Severe Loss Probabilities in Portfolio Credit Risk Models

Babbs, Simon H and Johnson, Andrew E

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Simon H Babbs[a,*] Andrew E Johnson[b]

[a] *The Options Clearing Corporation, Chicago, IL, USA and University of Warwick, UK*
[b] *Royal Bank of Canada, London, UK*

Abstract

We derive explicit sharp bounds on the distribution of the number of defaults from a pool of obligors with common probability of default and default correlation. These bounds are extremely wide, implying that default probabilities and default correlations only very loosely determine probabilities of severe portfolio losses. Our results quantify and thereby reinforce Gordy's (2002) statement that "Capital decisions … depend on higher moments".

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1. Introduction

Since the mid-1990s, a number of models have appeared for the measurement of credit risks in portfolios of loans, bonds, or other financial instruments. Prominent examples are KMV (see Kealhofer, 1995), CreditMetrics (see Gupton, Finger and Bhatia, 1997), CreditRisk+ (see CSFP, 1997), and CreditPortfolioView (see Wilson, 1997a,b). Models of this kind are widely used by banks to estimate their need for capital to support the credit risks of commercial lending, and underlie aspects of proposed reforms to regulatory capital requirements, "Basel II". Estimates of capital requirements are generally based on an extreme percentile of the portfolio loss distribution produced by the model. It is not uncommon for banks to base their internal estimates of capital needs upon a percentile comparable to the historical probability that a very highly rated corporate bond will not default within one year, say 99.90% or 99.95%.[2] In well-diversified portfolios, the severe losses estimated at these percentiles would likely only occur in exceptional economic downturns. Direct empirical evidence concerning portfolio losses of this rarity is, therefore, practically unavailable. Against this background, it is helpful to consider the question: "how robust are the models' estimates of the probabilities of severe losses?"

Various researchers (see Gordy, 2000, Koyluoglu and Hickman, 1998) have shown that all the models cited above, although formulated from a range of starting points, can be mapped into a

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[1] Views expressed in this paper are those of the authors and not necessarily those of any organisation to which they are affiliated.

[*] Corresponding author. The Options Clearing Corporation, 1 N. Wacker Drive, Chicago, IL 60606, USA.
email: sbabbs@theocc.com

[2] Cf Exhibit 32 in Hamilton, Cantor and Ou (2002).

common mathematical form – at least if attention is restricted to default losses, excluding mark-to-market movements on non-defaulted claims. This mapping is most readily expressed in the simplified case of a homogeneous[3] portfolio subject to a single systemic risk factor[4]; the probability, $P(j,n)$, that exactly *j* out of a total of *n* obligors will default can be expressed as

$$P(j,n) = \int_0^1 \binom{n}{j} x^j (1-x)^{n-j} dF(x) \qquad (1)$$

Here, *x* is the conditional default rate given the state of the economy, with probability distribution *F(x)*. Given the state of the economy, individual obligors default or survive independently of one another with probability *x*. Frey and McNeil (2002) call models in which (1) applies (simple cases of) "Bernouilli mixture models"; we prefer to emphasize the economics rather than the mathematics, and will use the alternative term "conditional default rate models". Moreover, for a homogeneous portfolio, it can be shown that the distribution of the conditional default rate depends on just two parameters, which – after an appropriate transformation – can be taken to be the unconditional default probability, *q*, of obligors in the portfolio, and their "default correlation", ρ, namely the unconditional correlation $\text{correl}\left[Y_j, Y_k\right],\ j \neq k$ where $Y_k = 1$ if the *k*th obligor defaults and zero otherwise. (For completeness, Appendix B derives the density of the conditional default rate for each of the cited models, and gives details of how to transform the parameters into *q* and ρ.)

Not only do the cited models share this structural similarity, but also, as various authors have shown, if the cited models are calibrated to common values of *q* and ρ, then they yield broadly similar estimates of the adverse tail of the loss distribution (see eg Finger, 1999, Koyluoglu and Hickman, 1998, and Frey and McNeil, 2003; see our Table 1 for illustrative numbers).

Extrapolating from these findings, one might be tempted to draw the naïve conclusions that perhaps all "reasonable" portfolio credit models would yield similar broadly adverse tails, and that the distribution of the number of defaults in a portfolio is quite well determined by its first two moments. The first part of this naïve conclusion is undermined by Gordy (2000) (pp142-3), who discusses a modified CreditRisk+ model, and by Frey and McNeil (2002) (pp1329-31), who use multivariate *t* variables in place of the multivariate Gaussian latent variables in CreditMetrics. These studies also illustrate the point that the distribution of the number of defaults even in a homogeneous portfolio is not wholly determined by its first two moments. Indeed Gordy warns (p145) that "Capital decisions … depend on higher moments … These cannot be estimated with any precision given available data."

The main practical contribution of this paper may be to reinforce the warning by Gordy, by quantifying the dependence upon higher moments. We do this by deriving explicit bounds for the probability that at least *m* out of *n* homogeneous obligors will default, given a common default probability *q* and a common pairwise default correlation ρ. We do this first in a very general model, in which a conditional default rate may not exist (ie for which (1) does not hold), and then for conditional default rate models. In purely mathematical terms, we provide explicit sharp bounds on the distribution of sums of dependent binomial random variables with common

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[3] By "homogeneous" we mean that, given any condition of the economy, the obligors will share a common default probability.

[4] In CreditPortfolioView, the risk factor is expressed as a linear combination of macroeconomic variables. In the other models, the risk factor(s) are latent, and may be one or many depending on a correlation structure.

marginal probabilities of occurrence and common covariances; these results may be of independent interest. Plugging plausible parameter values into our formulae, we show that the bounds are extremely wide; thus the first two moments of the number of defaults only extremely loosely constrain the probabilities of severe portfolio losses.

The structure of the rest of this paper is as follows. In Section 2, we formulate the problem and decide to restrict our attention to probability distributions with a certain symmetry property. Section 3 is devoted to determining the range of possible values that tail probabilities could take, consistent with given default probabilities and default correlations, but dropping other distributional assumptions. In Section 4, we obtain the range of possible tail probabilities under the restriction that the probability distributions possess a conditional default rate. Section 5 gives some illustrative numerical results, which reveal that the ranges we have determined are extremely wide. Section 6 contains brief concluding remarks. We relegate proofs to Appendix A.

2. Formulating the problem

Consider a portfolio of claims on *n* homogeneous obligors. For $i = 1,\ldots,n$ the variable X_i takes the value 1, to signify that the *i*th obligor defaults, with probability *q*; otherwise $X_i = 0$. For any distinct i, j the correlation between X_i and X_j equals $\rho \geq 0$. Given this information alone, what can we say about the distribution of the total number of defaults, $S_n \equiv X_1 + \ldots + X_n$?

We note immediately that the information determines the first two moments of the total number of defaults, since it is trivial to show that:

$$\mathrm{E}[S_n] = nq;\ \mathrm{var}[S_n] = n(1-q)q\{1+(n-1)\rho\} \tag{2}$$

It can also readily be shown that, for $n \leq 3$, all probability distributions of the default outcomes are symmetric in the sense that the probability of any outcome in which $S_n = m$ (corresponding to exactly *m* obligors defaulting) equals the probability of any other such outcome.

For $n > 3$, however, asymmetric solutions can occur. For example, for $n = 4$ with $q = 10\%$, $\rho = 5\%$, and defining $K = \sum_{j=1}^{4} 2^{j-1} X_j$ and $p(k) = prob\{K = k\}$, one solution is given by:

$$p(0) = 68.30\%;\ p(1) = p(2) = p(8) = 6.05\%;\ p(3) = p(9) = p(10) = 1.05\%;\ p(4) = 5.65\%;$$
$$p(5) = p(6) = p(12) = 1.45\%; p(7) = p(13) = p(14) = p(15) = 0.00\%; p(11) = 0.40\%$$

This solution is asymmetric, as $X_1 = X_2 = X_4 = 1$ is the only case with strictly positive probability corresponding to 3 or more defaults.

In the sequel, we will confine attention to "symmetrical" solutions, ie probability assignments that view as equally likely any two outcomes in which exactly the same number of obligors default.

3. General symmetrical solutions

We focus upon the possible range of tail probabilities consistent with given default probabilities and default correlations. For symmetrical solutions, these probabilities are a function of the number, *m*, of defaulting obligors, but not of their identities. Thus, the probability that at least *m* out of *n* obligors default is $T(m,n)$ given by

$$T(m,n) = \sum_{j=m}^{n} P(j,n) \tag{3}$$

while the applicable constraints are that $P(j,n)\geq 0,\ j=0,\ldots,n$ and – by (2) - that:

$$\sum_{j=0}^{n} P(j,n)=1;\ \sum_{j=0}^{n} j\,P(j,n)=nq;\ \sum_{j=0}^{n} j^2\,P(j,n)=nq\{1+(n-1)[q+(1-q)\rho]\} \tag{4}$$

To determine the range of possible tail probabilities we must maximize and minimize $T(m,n)$ subject to (4). It turns out that these maximization and minimization problems can be solved analytically:

PROPOSITION: 3.1 *For* $m\leq(n-1)(1-\rho)q$ *the maximum achievable value of* $T(m,n)$ *is* 1.

For $(n-1)(1-\rho)q<m<1+(n-1)[q+(1-q)\rho]$ *, the maximum achievable value of* $T(m,n)$ *is*

$$\left\{1+\frac{(n-1)(1-\rho)(1-q)}{m}\right\}q \tag{5}$$

achieved when $P(j,n)$ *is zero except at most for* $j\in\{0,m,n\}$**.**

Finally, for the most extreme values of m *, ie* $1+(n-1)[q+(1-q)\rho]\leq m\leq n$**,** *the maximum achievable value of* $T(m,n)$ *is:*

$$\max\left\{\frac{(k+1-nq)(k-nq)+nq(1-q)[1+(n-1)\rho]}{(m-k-1)(m-k)} : 0\leq \text{integer } k\in[G-1,G]\right\} \tag{6}$$

where

$$G=\frac{m-1-(n-1)[q+(1-q)\rho]}{m-nq}nq \tag{7}$$

and where the maximum is achieved when $P(j,n)$ *is zero except at most for* $j\in\{k^*,k^*+1,m\}$ *where* $k^*\in[G-1,G]$ *maximizes the expression inside* (6).

PROPOSITION: 3.2 *For* $m\leq 1+(n-1)(1-\rho)q$ *the minimum value of* $T(m,n)$ *is*

$$\min\left\{\frac{(nq-m+1)[2(k+1)-m-nq]-nq(1-q)[1+(n-1)\rho]}{(k-m+2)(k-m+1)}\text{: integer } k\in[H-1,H]<n\right\} \tag{8}$$

where

$$H=\frac{2-m+(n-1)[q+(1-q)\rho]}{nq+1-m}nq \tag{9}$$

and where the minimum is achieved when $P(j,n)$ *is zero except at most for* $j\in\{m-1,k^*,k^*+1\}$ *where* $k^*\in[H-1,H]$ *minimises the expression inside* (9).

For $1+(n-1)(1-\rho)q<m<2+(n-1)[q+(1-q)\rho]$ *the minimum value of* $T(m,n)$ *is:*

$$\left\{1-\frac{(n-1)(1-\rho)(1-q)}{n-m+1}\right\}q \tag{10}$$

achieved when $P(j,n)$ *is zero except at most for* $j\in\{0,m-1,n\}$**.**

Finally, for the most extreme values of m *, ie for* $2+(n-1)[\rho+(1-\rho)q]\leq m\leq n$*, the minimum achievable value of* $T(m,n)$ *is zero.*

We observe that the maximum tail probabilities obtained in this section correspond to highly pathological cases, since they correspond to a situation in which there are but three possible outturns for the number of defaults. However, the solutions corresponding to these maximum tail probabilities are limiting cases of solutions in which there are a greater number of possible outturns.

A more subtle observation would be that, while he KMV/CreditMetrics, CreditRisk+ and CreditPortfolioView models depend on distributional assumptions which may be unfounded, a superior model might be required to retain the property of being a conditional default rate model, ie if satisfying (1). At the very least, conditional default rate models are an important class. In the next section therefore, we consider the range of possible tail probabilities when the solution is constrained to correspond to a conditional default rate model.

4. Symmetrical solutions with a conditional default rate

We now restrict our attention to the family of solutions in which (1) holds. To eliminate pathological distributions, we further restrict ourselves to the family of solutions in which the distribution, $F(\bullet)$, of the conditional default rate, possesses a density, $F'(\bullet)$, except on a set of Lebesgue measure zero. We noted earlier that KMV, CreditMetrics, CreditRisk+ and CreditPortfolioView are all models of this kind. The constraints expressed in (4) still apply, exactly as stated.

It turns out that the maximum (minimum) tail probabilities in this family of solutions are obtained when the distribution $F(\bullet)$ takes the (limiting) form of two point masses:

PROPOSITION: 4.1 *For any $m > nq$, the maximum (minimum) achievable value of $T(m,n)$ is obtained when $F(\bullet)$ corresponds to a point mass b at some $x_1 \in [0,q)$, and a second point mass $1-b$ at $x_2 \in (q,1]$, where:*

$$b = \frac{\rho q(1-q)}{(q-x_1)^2 + \rho q(1-q)} \quad ; \quad x_2 = \frac{q - b x_1}{1-b}$$

and x_1 is chosen to maximize (minimize)

$$T(m,n) = \sum_{j=m}^{n} \binom{n}{j} \left[b x_1^j (1-x_1)^{n-j} + (1-b) x_2^j (1-x_2)^{n-j} \right]$$

subject to $x_2 \le 1$.

While it would be pathological for the distribution of the conditional default rate to consist of a small number of point masses, such solutions can be viewed as limiting cases of distributions admitting densities.

5. Numerical illustration

PROBABILITIES OF VARIOUS NUMBERS OF DEFAULTS

Number of Defaults	Minimum possible	KMV / CM	CR+	CPV	Maximum possible with CDR	Maximum possible
0	0	2.1%	5.1%	0.4%	59.3%	59.6%
100 or more	0.1%	14.4%	15.2%	13.0%	40.3%	48.8%
200 or more	0	3.4%	3.3%	3.3%	10.5%	14.1%
500 or more	0	0.05%	0.04%	0.11%	1.5%	1.8%
750 or more	0	0.0004%	0.0012%	0.0029%	0.7%	0.7%
All 1000	0	0.00000%	0.00000%	0.00000%	0.4%	0.4%

Table 1: Probabilities of various numbers of defaults in a portfolio of 1000 credits in which each credit has probability of default q=5%, and pairwise correlations of default are all ρ=7.66%. The value of q lies between the 5yr probabilities of default of credits rated Baa and Ba by Moody's and between the 1yr probabilities for Ba and B-rated credits. The value of ρ corresponds to an *asset* correlation of 25% in a KMV/CM-type model. As well as results for KMV / CreditMetrics (CM), CreditRisk+ (CR+) and CreditPortfolioView (CV) models, the table shows the minimum and maximum possible values for each of the probabilities (as given in Proposition: 3.1 and Proposition: 3.2) of various numbers of defaults. The column headed "CDR" gives the maximum possible probabilities in the family of solutions with a conditional default rate.

Table 1 provides illustrative results. We see that the range of possible tail probabilities is extremely wide, even if we demand that the model should possess a conditional default rate.[5]

Bearing in mind that the various columns in the table all correspond to a situation in which – by (4) - the expected number of defaults is 50, with standard deviation 60.7, the results in the table show starkly just how loosely the portfolio outcome is constrained by its first two moments.

6. Concluding remarks

We have obtained explicit sharp bounds on the distribution of the number of defaults in a homogeneous pool of obligors with common default probability and default correlation, both in general and also under the restriction that the model must possess a conditional default rate. Numerical illustration of our results reveals that the bounds are extremely wide. This suggests that the numerical estimates of tail probabilities for large losses in models of portfolio credit risk owe far less than one might have anticipated to their inputs relating to probability of default and correlations between obligors, and a very great deal to their particular distributional assumptions. An alternative formulation of this last point would be that tail probabilities can only be determined with any precision by the higher moments of the distribution. Our results both quantify and reinforce Gordy's (2000) remark that "Capital decisions … depend on higher

[5] It is interesting that the maximum possible solution has a finite probability for all obligors defaulting. Indeed, as $n \to \infty$, for q = 5% and ρ = 7.66%, the maximum possible probability of all n companies defaulting converges to 0.4%.

moments". As he went on to observe, "These cannot be estimated with any precision given available data." We hope that our findings will spur further research in this challenging but necessary area.

Appendix A

TECHNICAL LEMMA 1

(a) There exists a set of probabilities satisfying the constraints (4) and such that $P(j,n)=0\ \forall j \geq k$ if and only if $k \geq 2+(n-1)[q+(1-q)\rho]$

(b) There exists a set of probabilities satisfying the constraints (4) and such that $P(j,n)=0\ \forall j < k$ if and only if $k \leq (n-1)(1-\rho)q$

Proof (a) We commence by seeking to maximise $\sum_{j=1}^{k-1} j^2 P(j,n)$ subject to the constraints $\sum_{j=0}^{k-1} P(j,n)=1$ and $\sum_{j=0}^{k-1} j\,P(j,n)=nq$. To avoid dealing with inequality constraints, define $r_j \equiv \sqrt{P(j,n)}$. Setting up the problem with Lagrange multipliers as usual, we obtain the first and second order conditions:

$$\left.\begin{array}{ll}\{j^2+\lambda_0+\lambda_1 j\}r_j & =0\\ j^2+\lambda_0+\lambda_1 j & \leq 0\end{array}\right\},\quad j=0,\ldots,k-1$$

which together imply that the only non-zero values of r_j - and thus $P(j,n)$ - occur for $j \in \{0,k-1\}$. We can then solve the constraints Solving the constraints $\sum_{j=0}^{k-1} P(j,n)=1$ and $\sum_{j=0}^{k-1} j\,P(j,n)=nq$ directly and obtain $\sum_{j=0}^{k-1} j^2 P(j,n)=(k-1)nq$. But the value for the probability-weighted sum of squares falls short of that required by the constraints (4) unless $k \geq 2+(n-1)[q+(1-q)\rho]$. This proves necessity.

For sufficiency, note that $2+(n-1)[q+(1-q)\rho] > nq+1 \geq \lfloor nq \rfloor +1$. Thus the probability allocation $P(\lfloor nq \rfloor+1,n)=1-P(\lfloor nq \rfloor,n)=nq-\lfloor nq \rfloor$, which minimizes the variance – and hence, given the mean constraint, minimizes the probability-weighted sum of squares – is feasible for the maximization problem we have just considered. A weighted combination of this solution and of the maximizing solution will meet the sum of squares constraint in (4), and have all the probability mass attached to values of j in $\{0,\ldots,k-1\}$.

The proof of (b) define proceeds similarly, based on seeking to maximize $\sum_{j=k}^{n} j^2 P(j,n)$, which is achieved with the probability allocation $P(n,n)=1-P(k,n)=(nq-k)/(n-k)$. ■

Proof of Proposition: 3.1 By part (a) of Technical Lemma 1 above, it is easy to see that the maximum tail probability for $m \leq (n-1)(1-\rho)q$ is unity.

For larger values of m we proceed as follows. Then, once more defining $r_j \equiv \sqrt{P(j,n)}$, we are seeking to maximise $\sum_{j=m}^{n} r_j^2$ subject to:

$$\sum_{j=0}^{n} r_j^2 = 1; \quad \sum_{j=0}^{n} j r_j^2 = nq \equiv M; \quad \sum_{j=0}^{n} j^2 r_j^2 = nq\{1+(n-1)[\rho+(1-\rho)q]\} \equiv V + M^2$$

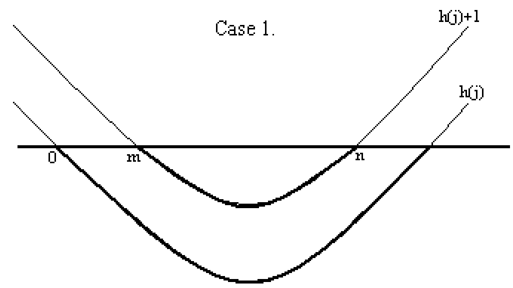
Setting up the problem with Lagrange multipliers as usual, we find that the first and second order conditions for a maximum are respectively:

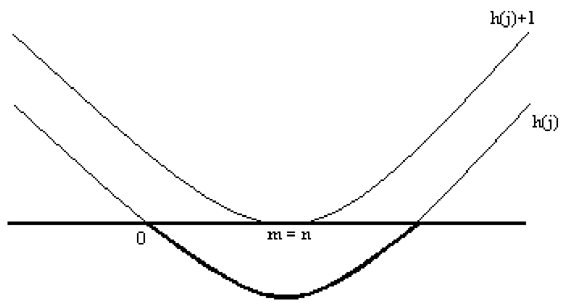
$$\{1_{j\geq m} + h(j)\} r_j = 0; \; 1_{j\geq m} + h(j) \leq 0$$

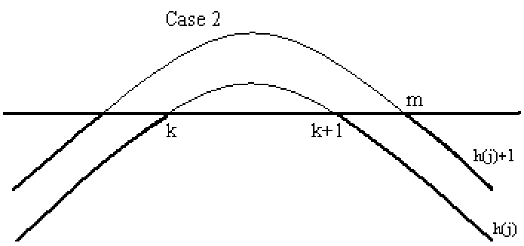
where $h(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2$

Recall that CM etc. provide a feasible solution in which no r_j vanishes. Hence, at the maximum, $\exists\, j \in \{m,\ldots,n\}: 1 + h(j) = 0$. However, $h(\bullet) \equiv -1$ cannot be a solution as this would imply there are no $j \in \{0,\ldots,m-1\}: h(j) = 0$, whereas, by Technical Lemma 1, we must have at least one non-zero probability with $j < m$. Thus we must have at least one solution to $1 + h(j) = 0$ for $j \in \{m,\ldots n\}$, and of $h(j) = 0$ for $j \in \{0,\ldots,m-1\}$.

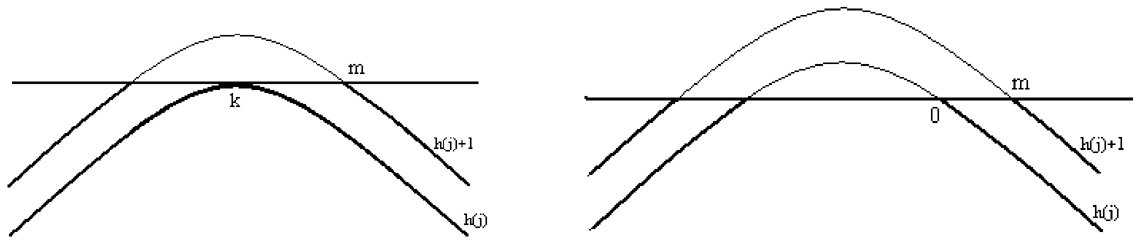
Consideration of the constraints now leads to the conclusion that possible solutions must involve r_j vanishing except at a set of values of j of the form $\{0, m, n\}$ or $\{k, k+1, m\}$. The graphs below may help the reader establish this for him/herself.[6]







[6] If $h(\bullet)$ were linear, then we would need $h(0) = 0$, $h(m) = -1$, leading to a special instance of Case 2.



Given that there at most three non-zero probabilities located at $j_1 > j_2 > j_3$ (say), the sole solution to the constraints is:

$$r_{j_1}^2 = \frac{(j_2 - M)(j_3 - M) + V}{(j_1 - j_2)(j_1 - j_3)};\ r_{j_2}^2 = \frac{(j_1 - M)(M - j_3) - V}{(j_1 - j_2)(j_2 - j_3)};\ r_{j_3}^2 = \frac{(j_1 - M)(j_2 - M) + V}{(j_1 - j_3)(j_2 - j_3)}$$

We now examine the cases identified above.

Case 1: $j_1 = n,\ j_2 = m,\ j_3 = 0$

Substituting these values implies that $V + M^2 > Mm$, ie $m < 1 + (n-1)[\rho + (1-\rho)q]$, must hold for all the probabilities to be non-negative. If this condition is satisfied, the corresponding candidate maximum of our objective function is

$$r_n^2 + r_m^2 = \frac{(m-M)(-M)+V}{(n-m)n} + \frac{(n-M)M - V}{(n-m)m} = \frac{(n+m)M - \{V + M^2\}}{mn} = \left\{1 + \frac{(n-1)(1-\rho)(1-q)}{m}\right\}q$$

Case 2: $j_1 = m,\ j_2 = k+1,\ j_3 = k < M$

For these values of j_1, j_2, j_3, only $r_m^2 = \dfrac{(k+1-M)(k-M)+V}{(m-k-1)(m-k)}$ contributes to $T(m,n)$.

The constraints that r_k, and r_{k+1} are non-negative imply

$$G - 1 = M - \frac{V}{m-M} - 1 \le k \le M - \frac{V}{m-M} = G$$

where G is given by (7). If $G < 0$, corresponding to $m < 1 + (n-1)[q + (1-q)\rho]$, this solution is unavailable. Otherwise, our candidate[7] values of k are any non-negative integers lying in the interval $[G-1, G]$, and:

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[7]For such k we will always have $r_m^2 \ge 0$. To see this note that r_m^2 is non-negative iff $k^2 + (1-2M)k + V + M^2 - M \ge 0$. If $V > \frac{1}{4}$, this quadratic has no roots and is always positive. If $V \le \frac{1}{4}$; it has roots at $M - \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4V} \approx M - 1 + V, M - V$. However, one can show that (excluding $q > \frac{1}{2}$ by assumption) $V \le \frac{1}{4}$ implies $nq < \frac{1}{2}$; in that case the only possible value for k is 0, and it is then easy to show that r_m^2 is positive.

$$P(m,n)=\frac{(k+1-nq)(k-nq)+nq(1-q)[1+(n-1)\rho]}{(m-k-1)(m-k)}$$

$$P(k+1,n)=\frac{(m-nq)(nq-k)-nq(1-q)[1+(n-1)\rho]}{(m-k-1)}$$

$$P(k,n)=\frac{(m-nq)(k+1-nq)+nq(1-q)[1+(n-1)\rho]}{(m-k)}$$

Unless G is an integer, there will be one such k. If G is an integer then $k=G-1$ and $k=G$ will both need to be considered and the non-negative k giving the larger value of $P(m,n)$ chosen.

The result follows. ■

Proof of Proposition: 3.2 By part (b) of Technical Lemma 1 above, it is easy to see that the minimum tail probability for $m \geq 2+(n-1)[q+(1-q)\rho]$ is zero.

For smaller values of m we consider the maximization of the complementary probability $\sum_{j=0}^{m-1} P(j,n)$ subject to the constraints (4). An analysis analogous to that in the proof of Proposition: 3.1 establishes that two cases arise, in each of which there are at most three non-zero probabilities, $P(j_1,n), P(j_2,n), P(j_3,n)$.

Case 1: $j_1=n$, $j_2=m-1$, $j_3=0$

Since each probability must, of course, be non-negative, this case can only obtain when $2+(n-1)[q+(1-q)\rho] \geq m \geq 1+(n-1)(1-\rho)q$. Moreover, only

$$r_n^2=\frac{(m-1-M)(-M)+V}{(n-m+1)n}=\frac{2+(n-1)[q+(1-q)\rho]-m}{n-m+1}q$$

contributes to the tail.

Case 2: $j_1=k+1$, $j_2=k \geq m$, $j_3=m-1$

For this case,

$$r_k^2=\frac{(k+1-M)(M-m+1)-V}{k-m+1}$$

which is certainly negative if $m \geq M+1$. We therefore restrict ourselves to $m < M+1$.

Now

$$r_{k+1}^2=\frac{(k-M)(m-1-M)+V}{k-m+2};\; r_{m-1}^2=\frac{(k+1-M)(k-M)+V}{(k-m+2)(k-m+1)}$$

and we can immediately note that the numerator in the expression for r_{m-1}^2 exceeds that in the expression for r_{k+1}^2, since $k \geq m$. Non-negativity of probabilities therefore requires precisely $k \in [H-1,H]$ where:

$$H \equiv M+\frac{V}{M+1-m}=\frac{2+(n-1)[q+(1-q)\rho]-m}{nq+1-m}nq$$

The obvious restriction $j_1 = k+1 \le n$ now entails $H \le n$, which, in turn, is equivalent to $m \le 1+(n-1)(1-\rho)q$.

We conclude that, for $m \le 1+(n-1)(1-\rho)q$ the minimum tail probability is

$$r_k^2 + r_{k+1}^2 = \frac{-(k-M)(M-m+1)+V}{k-m+2} + \frac{(k+1-M)(M-m+1)-V}{k-m+1}$$
$$= \frac{(nq-m+1)[2(k+1)-m-nq]-nq(1-q)[1+(n-1)\rho]}{(k-m+2)(k-m+1)}$$

where we select an integer $k \in [H-1, H]$ strictly less than *n*, to minimise this expression..

The result follows. ∎

Proof of Proposition: 4.1 Setting up the problem with Lagrange multipliers in the standard way, we define:

$$H \equiv T(m,n) - \lambda_0 \left\{ \int_{[0,1]} dF(x) - 1 \right\} - \lambda_1 \left\{ \int_{[0,1]} x dF(x) - q \right\} - \lambda_2 \left\{ \int_{[0,1]} x^2 dF(x) - \rho q(1-q) - q^2 \right\}$$

Then, for an extremum, we must satisfy the first order condition

$$\sum_{j=m}^{n} \binom{n}{j} x^j (1-x)^{n-j} = \lambda_0 + \lambda_1 x + \lambda_2 x^2$$

at any *x* at which $F'(x) \vee \Delta F(x) > 0$. But the LHS of the above condition is a polynomial of order *n* while the RHS is at most quadratic. Hence the condition can only be satisfied at a finite number of points – certainly a set of Lebesgue measure zero. Thus $F(\bullet)$ must be composed of a finite number of point masses.

For a maximum (minimum) each point mass must satisfy the following first and second order conditions with respect to its position:

$$\frac{d}{dx} \sum_{j=m}^{n} \binom{n}{j} x^j (1-x)^{n-j} = \lambda_1 + 2\lambda_2 x$$
$$\frac{d^2}{dx^2} \sum_{j=m}^{n} \binom{n}{j} x^j (1-x)^{n-j} \le (\ge)\ 2\lambda_2$$

Thus the point masses must occur at points at which the function $\frac{d}{dx} \sum_{j=m}^{n} \binom{n}{j} x^j (1-x)^{n-j}$ intersects the line $\lambda_1 + 2\lambda_2 x$ from below (above). Now, by Technical Lemma 2 below, $\frac{d}{dx} \sum_{j=m}^{n} \binom{n}{j} x^j (1-x)^{n-j}$ turns out to be a bell-shaped curve on $[0,1]$, and can therefore have at most two such intersections. Since meeting the constraints is easily shown to require at least two point masses, there must be exactly two, with one at some $x_1 < q$.

The rest of the proof is trivial. ∎

TECHNICAL LEMMA 2

$$\frac{d}{dx} \left[\sum_{j \ge m} \binom{n}{j} x^j (1-x)^{n-j} \right] = (n-m+1) \binom{n}{m-1} x^{m-1} (1-x)^{n-m}$$

Proof The proof is easy by induction. The lemma holds for $m = n$: $\frac{d}{dx}\left[\binom{n}{n}x^n\right] = nx^{n-1}$. Now suppose it is true for $m = k+1$. Then:

$$\frac{d}{dx}\left[\sum_{j\geq k}\binom{n}{j}x^j(1-x)^{n-j}\right] = \frac{d}{dx}\binom{n}{k}x^k(1-x)^{n-k} + (n-k)\binom{n}{k}x^k(1-x)^{n-k-1}$$

$$= \binom{n}{k}x^{k-1}(1-x)^{n-k-1}\left[k(1-x)-(n-k)x+(n-k)x\right] = k\binom{n}{k}x^{k-1}(1-x)^{n-k}$$

$$= (n-k+1)\binom{n}{k-1}x^{k-1}(1-x)^{n-k}$$

as required. ■

Appendix B[8]

We discuss CreditMetrics (CM), KMV, CreditRisk+ (CR+) and CreditPortfolioView (CPV). In CR+, the only credit event considered is default. By contrast, CM, KMV and CPV also consider the impact on portfolio value of upward and downward variations in obligors' credit quality short of default. We analyse the simplified case in which default is the sole credit event. We also confine our attention to a simplified portfolio, consisting of equally sized exposures to *n* essentially identical obligors, and assume that any variation in recovery rates in the event of default will average out; this leaves the number of defaults as the determinant of credit losses.

CREDITMETRICS AND KMV

Each obligor is assigned a "default probability". In CM, the default probability is a function of the obligor's credit rating, and is based on the frequency of past defaults by obligors initially having the same rating. In KMV, a default probability is assigned on a continuous scale, based on an analysis of the structure of the obligor's balance sheet and its stock price history, and of the frequency of past defaults by obligors initially having similar values of a key statistic in KMV's analysis: "distance from default".

In order to address the joint behaviour of multiple obligors, CM and KMV both assume that, for each obligor, the stock price and the credit quality are both driven by a single latent state variable. They further assume that the set of these state variables is multivariate Gaussian. This distributional assumption plays two key roles. Firstly, it implies that the correlation between movements in the obligors' stock prices ("asset correlation") equals the correlation between the latent variables determining credit quality. The asset correlation and the default probabilities then determine the correlation between the occurrences of defaults (the "default correlation"). Secondly, given the multivariate Gaussian assumption, the degree of association between defaults of <u>any number</u> of obligors is wholly determined by the individual default probabilities and by the correlations, even though correlation itself is, of course, <u>only a pairwise</u> measure of association.

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[8] Most of the material in this Appendix is a distillation of elements of Vasicek (1998), Koyluoglu and Hickman (1998), and Gordy (2000).

For a portfolio involving essentially identical obligors, let q denote the default probability of each obligor, and let α be the "asset correlation", assumed strictly positive, between stock price movements of any pair of distinct obligors. Then we can represent the latent variables determining the future credit quality of the *n* obligors by a multivariate Gaussian random variate, $Y_1,\dots,Y_n$, where: each Y_i is distributed standard univariate Normal; the *i*th obligor defaults if

$$Y_i \le w = \Phi^{-1}(q) \tag{11}$$

[$\Phi(\bullet)$ being the univariate standard Normal distribution function], and $\text{correl}\left[Y_i, Y_j\right] = \alpha,\ \forall i \ne j$

Defining X_i to equal one if the *i*th obligor defaults and zero otherwise, we have that:

$$prob\{X_i = 1\} = q \tag{12}$$

and the (pairwise) "default correlation", ρ, between the defaults of any two distinct obligors, is given by:

$$\rho \equiv \text{correl}\left[X_i, X_j\right] = \frac{prob\{Y_i \le w \text{ AND } Y_j \le w\} - q^2}{(1-q)q} = \frac{\int_{-\infty}^{w} \Phi\left(\frac{w-\alpha y}{\sqrt{1-\alpha^2}}\right) d\Phi(y)\ - q^2}{(1-q)q} \tag{13}$$

From now on, we assume that the default probability is less than 50%, and that "asset correlations" are strictly positive: $q < \frac{1}{2},\ \alpha > 0$. These conditions will almost invariably be satisfied in practice. With this assumption, it is easy to show from (13) that the relationship between α and ρ is strictly monotonic. Thus the model is effectively parameterised by q and ρ.

We now seek an expression for the distribution of the number of defaults. Given that a multivariate Normal distribution is wholly determined by its first two moments, we can express the latent variable for the *i*th obligor as:

$$L_i = \sqrt{\alpha}\, Z_0 + \sqrt{1-\alpha}\, Z_i\ ;\ \ i = 1,\dots,n \tag{14}$$

where $Z_0, Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$. It is now immediately clear that, conditional upon $Z_0 = z$, the occurrence (or non-occurrence) of default of the various obligors are the outcomes of Bernoulli trials with probability $\Phi\left(\frac{w-\sqrt{\alpha}\, z}{\sqrt{1-\alpha}}\right)$. Hence the probability that exactly *j* out of the *n* obligors default is $P(j,n)$ given by:

$$P(j,n) = \int_{\Re} \binom{n}{j} \Phi^j\left(\frac{w-\sqrt{\alpha}\, z}{\sqrt{1-\alpha}}\right)\left\{1-\Phi\left(\frac{w-\sqrt{\alpha}\, z}{\sqrt{1-\alpha}}\right)\right\}^{n-j} d\Phi(z)$$

which, by means of the change of variable $x = \Phi\left(\frac{w-\sqrt{\alpha}\, z}{\sqrt{1-\alpha}}\right)$, we may re-write as

$$P(j,n) = \int_0^1 \binom{n}{j} x^j (1-x)^{n-j} f(x)\,dx \tag{15}$$

where

$$f(x)=\frac{\sqrt{1-\alpha}\,\varphi\left(\dfrac{w-\sqrt{1-\alpha}\,\Phi^{-1}(x)}{\sqrt{\alpha}}\right)}{\sqrt{\alpha}\;\;\varphi\left(\Phi^{-1}(x)\right)}$$

$$=\sqrt{\frac{1-\alpha}{\alpha}}\exp\left\{\frac{(2\alpha-1)\left\{\Phi^{-1}(x)\right\}^2+2w\sqrt{1-\alpha}\Phi^{-1}(x)-w^2}{2\alpha}\right\} \tag{16}$$

CREDITRISK+

Each obligor is assigned a default probability, as an unconstrained user input. The default probability assigned to each obligor is regarded as the mean of a distribution of possible future "default rates". Conditional on the outturn default rate, the default (or non-default) of each obligor is assumed to be the outcome of a Bernoulli trial with probability set equal to the outturn default rate.

CR+ further assumes that each outturn default rate is drawn from a Gamma distribution.[9] A Gamma distribution is uniquely determined by its mean – the (unconditional) default probability – and its standard deviation – which CR+ calls "default volatility". This distributional assumption implies that the degree of association between the defaults of any number of obligors is wholly determined by the default probability and the "default volatility", even though the latter, being a standard deviation, itself applies, of course, only to an individual random variable.

With the notation employed earlier, we once more have (12), while the analogue of (13) is:

$$\rho=\frac{\int_{[0,1]} x^2\,dF(x)-q^2}{\mathrm{E}\left[X_i^2\right]-q^2}=\frac{\sigma^2}{\mathrm{E}\left[X_i\right]-q^2}=\frac{\sigma^2}{(1-q)q} \tag{17}$$

where $F(\bullet)$ is the distribution function of the conditional default rate. (Note that while CR+ assumes that $F(\bullet)$ is based on a Gamma distribution, (17) would also hold for any other distribution with variance σ.) It is obvious from (17) that the relationship between σ and ρ is monotonic. Thus we can think of CR+, like CM and KMV, as being parameterised by q and ρ.

Turning to the distribution of the number of defaults, it is immediate, given the structure of CR+, that (15) holds for this model also[10], but with $f(\bullet)$ replaced by the Gamma density:[11]

$$f(x)=\frac{1}{\Gamma\left(q^2/\sigma^2\right)\left\{\sigma^2/q\right\}^{q^2/\sigma^2}}\,x^{q^2/\sigma^2-1}\exp\left\{-qx/\sigma^2\right\} \tag{18}$$

---

[9] CR+ ignores the fact that the support of a Gamma density is not confined to [0,1]. For realistic parameter values, the effect of this is minute; moreover, it could easily be removed by truncating the density and rescaling so that it integrates to unity over the unit interval.

[10] We will use the exact distribution in this paper, in place of the approximations employed by CSFP (1997).

[11] Like CR+, we are ignoring the rescaling required by truncating the density to the interval [0,1]. To rescale, divide the RHS of (18) by its integral over that interval. For most reasonable parameter values, rescaling is insignificant.

CREDITPORTFOLIOVIEW

CPV assumes a distribution of possible future default rates according to a logit model based on macroeconomic factors. Conditional on the outturn default rate, the default (or non-default) of each obligor is assumed to be the outcome a Bernoulli trial with probability set equal to the outturn default rate. More specifically, the outturn default rate is assumed to take the form:

$$x = \frac{1}{1+e^y}$$

where y is a "macroeconomic index" expressed as a weighted sum of macroeconomic variables (and possibly lags thereof), with Normally distributed innovations. Thus y is itself Normally distributed.

It follows that $P(m,n)$, the probability that exactly m out of n obligors default, is given by (15) for this model also, with the conditional default rate having density $f(\bullet)$ over $(0,1)$ given by:

$$f(x) = \frac{1}{(1-x)x\sigma}\varphi\left(\frac{1}{\sigma}\left[\ln\left\{\frac{1}{x}-1\right\}-\mu\right]\right) \tag{19}$$

where μ,σ are respectively the mean and standard deviation of the macroeconomic index, y. The unconditional default probability, and (pairwise) default correlations are given by:

$$q = \int_0^1 x\, f(x)dx \tag{20}$$

$$\rho = \frac{\int_0^1 x^2 f(x)dx - q^2}{(1-q)q} \tag{21}$$

We may regard CPV also as parameterised in terms of q and ρ, because, at least for reasonable combinations of (q,ρ), the mapping which (20) and (21) define from the pair (μ,σ) to (q,ρ) has a unique inverse. To see this consider Figure 1, in which, for various levels of q, we have computed the unique[12] value of μ (as a function of σ) which satisfies (20), and then plotted the outcome of (21). The curves corresponding to various levels of q are all monotonic, and rise through all reasonable associated values of ρ. Given a (q,ρ) pair, therefore, there is a unique point at which the horizontal line corresponding to ρ cuts the curve corresponding to q. This determines σ, and μ is then determined in turn as the unique value that satisfies (20).

---

[12] It is easy to show that $\int_0^1 x\, f(x)dx$ is strictly monotonically decreasing in μ.

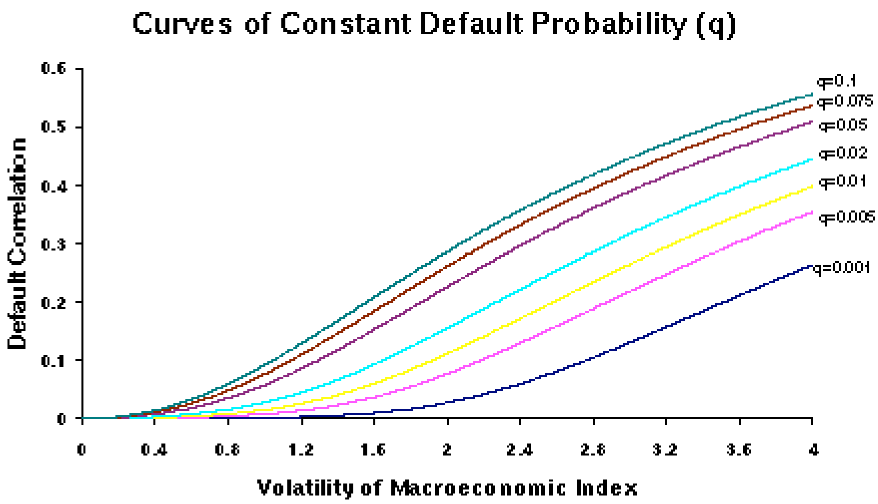

Figure 1: Graphs of default correlation (ρ) plotted against the volatility of the macroeconomic index (σ). For each plot, and for each value of σ, the (unique) value of μ is found which gives the required default probability *q*; then ρ is calculated from (21).

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