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Can consumption spillovers be a source of equilibrium indeterminacy?

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Abstract

In this paper, we show that consumption externalities are a source of equilibrium indeterminacy in a growth model with endogenous labor supply. In particular, when the marginal rate of substitution between own consumption and the others' consumption is constant along the equilibrium path, the equilibrium does not exhibit indeterminacy. In contrast, when that marginal rate of substitution is not constant, the equilibrium may exhibit indeterminacy even if the elasticity of the labor demand is smaller than the elasticity of the Frisch labor supply.

JEL classification codes: D91, E62, O40.

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1. Introduction

In this paper we study how the introduction of average consumption in the utility function modifies the determinacy of the equilibrium path of the one sector growth model. That average consumption generates an externality resulting on either an increase or a reduction in the felicity that each individual obtains from his own consumption. According to Dupor and Liu (2003), this means that individuals could exhibit either altruism or jealousy. Moreover, consumption externalities may also increase or reduce the marginal rate of substitution between own consumption and leisure. Thus, we will consider a model encompassing the "keeping-up with the Joneses" and the "running away from the Joneses" features considered by Dupor and Liu (2003).

Several authors have studied the uniqueness of the equilibrium path of the one-sector growth model when the labor supply is endogenous and production externalities are introduced. In particular, Benhabib and Farmer (1994) show that the equilibrium of the model with separable instantaneous utility may exhibit indeterminacy when the labor supply and the labor demand cross with the wrong slopes. If the labor supply is upward sloping, the condition for indeterminacy will require a sufficiently large degree of returns to labor so that the labor demand ends up being upward sloping. However, Bennett and Farmer (2000) argue that the required degree of returns to labor is not plausible. These authors show that if preferences are non-separable between consumption and leisure, then indeterminacy can arise when the labor demand and the labor supply cross with the normal slopes. In this case, the necessary condition for indeterminacy is that the elasticity of the labor demand is larger than the elasticity of the Frisch labor supply.[1] Thus, if the production function exhibits non-increasing returns to labor, indeterminacy requires that the Frisch labor supply has a negative elasticity, i.e., it must be downward sloping. However, as shown by Hintermaier (2003), the indeterminacy condition obtained by Bennett and Farmer (2000) implies that the utility function is not concave. In fact, Hintermaier shows that, if the utility function is concave, the Frisch labor supply will be upward sloping and the equilibrium of the one sector growth model with production externalities will not exhibit indeterminacy.[2]

In this paper, we will analyze whether the introduction of consumption externalities can cause the indeterminacy of the equilibrium path under a concave utility function and a production function that does not exhibit increasing returns to labor. Liu and Turnovsky (2005) show that consumption externalities do not generate indeterminacy of the equilibrium path when the labor supply is exogenous. Therefore, we will assume instead that the labor supply is endogenous and we will show that, in this case, consumption externalities can give rise to equilibrium indeterminacy. In particular, we will show that the indeterminacy of the equilibrium path depends on the restricted homotheticity property of the utility function (RH property henceforth). We say that the utility function satisfies this property when the marginal rate of substitution (MRS) between consumption and consumption spillovers is constant along the equilibrium path. In this case, the equilibrium does not exhibit indeterminacy. In contrast, when the utility function does not satisfy the RH property, the equilibrium may exhibit

[1] The Frisch labor supply is defined as the labor supply resulting from keeping the marginal utility of consumption constant.

[2] Lloyd-Braga et al. (2006) extend this analysis to technologies with factor-specific external effects.

indeterminacy even though the elasticity of the Frisch labor supply is larger than the elasticity of the labor demand.

Consumption externalities are a source of equilibrium indeterminacy if the equilibrium interest rate rises when agents coordinate to increase their savings. In this case, starting with an arbitrary equilibrium path, another can be constructed by increasing savings because the rate of return of capital increases accordingly so as to justify its higher rate of accumulation. We show that this positive relationship between saving and interest rate may arise when an increase in the amount of saving causes an increase in the next period equilibrium employment, which in turn requires that consumption externalities affect the labor supply. Depending on the effect of consumption externalities on the labor supply, we can distinguish two regions of indeterminacy. In one of these regions, the indeterminacy condition of Bennett and Farmer (2000) does not hold and the Frisch labor supply can even be upward slopping. In the other region, the Bennett and Farmer condition holds even though the utility function is concave. Thus, consumption externalities make a downward slopping Frisch labor supply compatible with a concave utility function. We conclude that indeterminacy can only arise when consumption externalities modify the Frisch labor supply, which requires that the utility function is non-separable between consumption and leisure. In fact, we prove that the kind of separability that prevents indeterminacy from arising is the one implied by the RH property.

The result that the only presence of consumption externalities may be a source of equilibrium indeterminacy in the one-sector growth model is in contrast with the negative result obtained by Guo (1999), who showed that consumption externalities are not a source of equilibrium indeterminacy. However, Guo considers in his analysis an instantaneous utility function that satisfies the RH property and, in this case, consumption externalities do not cause equilibrium indeterminacy. Weder (2000) also considers a model with consumption externalities and an utility function that satisfies the RH property. In his model productive externalities are thus needed to obtain indeterminacy of the dynamic equilibrium.

The rest of the paper is organized as follows. Section 2 describes the economy and characterizes the equilibrium path. Section 3 analyzes the uniqueness of the equilibrium path. Section 4 studies the mechanism that causes equilibrium indeterminacy. Finally, Section 5 concludes the paper. All the proofs appear in the Appendix.

2. The economy

We consider an infinite horizon, continuous time, one-sector model with capital accumulation. The economy consists of competitive firms and a representative household.

2.1. Firms

We assume that the unique good of this economy is produced by means of a neoclassical production function with constant returns to scale. For simplicity in the exposition, and without loss of generality, we consider a Cobb-Douglas production function. Hence, per capita output is given by $y = Ak^{\alpha}l^{1-\alpha}$, with $\alpha \in (0, 1)$ and where k and l are the per capita stock of capital and the employment, respectively. The depreciation rate of capital is $\delta \in (0,1)$. As firms behave competitively, profit maximization implies that the rental prices of the two inputs equal their marginal productivities,

$$r = \alpha A k^{\alpha-1} l^{1-\alpha} - \delta, \tag{2.1}$$

and

$$w = (1-\alpha) A k^{\alpha} l^{-\alpha}. \tag{2.2}$$

2.2. Household

We assume that the representative household is endowed in each period with one unit of time that can be devoted either to supply the amount l of labor or to enjoy the amount $1-l$ of leisure. The objective of the household is to maximize

$$\int_0^\infty e^{-\rho t} u\left(c, \overline{c}, 1-l\right) dt, \tag{2.3}$$

where c is the own consumption, $\overline{c}$ is the average consumption in the economy, and $\rho > 0$ is the subjective discount rate. The instantaneous utility function is twice continuously differentiable and satisfies the following properties: $u_1\left(c, \overline{c}, 1-l\right) > 0$, $u_{11}\left(c, \overline{c}, 1-l\right) < 0$, $u_3\left(c, \overline{c}, 1-l\right) > 0$, $u_{33}\left(c, \overline{c}, 1-l\right) < 0$, $\lim_{c \to \infty} u_1\left(c, \overline{c}, 1-l\right) = 0$, $\lim_{c \to 0} u_1\left(c, \overline{c}, 1-l\right) = \infty$, and

$$u_{11}\left(c, \overline{c}, 1-l\right) u_{33}\left(c, \overline{c}, 1-l\right) \geq \left[u_{13}\left(c, \overline{c}, 1-l\right)\right]^2, \tag{2.4}$$

for all $\overline{c} > 0$.[3] Condition (2.4) implies that the utility function is jointly concave with respect to consumption and leisure which, together with the other assumptions, guarantees that the solution to the household's maximization problem is interior. We also assume that consumption and leisure are normal goods.

The introduction of average consumption implies that consumption spillovers affect the household's utility. In particular, preferences exhibit jealously when $u_2\left(c, \overline{c}, 1-l\right) < 0$, whereas they display admiration (or altruism) when $u_2\left(c, \overline{c}, 1-l\right) > 0$. Following Dupor and Liu (2003) and Liu and Turnovsky (2005), we will assume that

$$u_1\left(c, \overline{c}, 1-l\right) + u_2\left(c, \overline{c}, 1-l\right) > 0.$$

According to Dupor and Liu (2003), preferences correspond to the "keeping-up with the Joneses" formulation when the marginal rate of substitution between own consumption and leisure raises with average consumption and correspond to the "running away from the Joneses" formulation when that marginal rate of substitution decreases.

The representative household maximizes (2.3) subject to the budget constraint

$$wl + rk = c + \dot{k}. \tag{2.5}$$

Let us denote by λ the Lagrangian multiplier of this maximization problem. Then, the first order conditions are

$$e^{-\rho t} u_1\left(c, \overline{c}, 1-l\right) = \lambda, \tag{2.6}$$

[3] From now on, the subindex of a function will refer to the position of the argument with respect to which the partial derivative is taken.

$$e^{-\rho t}u_3\left(c,\overline{c},1-l\right)=\lambda w, \tag{2.7}$$

$$r=-\frac{\dot{\lambda}}{\lambda}, \tag{2.8}$$

and the transversality condition is

$$\lim_{t\to\infty}\lambda k=0. \tag{2.9}$$

Combining (2.6) and (2.7), we obtain

$$\frac{u_3\left(c,\overline{c},1-l\right)}{u_1\left(c,\overline{c},1-l\right)}=w. \tag{2.10}$$

Equation (2.6) shows that the Lagrangian multiplier is equal to the discounted marginal utility of private consumption, (2.8) is the Keynes-Ramsey equation that shows the intertemporal trade-off between consuming today and consuming in the future, and (2.10) drives the intratemporal trade-off between consumption and leisure. Therefore, (2.10) characterizes the labor supply.

2.3. The competitive equilibrium

We are going to obtain the dynamic equations characterizing the equilibrium path. To this end, we combine (2.2) and (2.10) to get

$$\frac{u_3\left(c,\overline{c},1-l\right)}{u_1\left(c,\overline{c},1-l\right)}=(1-\alpha)Ak^{\alpha}l^{-\alpha}. \tag{2.11}$$

After evaluating the previous equation at a symmetric equilibrium (i.e., when $c=\overline{c}$), we obtain an equation that implicitly defines the mapping $c=c\left(k,l\right)$ from capital and employment to consumption. Let us differentiate equation (2.11) with respect to time and evaluate it at a symmetric equilibrium to obtain

$$\left[\phi\left(k,l\right)+\sigma\left(k,l\right)\right]\left(\frac{\dot{c}}{c}\right)+\left[\varepsilon\left(k,l\right)-\eta\left(k,l\right)+\alpha\right]\left(\frac{\dot{l}}{l}\right)=\alpha\left(\frac{\dot{k}}{k}\right), \tag{2.12}$$

where

$$\sigma\left(k,l\right)=-\left(\frac{u_{11}+u_{12}}{u_1}\right)c\left(k,l\right);$$

$$\varepsilon\left(k,l\right)=\left(\frac{u_{13}}{u_1}\right)l;$$

$$\phi\left(k,l\right)=\left(\frac{u_{13}+u_{23}}{u_3}\right)c\left(k,l\right);$$

and

$$\eta\left(k,l\right)=\left(\frac{u_{33}}{u_3}\right)l.$$

Note that $\sigma\left(k,l\right)$ is the inverse of the intertemporal elasticity of substitution and we assume that $\sigma\left(k,l\right)>0$. This requires that $u_{11}+u_{12}<0$. Moreover, we assume that

$$\left[\varepsilon\left(k,l\right)-\eta\left(k,l\right)\right]\left[\phi\left(k,l\right)+\sigma\left(k,l\right)+\chi\left(k,l\right)\right]>0, \tag{2.13}$$

where

$$\chi(k,l) = \left(\frac{u_{12}}{u_1} - \frac{u_{23}}{u_3}\right) c(k,l).$$

Inequality (2.13) holds because both consumption and leisure are assumed to be normal goods.[4] Note also that, if $\chi(k,l) > 0$, then preferences exhibit the "keeping up with the Joneses" feature, whereas they exhibit the "running away from the Joneses" feature when $\chi(k,l) < 0$.

We next combine equations (2.1), (2.6) and (2.8) evaluated at a symmetric equilibrium to get

$$\alpha A k^{\alpha-1} l^{1-\alpha} - \delta - \rho = \sigma(k,l)\left(\frac{\dot{c}}{c}\right) + \varepsilon(k,l)\left(\frac{\dot{l}}{l}\right). \tag{2.14}$$

Moreover, using (2.12) and (2.14), we get

$$\frac{\dot{l}}{l} = \frac{\alpha\left(\frac{\dot{k}}{k}\right) - \left[\frac{\phi(k,l)+\sigma(k,l)}{\sigma(k,l)}\right]\left[\alpha A k^{\alpha-1} l^{1-\alpha} - \delta - \rho\right]}{\alpha + \zeta(k,l)}, \tag{2.15}$$

where

$$\zeta(k,l) = -\eta(k,l) - \left[\frac{\phi(k,l)}{\sigma(k,l)}\right]\varepsilon(k,l)$$

is the price elasticity of the Frisch labor supply. Recall that the Frisch labor supply is the labor supply obtained when the marginal utility of consumption is kept constant. Thus, to obtain that elasticity just note from (2.10) that the labor supply evaluated at a symmetric equilibrium can be written as

$$w(l,\overline{u}_1) = \frac{u_3\left[c(\overline{u}_1,l), c(\overline{u}_1,l), 1-l\right]}{\overline{u}_1}, \tag{2.16}$$

where the upper bar in the marginal utility of consumption means that we keep it constant, and the function $c(\overline{u}_1,l)$ is obtained implicitly from

$$u_1(c,c,1-l) - \overline{u}_1 = 0. \tag{2.17}$$

Then, the elasticity of the Frisch labor supply is

$$\zeta(k,l) = \frac{l w_1(l,\overline{u}_1)}{w(l,\overline{u}_1)}.$$

Finally, from (2.1), (2.2), and (2.5), we obtain the resource constraint

$$\dot{k} = A k^{\alpha} l^{1-\alpha} - \delta k - c(k,l), \tag{2.18}$$

where $c(k,l)$ is implicitly defined in (2.11).

Given an initial condition k_0, a competitive equilibrium is a path of employment and capital that solves the system of differential equations formed by (2.15) and (2.18)

[4]Inequality (2.13) follows from applying the implicit function theorem in (2.11) and setting $\left.\frac{\partial c}{\partial(1-l)}\right|_{\frac{u_3}{u_1}} > 0$.

with $l \in (0,1)$ and that satisfies the transversality condition (2.9). Note that l is now the control variable, whereas k is the state variable.

Let us index the different interior steady states by i and let l_i^* be a steady state value of employment. Then, according to (2.15), a steady state value of capital will be given by

$$k_i^* = k(l_i^*) \equiv \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{\alpha-1}} l_i^*. \tag{2.19}$$

Therefore, using (2.18), we immediately see that the steady state values of employment l_i^* must solve the following equation:

$$Q(l) \equiv A\left[k(l)\right]^{\alpha} l^{1-\alpha} - \delta k(l) - c(k(l), l) = 0. \tag{2.20}$$

We denote the steady state values of the variables by means of a star and, hence, $c_i^* = c(k_i^*, l_i^*)$, $\sigma_i^* = \sigma(k_i^*, l_i^*)$, $\varepsilon_i^* = \varepsilon(k_i^*, l_i^*)$, $\phi_i^* = \phi(k_i^*, l_i^*)$, $\eta_i^* = \eta(k_i^*, l_i^*)$, $\chi_i^* = \chi(k_i^*, l_i^*)$, $\zeta_i^* = \zeta(k_i^*, l_i^*)$ and $y_i^* = y(k_i^*, l_i^*)$ are the values of the corresponding variables at the steady state i.

The existence and uniqueness of interior steady states depend on the properties of the mapping $c(k(l), l)$. In absence of externalities, the assumption on the normality of consumption and leisure implies by definition that consumption is a decreasing function of labor. Hence, since the net output $y - \delta k$ is an increasing function of employment, equation (2.20) has at most a solution l^* in the open interval $(0,1)$. Moreover, observe that the unique steady state satisfies that $Q'(l^*) > 0$. This property implies that net investment increases with employment around the steady state, which is equivalent to say that the elasticity of consumption with respect to gross output is smaller than one at the steady state.[5]

Condition (2.13) implies that the demand of consumption depends negatively on employment. However, in the presence of consumption externalities, this condition does not impose any restriction on the equilibrium relationship between consumption and employment. More precisely, in that case equation (2.11) can implicitly define consumption as an increasing function of employment or even as a set-valued mapping of consumption to employment. In these cases, equation (2.20) can have multiple solutions l_i^* in the open interval $(0,1)$, and the relationship between net investment and employment at these interior steady states is ambiguous.

From now on, we will assume that $Q'(l^*) > 0$. Although condition (2.13) does not imply that in equilibrium consumption decreases with employment, we assume that the increase in consumption is smaller than the increase in net production so that the amount of net investment raises with employment. However, this condition does not impede the existence of multiple interior steady states when (2.11) defines a set-valued mapping from employment to consumption. To see this, consider the following

[5] Constant returns to scale imply that output depends linearly on employment at the steady state. By using this property and (2.20), we get that the elasticity of consumption with respect to output at the steady state is smaller than one if and only if $Q'(l^*) > 0$.

instantaneous utility function:[6]

$$u(c,\overline{c},1-l)=\frac{\left(c\overline{c}^{\psi}\right)^{1-v}\left(1-l+\mu\overline{c}^{2}\right)^{\theta(1-v)}}{1-v}, \tag{2.21}$$

which always satisfies (2.13), and is jointly concave with respect to consumption and leisure when $\nu>\theta/(1+\theta)$ and $\theta>0$. The next result characterizes the existence of interior steady states for the economy with this particular utility.

Proposition 2.1. *Assume that the utility function has the functional form (2.21), and define*

$$a_1=\left[\frac{\theta}{(1-\alpha)A}\right]\left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{\alpha}{1-\alpha}},$$

$$a_2=\left[\frac{\delta(1-\alpha)+\rho}{\alpha}\right]\left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{\alpha-1}},$$

$\overline{\mu}=\left[(1+a_2a_1)/2a_2\right]^2$ *and* $\underline{\mu}=a_1/a_2$, *where* $\overline{\mu}>\underline{\mu}>0$. *Then, there are no interior steady states either if* $a_2a_1>1$ *and* $\mu>\overline{\mu}$ *or if* $a_2a_1<1$ *and* $\mu\geq\underline{\mu}$. *There are two interior steady states if* $a_2a_1>1$ *and* $\underline{\mu}\leq\mu<\overline{\mu}$. *Otherwise, there is a unique interior steady state.*

By using (2.10), it is easy to show that the elasticity of the labor supply with respect to the wage is equal to a_1a_2 in the economy with the utility function (2.21). Then, according to Proposition 2.1, there is a unique interior steady state when this elasticity is lower than one, whereas multiple interior steady states may exits when this elasticity is larger than one and the intensity of consumption externalities, measured by the parameter μ, is sufficiently large.

Table 1 provides the steady state values of employment of this economy for several numerical examples. We construct these examples as follows. First, we set $\alpha=0.4$, $\rho=0.04$ and $\delta=0.04$, so that our economy replicates at the steady state a labor income share of 0.6, a consumption-output ratio of 0.8 and a net interest rate of 0.04. Second, we set the efficiency parameter $A=1$. Third, the parameters v and ψ do not affect the steady state value of employment, but their value should be set so that the following conditions are satisfied: (i) the utility function is concave; (ii) $u_1+u_2>0$; and (iii) $\sigma>0$. The numerical examples in Table 1 satisfy these conditions when $v=1$ and $\psi>1.7$. Finally, the parameters θ and μ are set to replicate the different configurations of interior steady states given in Proposition 2.1. In particular, we consider two different values of θ : (i) $\theta=2.5$, for which the elasticity of labor supply is larger than one; and (ii) $\theta=0.25$, for which the elasticity of labor supply is smaller than one. When $\theta=2.5$

[6] The utility function (2.21) could be generalized to

$$u(c,\overline{c},1-l)=\frac{\left(c\overline{c}^{\psi}\right)^{1-v}\left(1-l+\mu\overline{c}^{\omega}\right)^{\theta(1-v)}}{1-v}.$$

We have considered the particular case with $\omega=2$ since this allows us both to have a reasonable number of subcases in the statement of Proposition 2.1 and to write the statement in tems of the deep parameters of the model. Obviously, the case $\omega=1$ is much more simple but generates only one region of indeterminacy instead of the two indeterminacy regions appearing when $\omega=2$ (see Proposition 3.3).

two interior steady states exist if $\mu > 0.6$. For the case with $\theta = 0.25$, there is a unique interior steady state if $\mu < 0.06$, whereas no interior steady states exists when $\mu > 0.06$. In these examples, all the interior steady states satisfy the condition $Q'(l^*) > 0$.

[Insert Table1]

3. Existence of local indeterminacy

In this section we show that consumption externalities can be a source of equilibrium indeterminacy and we will also provide a sufficient condition on the instantaneous utility function that guarantees the uniqueness of the equilibrium path. For that purpose, we first linearize the law of motions (2.15) and (2.18) around each interior steady state to find the local stability properties of these steady states. In this way, we obtain in the Appendix that the trace and the determinant of the Jacobian matrix J of the linearized dynamic system are respectively given by

$$Tr(J) = -\left(\frac{\delta+\rho}{\alpha+\zeta}\right) N(k,l), \tag{3.1}$$

and

$$Det(J) = -(1-\alpha)(\delta+\rho)\left[(1-\alpha)\delta+\rho\right]\left[\frac{\phi+\sigma+\varepsilon-\eta}{\alpha\sigma(\alpha+\zeta)}\right], \tag{3.2}$$

where

$$N(k_i^*, l_i^*) = (1-\alpha)\left(\frac{\phi^*}{\sigma^*}\right) - \left[\frac{(1-\alpha)\delta+\rho}{\delta+\rho}\right]\left(\frac{\varepsilon^*}{\sigma^*}\right) - \left(\frac{\rho}{\delta+\rho}\right)(\alpha+\zeta^*).$$

By using (3.1) and (3.2), we directly obtain the following result on the stability properties of the steady state equilibria:

Proposition 3.1. *Given any steady state i,*

(a) the steady state is unstable when one of the following two sets of conditions holds:

(i) $\alpha+\zeta_i^* < 0$, $\phi_i^*+\sigma_i^*+\varepsilon_i^*-\eta_i^* \geq 0$ and $N(k_i^*, l_i^*) \geq 0$;
(ii) $\alpha+\zeta_i^* > 0$, $\phi_i^*+\sigma_i^*+\varepsilon_i^*-\eta_i^* \leq 0$ and $N(k_i^*, l_i^*) \leq 0$.

(b) the steady state is locally saddle-path stable when one of the following two sets of conditions holds:

(i) $\alpha+\zeta_i^* < 0$ and $\phi_i^*+\sigma_i^*+\varepsilon_i^*-\eta_i^* < 0$;
(ii) $\alpha+\zeta_i^* > 0$ and $\phi_i^*+\sigma_i^*+\varepsilon_i^*-\eta_i^* > 0$.

(c) the steady state is locally stable when one of the following two sets of conditions holds:

(i) $\alpha+\zeta_i^* < 0$, $\phi_i^*+\sigma_i^*+\varepsilon_i^*-\eta_i^* > 0$ and $N(k_i^*, l_i^*) < 0$;
(ii) $\alpha+\zeta_i^* > 0$, $\phi_i^*+\sigma_i^*+\varepsilon_i^*-\eta_i^* < 0$ and $N(k_i^*, l_i^*) > 0$.

The dynamic equilibrium exhibits local indeterminacy when the steady state is locally stable and, thus, Proposition 3.1 shows that consumption spillovers can make the dynamic equilibrium locally indeterminate. This result is obtained when three reasonable assumptions are imposed: (i) the utility function is concave, (ii) consumption and leisure are normal goods, and (iii) the production function exhibits constant returns to scale. Moreover, indeterminacy may arise in two different regions of the parameter space that are separated by the equation $\zeta_i^* = -\alpha$. The right hand side of this equation is the elasticity of the labor demand and the left hand side is the elasticity of the Frisch labor supply at the steady state.

In the first region, indeterminacy arises when $\zeta_i^* < -\alpha < 0$ and, hence, the Frisch labor supply has a negative slope. A crucial contribution of this result is that indeterminacy in this region is possible with a concave utility function. Note that, since $u_{11} + u_{12} < 0$, the inequality $\zeta_i^* < 0$ implies that

$$u_{13}u_{23} - u_{12}u_{33} > u_{33}u_{11} - (u_{13})^2 > 0,$$

where the last inequality follows from the concavity condition (2.4). If consumption externalities are not present, then $u_{12} = u_{23} = 0$ and the two inequalities cannot be simultaneously satisfied. Thus, consumption spillovers make compatible the existence of a downward-sloping Frisch labor supply with a concave utility function. This is in stark contrast with the indeterminacy configuration derived by Bennett and Farmer (2000). These authors show that the equilibrium exhibits indeterminacy in a model without consumption externalities when production externalities are sufficiently large and $\zeta_i^* < -\alpha$. However, as Hintermaier (2003) shows, in their model indeterminacy requires that the utility function be non-concave. Therefore, our result complements the uniqueness result obtained by Hintermaier (2003) in a model without consumption externalities. This author shows for the case of a Cobb-Douglas production function that, if utility function is concave, then the equilibrium does not exhibit indeterminacy even though production externalities are present.

In the second region, indeterminacy arises when $\zeta_i^* > -\alpha$ and, hence, the Frisch labor supply may be upward sloping. Note that the related literature says that indeterminacy from production externalities requires a downward-sloping Frisch labor supply function. However, we show that consumption spillovers can lead to equilibrium indeterminacy even when this supply function is upward sloping. Observe that in the case with $\zeta_i^* > -\alpha$, indeterminacy arises when $\phi_i^* + \sigma_i^* + \varepsilon_i^* - \eta_i^* < 0$. This condition can only be satisfied if consumption spillovers are introduced. To see this, let us assume that there are no consumption externalities. In this case, the concavity condition (2.4) is given by $\varepsilon_i^*\phi_i^* + \sigma_i^*\eta_i^* \leq 0$ and condition (2.13) becomes $(\phi_i^* + \sigma_i^*)(\varepsilon_i^* - \eta_i^*) > 0$. Thus, if we assume that $\varepsilon_i^* < \eta_i^*$, then $\phi_i^* + \sigma_i^* < 0$. This means that $\phi_i^* < 0$ as $\sigma_i^* > 0$. Moreover, since there are no consumption spillovers, we end up having $u_{23} = 0$ and $\phi_i^* < 0$, which means that $u_{13} < 0$. This in turn implies that $\varepsilon_i^* < 0$. In this case, the concavity condition (2.4) only holds when $\varepsilon_i^* > \eta_i^*$, which contradicts our initial assumption. This means that $\varepsilon_i^* > \eta_i^*$ and condition (2.13) implies that $\phi_i^* + \sigma_i^* > 0$. Therefore, the indeterminacy condition $\phi_i^* + \sigma_i^* + \varepsilon_i^* - \eta_i^* < 0$ is not satisfied in absence of consumption externalities.

The uniqueness of the equilibrium path will depend on the assumptions made on the utility function. In what follows, we provide a sufficient condition on the utility function that implies the uniqueness of the equilibrium path. To this end, we define the concept of restricted homotheticity (RH). We say that the utility function satisfies the RH property if the marginal rate of substitution between consumption and average consumption is constant along the equilibrium path (i.e., when $c = \overline{c}$). This means that

$$\frac{u_2(c, c, 1-l)}{u_1(c, c, 1-l)} = \xi, \tag{3.3}$$

for some constant ξ. The RH property requires the following conditions at a symmetric equilibrium (with $c = \overline{c}$): (i) the marginal utilities u_1 and u_2 must be homogenous of the same degree with respect to consumption; and (ii) the utility u must be either additively or multiplicatively separable between consumption and leisure. The next result shows that consumption externalities do not give raise to equilibrium indeterminacy when the utility function satisfies the RH property:

Proposition 3.2. *The equilibrium does not exhibit indeterminacy when the utility function satisfies the RH property.*

As can be seen in the proof of Proposition 3.2, the RH property makes the dynamic equations characterizing the competitive equilibrium equivalent to those of the standard Ramsey model with no consumption externalities. Thus, the lack of equilibrium indeterminacy of the latter model is inherited by its counterpart with consumption externalities satisfying the RH property.

We next illustrate the results in Propositions 3.1 and 3.2 by using the functional form (2.21) of the utility function. We will look at two cases: (i) $\mu = 0$ and (ii) $\mu \neq 0$. In the first case, the utility function satisfies the RH property, whereas this property does not hold in the second case. From Proposition 2.1 we know that in the case with $\mu = 0$ there exists a unique interior steady state with a level of employment given by

$$l^* = \frac{(1-\alpha)(\delta+\rho)}{(1-\alpha)(\delta+\rho) + \theta\left[\delta(1-\alpha)+\rho\right]}.$$

Moreover, Proposition 3.2 ensures that this steady state is never locally indeterminate.

We now analyze the economy defined by the utility function (2.21) with $\mu \neq 0$. As was proved in Proposition 2.1, in this case two interior steady states may emerge, which would be given by

$$l_1^* = \left[(1 + a_1 a_2 + \Delta)/2\mu\,(a_2)^2\right]$$

and

$$l_2^* = \left[(1 + a_1 a_2 - \Delta)/2\mu\,(a_2)^2\right],$$

where

$$\Delta = \left[(1 + a_1 a_2)^2 - 4\mu\,(a_2)^2\right]^{1/2}.$$

The next proposition gives the necessary and sufficient conditions under which the dynamic equilibrium is locally indeterminate around these steady states.

Proposition 3.3. *Assume the instantaneous utility function (2.21) with $\mu \neq 0$, and define*

$$\underline{\psi}_i = \frac{v}{1-v} - \frac{1 - \alpha - \frac{\theta}{a_1 a_2} + \frac{\alpha}{(1-v)} + \alpha\chi_i\left[a_1 a_2 - \frac{1}{1-v}\right]}{\frac{1+\alpha a_1 a_2}{\theta} - \frac{1}{a_1 a_2}},$$

and

$$\overline{\psi}_i = \frac{v}{1-v} - \theta\left(\frac{1+\alpha a_2 a_1 \chi_i}{1+\alpha a_1 a_2}\right)$$

for $i = 1, 2$, *and with* $\chi_1 = \left[\left(1 + a_1 a_2 + \frac{\Delta}{a_2}\right)/2a_1 a_2\right]$ *and* $\chi_2 = \left[\left(1 + a_1 a_2 - \frac{\Delta}{a_2}\right)/2a_1 a_2\right]$. *Hence,*

(a) The steady state given by l_1^* *is locally stable either if* $v < 1$ *and* $\psi \in \left(\underline{\psi}_1, \overline{\psi}_1\right)$, *or if* $v > 1$ *and* $\psi \in \left(\overline{\psi}_1, \underline{\psi}_1\right)$.

(b) The steady state given by l_2^* *is locally stable either if* $v < 1$ *and* $\psi \in \left(\overline{\psi}_2, \underline{\psi}_2\right)$, *or if* $v > 1$ *and* $\psi \in \left(\underline{\psi}_2, \overline{\psi}_2\right)$.

As shown in Proposition 2.1, when the utility function (2.21) does not satisfy the RH property, two steady states may exist when the elasticity of the labor supply is larger than one. Proposition 3.3 shows how the stability properties of each steady state depend on the intensity of the consumption externality, measured by the parameter ψ. Note that the steady states can be locally stable and, in this case, the dynamic equilibrium exhibits local indeterminacy. However, the indeterminacy region is different in each steady state. From the proof of this proposition, it can be seen that steady state given by l_1^* exhibits indeterminacy only when $\alpha + \zeta_1^* > 0$, whereas steady state given by l_2^* exhibits indeterminacy only when $\alpha + \zeta_2^* < 0$.

Table 2 shows the value of the characteristic roots in each steady state when different values of the parameters μ, v and ψ are considered. The other parameters are set as in the economy of Table 1. The steady state equilibrium is saddle-path stable when the two roots have a different sign, and it is unstable when the real part of the two roots is positive. When the real part of the two roots takes a negative value, the dynamic equilibrium exhibits indeterminacy. Table 2 shows several examples of equilibrium indeterminacy. In particular, when $\theta = 2.5$, $v = 0.75$, $\mu = 0.7$ and $\psi = 0.9$, steady state 2 is locally stable. In this case, $l_2^* = 0.32$ and the intertemporal elasticity of substitution is equal to 16.4. Steady state 1 is also locally stable when $\theta = 2.5$, $v = 4, \mu = 0.85$ and $\psi = -4.4$. In this case, $l_1^* = 0.51$ and the intertemporal elasticity of substitution is equal to 0.67. These examples show that the equilibrium can exhibit local indeterminacy when the value of the parameters is plausible and the value of the intertemporal elasticity of substitution is low.

[Insert Table 2]

4. Labor supply and indeterminacy

In this section, we explain the economic mechanism underlying the indeterminacy result described in the previous section. We start by considering the intertemporal trade-off between consuming today and consuming in the future. In particular, we assume that the economy is in an equilibrium path and agents coordinate into a reduction of current consumption that reduces current utility and increases future utility through a larger amount of saving. Obviously, the increase in savings can only be an equilibrium decision if the interest rate increases. We proceed to show that consumption externalities can cause a complementarity between the current amount of saving and the next period amount of employment, which may lead to the necessary increase in the interest rate that justifies the larger amount of savings. When this occurs, then the equilibrium may exhibit local indeterminacy.

Note that equation (2.14) implies that consumption externalities modify the path of savings either by changing the intertemporal elasticity of substitution or by modifying the labor supply. As Liu and Turnovsky (2005) have already shown that the equilibrium path is unique when the labor supply is exogenous, we will focus the analysis on the effects of consumption externalities on the Frisch labor supply. We distinguish two different effects on this labor supply. First, consumption externalities modify the slope of the Frisch labor supply. Second, they also distort the effect that changes in private consumption have on this labor supply.

In order to understand the effect of an increase in consumption on the labor supply, we use (2.16) to obtain the elasticity of the Frisch labor supply with respect to the marginal utility of private consumption, that is,

$$\left(\frac{\partial l}{\partial \overline{u}_1}\right)\left(\frac{\overline{u}_1}{l}\right) = -\left[\frac{\left(u_{31}+u_{32}\right)\left(\frac{\partial c}{\partial \overline{u}_1}\right) - \frac{u_3}{\overline{u}_1}}{\left(u_{31}+u_{32}\right)\left(\frac{\partial c}{\partial l}\right) - u_{33}}\right]\left(\frac{\overline{u}_1}{l}\right),$$

where

$$\frac{\partial c}{\partial \overline{u}_1} = \frac{1}{u_{11}+u_{12}},$$

and

$$\frac{\partial c}{\partial l} = \frac{u_{13}}{u_{11}+u_{12}},$$

as follows from (2.17). Then, using the definitions of ϕ, σ and ζ, we obtain that in a steady state this elasticity is

$$\left(\frac{\partial l}{\partial \overline{u}_1}\right)\left(\frac{\overline{u}_1}{l}\right) = \frac{\phi_i^* + \sigma_i^*}{\sigma_i^* \zeta_i^*}. \tag{4.1}$$

Moreover, after applying the implicit function theorem to (2.11), we obtain from (2.20) that

$$Q'\left(l_i^*\right) = \left(\frac{c_i^*}{l_i^*}\right)\left(\frac{\phi_i^* + \sigma_i^* + \varepsilon_i^* - \eta_i^*}{\phi_i^* + \sigma_i^*}\right).$$

As we have assumed that $Q'\left(l_i^*\right) > 0$, the signs of $\phi_i^* + \sigma_i^*$ and of $\phi_i^* + \sigma_i^* + \varepsilon_i^* - \eta_i^*$ coincide. This implies that the sing of $\phi_i^* + \sigma_i^*$ is different in the the two regions for which Proposition 3.1 states the existence of local indeterminacy. We proceed to explain the mechanism driving the complementarity between saving and employment and, thus, the positive effect of saving on the interest rate, in each region of indeterminacy.

In the first region of indeterminacy, we have $\phi_i^* + \sigma_i^* + \varepsilon_i^* - \eta_i^* > 0$ and $\zeta_i^* < -\alpha$. This means that $\phi_i^* + \sigma_i^* > 0$ and $\zeta_i^* < 0$. and, hence, the elasticity of the labor supply with respect to the marginal utility of consumption, which is given by (4.1), is negative. As the increase in savings raises future consumption and then reduces next period marginal utility, the Frisch labor supply increases in the next period. The increase in the labor supply raises the equilibrium amount of employment since $\zeta_i^* < -\alpha$. This increases the next period interest rate because the marginal product of capital increases with employment. We have then explained the complementarity between savings and employment, and the corresponding positive relationship between saving and interest rate, in the first region. When this complementarity is sufficiently strong, it results in equilibrium indeterminacy. Note that, in this region, indeterminacy arises because the equilibrium amount of leisure decreases as consumption increases. If consumption externalities were not introduced, condition (2.13) would imply that leisure raises with consumption and the equilibrium would be unique.

In the second region of indeterminacy, we have $\phi_i^* + \sigma_i^* + \varepsilon_i^* - \eta_i^* < 0$ and $\zeta_i^* > -\alpha$. , which implies that $\phi_i^* + \sigma_i^* < 0$. However, ζ_i^* can be either positive or negative and, therefore, we can distinguish two cases in this region. Assume first that the price elasticity of Frisch labor supply is positive, i.e., $\zeta_i^* > 0$. In this case, the elasticity of the labor supply with respect to the marginal utility of consumption given by (4.1) is negative. Then, the labor supply increases as the amount of saving increases, due to the reduction in the next period marginal utility of consumption. As $\zeta_i^* > 0$, the increase in the labor supply raises the amount of employment in the next period. This makes the interest rate increase in the next period and we end up obtaining a positive relationship between saving and interest rate. Thus, in this case, indeterminacy is also explained by the reduction in the equilibrium amount of leisure when consumption raises.

In contrast, when $\zeta_i^* \in (-\alpha, 0)$, the elasticity of labor supply with respect to marginal utility of consumption given by (4.1) is positive. In this case the increase in saving then reduces the marginal utility of consumption, so that the next period labor supply declines. However, since the Frisch labor supply has a negative slope, but its price elasticity is larger than the elasticity of labor demand, the decrease in labor supply is followed by a decrease in the equilibrium rate of wages that finally drives employment up. This causes the increase in the interest rate resulting in indeterminacy.

Consumption externalities may then cause equilibrium indeterminacy because they distort the labor market by altering the marginal rate of substitution between consumption and leisure at the equilibrium. In fact, if there were no consumption externalities, condition (2.13) would imply that leisure increases with consumption. This means that, as a result of the increase in savings, next period labor supply would decrease. The reduction in the labor supply would imply a reduction in the amount of employment, because the Frisch labor supply is upward slopping when the utility function is concave and there are no consumption externalities. Obviously, the reduction in employment implies that next period interest rate would decline as agents increase savings. This shows that there is a unique equilibrium path in this economy when there are no consumption externalities.

Condition (2.13) implies that leisure and consumption are normal goods. However, indeterminacy arises when employment increases with consumption along an equilibrium path. It follows that the equilibrium can only exhibit indeterminacy if consumption externalities modify the labor supply. This requires that preferences are non-separable between consumption and leisure. In fact, as shown in Proposition 3.2, the kind of separability that prevents indeterminacy from arising is the one implied by the RH property. To see this, suppose that consumption externalities have been internalized, which implies that the utility function satisfies $\widehat{u}(c, 1-l) \equiv u(c, c, 1-l)$.

Then, the labor supply is characterized by

$$w = \frac{\widehat{u}_3}{\widehat{u}_1} = \frac{u_3}{u_1 + u_2}.$$

When the utility function satisfies the RH property, this equation can be rewritten as

$$w\,(1+\xi) = \frac{u_3}{u_1}.$$

Note that the Frisch labor supply obtained from this equation has both the same slope and the same elasticity with respect to the marginal utility of consumption than the labor supply obtained from equation (2.11). This shows that consumption externalities do not modify the labor supply when the RH property is satisfied, which explains the uniqueness of the equilibrium path in this case.

5. Concluding remarks

In this paper we have analyzed the uniqueness of the dynamic equilibrium of a one-sector growth model when we assume that average consumption affects individuals felicity, that is, when consumption externalities are present. We assume that the utility function of this model is concave, consumption and leisure are normal goods, and the production function does not exhibit increasing returns. With these plausible assumptions, we show that multiple steady states may exist in this economy with consumption externalities. We also show that the equilibrium may exhibit indeterminacy when consumption externalities are introduced. In particular, we show that the equilibrium path is unique if the utility function satisfies the RH property. In contrast, if the RH condition does not hold, the equilibrium can exhibit indeterminacy.

We have shown that there are two different regions of indeterminacy. In the first one, the elasticity of the labor demand is smaller than the elasticity of the Frisch labor supply; whereas the opposite relation holds in the second region. Therefore, when consumption externalities are introduced, the equilibrium can exhibit indeterminacy even if the elasticity of the labor demand is smaller than the elasticity of the Frisch labor supply.

Alonso-Carrera et. al. (2006) show that the equilibrium is inefficient when habit adjusted consumption and consumption externalities are not perfect substitutes, which suggests that the interaction between habits and externalities is another potential source of equilibrium indeterminacy. Moreover, Chen (2006) shows that the economy may exhibit multiple steady states when a process of habit formation is introduced. Therefore, to study how the interaction between habit formation and consumption externalities affects the uniqueness of the dynamic equilibrium seems a promising line of future research.

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Appendix

The linearized dynamic system. Throughout this analysis the variables of the model are evaluated at a steady state. The Jacobian matrix associated with the system of differential equations (2.15) and (2.18) around the steady state is

$$J = \begin{pmatrix} \frac{\partial \dot{l}}{\partial l} & \frac{\partial \dot{l}}{\partial k} \\ \frac{\partial \dot{k}}{\partial l} & \frac{\partial \dot{k}}{\partial k} \end{pmatrix} = \begin{pmatrix} l\left[\frac{\left(\frac{\alpha}{k}\right)\left(\frac{\partial \dot{k}}{\partial l}\right)-\left(\frac{\phi+\sigma}{\sigma}\right)y_{kl}}{\alpha+\zeta}\right] & l\left[\frac{\left(\frac{\alpha}{k}\right)\left(\frac{\partial \dot{k}}{\partial k}\right)-\left(\frac{\phi+\sigma}{\sigma}\right)y_{kk}}{\alpha+\zeta}\right] \\ y_l - c_l & y_k - \delta - c_k \end{pmatrix},$$

where

$$c_l = \frac{\partial c\,(k,l)}{\partial l} = \left(\frac{\eta-\varepsilon-\alpha}{\phi+\sigma}\right)\left(\frac{c}{l}\right),$$

$$c_k = \frac{\partial c\,(k,l)}{\partial k} = \left(\frac{\alpha}{\phi+\sigma}\right)\left(\frac{c}{k}\right),$$

and y_k, y_l, y_{kk} and y_{kl} represent the partial derivatives of the production function, with the subindex denoting the argument with respect to which the partial derivative is taken. The determinant of the Jacobian matrix is

$$Det(J) = -\left(\frac{l}{\alpha+\zeta}\right)\left(\frac{\phi+\sigma}{\sigma}\right)y_{kk}\left[(y_k-\delta-c_k)\left(\frac{y_{kl}}{y_{kk}}\right)-(y_l-c_l)\right].$$

Constant returns to scale imply that $y_{kl}l = -y_{kk}k$. Then, using the expressions for c_l and c_k, we obtain

$$Det(J) = \left(\frac{l}{\alpha+\zeta}\right)\left(\frac{\phi+\sigma}{\sigma}\right)y_{kk}\left[\left(\frac{k}{l}\right)(y_k-\delta)+y_l-\left(\frac{\eta-\varepsilon}{\phi+\sigma}\right)\left(\frac{c}{l}\right)\right].$$

As constant returns to scale also imply that $y = y_k k + y_l l$, the Cobb-Douglas functional form for the production function and (2.20) make the previous equation to simplify to (3.2). The trace of the Jacobian matrix is

$$Tr(J) = -\left(\frac{1}{\alpha+\zeta}\right)\left[\left(\frac{\phi+\sigma}{\sigma}\right)l y_{kl} - \left(\frac{\alpha l}{k}\right)(y_l-c_l)-(y_k-\delta-c_k)(\alpha+\zeta)\right].$$

By using the expressions for c_l and c_k, and the Cobb-Douglas production function and (2.20),we obtain (3.1).

Proof of Proposition 2.1. Consider the utility function (2.21). Then, using (2.19), we obtain

$$k\,(l) = \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{\alpha-1}} l, \tag{A.1}$$

and using (2.11) and (A.1), we obtain

$$l\,(c) = \mu c^2 - a_1 c + 1, \tag{A2}$$

where

$$a_1 = \frac{\theta}{(1-\alpha) A \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{\alpha}{\alpha-1}}}.$$

Finally, using (2.20), it can be shown that the steady-states are the roots of the following function:

$$Q(c) \equiv \mu c^2 - \left(\frac{a_2 a_1 + 1}{a_2}\right) c + 1,$$

where

$$a_2 = \left[\frac{\delta(1-\alpha)+\rho}{\alpha}\right] \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{\alpha-1}}.$$

The two roots of $Q(c)$ are:

$$c_1 = \frac{1 + a_2 a_1 + \Delta}{2\mu a_2}, \tag{A3}$$

$$c_2 = \frac{1 + a_2 a_1 - \Delta}{2\mu a_2}, \tag{A4}$$

with $\Delta = \left[(1+a_2 a_1)^2 - 4\mu (a_2)^2\right]^{1/2}$. Note that if $\mu > 0$ then $c_1 > c_2 > 0,$ whereas if $\mu < 0$ then $c_1 < 0$ and $c_2 > 0$. Note also that we can define $l_i = l(c_i)$. Obviously, an steady state requires that $l_i \in (0,1)$ and

$$\left(\frac{1+a_2 a_1}{a_2}\right)^2 \geq 4\mu$$

On the one hand, the second condition implies that $\mu \leq \overline{\mu}$ where

$$\overline{\mu} = \left(\frac{1+a_2 a_1}{2a_2}\right)^2.$$

On the other hand, in order to set conditions that guarantee that $l_i \in (0,1)$, we must differentiate between the two candidates to steady state. First, $l_1 < 1$ implies that

$$\mu c_1^2 - a_1 c_1 + 1 < 1$$

and, using (A3), we obtain

$$\sqrt{(1+a_2 a_1)^2 - 4\mu a_2^2} < a_1 a_2 - 1.$$

Note that if $a_1 a_2 < 1$ then $l_1 > 1$ and if $a_1 a_2 > 1$ then $l_1 < 1$ when $\mu > \underline{\mu}$, where $\underline{\mu} = \frac{a_1}{a_2}$. Second, $l_1 > 0$ implies that $\mu c_1^2 - a_1 c_1 > -1$ and, using (A3), we can rewrite this inequality as

$$\frac{1+a_2 a_1 + \sqrt{(1+a_2 a_1)^2 - 4\mu a_2^2}}{\mu} > 0.$$

Obviously, if $\mu > (<) 0$ then $l_1 > (<) 0$. Next, $l_2 < 1$ when $\mu c_2^2 - a_1 c_2 + 1 < 1.$ Using (A4), this inequality can be rewritten as

$$-\sqrt{(1+a_2 a_1)^2 - 4\mu a_2^2} < a_1 a_2 - 1$$

Note that $l_2 < 1$ when either $a_1 a_2 > 1$ or when $a_1 a_2 < 1$ and $\underline{\mu} > \mu$. Finally, $l_2 > 0$ when $\mu c_2^2 - a_1 c_2 > -1$, which is always satisfied as it can be shown using (A4). ■

Proof of Proposition 3.2. Let us find the command optimum allocation of our model when consumption externalities are internalized by a social planner. For this problem, the resource constraint is (2.18), which is the same that for the competitive economy. The instantaneous utility faced by the planner will be

$$\widehat{u}(c, 1-l) = u(c, c, 1-l). \tag{A.1}$$

The utility $\widehat{u}$ is increasing in c under our assumptions on u since $\widehat{u}_1 = u_1 + u_2 > 0$. Moreover, the planner's marginal utility is decreasing in c. To see this, use the RH property to compute

$$\widehat{u}_{11} = (1+\xi)(u_{11} + u_{12}) < 0,$$

and observe that $\sigma > 0$ implies that $u_{11} + u_{12} < 0$, while $u_1 + u_2 > 0$ together with the RH property (i.e., $u_1 + u_2 = (1+\xi)\, u_1$) implies that $1 + \xi > 0$.

The first order conditions for the planner's problem are

$$e^{-\rho t} u_1 (c, c, 1-l)(1+\xi) = \lambda, \tag{A.2}$$

$$e^{-\rho t} u_3 (c, c, 1-l) = \lambda (1-\alpha) A k^{\alpha} l^{-\alpha}, \tag{A.3}$$

and

$$\alpha A k^{\alpha-1} l^{1-\alpha} - \delta = -\frac{\dot{\lambda}}{\lambda}.$$

Dividing (A.3) by (A.2) we obtain

$$\frac{u_3 (c, c, 1-l)}{u_1 (c, c, 1-l)} = (1+\xi)(1-\alpha) A k^{\alpha} l^{-\alpha}.$$

If we differentiate the previous equation with respect to time in order to express it in terms of the growth rates of c, l and k, the term $(1+\xi)$ disappears and the resulting equation turns out to be (2.15). Therefore, the dynamic equations characterizing the planner's solution are (2.15) and (2.18), which characterize also the solution of the original model where the consumption spillovers are not internalized. Finally, notice that the planner's solution coincides with the solution of the standard Ramsey model without consumption externalities when the instantaneous utility function faced by individuals is (A.1). Since it is well known that the Ramsey model with endogenous labor supply does not exhibit indeterminacy (see, for instance, Turnovsky, 1995, Chapter 9), the model with consumption externalities satisfying the RH property does not display indeterminacy either. ■

Proof of Proposition 3.3. Assume that $\mu \neq 0$. Then, by using (2.21) we obtain that the following equalities hold at each interior steady state :

$$\sigma_i = v - \psi (1-v) - \theta (1-v) \chi_i,$$

$$\varepsilon_i = \theta (1-v) \left(\frac{l_i}{1 - l_i + \mu c_i^2} \right),$$

$$\phi_i = (1+\psi)(1-v) + [\theta(1-v)-1]\chi_i = 1-\chi_i-\sigma_i,$$

$$\eta_i = [\theta(1-v)-1]\left(\frac{l_i}{1-l_i+\mu c_i^2}\right),$$

and

$$\chi_i = \frac{2\mu c_i^2}{1-l_i+\mu c_i^2},$$

where l_i and c_i are the steady state value of employment and consumption, respectively. From some mechanical algebra we obtain that the interior steady state of this economy satisfies $1-l_i+\mu c_i^2 = a_1 c_i$ and $c_i = a_1 l_i$. We then get

$$\chi_i = \frac{2\mu c_i}{a_1},$$

$$\varepsilon_i = \theta(1-v)\left(\frac{1}{a_1 a_2}\right),$$

$$\eta_i = [\theta(1-v)-1]\left(\frac{1}{a_1 a_2}\right).$$

We proceed to obtain parameter conditions that define the different stability regions in Proposition 3.1. First, note that

$$\phi_i + \sigma_i + \varepsilon_i - \eta_i = 1 - \frac{2\mu c_i}{a_1} + \frac{1}{a_1 a_2}.$$

Replacing the starionary value of c_i in the previous equation, it follows that $\phi_i+\sigma_i+\varepsilon_i-\eta_i$ is negative in steady state given by l_1 and positive in steady state given by l_2. Next, $\alpha+\zeta_i$ can be rewritten as follows

$$\alpha+\zeta_i = \left(\frac{1}{a_1 a_2}\right)\left[1+\alpha a_1 a_2 - \theta(1-v)\left(\frac{1-\chi_i}{\sigma_i}\right)\right],$$

and, using the definition of σ_i, we obtain that

$$\alpha+\zeta_i = \left(\frac{1}{a_1 a_2}\right)\left\{\frac{[v-\psi(1-v)](1+\alpha a_1 a_2) - \theta(1-v)(1+\alpha a_1 a_2 \chi_i)}{\sigma_i}\right\}.$$

Hence, $\alpha+\zeta_i>0$ when either (i) $\psi<\overline{\psi}_i$ and $v<1$, or (ii) $\psi>\overline{\psi}_i$ and $v>1$. Finally, $N(k_i,l_i)$ can be rewritten as follows

$$N(k_i,l_i) = \left[\left(\frac{1-\chi_i}{\sigma_i}\right)\alpha+\zeta_i\right](1-\alpha) - \left[\alpha\frac{\varepsilon_i}{\sigma_i}+\alpha+\zeta_i\right]\left[\frac{(1-\alpha)\delta+\rho}{\delta+\rho}\right].$$

Using the definition of a_1 and a_2 in Proposition 2.1, we get

$$\left[\frac{(1-\alpha)\delta+\rho}{\delta+\rho}\right] = \frac{(1-\alpha)a_1 a_2}{\theta}.$$

Using this equation and the definitions of ζ_i and of ε_i, $N(k_i,l_i)$ simplifies to

$$
\begin{aligned}
N(k_i, l_i) &= (1-\alpha)\left(\frac{1-\chi_i}{\sigma_i}\right)\left[\alpha + (1-v) - \frac{\theta(1-v)}{a_1 a_2}\right] \\
&\quad - \frac{(1-\alpha)}{\alpha\theta}\left(\frac{(1-v)\theta}{\sigma_i} + a_1 a_2 + \frac{1}{\alpha} - \frac{\theta}{\alpha a_1 a_2}\right).
\end{aligned}
$$

Using the definition of σ_i, it can be shown that $N(k_i, l_i) > 0$ when either (i) $\psi > \underline{\psi}_i$ and $v < 1$, or (ii) $\psi < \underline{\psi}_i$ and $v > 1$. Otherwise, $N(k_i, l_i) < 0$. Using these conditions and Proposition 3.1, the statement of Proposition 3.4 follows. ■

$\theta = 2.5$		$\theta = 0.25$	
$a_1a_2 = 3.3$, $\underline{\mu} = 0.6$, $\overline{\mu} = 0.85$		$a_1a_2 = 0.33$, $\underline{\mu} = 0.06$, $\overline{\mu} = 0.08$	
$\mu = 0.5$	$l_2 = 0.28$	$\mu = -1$	$l_2 = 0.32$
$\mu = 0.7$	$l_1 = 0.8$ $l_2 = 0.32$	$\mu = -0.5$	$l_2 = 0.41$
$\mu = 0.8$	$l_1 = 0.62$ $l_2 = 0.36$	$\mu = -0.25$	$l_2 = 0.49$

Table 1. Steady states. The parameters in the economy take the following values: $\rho = 0.04$, $\alpha = 0.4$, $\delta = 0.04$ and $A = 1$.

	$\theta = 2.5,\ v = 0.75,\ \mu = 0.7$	
	Steady State 2	Steady State 1
$\psi = 0$	$\lambda_1 = 0.223$ $\lambda_2 = -0.12$	$\lambda_1 = 0.091$ $\lambda_2 = -0.36$
$\psi = 0.9$	$\lambda_1 = -0.12 + 0.8i$ $\lambda_2 = -0.12 - 0.8i$	$\lambda_1 = 0.0906$ $\lambda_2 = -0.16$
$\psi = 1.1$	$\lambda_1 = 2.02 - 0.3i$ $\lambda_2 = 2.02 + 0.3i$	$\lambda_1 = 0.0904$ $\lambda_2 = -0.142$

	$\theta = 2.5,\ v = 4,\ \mu = 0.85$	
	Steady State 2	Steady State 1
$\psi = 0$	$\lambda_1 = 0.07$ $\lambda_2 = -0.01$	$\lambda_1 = 0.001$ $\lambda_2 = 0.07$
$\psi = -4.4$	$\lambda_1 = 0.25$ $\lambda_2 = 0.0056$	$\lambda_1 = -0.845$ $\lambda_2 = -0.0134$
$\psi = -4.5$	$\lambda_1 = 0.21$ $\lambda_2 = 0.051$	$\lambda_1 = 0.733$ $\lambda_2 = -1.042$

$\theta = 0.25,\ \mu = 0.03$			
Steady State 2			
$v = 0.75$		$v = 4$	
$\psi = 0$	$\lambda_1 = 0.12$ $\lambda_2 = -0.079$	$\psi = 0$	$\lambda_1 = 0.0635$ $\lambda_2 = -0.023$
$\psi = 2$	$\lambda_1 = 0.0224$ $\lambda_2 = -0.169$	$\psi = -0.5$	$\lambda_1 = 0.07$ $\lambda_2 = -0.053$
$\psi = 2.76$	$\lambda_1 = -0.44 - 2.4i$ $\lambda_2 = -0.44 + 2.4i$	$\psi = -1$	$\lambda_1 = 0.0678$ $\lambda_2 = -0.032$

Table 2. Stability when $\mu \neq 0$