Colonel Blotto's Top Secret Files

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Abstract

Colonel Blotto "secret files" are opened and Information about the way that people play the game is revealed. The files rely on web-based experiments, which involve a tournament version of the Colonel Blotto Game. A total of 6,500 subjects from two diverse populations participated in the tournaments. The results are analyzed in light of a novel procedure of multi-dimensional iterative reasoning. According to the procedure, a player decides separately about different features of his strategy using iterative reasoning. Measuring the response time of the subjects assists in interpreting the reasoning procedure behind the choices. Common properties of the successful strategies in the tournament are exposed.

Keywords: Colonel Blotto, Multi-dimensional iterative reasoning, Response Time

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1. The Colonel Blotto Game

Imagine you are a colonel in command of an army during wartime. You and the colonel of the enemy's army each command 120 troops. Your troops will engage the enemy in 6 battles on 6 separate battlefields.

It is the night before the battles and each of you must decide how to deploy your forces across the 6 battlefields. In the morning, you will win a battle if the number of troops you have assigned to a particular battlefield is higher than that assigned by your opponent. In the case that you have both allocated the same number of troops to a particular battlefield, the outcome of the battle will be a loss for both of you.

Your deployment of troops will face that of each of the other participants in the tournament. Your total score will be the number of battles you win against all the other participants.

How will you deploy your 120 troops?

Please pause for a second and try to devise a strategy. How would you play such a game?

The first possibility that comes to mind is the simple strategy of allocating the 120 troops evenly across the six battlefields. This *instinctive strategy* is likely to be chosen by other participants as well. If you choose this strategy, you will score no points against these participants. If the marginal distribution of enemies' troops in each battlefields will be roughly symmetric around 20, the instinctive strategy will score at most 3 points on average, while you speculate that winning the tournament requires scoring more than 3 points.

This may lead you to consider concentrating your troops in only some of the battlefields. But in how many? By allocating your troops evenly across five battlefields, you will win 5 battles and score 5 points against any opponent who chooses the instinctive strategy. However, what if the other players have the same thought and concentrate their troops in only five fields? In that case it might be better for you to assign a larger number of troops to only four battlefields, hoping to score 4 points against these players. It is not obvious where to stop this chain of arguments.

Suppose you have decided to deploy your troops in a certain number of battlefields and to abandon the others. Should you completely abandon those battlefields? Other players might also abandon them and therefore assigning even a small number of troops to these fields would score some easy victories. Other players are likely to go through a similar reasoning process and you need to decide on the exact number of troops to be assigned to the almost abandoned battlefields.

You might also doubt that the six fields are treated symmetrically. Is it possible that other players will systematically deploy more troops in some fields than in others? If so, you will gain more points by assigning larger masses to specific fields.

As you can see, the game is quite complicated and numerous considerations arise. At this point you are probably hoping that a game-theoretical analysis of the situation will provide some guidance in formulating a strategy. In the multi-player tournament a player's target is to score more points than the others. However, you realize that for a distribution of pure strategies to be an equilibrium, it has to be a symmetric mixed strategy Nash equilibrium in a two player game, where each player maximizes his expected score. Sorry to disappoint you. The two player Blotto game corresponding to the described tournament, is not a constant-sum game (in the case of a tie on a particular battlefield neither player receives any points) and we are not aware of a game-theoretical analysis of this version of the Blotto game. In the classic version of the game, as suggested in Borel (1921), players who tie on a particular battlefield split the point. This related constant-sum version of the game has been analyzed in Hart (2008). In a game with 120 troops allocated to 6 fields, a symmetric mixed strategy Nash equilibrium treats the fields symmetrically and the marginal distribution of the troops in each battlefield is essentially uniform in the interval $[0, 40]$. Do you have any basis for the conjecture that this equilibrium of the related constant-sum game will be realized in the tournament of our version of the Blotto game?

It seems that the best way to proceed is by understanding how real people behave in the game. The current paper reports the results of a web-based experimental study of the Blotto tournament. The platform used in the experiment was the didactic website gametheory.tau.ac.il. Each subject participated only once. The subjects belong to two separate populations:

(i) **Classes**: Teachers of game theory courses occasionally assign their students virtual games and decision-theoretical problems from the site's bank of problems. The results obtained at the site are typically similar to those in laboratories experiments using monetary incentives (see Rubinstein (2007)).

The Blotto game was added to the site in July 2004. Students were asked to participate in a tournament against their classmates; we will be reporting mainly the aggregated data of all the tournaments. The only incentive provided to the subjects was that the three tournament winners in each class would have their names announced by the teacher. By the spring of 2009, 4605 students had participated in the game. They belong to 129 groups in 25 countries (Argentine, Australia, Belgium, Brazil, Brunei, Canada, China, Denmark, Finland, France, Ireland, Israel, Mexico, Moldova, Netherlands, Norway, Slovakia, Spain, Switzerland, Taiwan, Thailand, Turkey, UK, USA and Vietnam).

(ii) **Calcalist**: "Calcalist" is a Hebrew business daily published in Israel. In the eve of Passover 2009, we invited Calcalist's readers to experience Game Theory by playing three games posted on our website. The invitation was done through a newsletter, a link in a major news website, a link on "Calcalist on-line" and through

Calcalist printed version in which the games were described. 1928 readers chose to participate. Prior to the Blotto tournament, the readers played two other games: Arad's Tennis Coach problem (see Arad (2009)) and a novel game called "91-100", which will be described later in the paper. Both games naturally trigger k-level reasoning. It was promised to the readers that the names of the three tournament winners would appear in an article that will report the main findings.

The Students originated from a diverse range of countries. The Calcalist readers were mostly Israelis who have in common an interest in financial news but are quite heterogeneous in age and education. We were struck by the similarity of the results for the two quite distinct populations. We will report the results for the two populations side by side.

We are aware of the standard criticism of experimentation via the web without the use of monetary incentives. We don't want to get into the debate over the importance of the standard protocols of experimental economics. Our experiment was carried out on the web for two main reasons. First, the Blotto game has a huge number of possible strategies, as well as a very large number of chosen strategies, and thus obtaining a meaningful distribution of strategies requires thousands of subjects. The large size of our sample, 6533 subjects, enables us to study the variety of considerations that arise in a player's mind, which would be difficult to asses using a standard economics experiment's sample. Second, the web experiment allows us to confirm the robustness of the results by looking at both game theory students and readers of a financial newspaper who represent a less conventional class of subjects.

The Blotto game is endowed with a rich structure resulting from its framing and huge set of strategies. Our aim is to deepen the understanding of strategic behavior in such involved strategic situations. Though it is not clear that experimental behavior reflects the way that the game would be played in real life, the findings shed light on major considerations that arise in such a context. The task of identifying and interpreting common patterns of behavior in the data is challenging; The number of chosen strategies is immense and unlike some other well-known simple games, it is hard to imagine implementing attractive procedures, such as successive elimination of dominated strategies or k-level reasoning. Our thesis is that the large size of the strategy space and its structure force a player to consider the features of a strategy rather than concrete strategies. For example, a player might believe that the vast majority of subjects will totally abandon some battlefields. This would lead him to assign one troop to the battlefields that he intended to abandon, so that he will score a battlefield victory over the subjects who abandon that field completely. Such a consideration is not a standard "best-response" calculation since it does not rely on a concrete belief on the distribution of the other players' strategies. We will present evidence suggesting that some subjects utilize such multi-feature choice considerations (and some even do it iteratively). Overall, we find the analysis of subjects' behavior in this game to be a source of ideas for constructing novel models of strategic deliberations.

The main goal of the paper is to investigate traces of multi-feature thinking and in particular multi-dimensional steps of reasoning. To the best of our knowledge, this is the first attempt to identify a process involving several non-inclusive forms of iterative reasoning. We also open the "top secret files": We report the salient strategies, examine whether there are returns to time and finally expose the winning strategies. We promise a real surprise!

2. Popular strategies

A strategy is an allocation of the 120 troops across the six battlefields. The game has around 250 million different strategies. Nevertheless, the data indicates that there are a few strategies which are widely used. Nine of the strategies were chosen by at least one percent of the subjects each and are presented in Table 1. These strategies were together selected by around 30% of the subjects in each of the populations.

	Strategies						Classes		Calcalist	
	Battlefields						n=4605		n=1928	
#	1	2	3	4	5	6	%	Mean Score	%	Mean Score
1	20	20	20	20	20	20	11.4%	2.33	11.1%	2.09
2	30	30	30	30	0	0	4.4%	2.87	4.6%	2.86
3	0	0	30	30	30	30	3.4%	2.97	4.6%	2.91
4	120	0	0	0	0	0	1.9%	0.98	1.1%	0.99
5	21	21	21	21	21	15	1.5%	3.19	3.3%	2.80
6	24	24	24	24	24	0	1.4%	3.08	1.6%	2.90
7	0	30	30	30	30	0	1.3%	2.93	1.2%	2.86
8	40	40	40	0	0	0	1.2%	2.76	1.6%	2.79
9	0	24	24	24	24	24	1.0%	3.16	1.9%	2.94

Table 1

We refer to a set consisting of all strategies obtained by permuting a particular strategy (ignoring the labels of the battlefields) as a permutation. The game's strategy space splits into 400 thousand permutations. The eight most popular permutations (those chosen by at least 2% of the participants) are the same in both populations and were chosen by 41-45% of the participants. Table 2 presents those permutations.

	Permutation	Classes	Mean Score	Calcalist	Mean Score
		n=4605		n=1928	
1	20-20-20-20-20-20	11.4%	2.33	11.0%	2.09
2	30-30-30-30-0-0	11.2%	2.92	13.3%	2.90
3	120-0-0-0-0-0	4.7%	0.99	2.4%	0.99
4	40–40-40-0-0-0	4.1%	2.78	4.5%	2.81
5	30-30-20-20-10-10	2.9%	2.74	2.0%	2.66
6	24-24-24-24-24-0	2.8%	3.12	4.0%	2.94
7	30-30-29-29-1-1	2.2%	3.27	2.5%	3.18
8	21-21-21-21-21-15	2.0%	3.20	4.8%	2.81

Table 2

Table 1 also presents the average score for each of the strategies. Table 2 also presents the average score for all participants whose strategies were in the same permutation. As can be seen from Table 1, the most popular strategies were not very successful. The best-performing strategies in the tournament scored around 3.8 points on average while the most popular strategies scored well below that.

3. Multi-dimensional (iterative) reasoning

As mentioned in the introduction, the set of strategies in the Blotto game is immense. It is therefore implausible that subjects hold "single point" beliefs. In other words, it is not likely that subjects assume that most of the other participants will choose a particular strategy. If subjects hold non-single point beliefs, then calculating the best response is enormously difficult, even if the support of the belief consists of only a few strategies. Thus, it is reasonable to assume that players do not adopt a process which involves finding a best response to a well-defined belief.

We propose an alternative procedure that fits not only the Blotto game but also other games that are characterized by a rich structure. According to this procedure, a player has in mind some major dimensions (or features) of a strategy. What those dimensions are depends on the structure of the game. The player makes a decision for each dimension separately, based on his belief regarding the other players' choices of values in that dimension. He starts by considering the instinctive value for this dimension. He might end the process at this point and choose that value. Alternatively, he may *respond properly* to this instinctive value. That is, he may apply what we call a *proper response* operator, which assigns an approximation of a best response to each value of the dimension. The player might repeat this process iteratively by properly responding to the value obtained in the previous step. Thus, if he carries out step 1 of the reasoning in this dimension, he chooses the value that properly responds to the step 0 value, which is the instinctive choice of value in this dimension. If he implements step 2 of the reasoning, he chooses the value which is the proper response to the step 1 value. Higher steps of reasoning are defined in a similar manner. After making a decision in all dimensions, the player chooses a strategy that fits all chosen values. Note, that this strategy is not necessarily a best response to a particular belief on the other players' strategies.

We were inspired to think of this scheme reading subjects' explanations of their choices in the Blotto tournament. Actually, we feel that such a procedure is not alien to real life practices. Consider, for example, competition between two fashion clothing firms, A and B, producing a similar product. Each firm need to choose the price and the design of its product before it learns about the choice of the other firm. In the previous year, both firms chose a price of $12 and produced a similar design. In one of its meetings, A's management decides to reduce the price to $10 since it expects firm B to reduce its price to $11 (A speculates that B anticipates that A will not be altering the price). In a separate meeting, A's management decides to adopt a new and more modern design for the product since it expects firm B to stick to last year's design. Given these two decisions, the planning department must come up with a new design that can be cheaply produced and has a modern look. The outcome could be a provocative design made out of cheap material. Is this a best response to firm B's strategy? Not necessarily.

Note the similarity and difference between this procedure and the k-level reasoning approach (see Stahl and Wilson (1995)). A k-level model assumes that the population is partitioned into a collection of types, which differ in their depth of reasoning. Thus, a level-0 type is non-strategic and follows a simple decision rule. It is generally assumed that he randomizes uniformly (see Crawford and Irriberi (2007) and Arad (2009) for different specifications of the level-0 type, which take into account instinctive attraction to salience). A k-level type, for any $k \geq 1$, best responds to the belief that all other players are level $k-1$ types. In previous studies of other games, it was found that the most common types are level-1 and level-2 and it is rare to observe behavior consistent with a higher level.

Our procedure differs from k-level reasoning in two major aspects: First, it relates to the features of the strategies rather than to the strategies themselves. This enables the level of reasoning to vary across dimensions. Second, it uses a "proper response" operator, which is only roughly connected to the best response operator.

4. Multi-dimensional reasoning in the Blotto game

Before demonstrating how the multi-feature procedure can be plausibly applied in the Blotto game, it is worthwhile reviewing the difficulties involved in applying the standard k-level approach: We begin from the conjecture that the most prominent starting point for iterative reasoning is the instinctive deployment of troops, i.e., 20 to each battlefield. We take this to be the natural specification for level-0 behavior for two related reasons. First, it has the aesthetic features which make it the first strategy to come to mind (evidence for which is the particularly low response time associated with this strategy). Second, it passes the "coordination test": if two people are playing a coordination game in which they are rewarded only if they simultaneously choose the same allocation of 120 troops across the 6 battlefields, they would obviously choose the instinctive strategy.

The difficulty arises when trying to specify the typical actions of higher-level types. Unlike some other simple games, here there are many best responses to the level-0 strategy, which makes the specification of the level-1 type unclear. Consequently, it is not reasonable to assign a single-point belief to the level-2 type. Holding a complex belief, which takes into account all the possible level-1 strategies, is not plausible either. Furthermore, the calculation of a best response to a non-degenerate distribution of level-1 strategies is very difficult even if the belief contains only a few strategies in its support.

Taking the level-0 to be a uniform randomization over the strategy space (that is, assigning equal probabilities to all 250 million strategies) is of no benefit since the calculation of the best response to this behavior is extremely difficult.

Thus, we find it more likely that a subject in the Blotto game employs the multi-dimensional (iterative) reasoning process described in Section 3. In other words, he chooses his strategy after considering several dimensions of the strategy and applying iterative reasoning to each dimension separately.

We focus on three important features of a strategy: the number of reinforced battlefields, the unit digit in a single-field assignment and the order of the six single-field assignments.

(a) The number of reinforced battlefields.

Choosing to reinforce $k = 0, 1, .., 5$ battlefields means that the subject has decided to strengthen his forces in k battlefields by assigning to each of them a large number of troops which is necessarily above the average (20).

The description of the iterative process in a particular dimension requires specifying the starting point and the proper response operator. The starting point of the iterative process in this dimension is the reinforcement of 0 battlefields since the instinctive strategy, which assigns 20 troops to each battlefield, involves 0 reinforcements. This strategy is of course the only non-dominated strategy that involves 0 reinforcements.

As for the proper response function, we define the reinforcement of 5 battlefields to be a proper response to the reinforcement of 0 battlefields and the reinforcement of 4 battlefields to be a proper response to the reinforcement of 5 battlefields. However, it is not clear that the reinforcement of 3 battlefields can be considered as the third step of reasoning. It is worthwhile elaborating on this point.

The first iterative step is to reinforce 5 battlefields. (The straightforward strategy of this kind involves deploying 24 troops in 5 of the battlefields.) If used against the step-0 strategy it will win 5 battles and thus score the maximum number of points possible in this game. Furthermore, if a player believes that a vast majority of participants, but not all, will reinforce 0 battlefields, then reinforcing 5 battlefields is a necessary condition for him to win the tournament. Given his belief, reinforcing 4 battlefields or less does not guarantee winning the tournament: if even one player in the tournament reinforces 5 battlefields, that player will score an average of almost 5 points, which is higher than the average score achieved by anyone who reinforces 4 battlefields or less.

The second step is the reinforcement of 4 battlefields. (The straightforward strategy of this kind involves deploying 30 troops in each of 4 battlefields.) The second-step type in this dimension believes that a vast majority of players reinforce 5 battlefields. By reinforcing 4 battlefields, the player expects to score about 4 points against the step-1 strategies, while the step-1 strategies will score around 3 points against each other. Thus, he will expect to win the tournament as long as the proportion of step-0 types is not greater than the proportion of step-1 types. Reinforcing less than 4 battlefields yields at most an average score of 3 points and is not successful given his beliefs.

An automatic continuation of the iterative process may lead to the thought that reinforcing 3 battlefields is a proper response to the reinforcement of 4 battlefields. Indeed, reinforcing 3 battlefields would generally yield a score of at least 3 points against step-2 strategies, whereas the average score of step-2 strategies against themselves is at most 3 (due to the possibility of ties). However, if in addition to step-2 strategies there are strategies of lower steps, a step-2 strategy will have the advantage of scoring about 4 points against these strategies. In such cases, reinforcing 3 battlefields may turn out to be inferior overall. Thus, the third iterative step is not clear-cut. In any case, the iterative chain stops here. Reinforcing less than 3 battlefields is not optimal against strategies that involve reinforcing 3 or more battlefields.

Note that in the calculation of a proper response a player uses "an approximation of best response" to the lower step. He does so ignoring the other dimensions of the strategy and believing that if he reinforces less battlefields than his opponent, then he is likely to win in each of the reinforced battlefields. This argument makes sense since he has more resources for each reinforced battlefield. However, this is just an approximation since it does not take into account the possibility that the opponent's assignment in each reinforced battlefield can differ in size and some assignments may be very large.

We proceed into the analysis of the data. We will say that a subject has reinforced a battlefield if he assigned there more than 20 troops. This definition is somewhat arbitrary but it allows partitioning the strategies according to the number of reinforcements and captures the essence of concentrating troops in certain battlefields (for example the strategy 23-23-23-23-14-14 involves the reinforcement of 4 battlefields).

The following table presents the distribution of the number of reinforced battlefields in each population.

# fields		Classes			Calcalist		
>20		%	Score	RT (Stdev)	%	Score	RT (Stdev)
0	step 0	13%	2.20	114s (3.8)	12%	2.00	81s (4.9)
5	step 1	8%	3.20	182s (9.9)	14%	2.94	133s (7.4)
4	step 2	25%	3.11	194s (5.0)	32%	3.07	143s (3.3)
3		22%	2.98	189s (6.0)	22%	2.91	128s (4.6)
2		20%	2.64	161s (4.2)	13%	2.55	125s (5.3)
1		13%	1.75	144s (7.4)	7%	1.77	106s (7.2)

Table 3

Note that in both populations, 74% of the subjects who reinforced 5 battlefields assigned 24 troops to each, but almost half of the subjects who reinforced only 4 battlefields did not assign equal number of troops to each reinforced battlefield.

The data on response time supports our intuition regarding the structure of iterative reasoning in this dimension. The step-0 strategy is associated with exceptionally low response time, indicating that this choice is indeed instinctive. The step-1 and 2 strategies are associated with a relatively high response time, suggesting the use of a more complex deliberation process. The response time for the step-2 strategies is somewhat higher than that for step-1 strategies. The response time of strategies with 3 reinforced battlefields is high as well, suggesting that subjects who made this choice were involved in a complex reasoning process as well. It is possible that these subjects continued the iterative reasoning process intuitively, in an attempt to respond properly to step-2 strategies (though it is not clear that their choice is actually a proper response). Subjects who decided to reinforce only one or two battlefields spent significantly less time on the decision, a hint that those bad choices were made hastily.

In Table 3 there is a difference between the Students and the Calcalist readers: the Calcalist readers tended to reinforce 4 or 5 fields relatively more often and to reinforce 1 or 2 fields less often. This might be because the Calcalist readers are more sophisticated. Alternatively, having participated in the two other games (the 91-100 game and the tennis coach problem) prior to the Blotto game may have triggered deeper iterated reasoning in this dimension among the Calcalist readers.

Note that the response time of the Calcalist readers was generally lower than that of the Students. The difference may be explained by the fact that the Calcalist readers were presented with an Hebrew version of the game, which is much shorter than the English version presented to the students. Moreover, Hebrew is the Calcalist readers' mother tongue, whereas many of the students in the classes are not native English. Another possible explanation is that some Calcalist readers read the descriptions of the games in the printed version of the newspaper prior to entering the experiment's website. However, we have indications that a vast majority of the participants did not see the games before they entered the experiment website. In any case, all the relevant patterns regarding response time will turn out to be the same for both populations.

(b) *The unit digit in single-field assignments*

A non-complicated and somewhat instinctive allocation of 120 troops across 6 battlefields involves single-field assignments that are multiples of 10 troops. Thus, we consider the use of the unit digit 0 in all battlefields as reflecting step 0 in this strategy's feature.

The most efficient way to win a battlefield is by assigning to that battlefield one troop more than the opponent. Thus, if a player suspects that a vast majority of the opponents' single-field assignments involve the use of a certain unit digit, using often a unit digit greater by one can be considered a proper response. (Note that the unit digits across the battlefields are not independent since they must sum up to a multiple of ten.) Of course, the assumption in the background is that the player's choice of tens digit will frequently be the same as his opponents'. Thus, the first iterative reasoning step would be the use of the unit digit 1 in some of the battlefields. The second step would be using the unit digit 2 and so on. We doubt, though, that the unit digit 7, for example, reflects 7 steps of reasoning. Recall that even in the simplest games studied in the literature, level 3 is rarely found and level 4 and higher levels of reasoning are almost non-existent.

Table 4 presents the distribution of unit digits in all the single-field assignments:

	Unit digit									
	0	1	2	3	4	5	6	7	8	9
Classes	62%	10%	4%	2%	4%	12%	1%	1%	2%	4%
Calcalist	55%	14%	5%	2%	5%	11%	1%	1%	2%	4%

Table 4

A majority of single-field assignments involved the unit digit 0 (almost all of them were either 0, 10, 20, 30 or 40). The unit digit 1 is heavily used (primarily in the choices 1 and 21). The step 2 choice of the unit digit 2 is less frequent (it appears primarily in the choices of 2 and 22).

We found that 29%-38% of the subjects used the unit digits 1 or 2 in at least one single-field assignment; about half of them used these unit digits in at least three battlefields. These subjects spent significantly more time on deliberation than other subjects (the median response times was 214s vs. 150s in the classes and 153s vs. 113s in the Calcalist sample).

The unit digit 5 is the second most frequent unit digit. Three possible explanations come to mind: (i) Allocating in fives is a secondary instinctive way. (ii) The assignment 15 is the left over after deploying 21 troops in 5 battlefields. (iii) The number 25 can be used in order to beat the straightforward strategy that allocates 24 troops to each of 5 battlefields.

Note that in Table 4 we find again a difference between the Students and the Calcalist readers: the Calcalist readers used the unit digits 1 and 2 more frequently and the unit digit 0 less frequently. This difference is in the same spirit of the difference indicated above regarding the number of reinforced fields.

(C) *Order of single-fields assignments*

Once a player has chosen a particular partition of his 120 troops into 6 "divisions", he also needs to decide how to allocate the (perhaps) different-sized divisions among the 6 battlefields. Natural procedures of allocating the troops may treat the battlefields in a non symmetric way. For example, a player could allocate divisions successively, starting with allocating the strongest division to battlefield 1 and ending with allocating the weakest division to battlefield 6. Alternatively, he could concentrate the stronger divisions in the middle battlefields and the weaker in the edges (or the opposite). Since there is more than one intuitive way to allocate the divisions, the step 0 value in this dimension is not clear-cut.

The definition of a proper response is also not as intuitive as in the previous dimensions. Assume, for example, that a player believes that the other player is concentrating his troops primarily in the middle battlefields. One proper response would be to concentrate more troops in the middle battlefields and to assign a relatively small number of troops to the edges. Another plausible proper response would be to abandon the two central battlefields and assign more troops to all other battlefields in order to increase the chances of winning those battles.

The ambiguity in specifying a natural common starting point for iterated reasoning and in the definition of a proper response make the identification of steps of reasoning less appealing. In the following, we present some interesting patterns related to the order feature, while leaving aside the identification of steps of reasoning. The six battlefields are numbered 1 to 6. Naturally, we focus on two types of symmetry:

directional: Are battlefields 1,2,3 treated identically to battlefields 6,5,4 respectively?

positional: Is the pair of battlefields in the center (3 and 4) treated the same as the pair of battlefields in the edges (1 and 6) and as the pair of battlefields in the mid-positions (2 and 5)?

It should be mentioned that the Students and the Calcalist readers played versions of the game that differed in one framing detail: Thus, in the game played by the Students, the 6 battlefields were arranged vertically, with battlefield 1 on top and battlefield 6 on the bottom. In the game played by the Calcalist readers, the battlefields were arranged horizontally with battlefield 1 on the left and battlefield 6 on the right. However, this did not appear to have any effect on the results concerning this dimension.

The following graphs present the cumulative distribution of the number of troops assigned to each of the six battlefields. The graphs reveal that the marginal assignment of troops to each battlefield is far from being that induced by the uniform distribution over the interval [0,40], which is the game-theoretic prediction for the classical constant-sum Blotto game. Note the dramatic "jumps" around numbers like 20 and 30 and that only 10% (rather than 25%) of the choices in each field are above 30.

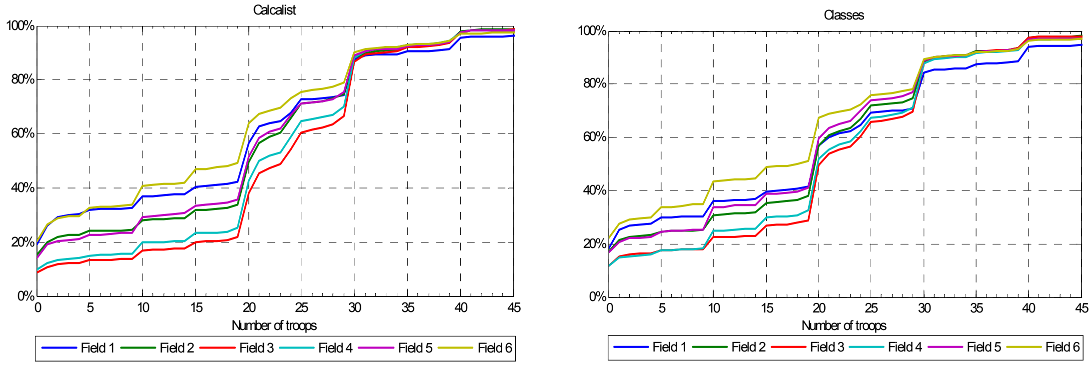

Figure 1: Cumulative distribution of single field assignments in the six fields

The cumulative distributions are essentially ordered identically in the two populations by first-order stochastic domination: 3,4,2,5,1,6. For convenience, we also present the 33^{rd}, 50^{th}, 67^{th} percentile points for each of the 6 battlefields:

	Classes						Calcalist					
Field	1	2	3	4	5	6	1	2	3	4	5	6
33^{rd} percentile	10	15	20	20	10	5	10	19	20	20	15	8
median	20	20	21	20	20	19	20	21	24	22	20	20
67^{th} percentile	25	24	27	25	24	20	24	25	30	28	24	21

Table 5

Most noticeable is the low number of troops assigned to the 6th battlefield and the high number assigned to battlefields 3 and 4. This is in line with some other experimental results which demonstrate a tendency of people to avoid the edges and to concentrate resources on the center positions (see, for example, Rubinstein, Tversky and Heller (1996)).

There is almost no distinction between right and left. Battlefields 2 and 5 are treated almost symmetrically and the distributions for battlefields 3 and 4 are also very close. The most significant directional asymmetry is that more troops are assigned to battlefield 1 than to battlefield 6, probably because allocations are sometimes executed from the first battlefield to the last and battlefield 6 is treated as the "residual".

As can be seen in Table 6, some of the most popular single-field assignments appear in a non-symmetric way in the six fields. For example, the assignments 0 and 1 are twice as frequent in battlefields 1 and 6 as in battlefields 3 and 4 and the frequency of the assignment of 30 to each of the battlefields 3 and 4 is much higher than for the other pairs.

	Classes						Calcalist					
Field	1	2	3	4	5	6	1	2	3	4	5	6
Assignment 0	18%	17%	12%	12%	17%	22%	19%	15%	9%	10%	14%	20%
Assignment 1	7%	4%	3%	3%	4%	5%	7%	5%	2%	3%	5%	6%
Assignment 30	13%	14%	18%	17%	12%	11%	13%	14%	20%	18%	14%	11%

Table 6

5. Support for the presence of multi-dimensional iterative reasoning

At this stage, we wish to introduce the novel "91-100 game". Calcalist readers (unlike the Students) played this game before playing the Blotto game. (For a more detailed analysis of several variants of the 91-100 game, see Arad and Rubinstein (2009).) The fact that the same subjects played both the 91-100 game and the Blotto game enables us to examine the correlation between their observed behavior in the two games. This can help in evaluating our interpretation of subjects' reasoning in the Blotto game. The data in this section is based solely on the Calcalist's subjects.

5.1 The 91-100 game

Following is a description of the game as presented (in Hebrew) to the Calcalist readers:

You and another person are playing a game in which each player requests an amount of money. The amount must be an integer between 91 and 100 shekels. Each player will receive the amount he requests. A player will receive an additional amount of 100 shekels if he asks for exactly one shekel less than the other player.

What amount of money would you ask for?

In this game it is hard to think of more than one dimension for a strategy. We find the game in particular suitable for studying (one-dimensional) k-level thinking for three reasons:

(i) *The level-0 type specification is intuitively appealing*

The choice of 100 is a natural anchor for an iterative reasoning process because the instinctive choice when choosing a sum of money between 91 and 100 shekels (100 is the salient number in this set and "the more money the better"). The choice of 100 is in fact not entirely naive. If a player does not want to take any risk or prefers to avoid competition, he might give up the attempt to win the additional 100 shekels and simply request the highest certain amount.

(ii) *Best-responding is easy*

Given the anchor 100, best-responding to any level-k action is very simple and leaves no room for errors.

(iii) *Robustness to the level-*0 *specification*

The type-1 action, i.e. choosing 99, is the unique best response to a wide range of reasonable beliefs including (a) all distributions in which 100 is the most frequent choice and (b) the uniform distribution and a class of beliefs that are close to it. This makes the analysis robust to the specification of the level-0 behavior.

Assuming that all players wish to maximize the expected amount of shekels they receive, the game has a unique symmetric mixed-strategy Nash equilibrium (which yields an expected payoff of 100).

Action	91	92	93	94	95	96	97	98	99	100
Equilibrium	55%	9%	8%	7%	6%	5%	4%	3%	2%	1%
Experiment	18%	19%					10%	21%	18%	14%

Table 7

As shown in Table 7, the behavior of subjects is very far from the Nash equilibrium. Most notable is the low percentage of subjects who chose 91 relative to the equilibrium prediction. The choice of 100, which in equilibrium appears only rarely, was chosen by 14% of the subjects. The actions 97-98-99 which seem to exhibit 1-2-3 levels of reasoning were chosen by 49% of the subjects, whereas in equilibrium they should have been chosen by only 9%. As noted above, higher levels of iterative reasoning are almost never observed in other studies of k-level reasoning. In our results, the actions 92-96 are also rare and appear much less often than expected by equilibrium.

In a parallel experiment of this game, about 160 Students provided ex-post explanations for their choices. An analysis of their explanations suggests that the actions 92-96 are generally not an outcome of 4- to 8-level of reasoning. It also validates the classification of 97-98-99 as the 1-2-3 levels of reasoning.

5.2 Correlation between behavior in "91-100" and "Blotto"

In this section, we seek support for our interpretation of some of the choices in the Blotto game as an outcome of multi-dimensional steps of reasoning. This is done by investigating the correlation between standard k-level iterative behavior in the 91-100 game and behavior in the Blotto game which seem to exhibit iterative reasoning in the various dimensions. More precisely, we examine the correlation between the choices 97-98-99 in the 91-100 game and either reinforcing 4 or 5 battlefields or using the unit digits 1 or 2 in the Blotto game . Since the choices 97-98-99 clearly reflect 1-2-3 levels of reasoning in the 91-100 game, evidence of correlation will provide support for our intuition that these Blotto game choices emerge from iterative reasoning.

Table 8 shows the distributions of choices in the 91-100 game as a function of the number of reinforced battlefields:

Fields with #>20		Action in 91-100			
		91	92-96	97-99	100
0	Step 0	26%	19%	34%	20%
5	Step 1	17%	13%	57%	13%
4	Step 2	16%	17%	55%	12%
3		16%	19%	51%	15%
2		20%	26%	41%	13%
1		14%	25%	41%	20%

Table 8

Subjects who did not reinforce any of the battlefields (i.e. 0 reinforcements) tended to choose 91 and 100 more often and to choose 97-99 dramatically less often than the other subjects. Of those who reinforced 4 or 5 battlefields, 55-57% chose 97-99 (σ =2%) whereas of those who reinforced 2 or less battlefields the proportion was only 38% (σ =2%). The behavior of subjects who reinforced 3 battlefields resembled more that of the subjects who reinforced 4 or 5 battlefields: 51% of them chose 97-99 (σ =2%). This finding supports our conjecture that strategies involving 3 battlefields reinforcements are the result of an intuitive continuation of the iterative reasoning process in this dimension.

Table 9 presents the mean number of battlefields with the unit digits 1 or 2 in the Blotto game as a function of the choices in the 91-100 game.

	Action in 91-100			
	91	92-96	97-99	100
Blotto	18%	19%	49%	14%
Mean # of fields with digit 1 or 2	0.86	1.00	1.22	1.06

Table 9

Table 9 demonstrates that the tendency to use the unit digits 1 and 2 is correlated with choosing 97-99 in the 91-100 game. Another way to see it: more than 55% of the subjects who used the unit digits 1 and 2 at least once chose 97-99, while only 45% of those who did not use those digits chose 97-99.

Recall that a total of 46% of the subjects in the Blotto game reinforced 4 or 5 battlefields and 38% of the subjects used the unit digits 1 and 2 at least once. The choices of 24% of the subjects exhibit iterated reasoning in both dimensions. We found that these subjects' behavior is correlated with the choices of 97-99 in the

91-100 game. In particular, 59% (σ =2%) of them chose 97-99, whereas among those who reinforced less than 3 battlefields and did not use the digits 1 or 2, only 39% (σ =2%) made those choices.

Incidentally, the choices of 97-99 are also correlated with a high score in the Blotto game. Table 10 demonstrates this by comparing the 91-100 choices of the highest-performing 20% in the Blotto game with the choices of the rest of the subjects.

	91	92-96	97-99	100
top 20%	10%	20%	60%	11%
the rest	20%	19%	46%	15%

Table 10

6. The Winning Strategies

A surprising result: The winning strategy in the classes' grand tournament and in the Calcalist tournament was the same: 2-31-31-31-23-2 (and needless to say, was chosen by two different people....). Furthermore, there was also significant overlap in the lists of the top 10 strategies in the two tournaments (see Table 11). Four strategies are common to both lists and, up to a permutation, 7 out of the top 10 strategies on the Classes list appear on the other list as well.

Classes' grand tournament								Calcalist tournament							
	1	2	3	4	5	6	Mean Score		1	2	3	4	5	6	Mean Score
1	**2**	**31**	**31**	**31**	**23**	**2**	3.83	1	**2**	**31**	**31**	**31**	**23**	**2**	3.77
2	3	31	31	31	21	3	3.80	2	2	32	31	31	22	2	3.76
3	3	31	3	31	31	21	3.76	3	2	23	31	31	31	2	3.75
4	**1**	**31**	**31**	**31**	**25**	**1**	3.76	4	1	1	32	32	32	22	3.72
5	**2**	**27**	**31**	**31**	**27**	**2**	3.75	5	**1**	**1**	**31**	**31**	**31**	**25**	3.71
6	2	31	23	31	31	2	3.74	6	**2**	**27**	**31**	**31**	**27**	**2**	3.71
7	**1**	**1**	**31**	**31**	**31**	**25**	3.73	7	2	31	1	31	31	24	3.70
8	2	21	32	32	2	31	3.72	8	**1**	**31**	**31**	**31**	**25**	**1**	3.70
9	1	1	31	31	25	31	3.71	9	1	25	31	31	31	1	3.69
10	1	31	31	25	31	1	3.69	10	1	1	34	31	31	22	3.69

Table 11

We were curious as to whether there is another strategy that was not chosen and could have done better than the others. We simulated 41,040 strategies in which all unit digits are 1,2 or 3 and found no such strategy.

The features of the winning strategy 2-31-31-31-23-2 are illuminated by the explanation provided by the Calcalist winner:

"In the first stage, I decided that I would "surrender" on two fronts, but not so easily. I thought that other people would decide to assign a few battalions to some of the fronts and perhaps would not deploy any battalions to other fronts. So I could win on an "abandoned" front at the inexpensive price of one battalion. Eventually, I decided to deploy two battalions on the weak fronts in order to overpower anyone who thought like me and placed one battalion on the weak fronts. It seems logical to me that the weak fronts would be on the edges. I was left with 116 battalions to allocate to four fronts, which is an average of 29 battalions per front. I decided to reinforce three of the four remaining fronts with two battalions - that is, to deploy 31 battalions - in order to defeat those who allocated the remaining battalions equally. In this way, I would also defeat those who allocated 30 battalions to each of the four central fronts."

Here are some of the features characterizing the ten leading strategies:

a) Two battlefields were essentially abandoned. In fact, all 30 leading strategies in the two tournaments used a low number of troops (1,2 or 3) in exactly two single-field assignments.

b) The most often almost abandoned battlefields are 1 and 6. This is profitable since these battlefields tended to be abandoned in the population much more than the middle ones.

c) Battlefields 2 and 5 were treated rather symmetrically (and, in particular, the strategy 2-23-31-31-31-2 does almost as well as the winning strategy).

d) 30+ troops are generally assigned to the middle battlefields. This is beneficial since the assignments to battlefields 3 and 4 tended to be the highest.

It is interesting to look at the winning strategies in the 11 tournaments of the largest classes, which contained at least 60 subjects. In 4 of these classes (Argentine (2) and Canada (2)), a permutation of 1-35-1-31-31-21 was the winning strategy. In other 4 (Switzerland (2), Thailand and Slovakia), a permutation of 31-1-31-1-31-25 was the winning strategy. In the remaining 3 large classes (in Switzerland and Argentina), the winning strategies were 31-31-31-21-3-3, 3-21-3-31-21-21 and 7-33-33-7-33-7. Note that 8 out of the 11 winning strategies belong to the same two permutations. All winning strategies in the 11 large classes, like the overall leading strategies (in the two grand tournaments), involved the reinforcement of 4 battlefields and the avoidance of multiples of ten. The winning strategies in the large classes performed well in the grand tournament as well. While the top 10 strategies in the grand tournament scored on average 3.7-3.83, the winning strategies in the large classes achieved an average score of 3.6-3.76 in the grand tournament.

7. Comments

Bibliographic notes

The classic Blotto game in its continuous version was explored analytically by Roberson (2006). The more difficult discrete case, with B troops allocated to K battlefields, was analyzed by Hart (2008). Both concluded that in an equilibrium, players treat the battlefields symmetrically and the marginal distribution of the troops in each battlefield is essentially uniform in the interval $[0, 2B/K]$. We are not aware of a game-theoretical analysis of the non-constant-sum version studied here.

The Blotto game has received widespread attention due to its interpretation within the political economics literature as a game between two presidential candidates who have to allocate their limited budgets to campaigns in the "battlefield" states. Myerson (1993) suggested another interpretation of the Blotto game as a vote-buying game.

Only a few experiments of Blotto games have been conducted. Partington reports in his website (http://www.amsta.leeds.ac.uk/~pmt6jrp/personal/blotto.html) on a Blotto game tournament conducted in 1990. In his version, the subjects had to allocate 100 troops across 10 battlefields. The winning strategy was 17-3-17-3-17-3-17-3-17-3.

Avrahami and Kareev (2009) report on an experiment of a "lottery version" of the constant-sum game. Each subject played 8 times in a row against a single player. In each round, once the two players have chosen their allocation of troops, one battlefield per player was randomly selected and the winner of the round was determined by comparing between the assignments in the two selected battlefields. This design prevents framing effects induced by the ordering of the battlefields. Among other things, the authors studied the case in which each player assigns 24 troops among 8 battlefields. In this case, the theory predicts that the marginal distribution of the assignment in each battlefield will be uniform in $[0, 6]$. In the vast majority of observations, 2-4 troops were assigned to each battlefield and a significant number of subjects allocated the troops homogeneously (3 troops to each battlefield). For another recent experiment of the game, see Chowdhury, Kovenock and Sheremeta (2009).

Time and Performance

Does an investment of more thought in the Blotto game translate into a better performance?

A standard regression affirms that response time contributes significantly and positively to performance in the Blotto game. Thus, for the Students sample we obtain the equation: $score = 2.07 + 0.12 ln(response \quad time)$ and for the Calcalist sample we obtain a similar equation: $score = 1.99 + 0.15ln(response\ time)$.

In order to better understand the correlation, we divided the two populations into ten deciles according to their response time. Figure 2 plots the average score and the two standard deviations for each decile. We find that the mean score of the three bottom deciles is dramatically lower than that of the two top deciles. On the other hand, the average performance in the 4^{th}-8^{th} deciles is almost identical.

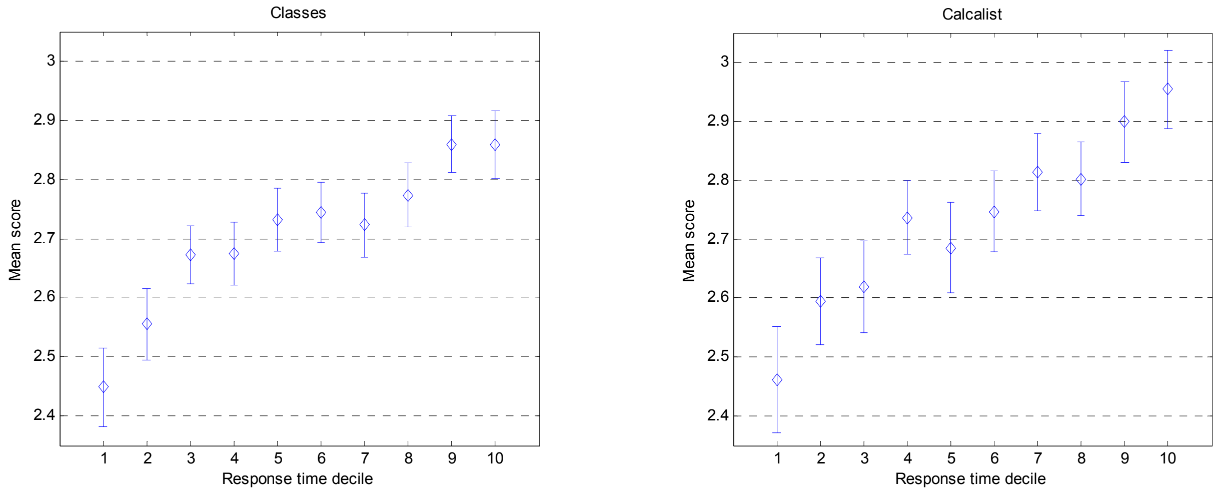

Figure 2: Mean score by response time deciles

The best response to the uniform distribution

In most of the literature on k-level reasoning, the level-0 behavior is taken to be a uniform distribution over the set of strategies. In the Blotto game, however, the uniform distribution is not a plausible description of making an arbitrary choice without the use of strategic reasoning. Nonetheless, we were curious to see what would be the best response if all the players in the tournament had chosen their strategies randomly according to a uniform distribution. For this purpose, we ran a simulation of the Blotto tournament with 9990 strategies drawn from the uniform distribution together with the five leading strategies and the five most popular strategies in the Calcalist tournament.

The winning strategy in this simulation, with a score of 3.54, was the homogeneous strategy, which assigns 20 troops to each field. Ex-post we understand that this is indeed the best strategy against this distribution. This is because when strategies are chosen from the uniform distribution, the marginal cumulative distribution of troops for each field is concave. None of the winning strategies in the experiment performs well in the simulated tournament. The winning strategy in the two experiments scored only about 3.10 in the simulation; more than 30% of the strategies performed better than this strategy. Nevertheless, it is difficult to think of the homogeneous strategy as a best response to the uniform distribution.

Attentiveness of subjects

Subjects were not forced to assign all the 120 troops across the battlefields. This was a device for checking their attentiveness. Among the students, only 5.4% of the subjects chose such a dominated strategy. Among Calcalist's readers, the proportion dropped to 3.0%.

Note also that only 12% of our subjects chose the instinctive homogenous strategy. To scale this fact, in the Avrahami and Kareev (2009) experiment – which was carried out in a laboratory with monetary incentives – 25% of the subjects chose the homogenous strategy in the first round of the game.

Gender effects

Calcalist readers (but not the students) were asked to report their gender. Only 10% of the readers were females. Males did significantly better with an average score of 2.75 (σ =0.01) as compared to 2.55 (σ =0.04) for females. This is in spite of the fact that females spent more time on the game (females' median response time was 145 vs. 125 for the males).

Experience in Game Theory

We asked the Calcalist readers whether they had ever taken a course in Game Theory and one-sixth of them had. One might expect that a course in game theory would improve one's ability to play games; especially a synthetic game like the one experimented here. We found that taking a Game Theory course has only a marginal effect: The average score of Game Theory graduates was 2.78 (σ =0.03) as compared to a close average score of 2.72 ($\sigma = 0.01$) for the others. In other words, a course in Game Theory may be entertaining, but there is no evidence that it helps in playing games.

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