Economics Bulletin

Volume 30, Issue 3

On the stability of network structures with public goods

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Abstract

This paper explores the formation of stable network structures in a model with public goods. The multiplicity of equilibria in the non-cooperative formulation of network formation games brings out further difficulties in analyzing stability of network structures. This contrasts with the cooperative game approach where payoffs for agents are predetermined and thereby the multiplicity of equilibrium issues are sidestepped. We took issue with the multiplicity of equilibrium effort levels exerted on a given network structure, and we suggested different stability definitions for such network structures under multiplicity of equilibria. We demonstrated how these stability notions work for the network structures with four agents where breaking and forming links is costless, and the cost of exerting effort level is linear.

Citation: Mehtap Işık and Unal Zenginobuz, (2010) ''On the stability of network structures with public goods'', *Economics Bulletin*, Vol. 30 no.3 pp. 2425-2436.
Submitted: Aug 12 2010. **Published:** September 17, 2010.

1 Introduction

We aim to explore the formation of network structures over which a public good is produced. We follow the network model by Bramoulle and Kranton (2007) in which two agents who are directly linked to each other share the public good produced. If an agent has no link, he produces the public good himself and no one else can utilize the good. Agents should exert effort to produce the public good. The benefit that an agent gets from the public good accrues from all the direct links that the agent has. Therefore, an agent will determine his optimal effort level depending on the network structure. Bramoulle and Kranton (2007) analyze certain properties of the Nash equilibria of effort levels exerted in a given network structure. They show that at least one Nash equilibrium of effort levels exist for a given network, and the equilibrium is not unique for most of the networks. Going beyond their analysis, we address stability of network structures. However, the multiplicity of the equilibrium effort profiles becomes an issue to be tackled in defining stability of network structures. In this paper, we suggest different stability notions to take into account the issue of multiple equilibria when agents decide on how much to contribute to provide a public good on a network.

Network models typically treat an agent's payoff in a network structure as fixed and given exogenously.[1] However, in a network model where agents provide a public good voluntarily, an agent's payoff is determined by the equilibrium of the corresponding game. Ballester et al. (2006) and Calvo-Armengol and Jackson (2006) analyze the formation of networks of that type, but in their models there exist unique Nash equilibrium of the actions taken by the agents in the network structure. In our network model with public goods, there may be multiple Nash equilibrium of effort profiles in a given network.

The plan of the paper is as follows. In the next section, we state the model and characterize the equilibrium of effort profiles for a given network structure. Then, we show that conditions given to guarantee uniqueness of equilibrium in a standard public good model with voluntary contributions (Cornes, Hartly and Sandler, 1999) are not sufficient on a network structure unless every agent is directly linked to all other agents. In the third section, we define stability of network structure in the presence of multiple equilibrium effort profiles in four different ways. In Section 4, we demonstrate the stability notions defined using an extended example in a society of four agents.

2 The Model

Agents from the set $N = \{1, 2, \cdots, n\}$ demand a public good. They can set links with each other. We denote by ij the *link* between agents i and agent j. A *network structure* that results from the links formed between agents is represented by a graph g. We use the notation $ij \in g$ to indicate that there exist a link between agent i and agent j in the network g. G^N stands for the set of all possible networks for society N. When every agent has direct links with all other agents in a network g, we call g *complete network*. The set of *neighbors* of agent i in network g is defined as the agents who are directly linked to agent i, $N_i^g = \{j : ij \in g\}$. $N_i^g \cup \{i\}$ is the *neighborhood* of agent i in network g. Each agent benefits from the public good produced by their neighbors.

In order to contribute to the provision of the public good, agent i in network g exerts

[1] A detailed and recent literature survey on economic and social networks is given by Jackson (2010). Jackson and Wolinsky (1996) and Jackson (2002) study the stability and efficiency of social and economic networks when self-interested agents form or break links.

effort $e_i^g \geq 0$. He benefits from public good as a result of his own effort $e_i^g \geq 0$ and the sum of efforts exerted by his neighbors in g, $e_{-i}^g = \sum_{j \in N_i^g} e_j^g$. Public good is produced from efforts in a one-to-one ratio. Agents enjoy the public good according to a strictly concave benefit function, $b(e)$, with $b : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and where $b(0) = 0$, $b'(0) > 0$ and $b''(0) < 0$. The cost of contribution for any agent is assumed to be linear: $C(e_i) = ce_i$, where $0 < c < b'(0)$. The overall effort profile $\mathbf{e}^g = \{e_i^g\}_{i=1}^g$ shows how much agent i contributes to the provision of public good in network g. Then, the net utility of agent i in network g is

$$U_i(g, \mathbf{e}^g) = b\left(e_i^g + e_{-i}^g\right) - ce_i^g. \tag{1}$$

We test stability of a given network structure for breaking of existing links and addition of new links one by one. In each network structure, agents decide non-cooperatively on their efforts in provision of the public good. Therefore, an equilibrium outcome of the game consists of a couple $(g, \mathbf{e}^g)$, where g is the graph of network structure formed and $\mathbf{e}^g$ is an effort profile exerted in g.

2.1 Stability of equilibrium effort profiles on fixed networks

Let e^* denote the aggregate effort level that satisfies $b'(e^*) = c$. In provision of the public good on a given network g, the problem of agent i is given by:

$$\max b\left(e_i^g + e_{-i}^g\right) - ce_i^g. \tag{2}$$

Observe that a Nash equilibrium effort profile $\mathbf{e}^g = (e_1^g, \cdots, e_i^g, \cdots, e_n^g)$ is such that,

$$e_i^g = \max\left\{0, e^* - e_{-i}^g\right\}. \tag{3}$$

Let $NE(g) = \{\mathbf{e}^g : \mathbf{e}^g \text{ is a Nash equilibrium effort profile in } g\}$ denote the Nash equilibrium effort profiles in a network g. $NE(g)$ is non-empty for any network g (Theorem 1, Bramoulle and Kranton, 2007).

Following Bramoulle and Kranton (2007), we call a Nash equilibrium effort profile $\mathbf{e}^g$ *stable equilibrium effort profile in* g if and only if there exist a positive number $\rho > 0$ such that for any vector ε satisfying $|\varepsilon_i| \leq \rho$ and $e_i^g + \varepsilon_i \rho$, for all i, the sequence $\mathbf{e}^g(n)$ defined by $\mathbf{e}^g(0) = \mathbf{e}^g + \varepsilon$ and $\mathbf{e}^g(n+1) = f(\mathbf{e}^g(n))$, where $f_i(\mathbf{e}^g)$ is the best response of agent i to profile $\mathbf{e}^g$, converges to $\mathbf{e}^g$ as n goes to infinity. In a stable effort profile in a given network, no agent changes his equilibrium effort level in response to changes in the efforts exerted by his neighbors in that network. Let $S(g) = \{\mathbf{e}^g : \mathbf{e}^g \text{ is a stable equilibrium effort profile in } g\}$ denote the set of stable equilibrium effort profiles in g. Note that $S(g)$ will be empty for some networks g (see Table A1 for an example).

2.2 Uniqueness of equilibrium effort levels on fixed networks

Cornes et al.(1999) provide uniqueness results for non-cooperative provision equilibrium of the standard public good game where everyone benefits from contribution of everyone else. Note that the complete network structure in our model corresponds to that standard case. Given their results, the assumptions of our model on the benefit and the cost functions lead to a unique Nash equilibrium effort profile in the complete network. However, these assumptions are not enough to generalize the uniqueness of the Nash equilibrium for all network structures of the same society.

Suppose $N = \{1,2,3,4\}$ and $b(e) = 2\sqrt{e}$ and $c(e) = \frac{1-\sqrt{2}}{\sqrt{2}}e^2 + \frac{1}{\sqrt{2}}$. The function $b(\cdot)$ and $c(\cdot)$ satisfy all the assumptions that guarantee the uniqueness for the complete network, i.e. $b(\cdot)$ is strictly increasing and concave, with $b(0) = 0$, and $c(\cdot)$ is strictly increasing and convex, with $b'(\infty) \leq c'(0)$ (see Cornes et al., 1999). In the unique Nash equilibrium effort profile in the complete network each agent exerts the same effort level $e = 0.37$. However, in the *circle structure,* where each agent has 2 links (see g_9 in Figure 1), there are three different Nash equilibrium effort profiles: $(0.47, 0.47, 0.47, 0.47)$, $(0,1,1,0)$, and $(1,0,0,1)$.

3 Stability of Network Structures

Sections 2.1 and 2.2 stated that there exists at least one Nash equilibrium effort profile in any given network in our model, but there will typically exist multiple equilibria. Restriction of effort profiles to Nash equilibrium effort profiles do not reduce the number of possible payoffs of agents to a unique value. Therefore, further restrictions will be required to make precise what is meant by the stability of a network structure.

Note that, since agents' efforts are strategic substitutes in our model, breaking of an existing link or formation of a new link will affect possibly all agents in a network structure. The agents whose approval is needed for breaking and/or forming links in a given network structure will play an important role in the formation of a new network.

In the subsections below where we define four different stability notions, we impose different requirements related to how an agent evaluates possible payoffs that will arise when he breaks or forms a link, as well as to whose approval is needed to break or form links.

3.1 Strongly Pairwise Stability

In the first definition, we assume that the formation of a new link requires just the approval of two agents who form the link. Also, an agent can violate an existing link freely, without the consent of the other agent who is part of that link. We also assume that an agent will break an existing link, or agree to form a new link, provided that there exists at least one Nash equilibrium effort profile in the network structure that will follow immediately after his action and gives him higher utility. In other words, we are assuming optimistic agents who are also very myopic in the sense that they consider only the best possible outcome after their own deviation. An opportunity for a gain in a possible network structure is enough for an agent to form or break a link. The agent does not care whether the immediately profitable Nash equilibrium effort profile in the new network structure is going to be stable.

Formally, we call the network structure g with the equilibrium effort profile $\mathbf{e}^g$ as *strongly pairwise stable (SPS)* if and only if

1. $\forall\ ij \in g, \forall\ \widetilde{e}^{g-ij} \in NE(g - ij)$, we have $U_i(g, \mathbf{e}^g) \geq U_i(g - ij, \widetilde{\mathbf{e}}^{g-ij})$ and $U_j(g, \mathbf{e}^g) \geq U_j(g - ij, \widetilde{\mathbf{e}}^{g-ij})$;

and

2. $\forall\ ij \notin g$, if $\exists\ \widetilde{e}^{g+ij} \in NE(g + ij)$ such that $U_i(g, \mathbf{e}^g) < U_i(g + ij, \widetilde{\mathbf{e}}^{g+ij})$, then $U_j(g, \mathbf{e}^g) > U_j(g + ij, \widetilde{\mathbf{e}}^{g+ij})$.

3.2 Weakly Pairwise Stability

In the second definition, approval requirements for adding or breaking a link are the same as in the previous definition. On the other hand, in deciding to break or form a link, an agent looks at whether a stable as well as a profitable Nash equilibrium effort profile exists in the new network structure that will ensue one-step after his move.

Formally, we call the network structure g with the equilibrium effort profile $\mathbf{e}^g$ as *weakly pairwise stable (WPS)* if and only if

1. $\mathbf{e}^g$ is a stable equilibrium effort profile in network g (i.e., $\mathbf{e}^g \in S(g)$);

2. $\forall\ ij \in g,\ \forall\ \widetilde{e}^{g-ij} \in S(g-ij)$, we have
$U_i(g, \mathbf{e}^g) \geq U_i(g-ij, \widetilde{\mathbf{e}}^{g-ij})$ and $U_j(g, \mathbf{e}^g) \geq U_j(g-ij, \widetilde{\mathbf{e}}^{g-ij})$;

and

3. $\forall\ ij \notin g$, if $\exists\ \widetilde{e}^{g+ij} \in S(g+ij)$ such that $U_i(g, \mathbf{e}^g) < U_i(g+ij, \widetilde{\mathbf{e}}^{g+ij})$, then
$U_j(g, \mathbf{e}^g) > U_j(g+ij, \widetilde{\mathbf{e}}^{g+ij})$.

3.3 Approval Stability

In the third definition, we consider that an agent can freely break one of the existing links, but he will need the approval of all agents in order to form a link. Formally, we call the network structure g with the equilibrium effort profile $\mathbf{e}^g$ as *approval stable (AS)* if and only if

1. $\forall\ ij \in g,\ \forall\ \widetilde{e}^{g-ij} \in NE(g-ij)$, we have
$U_i(g, \mathbf{e}^g) \geq U_i(g-ij, \widetilde{\mathbf{e}}^{g-ij})$ and $U_j(g, \mathbf{e}^g) \geq U_j(g-ij, \widetilde{\mathbf{e}}^{g-ij})$;

and

2. $\forall\ ij \notin g$, if $\exists\ \widetilde{e}^{g+ij} \in NE(g+ij)$ such that $U_i(g, \mathbf{e}^g) < U_i(g+ij, \widetilde{\mathbf{e}}^{g+ij})$, then
$\exists\ k \in N = \{1, 2, \cdots, n\}$ such that $U_k(g, \mathbf{e}^g) > U_k(g+ij, \widetilde{\mathbf{e}}^{g+ij})$.

3.4 Full Approval Stability

In the fourth and last definition, agents in a given network structure g need the consent of the whole society to break as well as to form links. Formally, we call the network structure g with the equilibrium effort profile $\mathbf{e}^g$ as *full approval stable (FAS)* if and only if

1. $\forall\ ij \in g$, if $\exists\ \widetilde{e}^{g-ij} \in NE(g-ij)$ such that $U_i(g, \mathbf{e}^g) < U_i(g-ij, \widetilde{\mathbf{e}}^{g-ij})$, then
$\exists\ k \in N = \{1, 2, \cdots, n\}$ such that $U_k(g, \mathbf{e}^g) > U_i(g-ij, \widetilde{\mathbf{e}}^{g-ij})$;

and

2. $\forall\ ij \notin g$, if $\exists\ \widetilde{e}^{g+ij} \in NE(g+ij)$ such that $U_i(g, \mathbf{e}^g) < U_i(g+ij, \widetilde{\mathbf{e}}^{g+ij})$, then
$\exists\ k \in N = \{1, 2, \cdots, n\}$ such that $U_k(g, \mathbf{e}^g) > U_k(g+ij, \widetilde{\mathbf{e}}^{g+ij})$.

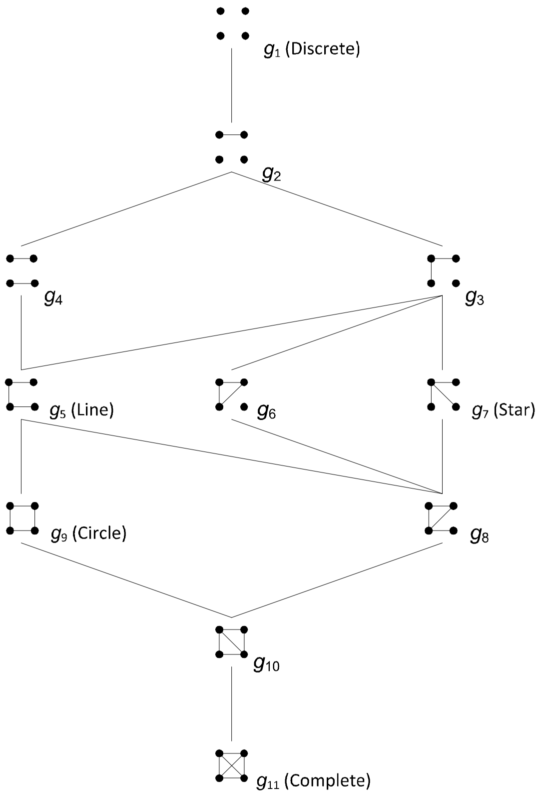

Figure 1: Network tree of four agents

4 An Example

We consider a society of four people. Note that there exist 11 possible network structures in that case. In Figure 2, we list the graphs that represent possible network structures formed by four agents. A line between two types of network structures in Figure 2 represents that it is possible to form one network structure from the other through breaking an existing link or adding a new link.

We test stability of network structures under the definitions that were given in the previous section. Table 1 summarizes the results (details are given in the Appendix). A network structure in a row is stable according to a stability definition in a column when an equilibrium effort profile in the set given in the cell corresponding to that row and that column is exerted. When there is no equilibrium effort profile that makes the network structure stated in a row stable according to the definition in a column, we leave the cell corresponding to that row and that column empty.

We observe from the first column that the set of strongly pairwise stable network structures (SPS) is empty. The second column shows that there does exist stable network structures, including the discrete graph, under the weakly pairwise stability (WPS) notion. Moreover, requiring approvals of all agents in forming links leads to existence of stable network structures (third column). The more stringent is the approval requirement, the higher will be the number of stable structures (fourth column).

Table I: Stable network structures

	SPS	WPS	AS	FAS
g_1 (Discrete)		$\{(e^*,e^*,e^*,e^*)\}$		
g_2				
g_3		$\{(0,e^*,e^*,e^*)\}$		
g_4			$\{(e_1^{g_4},e^*-e_1^{g_4},e_3^{g_4},e^*-e_3^{g_4}): 0\leq e_1^{g_4},e_3^{g_4}\leq e^*\}$	$\{(e_1^{g_4},e^*-e_1^{g_4},e_3^{g_4},e^*-e_3^{g_4}): 0\leq e_1^{g_4},e_3^{g_4}\leq e^*\}$
g_5 (Line)				$\{(e_1^{g_5},e^*-e_1^{g_5},0,e^*): 0\leq e_1^{g_5}\leq e*\}$
g_6				$\{(e_1^{g_6},e_2^{g_6},e_3^{g_6},e^*): 0\leq e_1^{g_6}+e_2^{g_6}+e_3^{g_6}\leq e^*\}$
g_7 (Star)		$\{(0,e^*,e^*,e^*)\}$		$\{(e^*,0,0,0)\}$
g_8				$\{(e_1^{g_8},e^*-e_1^{g_8},0,e^*), (0,0,e^*,0): 0\leq e_1^{g_8}\leq e^*\}$
g_9 (Circle)				$\{(e^*/3,e^*/3,e^*/3,e^*/3), (e^*,0,0,e^*)\}$
g_{10}		$\{(0,e^*,e^*,0)\}$		$\{(e_1^{g_{10}},0,0,e*-e_1^{g_{10}}): 0\leq e_1^{g_{10}}\leq e^*\}$
g_{11} (Complete)				$\{(e_1^{g_{11}},e_2^{g_{11}},e_3^{g_{11}},e_4^{g_{11}})\ 0\leq e_1^{g_{11}}+e_2^{g_{11}}+e_3^{g_{11}}+e_4^{g_{11}}\leq e^*\}$

4.1 Concluding Remarks

Our main aim was to explore the formation of stable network structures in a model with public goods. The multiplicity of equilibria in the non-cooperative formulation of network formation games brings out further difficulties in analyzing stability of network structures. This contrasts with the cooperative game approach where payoffs for agents are predetermined and thereby the multiplicity of equilibrium issues are sidestepped. We took issue with the multiplicity of equilibrium effort levels exerted on a given network structure, and we suggested different stability definitions for such network structures under multiplicity of equilibria.

We demonstrated how these stability notions work for the network structures with four agents where breaking and forming links is costless, and the cost of exerting effort is linear. In that example we observed the following:

1. In a given network structure when all other agents keep their existing links, there will always exist an equilibrium effort profile that will provide a higher utility for an agent who breaks an existing link or a pair of agents who form a new link. Thus, if a stability notion requires that no such profitable deviation exists, then there will be no stable network structure (the SPS case);

2. If the stability notion requires that there exist a stable equilibrium effort profile in providing the public good that will be profitable for an agent who breaks an existing link or a pair of agents who form a new link (the WPS case), then besides the discrete, and the star network structures, g_3 (in which one agent has no link

and the others are linked in a line form) and g_{10}(in which two agents are linked to all others and two agents are linked only two agents) turn out to be stable;

3. If all agents' approvals are required in forming a new link but no such approval is required when breaking links, only g_4 in which all agents have one link becomes stable (the AS case);

4. If all agents' approvals are required in both forming a new link and breaking existing links, then the number of stable networks increases as it is expected (the FAS case).

Given a network structure, Bramoulle and Kranton (2007) define the *efficient effort profile* as the one that maximizes the sum of utilities of all agents. Following that definition, the efficient network structure can be defined as the one on which (the efficient effort profile is exerted and) the sum of utilities is maximized over all possible network structures. Bramoulle and Kranton (2007) prove that the Nash equilibrium effort profiles of a given network structure cannot be efficient. Therefore, it is straightforward to conclude that none of the stability notions given in the previous sections give rise to efficient network structures. The main result of the literature on the network formation holds in our case as well, i.e. there is a strong tension between efficiency and stability. On the other hand, in a full approval stable (FAS) network structure, no one can improve himself by forming a new link or by violating an existing link without hurting someone else. In that sense, a full approval stable network structure is Pareto efficient, even though it does not maximize the sum of total utilities.

As a possible venue for future work, costly link formation may be considered. With links that are costless to form, agents can add as many links as they can. However, substitutability of the efforts (in providing the public good) leads to the underprovision, and hence, inefficiency. The costliness of links may provide a balance against underprovision and may work towards efficiency.

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5 Appendix

In the following subsections, we explain how the network structures formed by four agents are analyzed according to the stability definitions given in section 3. Note that Figure 1 presents what kind of network structures can be obtained from a given network structure through formation or breaking of links. Moreover, the set of Nash equilibrium and stable equilibrium effort profiles on the network structures are given in the Table A1.

Table II: Nash equilibrium and stable equilbrium effort profiles

	Nash equilibrium effort profiles	Stable equilibrium effort profiles
g_1 (Discrete)	$\{(e^*,e^*,e^*,e^*)\}$	$\{(e^*,e^*,e^*,e^*)\}$
g_2	$\{(e_1^{g_2},e^*-e_1^{g_2},e^*,e^*):0\leq e_1^{g_2}\leq e^*\}$	$\varnothing$
g_3	$\{(e^*,0,0,e^*),(0,e^*,e^*,e^*)\}$	$\{(0,e^*,e^*,e^*)\}$
g_4	$\{(e_1^{g_4},e^*-e_1^{g_4},e_3^{g_4},e^*-e_3^{g_4}):0\leq e_1^{g_4},e_3^{g_4}\leq e^*\}$	$\varnothing$
g_5 (Line)	$\{(e_1^{g_5},e^*-e_1^{g_5},0,e^*):0\leq e_1^{g_5}\leq e^*\}$	$\varnothing$
g_6	$\{(e_1^{g_6},e_2^{g_6},e_3^{g_6},e^*):e_1^{g_6}+e_2^{g_6}+e_3^{g_6}=e^*\}$	$\varnothing$
g_7 (Star)	$\{(0,e^*,e^*,e^*),(e^*,0,0,0)\}$	$\{(0,e^*,e^*,e^*)\}$
g_8	$\{(e_1^{g_8},e^*-e_1^{g_8},0,e^*),(0,0,e^*,0):0\leq e_1^{g_8}\leq e^*\}$	$\varnothing$
g_9 (Circle)	$\{(e^*,0,0,e^*),(e^*/3,e^*/3,e^*/3,e^*/3)\}$	$\{(e^*,0,0,e^*)\}$
g_{10}	$\{(e_1^{g_{10}},0,0,e^*-e_1^{g_{10}}),(0,e^*,e^*,0):0\leq e_1^{g_{10}}\leq e^*\}$	$\{(0,e^*,e^*,0)\}$
g_{11} (Complete)	$\{(e_1^{g_{11}},e_2^{g_{11}},e_3^{g_{11}},e_4^{g_{11}}):e_1^{g_{11}}+e_2^{g_{11}}+e_3^{g_{11}}+e_4^{g_{11}}=e^*\}$	$\varnothing$

5.1 g_1 (Discrete)

g_1 with (e^*, e^*, e^*, e^*) is a weakly pairwise stable network structure. The only possible network structure that can be obtained from g_1 is g_2 in our model (One link at a time). There is no stable effort profile on g_2 (Table A1). Therefore, no agent in g_1 will add a new link according to the definition of weakly pairwise stability.

g_1 with (e^*, e^*, e^*, e^*) cannot be strongly pairwise stable, approval stable or full approval stable: Agents 1 and 2 have the utility $b(e^*) - c(e^*)$ on g_1. When they add link 12, g_2 is obtained (Figure 1). In g_2, with an equilibrium profile $(e_1, e^* - e_1, e^*, e^*)$ agents 1 and 2 achieve utilities $b(e^*) - c(e_1)$ and $b(e^*) - c(e_2)$, respectively, where $e_1 + e_2 = e^*$. $b(e^*) - c(e_i) > b(e^*) - c(e^*)$ for $i = 1, 2$. Hence, g_1 with (e^*, e^*, e^*, e^*) cannot be strongly pairwise stable. Besides, with $\mathbf{e}^{g_2} = (e_1^{g_2}, e^* - e_1^{g_2}, e^*, e^*)$ agents 3 and 4 achieve the same utilities as in g_1 with (e^*, e^*, e^*, e^*). Therefore, they do not object the formation of the link 12. g_1 is not approval or full approval stable.

5.2 g_3

g_3 with $(0, e^*, e^*, e^*)$ is weakly pairwise stable. In g_3, adding link 34 leads to g_5, adding link 23 leads to g_6 and adding link 14 leads to g_7. However, there is no stable equilibrium effort profile on either g_5 or g_6. Although $(0, e^*, e^*, e^*)$ is a stable equilibrium effort profile on g_7, neither agent 1 nor agent 4 obtain a strict gain by adding link 14. breaking of links 12 or 13 leads to g_2 on which no stable equilibrium effort profile exist. Hence, according to the definition of weakly pairwise stability, g_3 with $(0, e^*, e^*, e^*)$ becomes weakly pairwise stable.

g_3 with $(0, e^*, e^*, e^*)$ cannot be strongly pairwise stable, approval stable or full approval stable: Agents 3 and 4 have the utility $b(e^*) - c(e^*)$ in g_3. When they add link 34, g_5 is obtained. In an equilibrium effort profile $\mathbf{e}^{g_5} = (e^*, 0, e_3, e^* - e_3)$, agents 3 and 4 achieve utilities $b(e^*) - c(e_3)$ and $b(e^*) - c(e_4)$, respectively, where $0 < e_3 < e^*$. $b(e^*) - c(e_i) > b(e^*) - c(e^*)$ for $i = 3, 4$. Hence, g_3 with $(0, e^*, e^*, e^*)$ cannot be strongly pairwise stable. Besides, with $\mathbf{e}^{g_5} = (e^*, 0, e_3, e^* - e_3)$, agents 1 and 2 achieve the same utilities as in $\mathbf{e}^{g_3} = (0, e^*, e^*, e^*)$. Therefore, they do not object the formation of the link 34. g_3 cannot be approval or full approval stable.

g_3 with $(e^*, 0, 0, e^*)$ cannot be strongly pairwise stable, approval stable, full approval stable or weakly pairwise stable. When agents 1 and 4 add the link 14, g_7 is obtained. With the Nash equilibrium effort profile, $\mathbf{e}^{g_7} = (e^*, 0, 0, 0)$, in g_7, the utility of agent 4 is $b(e^*)$ and the utility of agent 1 is the same as in g_3. Hence, g_3 with $(e^*, 0, 0, e^*)$ cannot be strongly pairwise stable. Besides, with $\mathbf{e}^{g_7} = (e^*, 0, 0, 0)$, in g_7, agents 2 and 3 do not become worse off so that they do not object the formation of the link 14. Thus, g_3 with $(e^*, 0, 0, e^*)$ cannot be approval or full approval stable. In the stable equilibrium effort profile $\mathbf{e}^{g_7} = (0, e^*, e^*, e^*)$, agent 1 achieves a higher utility, while agent 4 does not become worse off. So, according to the definition of weakly pairwise stability, g_3 with $(e^*, 0, 0, e^*)$ cannot be weakly pairwise stable.

5.3 g_4

g_4 with any one of the Nash equilibrium effort profiles $\mathbf{e}^{g_4} = (e_1^{g_4}, e^* - e_1^{g_4}, e_3^{g_4}, e^* - e_3^{g_4})$, where $0 \leq e_1^{g_4}, e_3^{g_4} \leq e^*$ is approval and full approval stable. The links 13, 14, 23, 24 are possible to formed. Formation of any one of these links leads to g_5. However, one can easily find a Nash equilibrium on g_5 in which at least one agent becomes worse off. So addition of the link is not approved by that agent. For example, if $e_1^{g_4}, e_3^{g_4} > 0$, with $\mathbf{e}^{g_5} = (0, e^*, 0, e^*)$, agents 2 and 4 object the addition of link 13. A similar argument works when $e_1^{g_4}, e_3^{g_4} = 0$. Besides, breaking link 12 or link 13 do not bring any benefit to whom breaks the link. An agent who breaks the link has to exert effort e^* in g_2.

On the other hand, g_4 with any one of the Nash equilibrium effort profiles $\mathbf{e}^{g_4} = (e_1^{g_4}, e^* - e_1^{g_4}, e_3^{g_4}, e^* - e_3^{g_4})$, where $0 \leq e_1^{g_4}, e_3^{g_4} \leq e^*$ cannot be defined as weakly pairwise, since $\mathbf{e}^{g_4}$ is not stable effort profile on g_4. Moreover, g_4 with $\mathbf{e}^{g_4}$ cannot be defined strongly pairwise stable. When agents 1 and 3 add the link 13, g_5 occurs. In g_4, agents 1 and 3 have utility $b(e^*) - c(e_1^{g_4})$ and $b(e^*) - c(e_3^{g_4})$, respectively. With the Nash equilibrium, $\mathbf{e}^{g_5} = (0, e^*, 0, e^*)$, agent 1 and agent 3 achieve utilities $b(e^*)$. So, adding 13 is profitable for agent 3 and 4, when $e_1^{g_4}, e_3^{g_4} > 0$. A similar argument works for $e^{g_4} = (e_1^{g_4}, e^* - e_1^{g_4}, e_3^{g_4}, e^* - e_3^{g_4})$, when $e_3^{g_4} = 0$ or $e_1^{g_4} = 0$. Thus, g_4 cannot be defined as strongly pairwise stable for any Nash equilibrium effort profile.

5.4 g_5 (Line)

g_5 with any one of the Nash equilibrium effort profiles $\mathbf{e}^{g_5} = (e_1^{g_5}, e^* - e_1^{g_5}, 0, e^*)$ where $0 \leq e_1^{g_5} \leq e^*$ is full approval stable, but not approval stable, not strongly pairwise stable or not weakly pairwise stable. First of all, $\mathbf{e}^{g_5}$ is not a stable equilibrium profile for any $0 \leq e_1^{g_5} \leq e^*$. Therefore, g_5 cannot be weakly pairwise stable. When agent 1 breaks link 13, with $\mathbf{e}^{g_4} = (0, e^*, 0, e^*)$ he achieves higher utility. Therefore, according to the definitions stated in section 3, g_5 can be neither strongly pairwise stable nor approval stable.

On the other hand, links 14, 24 or 23 are possible to be formed. If agents 1 and 4 add link 14, g_6 is obtained. Any Nash equilibrium effort profile in g_6 is of the form $\mathbf{e}^{g_6}$=$(e_1^{g_6}, e_2^{g_6}, e_3^{g_6}, e^*)$ where $e_1^{g_6} + e_2^{g_6} + e_3^{g_6} = e^*$. Utility of agent 4 in g_6 is as the same in g_5. Agent 1 would like to add link 14 only if $e_1^{g_5} < e_1^{g_6}$which implies $e_2^{g_6} + e_3^{g_6} > e_2^{g_5} + e_3^{g_5}$. Therefore, either agent 2 or agent 3 becomes worse off, and objects the formation of link 14. Thus, link 14 is not formed when the approval of all agents required. Formation of link 23 which leads to g_6 is not accepted by agent 3. Formation of link 24 leads to g_9. $(e^*/3, e^*/3, e^*/3, e^*/3)$ and $(e^*, 0, 0, e^*)$ are the Nash equilibrium effort profiles on g_9. However, g_9 with $(e^*/3, e^*/3, e^*/3, e^*/3)$ is not acceptable by agent 4. If $e_1^{g_5} > 0$, agent 1 objects g_9 with $(e^*0, 0, e^*)$. If $e_1^{g_5} = 0$, in g_9 with $(e^*0, 0, e^*)$, utilities of agents 2 and 4 are the same as in g_5 so that formation of link 24 becomes meaningless. Hence, link 24 is not formed when approval of all agents are required. Then, formation of no link is approved by all agents in the society. Links 12, 13 or 34 are possible to break. Agent 3 has utility $b(e^*)$ in g_5 with $\mathbf{e}^{g_5} = (e_1^{g_5}, e^* - e_1^{g_5}, 0, e^*)$. He cannot improve his utility. Agent 4 has only one link in g_5 where he has utility $b(e^*) - c(e^*)$, and he cannot improve his utility breaking his unique link. On the other hand, if $e_1^{g_5} > 0$, when agent 1 breaks link 13, with $\mathbf{e}^{g_4} = (0, e^*, 0, e^*)$ his utility increases to $b(e^*)$, but utility of agent 2 decreases. Agent 1 cannot break link 13. If $e_1^{g_5} = 0$, in g_4 with $(0, e^*, 0, e^*)$, utilities of agents 2 and 4 are the same as in g_5 so that breaking of link 13 becomes meaningless. Similarly, agent 1 cannot break link 12. Therefore, breaking of any link is not approved by all agents in the society. Thus, g_5 with any one of the Nash equilibrium effort profiles $\mathbf{e}^{g_5} = (e_1^{g_5}, e^* - e_1^{g_5}, 0, e^*)$ where $0 \leq e_1^{g_5} \leq e^*$ is full approval stable.

5.5 g_6

g_6 with anyone of the Nash equilibrium effort profiles $\mathbf{e}^{g_6} = (e_1^{g_6}, e_2^{g_6}, e_3^{g_6}, e^*)$ where $0 \leq e_1^{g_5} + e_2^{g_6} + e_3^{g_6} \leq e^*$ is full approval stable, but not approval stable, not strongly pairwise stable or not weakly pairwise stable. First, $\mathbf{e}^{g_6}$ is not a stable equilibrium profile. Therefore, g_6 cannot be weakly pairwise stable. If $e_2^{g_6} > 0$, when agent 2 breaks link 23, in $\mathbf{e}^{g_3}$ $= (e^*, 0, 0, e^*)$ he achieves higher utility. If $e_2^{g_6} = 0$, either $e_1^{g_5} > 0$ or $e_2^{g_6} > 0$. Then, the agent whose effort is greater than 0 in g_6 break one of his links, and he can achieve higher utility in g_3. Therefore, according to the definitions stated in section 3, g_6 can be neither strongly pairwise stable nor approval stable.

However, links 14, 24 or 34 are possible to be formed. When any one of these links is added, the network structure occurred can be represented by a graph of type g_8. A Nash equilibrium in g_8 is either of the type $\mathbf{e}^{g_8} = (e_1^{g_8}, e^* - e_1^{g_8}, 0, e^*)$ where $0 \leq e_1^{g_8} \leq e^*$ or of the type $(0, 0, e^*, 0)$. In g_8, any of these Nash equilibrium leaves at least one of the agents worse off than in g_6. For example, when $e_2^{g_6} > 0$, agent 2 objects the formation of link 34 with $\mathbf{e}^{g_8} = (e_1^{g_8}, e^* - e_1^{g_8}, 0, e^*)$. Therefore, formation of no link is approved by all agents in the society. Links 12, 13 and 23 leads to a network structure that can be represented by a graph of type g_3. However, any one of the Nash equilibrium effort profile $(e^*, 0, 0, e^*)$ and $(0, e^*, e^*, e^*)$ on g_3 makes an agent worse off than in g_6. For example, if $e_2^{g_6} > 0$ or $e_3^{g_6} > 0$, breaking of 23 is not approved by agent 1 with $(e^*, 0, 0, e^*)$ on g_3. Therefore, breaking of no link is approved by whole society. g_6 with any one of the Nash equilibrium effort profiles, $\mathbf{e}^{g_6} = (e_1^{g_6}, e_2^{g_6}, e_3^{g_6}, e^*)$ where $0 \leq e_1^{g_5} + e_2^{g_6} + e_3^{g_6} \leq e^*$, is full approval stable.

5.6 g_7 (Star)

g_7 with $\mathbf{e}^{g_7} = (0, e^*, e^*, e^*)$ is weakly pairwise stable, but not approval stable, not full approval stable and not strongly pairwise stable. When link 12 is formed, with $(0, e^*, 0, e^*)$ in g_8, utility of agent 3 increases, utilities of agent 4, 1 and 2 are the same as in g_7. Therefore, g_7 with $\mathbf{e}^{g_7} = (0, e^*, e^*, e^*)$ cannot be strongly pairwise stable, approval stable and full approval stable. However, $(0, e^*, e^*, e^*)$ is stable effort profile on g_7. When a link is formed in g_7, g_8 on which no stable equilibrium effort profile exist represents the new network structure. When a link is violated in g_7, g_3 represents the new network structure, but breaking of any link in g_7 is objected by agent 1 whose utility is $b(e^*)$ in g_7. Hence, g_7 with $\mathbf{e}^{g_7} = (0, e^*, e^*, e^*)$ is weakly pairwise stable.

g_7 with $\mathbf{e}^{g_7} = (e^*, 0, 0, 0)$ is full approval stable, but not weakly pairwise stable, not approval stable, and not strongly pairwise stable. First, $(e^*, 0, 0, 0)$ is not stable effort profile. Hence g_7 cannot be weakly pairwise stable in that equilibrium. Moreover, here, no one would like to add a new link. Agent 1 can benefit from breaking any one of links 14, 13 or 12. So, g_7 is not approval stable, and not strongly pairwise stable with $(e^*, 0, 0, 0)$. However, breaking any one of the links is not approved by at least one of agents 2, 3 or 4. Hence, g_7 with $\mathbf{e}^{g_7} = (e^*, 0, 0, 0)$ is full approval stable.

5.7 g_8, g_9 (Circle), g_{11} (Complete)

Using similar arguments as above, one can easily show that g_8, g_9 , g_{10} and g_{11} are defined as full approval stable with any Nash equilibrium effort profile exerted on each graph. However, these graphs are not stable according to other definitions.

5.8 g_{10}

g_{10} with $(0, e^*, e^*, 0)$ is weakly pairwise stable, but not strongly pairwise stable, not approval stable and not full approval stable. When link 23 is formed, in g_{10} with $(0, e^*/2, e^*/2, 0)$, agent 2 and 3 become better-off, while agents 1 and 4 do not become worse-off. Therefore, g_{10} with $(0, e^*, e^*, 0)$ is not approval stable, not full approval stable and not strongly pairwise stable. $(0, e^*, e^*, 0)$ is stable effort profile on g_{10}. When a link is violated on g_{10}, g_8 is obtained. When a link is formed on g_{10}, g_{11} is obtained. However, there is no stable effort profile on either g_8 or g_{11}. Thus, g_{10} with $(0, e^*, e^*, 0)$ is weakly pairwise stable.

On the other hand, g_{10} with $\left(e_1^{g_{10}}, 0, 0, e^* - e_1^{g_{10}}\right)$ is full approval stable, but not weakly pairwise stable, not strongly pairwise stable and not approval stable. First, $\left(e_1^{g_{10}}, 0, 0, e^* - e_1^{g_{10}}\right)$ is not a stable equilibrium profile for any $0 \leq e_1^{g_{10}} \leq e^*$. Therefore, g_{10} with $\left(e_1^{g_{10}}, 0, 0, e^* - e_1^{g_{10}}\right)$ cannot be weakly pairwise stable.When link 14 is violated, with $(0, e^*, 0, e^*)$ on g_9,either that both agents 1 and 4 becomes better off or that one of the agents 1 and 4 becomes better off, but the other does not become worse off. Moreover, agents 2 and 3 do not become worse-off in any case. Hence, g_{10} with $\left(e_1^{g_{10}}, 0, 0, e^* - e_1^{g_{10}}\right)$ is not weakly pairwise stable, not strongly pairwise stable and not approval stable. Formation of a new link is not beneficiary for the agents (Only agents 2 and 3 can add a link, but there is no incentive for them). Note that breaking of any link is not approved by at least one agent. So, g_{10} with $\left(e_1^{g_{10}}, 0, 0, e^* - e_1^{g_{10}}\right)$ is full approval stable.