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# Improved Approximation of the General Soft-Capacitated Facility Location Problem

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**March 2005**

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# Soft-Capacitated Facility Location Problem

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**ABSTRACT :**

An NP-hard variant of the single-source Capacitated Facility Location Problem is studied, where each facility is composed of a variable number of fixed-capacity production units. This problem, especially the metric case, has been recently studied in several papers. In this paper, we only consider the general problem where connection costs do not systematically satisfy the triangle inequality property. We show that an adaptation of the set covering greedy heuristic, where the sub-problem is approximately solved by a Fully Polynomial-Time Approximation Scheme based on cost scaling and dynamic programming, achieves a logarithmic approximation ratio of $(1+\varepsilon)H(n)$ for the problem, where n is the number of clients to be served, and H is the harmonic series. This improves the previous bound of $2H(n)$ for this problem.

Key Words:

- Facility Location
- Combinatorial optimization
- Set Covering
- Dynamic Programming
- Approximation

**RESUME :**

Nous étudions une variante du problème classique de localisation optimale d'entreprises avec capacités de livraison ou de production limitées, dans laquelle chaque centre de production se décompose en un nombre variable d'unités de production. Cette variante, NP-difficile également, a été récemment étudiée dans plusieurs articles, principalement dans le cas métrique. Dans ce document, nous traitons le problème général où les coûts de connexion des clients aux centres de livraison ne vérifient pas obligatoirement l'inégalité triangulaire. Nous proposons pour ce problème une heuristique adaptée de la méthode gloutonne pour le problème de Couverture d'Ensemble, où le sous-problème est traité par un schéma d'approximation utilisant une normalisation des coûts et la programmation dynamique. Nous montrons que cette heuristique permet d'approcher en temps polynomial le problème à rapport $(1+\varepsilon)H(n)$, où n est le nombre de clients devant être livrés et H est la série harmonique. Ce rapport améliore le précédent ratio de $2H(n)$ pour ce problème.

Mots-clés :

- Localisation d'Entreprises
- Optimisation Combinatoire
- Couverture d'Ensemble
- Programmation Dynamique
- Approximation

JEL Classification Code : C44, C61

# Improved approximation of the general Soft-Capacitated Facility Location Problem

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**Abstract**

An NP-hard variant of the single-source Capacitated Facility Location Problem is studied, where each facility is composed of a variable number of fixed-capacity production units. This problem, especially the metric case, has been recently studied in several papers. In this paper, we only consider the general problem where connection costs do not systematically satisfy the triangle inequality property. We show that an adaptation of the set covering greedy heuristic, where the subproblem is approximately solved by a Fully Polynomial-Time Approximation Scheme based on cost scaling and dynamic programming, achieves a logaritmic approximation ratio of $(1+\epsilon)H(n)$ for the problem, where $n$ is the number of customers to be served and $H$ is the harmonic series. This improves the previous bound of $2H(n)$ for this problem.

**Key-words**: facility location, set covering, dynamic programming, FPTAS.

## 1 Introduction

The classical single-source Capacitated Facility Location Problem (CFLP) consists in assigning a set of $n$ customers with known demands to a set of $m$ possible facilities so that each customer is assigned to a single facility without violating capacities of open facilities, while minimizing the sum of the construction cost of selected facilites and the connexion cost of customers to facilities. In this paper, we consider a variant of CFLP where each facility, if open, can be composed of a variable number (to determine) of fixed-size production units. This problem, known as the Soft-Capacitated Facility Location Problem (SCFLP), was first introduced in [10]. It arises indeed in many industrial applications, as production is often structured by production lines or teams whose number is a decision to make. For large instances of hard problems, the design of heuristics that are both fast and efficient is a challenge. In this field, the polynomial approximation theory has received much attention in the last two decades. The aim is to develop a $\rho$-approximation of the problem, i.e., a polynomial-time algorithm that finds a feasible solution whose objective function is always

*also LIPN, UMR-CNRS 7030, Université Paris XIII, France within a factor $\rho$ of the optimum, so that $\rho$ is as small as possible. The best-known approximation ratio for the metric version of CFLP is $6(1 + \epsilon)$ and was produced by Chudak and Williamson [2]. This ratio is obtained by improving the analysis of the local search heuristic of Korupolu, Plaxton and Rajamaran for this problem [14]. The first constant approximation ratio for the metric uncapacitated problem (UFLP) was found by Shmoys, Tardos and Aardal [18]. Their method, achieving an approximation ratio of 3.16, is based on LP-rounding. This ratio has been repeatedly improved then until the greedy algorithm of Mahdian, Ye and Zhang [16] which provides an approximation ratio of 1.52 for UFLP. The metric version of SCFLP was shown by Jain, Mahdian and Saberi to admit a 3-approximation by a combination of a primal-dual greedy process and lagrangian relaxation [12]. This ratio was recently improved by the same authors to a 2-approximation [17]. Note that in many real cases connection costs are transportation costs which do not only depend on the distance in kilometers between customers and facilities, but also depend on the quantity delivered to the customer. Therefore, connection costs do not generally satisfy the triangle inequality, and approximating the general (non-metric) problem is a real issue. The general SCFLP is approximable within ratio $2H(n)$, where $H(n) = \sum_{1 \leq i \leq n} 1/i$ (see [10]). This comes from the fact that a $\rho$-approximation for UFLP provides a $2\rho$-approximation for SCFLP, and UFLP was shown to be approximable within ratio $H(n)$ by Hochbaum [8]. The algorithm of [8] for UFLP relies on an exponential-size Set Covering reformulation of UFLP and the fact that the exponential set of candidate subsets can be reduced to an equivalent set of polynomial size. Since the Set Covering Problem (SCP) is approximable within ratio $H(n) \leq 1 + \ln n$ [3], the result also holds for UFLP. The bound of $O(\ln n)$ is asymptotically tight for SCFLP since the problem is linked by an approximation-preserving reduction with SCP and SCP cannot be approximated within a ratio better than $\ln n - \ln \ln n$ [4]. We improve the ratio of $2H(n)$ for SCFLP to $(1+\epsilon)H(n)$ by an algorithm running in time $O(mn^4/\epsilon)$. This algorithm also uses an exponential-size Set Covering reformulation of SCFLP, and a FPTAS based on cost scaling and rounding and dynamic programming for the subproblem of the SCP greedy heuristic. In our approach, we do not restrict a priori the collection of subsets in the SCP reformulation and do not exactly solve the subproblem. The SCFLP is formally stated and reformulated as a SCP in section 2. The adaptation of the SCP classical greedy process to SCFLP is presented in section 3. The subproblem of the greedy heuristic for SCFLP is shown to admit a Fully Polynomial Time Approximation Scheme (FPTAS) in section 4. Section 5 concludes the paper.

# 2 Problem statement and reformulation

The Soft-Capacitated Facility Location Problem (SCFLP) is stated as follows. The set of customers to be served is denoted by $I = \{1, \ldots, n\}$, whereas the set of possible locations for facilities is $J = \{1, \ldots, m\}$. For $(i, j) \in I \times J$, $c_{ij}$ is the connection cost between customer $i$ and location $j$, $d_i$ is the demand of customer $i$, $f_j$ (resp. $K_j$) is the construction cost (resp., capacity) of a production line on location $j$. The Integer Linear Programming model corresponding to SCFLP is the following:

$$\text{Minimize} \sum_{j\in J} f_j y_j + \sum_{(i,j)\in I\times J} c_{ij}x_{ij} \quad (1)$$

$$\text{s.t.} \quad \sum_{j\in J} x_{ij} = 1 \quad \text{for } i \in I \quad (2)$$

$$\text{(SCFLP)} \quad \sum_{i\in I} d_i x_{ij} \leq K_j y_j \quad \text{for } j \in J \quad (3)$$

$$y_j \in \mathbf{N}, x_{ij} \in \{0,1\} \quad (4)$$

where integer variables $y_j$ indicate the number of production lines settled in facility $j \in J$, and binary variables $x_{ij}$ indicate whether customer $i \in I$ is assigned to location $j \in J$ or not. The objective (1) minimizes the total cost of the location. The semi-assignment constraints (2) express single-source supplying. Constraints (3) express restricted capacities of facilities. The difference between SCFLP and the classical CFLP is that variables $y_j$ are not binary but integer (and unbounded). SCFLP is NP-hard, since the Set Covering Problem (SCP), which is NP-hard [5], reduces to it. Given a set $C$ of elements and a collection $\mathcal{S} = \{S_1, \ldots, S_m\}$ of subsets of $C$ with cost $c(S)$ for $S \in \mathcal{S}$, SCP consists in finding a minimum cover of $C$, i.e., a subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $\cup_{S\in\mathcal{S}'} S = C$ and total cost $\sum_{S\in\mathcal{S}'} c(S)$ is minimum. The polynomial reduction is built as follows: set $I = C$, $J = \mathcal{S}$, $K_j = n$, $f_j = c(S_j)$ for all $j \in J$, $d_i = 1$ for all $i \in I$, and $c_{ij} = 0$ if $i \in S_j$, $M$ otherwise, with $M > \sum_{j\in J} f_j$. Then, there is a SCFLP solution of cost at most $c$ if and only if there is a cover of cost at most $c$ in the transformed set covering instance.

In general, $c_{ij}$ is a transportation cost which depends on the distance in kilometers $\delta_{ij}$ between $i$ and $j$, the demand $d_i$ of customer $i$, expressed in tons for instance, and the unitary transportation cost $\mu$ expressed in currency units per ton and kilometer. If transportation costs are linear, then $c_{ij} = \delta_{ij} d_i \mu$. In that case, the triangle inequality is rarely verified: consider the following example of two facilities $j$ and $j'$ and two customers $i$ and $i'$ such that $\delta_{ij} = \delta_{i'j'} = 50$, $\delta_{ij'} = \delta_{i'j} = 30$, $d_i = 1000$ and $d_{i'} = 100$. Then, $c_{ij} = 50000\mu$ and $c_{ij'} + c_{i'j'} + c_{i'j} = 38000\mu$, so we do not have $c_{ij} \leq c_{ij'} + c_{i'j'} + c_{i'j}$. The best-known ratio for the general (non-metric) SCFLP relies on a reduction to the uncapacitated problem UFLP. The formulation of UFLP is: minimize (1) under constraints (2) and $y_j \geq x_{ij}$ for all $i, j \in I \times J$, where variables $y_j$ are binary. The result mentionned in section 1, according to which a $\rho$-approximation for UFLP provides a $2\rho$-approximation for SCFLP [10], is obtained by replacing connection costs $c_{ij}$ by $c_{ij} + d_i(f_j/K_j)$ in UFLP. The approximation result of $2H(n)$ for SCFLP is achieved by applying Hochbaum's approach to UFLP with the modified connection costs. Our improvement of this bound is achieved by reformulating SCFLP as a particular SCP. The key idea is that approximately solving the subproblem of the exact SCFLP problem reveals to be better than exactly solving the subproblem of the approximate UFLP model. We introduce now the SCP reformulation of SCFLP.

**Definition 1.** Let $\mathcal{I}$ be an arbitrary instance of SCFLP. We denote by $\gamma(\mathcal{I})$ the transformed Set Covering instance of $\mathcal{I}$ such that:

(i) $C = I$ is the set of elements to cover,

(ii) $\mathcal{S} = \{S_{L,j} : L \subseteq I, j \in J\}$ is the collection of subsets,

(iii) each subset $S_{L,j} \in \mathcal{S}$ covers $L$ and has cost $c(S_{L,j}) = \lceil \sum_{i \in L} d_i / K_j \rceil f_j + \sum_{i \in L} c_{ij}$

**Proposition 1.** *Solving SCFLP on an arbitrary instance $\mathcal{I}$ is equivalent to solve SCP on $\gamma(\mathcal{I})$, i.e., every SCFLP-solution of cost at most c for $\mathcal{I}$ can be transformed in polynomial time in a cover of cost at most c for $\gamma(\mathcal{I})$ .*

**Proof.** Let $\{y_j, x_{ij}\}$ be a solution of VFCLP on $\mathcal{I}$ with cost $c$. Then, the collection of subsets $\{S_{L(j),j} : j \in J / y_j > 0\}$, where $L(j) = \{i \in I : x_{ij} = 1\}$, is a feasible cover in $\gamma(\mathcal{I})$. From (3) and (4) we have $\lceil \sum_{i \in I} d_i x_{ij} / K_j \rceil \leq y_j$ and we easily derive that the cost of the cover is at most $c$. Conversely, let $\mathcal{S}' = \{S_{L^t, j^t}, t = 1, \ldots, q\} \subset \mathcal{S}$ be a feasible cover for $\gamma(\mathcal{I})$. Set $Q^1 = L^1$ and $Q^t = L^t \setminus \cup_{1 \leq h \leq l-1} L_{DP}$ for $t = 2, \ldots, q$. Set $x_{ij^l} = 1$ for all $i \in Q^t$, $t = 1, \ldots, q$, set all other $x$-variables to zero, and $y_j = \lceil \sum_{i \in I} d_i x_{ij} / K_j \rceil$ for $j \in J$. This solution satisfies (2-4) and thus is indeed a feasible solution of VFCLP. We get

$$\begin{aligned}
\sum_{j \in J} f_j y_j + \sum_{(i,j) \in I \times J} c_{ij} x_{ij} &\leq \sum_{t=1}^{q} \left( \lceil \frac{\sum_{i \in Q^t} d_i}{K_{j^t}} \rceil f_{j^t} + \sum_{i \in Q^t} c_{ij} \right) \quad \text{as } y_j \leq \sum_{t : j^t = j} \lceil \frac{\sum_{i \in Q^t} d_i}{K_{j^t}} \rceil \\
&\leq \sum_{t=1}^{q} \left( \lceil \frac{\sum_{i \in L^t} d_i}{K_{j^t}} \rceil f_{j^t} + \sum_{i \in L^t} c_{ij} \right) \quad \text{as } Q^t \subseteq L^t \\
&= c(\mathcal{S}')
\end{aligned}$$

which completes the proof. □.

# 3 Greedy heuristic and worst-case analysis

Since SCFLP reduces to SCP by proposition 1, we consider the best polynomial-time algorithm for SCP, i.e., the *Greedy* heuristic which picks at each step a subset $S^* \in \mathcal{S}$ minimizing the ratio 'cost over number of new covered elements'. If $U$ denotes the set of elements that remains to cover at current step, the *subproblem* of the *Greedy* heuristic is formally described as finding

$$r^*(U) = \min_{S \in \mathcal{S}} \frac{c(S)}{|S \cap U|} \tag{5}$$

This iterative search terminates when $U = \emptyset$. The *Greedy* heuristic was shown by Chvátal to guarantee an approximation ratio of $H(\Delta) \leq 1 + \ln \Delta$, where $\Delta = \max_{S \in \mathcal{S}} |S|$ [3]. Nevertheless, this heuristic cannot be directly applied to the SCP instance $\gamma(\mathcal{I})$, given an instance $\mathcal{I}$ of SCFLP, since the number $|\mathcal{S}|$ of candidate subsets in $\gamma(\mathcal{I})$ is equal to $|J| 2^{|I|} = m 2^n$, which is exponential in $n$ (hence the reduction of definition 1 is not a polynomial Karp-reduction [6]). Therefore, enumeration of $\mathcal{S}$ for solving subproblem (5) is prohibited. We first use the fact that that if subproblem (5) is approximable within ratio $(1 + \epsilon)$ then the logaritmic approximation ratio of *Greedy* is conserved (proposition 2). Then we prove that the subproblem for SCFLP, which is NP-hard, admits indeed a polynomial-time $(1 + \epsilon)$-approximation despite the exponential number of subsets in $\gamma(\mathcal{I})$ (Proposition 3). For proposition 2, we need the following lemma that reformulates for our needs a part of the proof of [3].

**Lemma 1.** [3] *Let $\mathcal{S}' = \{S_1, \ldots, S_q\}$ be a feasible cover of $C$ for SCP. For $S \in \mathcal{S}$, let $S^1 = S$ and $S^t = S \setminus \cup_{1 \le h \le t-1} S_h$ for $t = 2, \ldots, q$. Moreover, set $t_S = \max\{t : S^t \neq \emptyset\}$. Then we have*

$$c(\mathcal{S}') \le \sum_{S \in \mathcal{S}^{opt}} \left( \sum_{t=1}^{t_S} \left(|S^t| - |S^{t+1}|\right) \left( \frac{c(S_t)}{|S_t^t|} \right) \right) \tag{6}$$

*where $S^{opt}$ is an optimal cover.*

**Proposition 2.** *Consider an instance $(C, \mathcal{S})$ of the Set Covering problem. If the subproblem (5) can be approximated within ratio $1+\epsilon$ by some polynomial-time algorithm $A$, then the associated greedy heuristic Greedy$(A)$, where $A$ is applied to the subproblem, approximates the Set Covering instance within ratio $(1+\epsilon)H(\Delta)$, where $\Delta = \max_{S \in \mathcal{S}} |S|$.*

**Proof.** The proof simply adapts Chvàtal's one. Let $\mathcal{S}' = \{S_1, \ldots, S_q\}$ denote the cover constructed by *Greedy*$(A)$ in chronological order $1, \ldots, q$. Since $A$ is a $(1+\epsilon)$ approximation for the subproblem, the subset $S_t^t$ defined as in lemma 1 satisfies $c(S_t)/|S_t^t| \le (1+\epsilon)(c(S)/|S^t|)$ for all $S \in \mathcal{S}$. Plugging that inequality into (6) leads to

$$\begin{aligned} c(\mathcal{S}') &\le (1+\epsilon) \sum_{S \in \mathcal{S}^{opt}} c(S) \left( \sum_{t=1}^{t_S} \frac{|S^t| - |S^{t+1}|}{|S^t|} \right) \\ &\le (1+\epsilon) \sum_{S \in \mathcal{S}^{opt}} c(S) \sum_{i=1}^{|S|} \frac{1}{i} \\ &\le (1+\epsilon) \left( \sum_{i=1}^{\Delta} \frac{1}{i} \right) c(\mathcal{S}^{opt}) \\ &= (1+\epsilon) H(\Delta) c(\mathcal{S}^{opt}) \quad \Box \end{aligned}$$

We now go back to the original facility location problem SCFLP. Set

$$w_j(L) = \left( \lceil \sum_{i \in L} d_i / K_j \rceil f_j + \sum_{i \in L} c_{ij} \right) \tag{7}$$

$$r_j(L) = w_j(L)/|L| \tag{8}$$

$$r_j^*(U) = \min_{L \subseteq U} r_j(L) \tag{9}$$

The adaptation of the set covering *Greedy* heuristic for SCFLP is described in algorithm 1. The transfer of the *Greedy* ratio $H(n)$ $(\le 1 + \ln n)$ to SCFLP depends on the approximability of subproblem (9), which is NP-hard. Indeed, the subproblem for a fixed $j \in J$ can be reformulated as the following fractional Integer Program:

$$\begin{aligned} &\text{Minimize} && \left( f_j y + \sum_{i \in U} c_{ij} x_i \right) \Big/ \left( \sum_{i \in U} x_i \right) \\ &\text{s.t.} && \sum_{i \in U} d_i x_i \le K_j y \\ &(\text{SP})_j && \sum_{i \in U} x_i \ge 1 \\ & && y \in \mathbf{N}, x_i \in \{0, 1\} \end{aligned}$$

**Algorithm 1** /Greedy heuristic for SCFLP/
*Begin*
$U \leftarrow I$
*Repeat*
*For* $j \in J$ *do* find a best-possible approximation of ratio $r_j^*(U)$ of (9)
$r^*(U) = \min_{j \in J} r_j^*(U)$
Let $(L^*, j^*)$ be the optimal pair for $r^*(U)$
$y_{j^*} := 1, \; x_{ij^*} := 1 \;$ for $i \in L^*$
$U \leftarrow U \setminus \{L^*\}$
*Until* $U = \emptyset$
output $y, x$
*End*

Consider the restriction SP'$_j$ of SP$_j$ where $K_j \geq d_i$ for all $i \in U$, $\sum_{i \in U} d_i > K_j$, and $f_j > |U| \sum_{i \in U} c_{ij}$. The latter inequality induces that $y = 1$ for the optimal solution of SP'$_j$. Hence SP'$_j$ is equivalent to maximizing $(\sum_{i \in U} x_i)/(f_j + \sum_{i \in U} c_{ij} x_i)$ under the knapsack constraint $\sum_{i \in U} d_i x_i \leq K_j$, which is an NP-hard Binary Fractional Knapsack Problem (BFKP) studied by Billionnet in [1]. Since SP'$_j$ is a subcase of SP$_j$, the latter problem is NP-hard. The rest of the paper is devoted to showing that the subproblem (9) of finding $r_j^*(U)$ for $j \in J$ admits a Fully Polynomial-Time Approximation Scheme (FPTAS).

## 4 A FPTAS for the subproblem

The algorithm for approximating optimal ratio $r_j^*(U)$ of (9) is a two-phase algorithm. In the first step, a 2-approximation of the optimal ratio is found. In the second step, costs are scaled and rounded as in the approximation algorithms of Ibarra and Kim [9] and Lawler [15] for the Knapsack Problem or the algorithm of Hassin for the Constrained Shortest Path Problem [7], and a Dynamic Programming procedure is applied. Before describing more formally the algorithm, we need to introduce the following two lemmas.

**Lemma 2.** *Set* $\alpha_i^j = d_i(f_j/K_j) + c_{ij}$, *and let* $S_p = \{\alpha_{i_1}^j, \ldots, \alpha_{i_p}^j\}$, *for* $p = 1, \ldots, |U|$, *be the sorted list of* $p$ *smallest* $\alpha_i^j$ *values, i.e.* $\alpha_{i_l}^j \leq \alpha_{i_{l+1}}^j$ *for* $l = 1, \ldots, |U| - 1$. *Set* $S_q = arg \min_{1 \leq p \leq |U|} r_j(S_p)$. *Then* $r_j(S_q)/r_j^*(U) \leq 2$.

**Proof.** We note $L^*$ the optimal subset associated with $r_j^*(U)$ and $v(L) = \sum_{i \in L} \alpha_i^j$ for $L \subseteq U$. Then we have $v(L) \leq w(L) \leq v(L) + f_j$ for all $L \subseteq U$ (see (7) for the definition of

$w_j$). It comes:

$$\begin{aligned} r_j(S_q) \le r_j(S_{|L^*|}) &= w(S_{|L^*|})/|L^*| \\ &\le (v(S_{|L^*|}) + f_j)/|L^*| \\ &\le (v(L^*) + f_j)/|L^*| \quad \text{as } v(S_{|L^*|}) = \min_{|L|=|L^*|} v(L) \\ &\le (w(L^*) + f_j)/|L^*| = r_j^*(U) + f_j/|L^*| \\ &\le 2r_j^*(U) \quad \square \end{aligned}$$

**Lemma 3**. *Given $j \in J$ and non-negative real values $B$, $\hat{f}_j$ and $\hat{c}_{ij}$ for $i \in U$, the problem of minimizing*

$$\hat{r}_j(L) = \frac{\lceil \sum_{i \in L} d_i/K_j \rceil \hat{f}_j + \sum_{i \in L} \hat{c}_{ij}}{|L|} \tag{10}$$

*over subsets $L \subseteq U$ under the constraint $\hat{r}_j(L) \le B$ can be solved in time $O(|U|^3 B)$ by a Dynamic Programming procedure.*

**Proof.** Set $U = \{i_1, \ldots, i_{|U|}\}$ and $U_l = \{i_1, \ldots, i_l\}$ for $l = 1, \ldots, |U|$. Set

$$d^*(\hat{w}, l, p) = \min_{L \subseteq U_l} \{\sum_{i \in L} d_i / \quad \lceil \sum_{i \in L} d_i/K_j \rceil \hat{f}_j + \sum_{i \in L} \hat{c}_{ij} = \hat{w}; |L| = p\}$$

for $\hat{w} \in \{1, \ldots, |U|B\}, l \in \{1, \ldots, |U|\}, p \in \{0, \ldots, l\}$. This can be calculated by setting:

$$\begin{aligned} d^*(\hat{w}, l, 0) &= \begin{cases} 0 \text{ if } \hat{w} = 0 \\ +\infty \text{ otherwise} \end{cases} \qquad \text{for } l = 1, \ldots, |U| \\ d^*(\hat{w}, 1, 1) &= \begin{cases} d_1 \text{ if } \hat{w} = \hat{c}_{1j} + \lceil d_1/K_j \rceil \hat{f}_j, \\ +\infty \text{ otherwise} \end{cases} \end{aligned}$$

and for other triples $(\hat{w}, l, p)$,

$$\begin{aligned} d^*(\hat{w}, l, p) = \min(&d^*(\hat{w}, l-1, p), \\ &\left(d^*(\hat{w} - \hat{c}_{i_l j} - \lfloor d_{i_l}/K_j \rfloor \hat{f}_j, l-1, p-1) + d_{i_l}\right) z^0(\hat{w}, l, p), \\ &\left(d^*(\hat{w} - \hat{c}_{i_l j} - (\lfloor d_{i_l}/K_j \rfloor + 1)\hat{f}_j, l-1, p-1) + d_{i_l}\right) z^1(\hat{w}, l, p)) \end{aligned}$$

where, for $k = 0, 1$,

$$z^k(\hat{w}, l, p) = \begin{cases} 1 \text{ if } \quad \lceil (d^*(\hat{w} - \hat{c}_{i_l j} - \lfloor d_{i_l}/K_j \rfloor - k, l-1, p-1) \ + d_{i_l})/K_j \rceil \\ \qquad = \lceil d^*(\hat{w} - \hat{c}_{i_l j} - \lfloor d_{i_l}/K_j \rfloor - k, l-1, p-1)/K_j \rceil \ + \lfloor d_{i_l}/K_j \rfloor + k \\ +\infty \text{ otherwise} \end{cases}$$

We thus look for

$$\min_{\hat{w}, p \ge 1} \{\hat{w}/p : d^*(\hat{w}, n, p) < \infty\}$$

The complexity order of this Dynamic Programming procedure is the produce of the ranges of the three integer indexes $\hat{w}$, $l$ and $p$, hence the whole process runs in $O(|U|^3 B)$.$\square$

**Algorithm 2** /FPTAS for the subproblem/
*Begin*
Step 1. Let $\{\alpha^j_{i_1}, \ldots, \alpha^j_{i_{|U|}}\}$ be the list of coefficients $\alpha^j_i = d_i(f_j/K_j) + c_{ij}$ sorted by non-decreasing order
$S_p := \{\alpha^j_{i_1}, \ldots, \alpha^j_{i_p}\}$ for $p = 1, \ldots, |U|$
Compute $S_q = arg\min_{1\leq p\leq |U|} w(S_p)/p$
$R := r_j(S_q)$
Step 2. Set $\hat{f}_j = \lfloor f_j/(\epsilon R/4)\rfloor$ and $\hat{c}_{ij} = \lfloor c_{ij}/(\epsilon R/4)\rfloor$
Output subset $L_{DP}$ returned by the Dynamic Programming procedure of lemma 3 with upper bound $B = 2/\epsilon$
*End*

We now introduce algorithm 2 which approximates optimal ratio $r^*_j(U)$.

**Proposition 3.** *Algorithm 2 is a $(1+\epsilon)$-approximation of $r^*_j(U)$ running in $O(|U|^3/\epsilon)$.*

**Proof.** Combining $f_j \geq (\epsilon R/4)\hat{f}_j$ and $c_{ij} \geq (\epsilon R/4)\hat{c}_{ij}$ we obtain that

$$r_j(L^*) \geq (\epsilon R/4)\hat{r}_j(L^*) \geq (\epsilon R/4) \min_{L\subseteq U} \hat{r}_j(L) = (\epsilon R/4)\hat{r}_j(L_{DP})$$

Since $r_j(L^*) \leq 2R$ we get that

$$\hat{r}_j(L_{DP}) \leq 2/\epsilon \qquad (11)$$

which justifies that the upper bound $B$ is set to $2/\epsilon$ in $DP$. Now, we have:

$$\begin{aligned} r_j(L_{DP}) &= \frac{\lceil \sum_{i\in L_{DP}} d_i/K_j\rceil f_j + \sum^h_{i\in L_{DP}} c_{ij}}{|L_{DP}|} \\ &\leq \frac{\lceil \sum_{i\in L_{DP}} d_i/K_j\rceil \left(\lfloor f_j/(\epsilon R/4)\rfloor + 1\right)(\epsilon R/4) + \sum_{i\in L_{DP}} \left(\lfloor c_{ij}/(\epsilon R/4)\rfloor + 1\right)(\epsilon R/4)}{|L_{DP}|} \\ &= (\frac{\epsilon R}{4})\left(\hat{r}_j(L_{DP}) + \frac{|L_{DP}|+1}{|L_{DP}|}\right) \\ &\leq (R/2)(1+\epsilon) \qquad \text{by (11)} \\ &\leq r^*_j(U)(1+\epsilon) \qquad \text{by lemma 2} \end{aligned}$$

The complexity of step 1 of algorithm 2 is the time of sorting coefficients $\alpha^j_i$ for $i \in U$, which can be done in time $|U|\ln|U|$. The complexity of step 2 is $O(|U|^3 B) = O(|U|^3/\epsilon)$. Hence, the overall complexity of algorithm 2 is $O(|U|^3/\epsilon)$. □.

# 5 Conclusion

From propositions 1, 2 and 3 we derive the main result of the paper.

**Theorem 1.** *Algorithm 1 combined with FPTAS Algorithm 2 for subproblem (9) approximates SCFLP within ratio* $(1+\epsilon)H(n)$ *in computational time* $O(mn^4/\epsilon)$.

Since $H(n) \leq 1 + \ln n$, the gap to the inapproximability bound $\ln n$ of Feige [4] is reduced as close as possible. We can note that an adaptation of the partitioning algorithm of [7] to the SCFLP case would solve the subproblem in time $O((n^4/\epsilon)\log(n/\epsilon))$, which is significantly higher than $O(n^3/\epsilon)$. Finally, we could address the question whether such techniques could be applied to the classical Capacitated Facility Location Problem (CFLP) with hard capacities and obtain a $O(\ln n)$ ratio for this problem, which is to our knowledge an open problem. An important obstacle is that there is no trivial reformulation of CFLP as an equivalent SCP as in definition 1 since for CFLP, a cover cannot be systematically transformed into a feasible partition of inferior cost (joining two subsets, i.e., two sets of customers corresponding to the same facility $j$ may exceed the hard capacity of $j$). Another interesting issue is whether our algorithm could improve the best-known ratio of 2 for the metric SCFLP [17]. This is quite possible since a slightly-modified version of the greedy SCP-type procedure of Hochbaum, where opening cost is set to zero once a facility is open, was proved to achieve a pretty good approximation ratio of 1.81 for the metric UFLP [13]. This leaves several open problems for future research.

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