. The other one may lie above the unstable manifold $E_1^u$ which approaches $e_0$, then switches to mode 2 at $g_0$ and switches back to mode 1 at $d_0$. For $e_0 < U_1$, there always exists one candidate that lies on the stable manifold $E_1^s$. Further, we observe that trajectories starting at $d_{k-1}$ and ending at $e_k$ intersect the line $x = x_0$ at two points or at one point (tangential intersection). So, depending upon the location of $x_0$ we have either $2L$ or $2L+1$ candidates, where $L$ represents the number of paths that intersect the section $x = x_0$.

- For $x_0 < x_{eq}^1$, if $e_0 < U_1$ then there exists one candidate that lies on the stable manifold $E_1^s$. Further, if $e_0 > U_1$ there may exist an additional candidate that reaches $e_0$ switches to mode 2 at $g_0$ and returns to mode 1 at $d_0$.




(a) Phase portrait with $b = 1$, $c = 0.6$, $r = 0.03$ and $\Delta = 1.6$

(b) Phase portrait with $b = 1$, $c = 0.15$, $r = 0.03$ and $\Delta = 1.6$

**Figure 5.8** – Optimal candidates for various initial states. Thick gray lines indicate stable and unstable manifolds, dashed line indicates the switching surface. Thick dark lines indicate the optimal candidates.

| Initial State | Candidate | | | | Objective |
|---|---|---|---|---|---|
| | starting at $(x_0,\ a_0)$ | | # of cycles | reaches steady state in mode # | $N=4$ optimal management |
| 0.50 | (0.50, 1.1425) | i | 0 | 1 | -65.5150 |
| 1.06 | (1.06, 0.8733) | ii | 0 | 1 | -66.0710 |
| | (1.06, 0.9420) | iii | 1 | 1 | -66.4980 |
| | (1.06, 1.1998) | iv | 1 | 1 | -66.4940 |
| 2.50 | (2.50, 0.2844) | v | 0 | 1 | -70.6235 |
| | (2.50, 0.2851) | | 1 | 1 | -71.0171 |
| | (2.50, 0.2859) | | 2 | 1 | -71.3826 |
| | ⋮ | | ⋮ | ⋮ | ⋮ |
| | (2.50, 0.3003) | | 38 | 1 | -78.1329 |
| | (2.50, 0.3005) | vi | 39 | 1 | -78.2385 |
| | ⋮ | | ⋮ | ⋮ | ⋮ |

**Table 5.1** – Numerical analysis of optimal candidates for the parameters $b=1$, $c=0.6$, $r=0.03$ and $\Delta=1.6$.

Figure 5.8(a) illustrates the phase plane of the optimal vector field (5.9-5.12) for the choice of parameters $b=1$, $c=0.6$, $r=0.03$, $\Delta=1.6$ and $N=4$. For this choice of parameters, we have $b\Delta=1.6>1$, $U_1=5.03493$ and $e_0=4.9259$. The optimal candidates are obtained by following the previous discussion. Optimal candidates starting at various initial states are illustrated, in small roman letters, and the benefit of each player along these trajectories is calculated according to equations (5.15), in cooperation, and (5.17), in noncooperation. First, we consider the optimal management case and the results are given in table 5.1 and illustrated in figure 5.8(a). For initial state $x_0=1.06$, we obtain three candidates labeled as (ii), (iii) and (iv). When following the paths (iii) and (iv), the agents increase the level of nutrient loading till the lake switches to mode 2 and instantaneously drop the levels to be able to switch back to mode 1 along the stable manifold $E_1^s$. When the agents start in mode 2, i.e., $x_0>1.6$, there exist countably infinite candidates each undergoing a finite number of cycles before reaching the steady state in mode 1. For instance, on path (vi) the agents can alter the nutrient levels, i.e., a decrease and increase cycle, 39 times before reaching the steady state in mode 1. The results can alter once the costs associated with switchings are considered. For the open loop Nash equilibrium, following lemma 5.4.1, we analyze the optimal field with $c=0.6$ replaced with $\frac{c}{N}=0.15$. The phase plane for the optimal vector field with $c=0.15$, ceteris paribus, is illustrated in figure 5.8(b). We notice that only steady state in mode 2 can be achieved by the agents. So, the trajectory

(vii) corresponds to the open loop Nash equilibrium path with $x_0 = 2.5$ and the welfare parameter set to $c = 0.6$. We notice that the mode of play induces a bifurcation in the optimal vector field; that is when players play cooperatively steady state in mode 1 is attained whereas a noncooperative behavior leads to the steady state in mode 2. Further, it is clear from the tables 5.1 and 5.2 that each player receives greater benefits in cooperation. Now, consider the effect of reducing the $c$ from 0.6 to 0.15 on optimal management. Since, in the later case the players incur less costs, towards cleaning activities, there is an incentive for increasing the nutrients and as a result the optimal vector field results in the steady state in mode 2.

| Initial State | Candidate starting at $(x_0,\ a_0)$ | # of cycles | reaches mode # steady state in mode # | Objective $N=4$ optimal management | $N=4$ ONLE for $c=0.6$ |
|---|---|---|---|---|---|
| 0.50 | (0.50, 3.500000) | - | - | -11.1676[11] | |
| 1.70 | (1.70, 1.635331) | 0 | 2 | -63.3300 | |
| 2.50 | (2.50, 1.398072) vii | 0 | 2 | -63.8598 | -150.3192 |

**Table 5.2** – Numerical analysis of optimal candidates for the parameters $b = 1$, $c = 0.15$, $r = 0.03$ and $\Delta = 1.6$

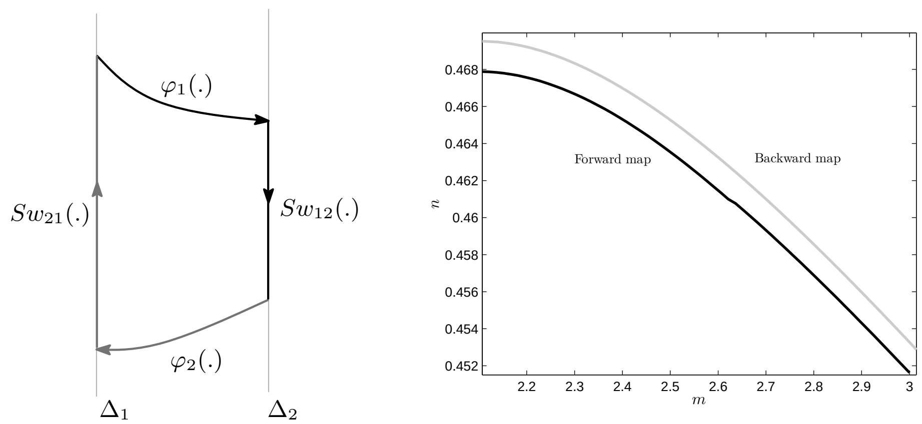
## 5.4.2 Hysteresis

In the previous section we notice that some candidates undergo multiple switches before reaching a steady state in mode 1. During each cycle, a candidate leaving mode 1 reenters the same mode instantaneously following the switching sequence $e_k \to g_k \to d_k,\ k \geq 0$. These control actions are still admissible in the class $\mathscr{U}$. The successors of $g_k$, in figure 5.7, are $e_k$ and $d_k$. If we consider the path $g_k \to e_k$ instead of $g_k \to d_k$ we obtain a limit cycle $\cdots g_k \to e_k \to g_k \to e_k \cdots$. However, all these multiple (indefinite) switchings happen simultaneously at the switching instant $\tau$ and these (ill posed) candidates are not considered in our analysis. These candidates, though remain on the switching surface, are a result of switching rules obtained from the necessary conditions. The conditions formulated in theorem 5.2.1 do not include Filippov type solutions [44] due to assumption 5.2.1. A hysteresis approximation, as given in section 5.3, would be useful to avoid the simultaneous multiple switchings. Sometimes, a hysteresis behavior could be inherent to the economic or ecosystem dynamics (5.5), see [32] and [91]. The analysis of the optimal vector field

[11] The candidate starting at $(0.5,\ 3.5)$ switches to mode 2 at time $t = 0.4732$ and the system (5.9-5.12) has a finite escape time at $t = 0.7844$. So, the objective is computed in the interval [0, 0.7844). We discard this candidate as assumption 5.2.1 is violated, i.e., the control policy along this candidate is unbounded.

remains unaltered except for the switching rules where $\Delta$ in (5.11) and (5.12) should be replaced with $\Delta_2$ and $\Delta_1$ respectively. For initial states in the range $\Delta_1 < x_0 < \Delta_2$, the objective along a particular candidate depends upon whether $x_0$ is in mode 1 or mode 2.

Next, we discuss the issue of existence of closed orbits in the optimal dynamics with hysteresis. Poincaré - Bendixson criterion is generally used to analyze the existence of closed orbits in a smooth vector field, and it involves computation of divergence. At present, it is unclear as to how the notion of divergence of a switching vector field [12] can be defined. Firstly, a closed orbit cannot occur within either of the modes 1 and 2 as the divergence of the optimal vector field, in state-adjoint coordinates, is equal to $r > 0$. So, a closed orbit may occur during mode transitions. Let us denote the map $\varphi_i(.)$ to be an orbit of the vector field in mode $i$. Let the adjoint jump and Hamiltonian continuity properties, see item (c) of theorem 5.2.1, be abstracted as a switching map $Sw_{ij}(.)$ during a transition from mode $i$ to mode $j$. To see if there exists a closed orbit, we construct the forward and backward maps as $n = Sw_{12}(\varphi_1(m))$ and $m = Sw_{21}(\varphi_2(n))$ and look for intersection points in the $(m,n)$ plane. Figure 5.9(b) shows one such instance for parameter values $b = 1$, $c = 0.15$, $r = 0.03$, $\Delta_1 = 2.1$ and $\Delta_2 = 2.2$. The forward and backward maps never intersect hinting that there may not be a closed orbit. At this point we do not have conclusive evidence, however.




(a) (b)

**Figure 5.9** – Forward and backward maps to analyze closed orbits

## 5.5 Conclusions

In this chapter we analyze the shallow lake model in the presence of threshold effects. We approximate the nonlinearities in the shallow lake dynamics with simple and hysteresis switching. Assuming symmetry in agents' actions, we solve the associated optimal management problem using relevant necessary conditions, which resulted in an optimal

[12]See [74] for some work in this direction.

switching vector field. We observe that the variation of switching surface induces bifurcations in this vector field. Further, we notice, that agents' mode of play can also induce bifurcations; that is, cooperation can lead to 'clean' steady state and noncooperation can lead to 'turbid' steady state. These observations, though incomplete, do agree with previous studies on the classical shallow lake model.

The main objective of this chapter was to highlight, with an example, the key observations when smooth nonlinear models are approximated with simple discontinuous functions. This approximation leads to simple optimal dynamics within each mode and a complex jump rule near the switching surface. Further, the analysis may lead to many optimal candidates depending upon the initial state. An attempt towards a complete analysis raises several interesting questions. It was shown in [108], that existence of Skiba points is closely related to heteroclinic connections[13] in the optimal vector field. A piecewise approximation of the convex-concave production function [100] also results in multiple steady states, then to see whether such points exist in the optimal switching dynamics would be interesting. A closed orbit, if it exists, would result in policies where players increase and decrease the nutrient levels in a sustainable way. So, to characterize the switched optimal control problems, of the type (5.1-5.3), that admit closed orbits as an optimal solution would be insightful. For noncooperation, the present analysis is restricted to symmetric open loop Nash equilibrium policies. Using feedback policies for the optimal management problem may involve some computational burden, for instance, see [25].

## 5.A Appendix

*Proof of Lemma 5.2.1:*. Since $(x^*(.),u^*(.))$ is an optimal admissible pair, it satisfies the necessary conditions given in theorem 5.2.1. As the number of switchings is finite, say $M$, there exists a sequence of switching instants associated with $k^*(.)$, which we denote as $\tau_1,\tau_2,\cdots,\tau_j,\cdots\tau_M$. Taking the total derivative of the Hamiltonian $H_{k^*(t)}(.)$ in the interval $t\in\left(\tau_j^+,\tau_{j+1}^-\right)$ we have:

$$
\begin{aligned}
\frac{dH_{k^*(t)}}{dt} &= \frac{\partial H_{k^*(t)}}{\partial t}+\frac{\partial H_{k^*(t)}}{\partial x^*}\dot{x}^*+\frac{\partial H_{k^*(t)}}{\partial \lambda_{k^*}}\dot{\lambda}_{k^*}+\frac{\partial H_{k^*(t)}}{\partial u^*}u^* \\
&= \frac{\partial H_{k^*(t)}}{\partial t} \quad \text{(last three terms vanish due to necessary conditions)} \\
&= -re^{-rt}g(x^*(t),u^*(t)) \\
H_{k^*(\tau_{j+1}^-)}-H_{k^*(\tau_j^+)} &= -r\int_{\tau_j^+}^{\tau_{j+1}^-} e^{-rt}g(x^*(t),u^*(t))dt.
\end{aligned}
$$

[13]This happens when a branch of an unstable manifold of an equilibrium point coincides with a branch of a stable manifold of a different equilibrium point.

Again from the necessary conditions we notice that in the last interval, i.e., $t \in [\tau_M^+, \infty)$, $(x^*(t), u^*(t))$ maximizes the objective $\int_{\tau_M^+}^{\infty} e^{-rt} g(x(t), u(t))dt$. The truncated trajectory $(x^*(t), u^*(t))$, $t \in [\tau_M^+, \infty)$ is an optimal admissible pair for the classical discounted infinite horizon optimal control problem

$$\begin{aligned}
&\max \int_{\tau_M^+}^{\infty} e^{-rt} g(x(t), u(t))dt \\
&\dot{x}(t) = f_{k^*(\tau_M^+)}(x(t), u(t)),\ x(\tau_M^+) = x^*(\tau_M^+) \\
&u(t) \in U,\ t \in [\tau_M^+,\ \infty).
\end{aligned}$$

The objective along the truncated trajectory is given by $\frac{1}{r} H_{k^*(\tau_M^+)}\Bigg(\tau_M^+, x^*(\tau_M^+), u^*(\tau_M^+), \lambda_{k^*(\tau_M^+)}(\tau_M^+)\Bigg)$[14]. The objective along $(x^*(t), u^*(t))$, $t \in [0, \infty)$ is then given by:

$$\begin{aligned}
\int_0^{\infty} e^{-rt} g(x^*(t), u^*(t))dt &= \int_0^{\tau_1^-} e^{-rt} g(x^*(t), u^*(t))dt + \sum_{j=1}^{M-1} \int_{\tau_j^+}^{\tau_{j+1}^-} e^{-rt} g(x^*(t), u^*(t))dt \\
&\quad + \int_{\tau_M^+}^{\infty} e^{-rt} g(x^*(t), u^*(t))dt \\
&= \frac{1}{r}\Big(H_{k^*(0)}(.) - H_{k^*(\tau_1^-)}(.) + H_{k^*(\tau_1^+)}(.) \cdots - H_{k^*(\tau_M^-)}(.) + H_{k^*(\tau_M^+)}(.)\Big) = \frac{1}{r} H_{k^*(0)}(.).
\end{aligned}$$

In the above result the costs associated with switching are assumed to be zero. ■

*Proof of Lemma 5.4.1:.* Optimal management involves solving the following optimization problem

$$\dot{x}(t) = \sum_i a_i(t) - bx(t) + \alpha, \quad \alpha = \begin{cases} 0 & x(t) < \Delta \\ 1 & x(t) > \Delta \end{cases},\ x(0) = x_0,$$

$$\max_{a_1, a_2, \cdots, a_N} J, \quad J = \int_0^{\infty} e^{-rt} \left( \sum_i \ln a_i(t) - Ncx^2(t) \right) dt.$$

The necessary conditions in the current value form are given by:

$$\text{mode } j \qquad H_j(.) = \left( \sum_i \ln(a_i) - Ncx^2 \right) + \lambda_j \left( \sum_i a_i - bx + j - 1 \right)$$

[14] Follows from the observation; the truncated candidate in the last interval satisfies the additional necessary condition that maximized Hamiltonian is zero when $t$ goes to infinity, see footnote 3.

$$\frac{1}{a_i^*} + \lambda_j = 0,\ i = 1,2,\cdots,N \text{ (follows from maximum condition)}$$
$$\dot{\lambda}_j(t) = (r+b)\lambda_j + 2cNx^*$$
$$\dot{a}_i^* = -(r+b)a_i^* + 2cNa_i^{*2}x^*.$$

Assuming symmetry, i.e., $a_i^* = \frac{a^*}{N}$, $i = 1,2,\cdots,N$, the optimal system is given by

$$\text{mode } j \qquad \dot{x}^* = a^* - bx^* + j - 1 \tag{5.14a}$$
$$\dot{a}^* = -(r+b)a^* + 2ca^{*2}x^*, \tag{5.14b}$$

and the optimal cost for a candidate starting at $(x_0, a_0)$, with $x_0$ in mode $j$, is given by:

$$J = \frac{1}{r}H_j(a_0,x_0) = \sum_i \ln\left(\frac{a_0}{N}\right) - Ncx_0^2 + \left(-\frac{N}{a_0}\right)(a_0 - bx_0 + j - 1)$$
$$= \frac{1}{r}\left(N\ln\left(\frac{a_0}{N}\right) - Ncx_0^2 + N\frac{bx_0 - j + 1}{a_0} - N\right).$$

So, each player receives a benefit of $J_i^{\text{opt}}$ given by:

$$J_i^{\text{opt}} = \frac{J}{N} = \frac{1}{r}\left(\ln\left(\frac{a_0}{N}\right) + \frac{bx_0 - j + 1}{a_0} - cx_0^2 - 1\right). \tag{5.15}$$

Let $(a_1^*, a_2^*, \cdots, a_N^*)$ denote the open loop Nash equilibrium, then agent $i$ solves the following optimization problem:

$$\dot{x}(t) = a_i + \sum_{k\neq i} a_k^*(t) - bx(t) + \alpha,\ \alpha = \begin{cases} 0 & x(t) < \Delta \\ 1 & x(t) > \Delta \end{cases},\ x(0) = x_0$$
$$\max_{a_i} J_i, \quad J_i = \int_0^\infty e^{-rt}\left(\ln a_i(t) - cx^2(t)\right)dt$$

$$\text{mode } j \qquad H_j^i(.) = \ln a_i - cx^2 + \lambda_j^i\left(a_i + \sum_{k\neq i} a_k^* - bx + j - 1\right)$$
$$\frac{1}{a_i^*} + \lambda_j^i = 0,\ i = 1,2,\cdots,N \text{ (follows from maximum condition)}$$
$$\dot{\lambda}_j^i = (r+b)\lambda_j^i + 2cx^*$$
$$\dot{a}_i^* = -(r+b)a_i^* + 2ca_i^{*2}x^*.$$

Again, assuming symmetry we have $a_i^* = a^*/N,\ i = 1, 2, \cdots, N$. So, the optimal system is given by

$$\begin{aligned} \text{mode } j \qquad & \dot{x}^* = a^* - bx^* + j - 1 && (5.16a)\\ & \dot{a}^* = -(r+b)a^* + 2\left(\frac{c}{N}\right) a^{*2} x^*, && (5.16b) \end{aligned}$$

and the benefit for the candidate, at equilibrium, starting at $(x_0, a_0)$, with $x_0$ in mode $j$, is given by

$$J_i^{\text{onle}} = \frac{1}{r} H_j^i(a_0, x_0) = \frac{1}{r}\left(\ln\left(\frac{a_0}{N}\right) + \frac{bx_0 - j + 1}{\frac{a_0}{N}} - cx_0^2 - N\right). \qquad (5.17)$$

■

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