Independent Random Matching

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Abstract

Random matching models with a continuum population are widely used in economics to study environments where agents interact in small coalitions. This paper provides foundations to such models. In particular, the paper establishes an existence result for random matchings that are universal in the sense that certain desirable properties are satisfied for any assignment of types to agents. The result applies to infinitely many types of agents, thus covering random matching models which are currently used in the literature without a foundation. Furthermore, the paper provides conditions guaranteeing uniqueness of random matching.

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1 Introduction

A substantial part of economics investigates the implications of a variety of frictions for allocations. An important case are frictions arising because social interactions occur in a decentralized fashion, i.e., in small groups of agents. Frequently, such contexts are modeled by assuming that there is random matching of agents. For instance, in several papers in monetary theory, markets with decentralized trade are modeled by assuming that agents are randomly matched in pairs; see, e.g., Kiyotaki and Wright (1989) or Lagos and Wright (2005). In labor economics, search frictions are modeled by assuming that workers and firms are randomly matched; see, e.g., Mortensen and Pissarides (1994). In game theory, pairwise random matching of agents is used as a framework to study environments with infrequent interaction; see, e.g., the papers on social norms by Kandori (1992), Okuno-Fujiwara and Postlewaite (1995), or Takahashi (2010).

Many models with random matching consider a continuum population, or, more precisely, a non-atomic probability space of agents. This specification of the population is taken as a justification for assuming that the random matching satisfies certain desirable properties which, in particular, should ensure that cross-sectional aggregate outcomes associated with the random matching are deterministic almost surely. Formal statements of desirable properties of random matching will be given in Section 2. Here we illustrate proportionality and mixing properties with a simple example.

Consider a continuum of agents who are randomly and bilaterally matched. Suppose that an agent is either a buyer or a seller. Proportionality means that for any agent the probability of being matched with a buyer (seller) is equal to the proportion of buyers (sellers) in the population. Mixing, on the other hand, is a property concerning sample functions and means that, almost surely, the proportion in the population of those agents of type t that are matched with agents of type t' is equal to the product of the proportions of agents of these types, where both t and t' can stand for "buyer" or "seller." Now a typical approach in random matching models with a continuum population amounts to taking it for granted that proportionality and some independence in the matching process are satisfied, and that, as a consequence, mixing should be satisfied as well. The prediction in the example is then that the proportions of matches between two buyers, two sellers, and a buyer and a seller are, respectively, p_1^2, p_2^2, and $2p_1p_2$ almost surely, denoting by p_1 the proportion of buyers, and by p_2 that of sellers.

This approach has some intuitive appeal. In fact, as noted for instance in Molzon and Puzzello (2010), proportionality and mixing properties as in the above example hold asymptotically for uniform random matching on finite populations when the number of agents becomes large. That is, these properties are consistent with the idea of a continuum population as a convenient idealization of large finite populations. Nevertheless, the approach above is problematic. Indeed, it is a well known (and rather trivial) fact that if one considers, as in the example, a continuum of random variables, indexed by an atomless probability space, then just assuming the random variables to be identically distributed and to satisfy independence conditions does not imply that, almost surely, the sample functions over the index probability space have a distribution equal to that of the random variables; actually, it need not even be the case that, almost surely, distributions of the sample functions are defined.

Taking up such issues, several papers have raised the question of existence of random matching, and have shown that it is indeed possible to have models with random matching such that some desired properties are satisfied. We refer to Aliprantis et al. (2006), Alós-Ferrer (1999), Alós-Ferrer (2002), Boylan (1992), Boylan (1995), Duffie and Sun (2007), Gilboa and Matsui (1992), and Molzon and Puzzello (2010).

Alós-Ferrer (1999) is the seminal paper about existence of random matching with a continuum population. In the present paper we continue the line of research initiated by Alós-Ferrer (1999). In particular, we provide an existence result which improves on that of Alós-Ferrer (1999) in several aspects. First, we establish existence of random matchings that are universal in the sense that certain desirable properties (see Section 2) are satisfied for any assignment of types to agents. Second, the random matching in our result satisfies independence properties which are natural if the population is modeled as a continuum. Third, our result applies to random matching models with infinitely many types of agents.

These improvements on the result of Alós-Ferrer (1999) are relevant to economic models with random matching. For instance, if a model with repeated random matching is to be constructed in a context dealing with the evolution of the frequencies of agents' types in a population, it is inconvenient to have the random matching depend on the type assignment. Moreover, for a wide class of random matching models with a continuum population, it is not possible to capture the relevant attributes of all the agents in a finite type space. This is the case, e.g., for the models described in Cavalcanti and Puzzello (2010), Green and Zhou (2002), Lagos and Wright (2005), Molico (2006), Shi (1997), Zhu (2005), where there are no upper bounds on money holdings or money holdings are perfectly divisible, or those described in Hofbauer et al. (2008), Oechssler and Riedel (2002), Sandholm (2001), van Veelen and Spreij (2009), where infinite strategy sets matter.

However, it is fair to mention here that, unlike Alós-Ferrer (1999), we do not take the probability space of agents to be the unit interval $[0,1]$ with Lebesgue measure. In fact, as shown in Alós-Ferrer (1999), with this choice of the space of agents there can be no random matching that is universal in the sense above. Existence of a probability space of agents that does not entail this restriction is part of our result on existence of random matching. In this sense, our result provides an alternative to that of Alós-Ferrer (1999), but is not an extension in the strictly formal sense.

Actually, in our result we can still have $[0,1]$ for the set of agents. It is just that the measure involved cannot be Lebesgue measure. However, in many contexts, a continuum population is assumed just to render individual agents negligible, that is, only non-atomicity of the measure on the population is important. In particular, then, if the set of agents is taken to be $[0, 1]$, any atomless probability measure on this set is as good as Lebesgue measure.

An existence result for random matching related to ours can be found in Duffie and Sun (2007). However, the approach in that paper restricts attention to random matching with finitely many types of agents. Further, the independence property required of the random matching in our existence result is stronger and more natural than that stated in the result of Duffie and Sun (2007). On the technical side, the approach of Duffie and Sun (2007) relies on nonstandard analysis, whereas ours stays in the framework of ordinary measure and set theory.

The literature on foundations of random matching so far has mainly focused on existence problems. Another interesting question concerns uniqueness of random matching. In fact, as shown in Molzon and Puzzello (2010), a random matching is not uniquely determined by measure preservation, proportionality, and mixing properties. However, in this paper we show that, in terms of distributions on the set of matchings, a random matching is uniquely determined by proportionality and independence properties.

The plan of the paper is as follows. In Section 2 we give basic definitions and present formal statements of measure preserving, proportionality, independence, and mixing properties of random matching. Section 3 addresses measurability issues arising with random matching. The main result of our paper is about existence of independent random matching and is stated and discussed in Section 4. Section 5 provides a uniqueness result. Most of the proofs are in Section 6. Note that our existence result is quite general and applies also to models with infinitely many types. Such models arise naturally in economics, as illustrated in Section 7 where some examples from the literature are discussed.

2 Properties of Random Matching

We start by introducing some basic definitions, and then state properties of random matching which are used in a variety of models.

Definition 1. Let X be a set. An *involution* on X is a bijection $f \colon X \to X$ which is self-inverse (i.e., such that the inverse f^{-1} satisfies $f^{-1} = f$); equivalently, an involution on X is a mapping $f \colon X \to X$ such that $f \circ f$ is the identity on X. A mapping $f \colon X \to X$ is said to be *fixed point free* if $f(x) \neq x$ for all $x \in X$.

Involutions provide a natural formalization for the notion of bilateral matching (e.g., Alós-Ferrer (1999), Aliprantis et al. (2006)). In this study, we focus on bilateral matchings where no agent remains unmatched. For short, we will call a bilateral matching simply a matching.

Definition 2. A *matching* on a set A of agents is a fixed point free involution on A.

We will now give the definition of random matching.

Definition 3. Let $(A, \mathcal{A}, \mu)$ be a probability space of agents and let (Ω, Σ, ν) be a sample probability space. A *random matching* is a mapping $f\colon A \times \Omega \to A$ such that

(a) $f(\cdot, y)$ is a matching on A for each $y \in \Omega$,

(b) the mappings $f(\cdot, y)\colon A \to A$ and $f(x, \cdot)\colon \Omega \to A$ are measurable for each $y \in \Omega$ and each $x \in A$.

Notation. In the context of Definition 3, we also write f_x for the function $f(x, \cdot)$, and f_y for $f(\cdot, y)$.

Definition 3 is general in the sense that it imposes only minimal measurability conditions needed to formulate our definitions and results. It leaves the door open to a variety of specific properties of random matching that are considered in economic models.

An essential part of random matching models is the specification of agents' types. The notion of type is meant to capture the payoff-relevant characteristics of agents. For instance, in several models of monetary theory, the type of an agent is simply given by his money holdings and by whether he is a buyer or a seller. In evolutionary game theory, types are identified with strategies (e.g., Kandori et al. (1993)).

In this paper, we consider abstract notions of type space and type assignment, defined as follows.

Definition 4. A *type space* is a measurable space $(T, \mathcal{T})$. Given a probability space $(A, \mathcal{A}, \mu)$ of agents, a *type assignment* is a measurable mapping θ from $(A, \mathcal{A}, \mu)$ to a type space $(T, \mathcal{T})$, and the corresponding type distribution is the distribution of θ, i.e., the probability measure on T given by $\tau(B) = \mu(\theta^{-1}(B))$ for each $B \in \mathcal{T}$.

We are now ready to formally present desirable properties of random matching. In naming them, we follow Alós-Ferrer (1999) if there is an analog in that paper (which is the case for (P1)–(P3) and (P5)–(P7) below). See (b)–(d) of Remark 4 at the end of this section for differences between the formalizations given here and that in Alós-Ferrer (1999).

Let $(A, \mathcal{A}, \mu)$ be a probability space of agents, (Ω, Σ, ν) a sample probability space, $f\colon A \times \Omega \to A$ a random matching, $(T, \mathcal{T})$ a type space, $\theta\colon A \to T$ a type assignment, and τ the corresponding type distribution.

(P1) "Measure preservation:" For all $y \in \Omega$, f_y is inverse-measure-preserving, i.e., $\mu(f_y^{-1}(E)) = \mu(E)$ for any $E \in \mathcal{A}$.

(P2) "General proportional law:" For all $x \in A$, f_x is inverse-measure-preserving, i.e., $\nu(f_x^{-1}(E)) = \mu(E)$ for any $E \in \mathcal{A}$.

(P3) "Strong mixing:" For any $E_1, E_2 \in \mathcal{A}$, $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for almost all $y \in \Omega$.

(P4) "General independence:" The family $\langle f_x \rangle_{x \in A}$ is stochastically independent; that is, the family $\langle \Sigma_x \rangle_{x \in A}$ is stochastically independent, writing Σ_x for the sub-σ-algebra of Σ generated by f_x.

(P5) "Atomless:" For any two $x, x' \in A$, the set $\{y \in \Omega\colon f_x(y) = x'\}$ is a ν-null set.

(P6) "Types proportional law:" For all $x \in A$, the mapping $\theta \circ f_x$ from Ω to T has distribution τ, i.e., $\nu\big((\theta \circ f_x)^{-1}(B)\big) = \tau(B)$ for any $B \in \mathcal{T}$.

(P7) "Types mixing:" For any $B_1, B_2 \in \mathcal{T}$, $\mu\big(\theta^{-1}(B_1) \cap (\theta \circ f_y)^{-1}(B_2)\big) = \tau(B_1)\tau(B_2)$ for almost all $y \in \Omega$.

(P8) "Independence in types:" The family $\langle \theta \circ f_x \rangle_{x \in A}$ of mapping from Ω to T is stochastically independent; that is, the family $\langle \Sigma_x^\theta \rangle_{x \in A}$ is stochastically independent, writing Σ_x^θ for the sub-σ-algebra of Σ generated by $\theta \circ f_x$.

Property (P1) states that, for all matchings, a given measurable set of agents must have the same measure as the set of their partners. This is simply a consistency requirement: It should not be the case that, say, 1/8 of the agents are matched with 5/8 of the agents. Note that if the population is finite, (P1) is automatically satisfied for the normalized counting measure. The other properties should be considered in view of continuum populations. Property (P2) says that for any agent the probability of being paired to an agent belonging to a measurable set E in the population A is equal to the proportion of the agents from E in the total population. This property is usually seen as saying that the random matching is uniform over the agents. In applications, it plays an important role for expected payoff equations. Property (P3) states that the measure of the set of those agents in a given measurable set $E_1 \subset A$ that are matched with agents belonging to a measurable set $E_2 \subset A$ is equal to the product of the measures of E_1 and E_2 for almost all matchings. In other words, it states that given any non-negligible measurable set $E_1 \subset A$, the proportion in E_1 of the agents that are matched with agents belonging to another given measurable set $E_2 \subset A$ is equal to the proportion of the agents from E_2 in the total population, almost surely in Ω. This property is sometimes interpreted as manifestation of a law of large numbers. In the next section we show that under some condition this view can be justified. The intuition behind property (P4) is that, for finitely many distinct agents in a continuum population, the events that these agents have partners in any given measurable sets should be independent, as finite sets in a continuum population (specified as atomless probability space) are negligible.[1]

[1]The fact that matching agent x_i with agent x_j implies x_j is matched with x_i does not mean a contradiction to (P4) if the space of agents is atomless and the random matching satisfies (P2), because any two null sets in the sample space are trivially stochastically independent.

Of course, (P4) cannot be satisfied if the population is finite. However, considering uniform random matching on finite populations and letting the number of agents go to infinity, it may be seen by calculations that, for any fixed integer $k \geq 2$, the deviation from independence that appears for any sets of k agents vanishes asymptotically.[2] Thus, in a model with a continuum population, viewed as idealization of a large finite set of negligible agents, (P4) may be seen as a natural property of random matching. Property (P5) states that the probability that any two given agents are matched is zero. This property may also be seen as natural in a random matching model with a continuum population. Moreover, this property is important as it captures the notion of "anonymity" (see Aliprantis et al. (2006)). Properties (P6), (P7), and (P8) have meanings similar to those of (P2), (P3), and (P4) respectively. For instance, (P6) says that for any agent the probability of being paired to an agent whose type belongs to a measurable set B in the type space is equal to the proportion of the agents whose types belong to B.

We note the following simple fact for later reference.

Remark 1. For any type assignment, (P2) implies (P6), (P3) implies (P7), and (P4) implies (P8). If $(A, \mathcal{A}, \mu)$ is atomless, then (P2) also implies (P5).[3]

The proposition below summarizes converse implications. It shows, in particular, that for a random matching to be universal in the sense that (P6)–(P8) are satisfied for every possible type assignment, it is necessary that the general properties (P2)–(P4) be satisfied. Actually, the matter reduces to finite type spaces.

Proposition 1. *Let $(A, \mathcal{A}, \mu)$ be a probability space of agents, (Ω, Σ, ν) a sample probability space, and $f\colon A \times \Omega \to A$ a random matching.*

(a) *If f satisfies* (P8) *for every type assignment with a finite type space, then f satisfies* (P4)*.*

(b) *If f satisfies* (P7) *for every type assignment with a finite type space, then f satisfies* (P3)*.*

(c) *If f satisfies* (P6) *for every type assignment with a finite type space, then f satisfies* (P2)*.*

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[2]Indeed, to capture also finite populations with an odd number of agents, modify Definition 1 to require that at most one agent remains unmatched. Then for each integer $n > 0$, let A^n be a finite population with n agents, and I^n the set of all matchings on A^n. Let P^n be the normalized counting measure on I^n. Suppose that the random matching is uniform for each n, i.e., that all elements of I^n are equally likely. Then, for each n, randomness of matching is described by P^n. Fix an integer $k > 0$. For each $n > k$, let $A_1^n, \ldots, A_k^n$ be any subsets of A^n, let $x_1^n, \ldots, x_k^n$ be any distinct agents in A^n, and for each $i = 1, \ldots, k$, let F_i^n be the set of those elements of I^n which match agent x_i^n with an agent belonging to A_i^n. A straightforward but a bit messy calculation shows that $\left| P^n(\bigcap_{i=1}^k F_i^n) - \prod_{i=1}^k P^n(F_i^n) \right| \to 0$ as $n \to \infty$, i.e., the deviation from independence that appears for any sets of k agents vanishes asymptotically.

[3]To see that (P4) implies (P8), simply note that, for any $x \in A$, measurability of θ implies that the σ-algebra generated by $\theta \circ f_x$ is included in that generated by f_x.

The proof is elementary. The argument is given in Section 6.3 for completeness.

We close this section with three remarks. The first two concern the positions of the quantifiers in (P3) and (P7), the third concerns the relationship between our setting of random matching and that in Alós-Ferrer (1999).

Remark 2. Interchanging the positions of the quantifiers in (P7), one obtains the following substantially stronger property.

(P7') For almost all $y \in \Omega$, $\mu\big(\theta^{-1}(B_1) \cap (\theta \circ f_y)^{-1}(B_2)\big) = \tau(B_1)\tau(B_2)$ for any two $B_1, B_2 \in \mathcal{T}$.

However, if the σ-algebra of the type space $(T, \mathcal{T})$ is countably generated, then (P7) and (P7') are equivalent. Indeed, it follows by a straightforward monotone class argument that if $\mathcal{T}$ is countably generated then (P7) implies (P7'); see Section 6.4 for details. Actually, in most applications the σ-algebra of the type space is countably generated. In fact, this property is satisfied whenever the type space is a Polish space with its Borel σ-algebra, and in particular, of course, whenever the type space is finite.

Remark 3. In view of the previous remark, it might be tempting to also take the following strengthening of (P3) into consideration.

(P3') For almost all $y \in \Omega$, $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for any two $E_1, E_2 \in \mathcal{A}$.

However, it is trivial that this is false for any random matching f on any probability space $(A, \mathcal{A}, \mu)$ of agents which is non-trivial in the sense that μ does not take only the values 0 and 1 (regardless of whether or not any other of the properties listed so far are satisfied). Indeed, given any $y \in \Omega$, take E_2 to be a member of $\mathcal{A}$ with $0 < \mu(E_2) < 1$, and then take $E_1 = f_y^{-1}(E_2)$ if $\mu(f_y^{-1}(E_2)) > 0$ and $E_1 = A$ otherwise, obtaining a pair of members of $\mathcal{A}$ for which the equality in (P3') does not hold. (In fact, there is no non-trivial probability space on which there is a measurable mapping to itself satisfying the equality in (P3') for all pairs of measurable subsets.)

Remark 4. As noted in the introduction, research on issues of existence of random matching with a continuum population was initiated by Alós-Ferrer (1999), so some discussion of the relationship between his approach and what has been presented in this section of our paper is perhaps in order. (Concerning the choice of the space of agents, see Section 4.)

(a) One minor difference between the approach in our paper and that in Alós-Ferrer (1999) concerns the general definition of random matching. While in our paper a random matching is defined as a mapping whose domain is the product of the agent space with an abstract sample space, in Alós-Ferrer (1999) a random matching is defined as a probability measure on the set of matchings, actually on the set of those matchings which are measurable. However, given a random matching according to this latter approach, one may view the set of measurable matchings together with the measure on top of it as the sample space and then has a random matching according to our definition, the mapping f of our definition now being concretely given by $f(x, y) = y(x)$, y a measurable matching, x a point in the agent space, the only qualification being that it need not be true that the mappings $f(x, \cdot)$ are measurable as required in our definition. In Alós-Ferrer (1999), a measurability property is imposed on these mappings implicitly in the proportionality properties stated for a random matching, but not as part of the general definition of random matching. Actually, we could also have formulated our setting in such a way that measurability of the functions $f(x, \cdot)$ is not part of the definition of random matching, but we have found it convenient to have measurability of these functions out in the open prior to the statement of the specific properties of a random matching, in particular in view of the independence properties we consider in our paper. Apart from this aspect, a random matching according to the definition in Alós-Ferrer (1999) can be viewed as random matching according to our definition. Thus, in principle, our definition of random matching encompasses that of Alós-Ferrer (1999). For reasons of notational flexibility, we have chosen to work with an abstract sample space in our paper.

(b) Regarding the specific properties of random matching, we first note that in Alós-Ferrer (1999) the properties "types proportional law" and "types mixing" are stated in terms of singleton subsets of the type space. That is, translated into our setting and notation, only singletons are taken for the sets B, B_1, and B_2 in the statements of properties (P6) and (P7) respectively. Of course, this is equivalent to the way these properties are actually stated in our paper if, as in Alós-Ferrer (1999), the type space is finite (and its σ-algebra contains the singleton subsets).

(c) Concerning the properties "measure preservation" and "strong mixing," in our paper their statements involve inverse images of measurable subsets of the agent space, while in Alós-Ferrer (1999) they are formulated in terms of direct images. That is, to use our notation, what is $f_y^{-1}(E)$ in our statements of (P1) and (P3) is $f_y(E)$ in the corresponding statements in Alós-Ferrer (1999). However, by the very definition of random matching, the functions f_y are involutions, i.e., $f_y^{-1}(E)$ and $f_y(E)$ are the same, so it is just a matter of taste and habit which form one prefers to work with.

(d) Actually, the way in which in Alós-Ferrer (1999) the quantifiers in the statement of the "strong mixing" property are placed leaves it unclear whether in Alós-Ferrer (1999) this property is meant in the sense of (P3) or in that of (P3') as formulated in Remark 3. However, as noted in that remark, it is trivial that (P3') fails for any random matching with a non-trivial space of agents.

(e) Our statements of the properties "general proportional law" and "atomless" are as in Alós-Ferrer (1999) apart from notation. Independence properties, as for instance (P4) and (P8) of our paper, are not considered in Alós-Ferrer (1999).

3 Joint measurability issues

Let $(A, \mathcal{A}, \mu)$ be a probability space of agents, (Ω, Σ, ν) a sample probability space, $f\colon A \times \Omega \to A$ a random matching, $(T, \mathcal{T})$ a type space, and $\theta\colon A \to T$ a type assignment. Further, let λ denote the product measure on $A \times \Omega$ defined from μ and ν, and Λ its domain.

Joint measurability issues with random matching arise in contexts like the following. Suppose that $r\colon T \times T \to \mathbb{R}$ is a bounded $\mathcal{T} \otimes \mathcal{T}$-measurable function with the interpretation that if an agent of type t is matched with an agent of type t' then the former agent gets a reward $r(t, t')$. Let $R\colon A \times \Omega \to \mathbb{R}$ denote the corresponding reward process, i.e., the process defined by setting $R(x, y) = r(\theta(x), \theta(f(x, y)))$ for $x \in A$ and $y \in \Omega$. In such a situation, it is natural that one would like to talk about an expected aggregate reward. Moreover, one would like to be able to express the expected aggregate reward in terms of repeated integrals with respect to the factor measures μ and ν; in particular, one might want to relate it to cross-sectional aggregate rewards. For these purposes, it would be ideal if the random matching f were jointly measurable, i.e., $(\Lambda, \mathcal{A})$-measurable.[4] If this is the case, then for any given θ and r as above, the process R is Λ-measurable, so that the expected aggregate reward is defined as the integral of R with respect to λ and can be computed in terms of repeated integrals according to Fubini's theorem.

Unfortunately, joint measurability may conflict with other desirable properties of random matching. Since the concern of this paper is random matching on continuum populations, we will just give a short argument showing that if the probability space $(A, \mathcal{A}, \mu)$ of agents is atomless, then it is impossible for the random matching f to be $(\Lambda, \mathcal{A})$-measurable if (P2) and (P4) are satisfied.

To see this, suppose the contrary. Then since $(A, \mathcal{A}, \mu)$ is atomless, we can select a measurable function $\theta\colon A \to \{0, 1\}$ with distribution $(1/2, 1/2)$. Let g be the composition $g = \theta \circ f$. Then g is Λ-measurable, and (P2) and (P4) imply that for the family $\langle g_x \rangle_{x \in A}$ of sections of g we have $\int_\Omega |g_x - g_{x'}|\, d\nu = 1/2$ for any two distinct x, $x' \in A$. Now by a standard fact,[5] since g is bounded and Λ-measurable, there is a null set $N \subset A$ such that the set $\{g_x^\bullet : x \in A \setminus N\}$ is a separable subset of $L_1(\nu)$, writing $g_x^\bullet$ for the ν-equivalence class of g_x, $x \in A$. However, this contradicts the above conclusion about the family $\langle g_x \rangle_{x \in A}$ because, $(A, \mathcal{A}, \mu)$ being atomless, $A \setminus N$ is uncountable.

Fortunately it will turn out that joint measurability as a condition on random matching can be relaxed into a condition that practically does the same job as joint measurability, but is not inconsistent with a combination of (P2) and (P4) (and the other general properties). The suitable concept in this regard is provided by the notion of a Fubini extension of a product measure, a notion

[4] As usual, if Σ and T are σ-algebras on sets X and Y respectively, "(Σ, T)-measurable" for a function $f\colon X \to Y$ means $f^{-1}(E) \in \Sigma$ for each $E \in \mathrm{T}$; in case $Y = \mathbb{R}$, "Σ-measurable" means $f^{-1}(E) \in \Sigma$ for each Borel set $E \subset \mathbb{R}$.

[5] See, e.g., Fremlin (2003, 418S) and Fremlin (2001, 245X(h)).

introduced by Sun (2006) into the economics literature. Here is a formal definition.

Definition 5. Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $(X \times Y, \Lambda, \lambda)$ the corresponding product probability space. Let $\bar{\lambda}$ be a probability measure on $X \times Y$, and $\bar{\Lambda}$ its domain. Then $\bar{\lambda}$ is said to be a *Fubini extension* of λ if (a) $\bar{\Lambda} \supset \Lambda$ and (b) for each $H \in \bar{\Lambda}$—denoting by χH the characteristic function of H—the integrals $\iint \chi H(x,y)\, d\nu(y)\, d\mu(x)$ and $\iint \chi H(x,y)\, d\mu(x)\, d\nu(y)$ are well-defined and $\iint \chi H(x,y)\, d\nu(y)\, d\mu(x) = \bar{\lambda}(H) = \iint \chi H(x,y)\, d\mu(x)\, d\nu(y)$.

Note that (a) and (b) in this definition imply that $\bar{\lambda}$ agrees with λ on Λ, so $\bar{\lambda}$ is indeed an extension of λ. The definition implies in particular that the *conclusion* of Fubini's theorem about repeated integrals with respect to the factor measures μ and ν continues to hold for $\bar{\lambda}$-integrable real-valued functions.

Now in the context above, suppose there is a Fubini extension $\bar{\lambda}$ of λ such that the random matching f is $(\bar{\Lambda}, \mathcal{A})$-measurable, writing $\bar{\Lambda}$ for the domain of $\bar{\lambda}$. Then, for any $\theta\colon A \to T$ and $r\colon T \times T \to \mathbb{R}$ as above, the reward process R is $\bar{\Lambda}$-measurable, and thus the expected aggregate reward can be defined as the integral of R against $\bar{\lambda}$. This still gives a meaningful notion of expected aggregate reward, because, by the definition of Fubini extension, $\bar{\lambda}$ preserves its ties with λ in such a way that this integral can be expressed in terms of repeated integrals against the factor measures μ and ν. In particular, the expected aggregate reward, so defined, does not depend on the particular choice of the Fubini extension $\bar{\lambda}$, subject to the requirement that f be $(\bar{\Lambda}, \mathcal{A})$-measurable.[6]

Part of our existence result for random matching (to be stated in the next section) is that this kind of generalized joint measurability property can indeed be satisfied simultaneously with all the properties of random matching listed in Section 2. Here we note that if the probability space of agents is atomless and the random matching satisfies all of (P2), (P3), and (P4) then, in fact, an appropriate Fubini extension of the product of the measures on the agent space and the sample space must exist.

Proposition 2. *Let $(A, \mathcal{A}, \mu)$ be an atomless probability space of agents, (Ω, Σ, ν) a sample probability space, and $f\colon A \times \Omega \to A$ a random matching. Let λ be the product probability measure on $A \times \Omega$ defined from μ and ν. If f satisfies* (P2) *to* (P4)*, then λ has a Fubini extension $\bar{\lambda}$ such that f is $(\bar{\Lambda}, \mathcal{A})$-measurable, writing $\bar{\Lambda}$ for the domain of $\bar{\lambda}$.*

See Section 6.2 for the proof.

In the previous section we mentioned that property (P3) is sometimes viewed as manifestation of a law of large numbers. Now the notion of Fubini extension also provides the framework in which this view may be justified, in the sense that (P3) may be derived as a conclusion from (P2) and (P4) (note that given any

[6]For a recent application of the notion of Fubini extension outside the scope of random matching models, see Sun and Yannelis (2008). A general result on existence of (proper) Fubini extensions can be found in Podczeck (2010).

random matching $f\colon A \times \Omega \to A$, these latter two properties together mean that the family $\langle f_x \rangle_{x \in A}$ is i.i.d.). In fact, the following holds.

Proposition 3. *Let $(A, \mathcal{A}, \mu)$ be an atomless probability space of agents, (Ω, Σ, ν) a sample probability space, and $f\colon A \times \Omega \to A$ a random matching. Let λ be the product probability measure on $A \times \Omega$ defined from μ and ν. Suppose:*

(i) *There is a Fubini extension $\bar{\lambda}$ of λ such that f is $(\bar{\Lambda}, \mathcal{A})$-measurable, writing $\bar{\Lambda}$ for the domain of $\bar{\lambda}$.*

(ii) *f satisfies* (P2) *and* (P4)*.*

Then f satisfies (P3).[7]

Note that Propositions 2 and 3 in combination say that if a random matching on a continuum population satisfies (P2) and (P4), then (P3) and (i) of Proposition 3 are equivalent properties. Thus, for a "large numbers" interpretation of the strong mixing property to be valid, it is both necessary and sufficient that the random matching have the measurability property stated in (i) of Proposition 3.

As the proof of Proposition 3 is short and illustrative for the role of a Fubini extension, it will be given here.

Proof of Proposition 3. Fix any E_1, $E_2 \in \mathcal{A}$ and pick any $B \in \Sigma$. Note first that by (i), we have $(E_1 \times B) \cap f^{-1}(E_2) \in \bar{\Lambda}$, and therefore, from the definition of Fubini extension, the integrals $\int_{E_1} \nu(B \cap f_x^{-1}(E_2))\, d\mu(x)$ and $\int_B \mu(E_1 \cap f_y^{-1}(E_2))\, d\nu(y)$ are well-defined and equal. Write Σ_B for the sub-σ-algebra of Σ generated by B, and Σ_x for that generated by f_x, $x \in A$. Now (P4) says that the family $\langle \Sigma_x \rangle_{x \in A}$ is stochastically independent. Using Fremlin (2008, 5A6-272W), it follows that there is a countable $D \subset A$ such that for each $x \in A \backslash D$, Σ_B and Σ_x are stochastically independent. As $(A, \mathcal{A}, \nu)$ is atomless, this means Σ_B and Σ_x are stochastically independent for almost all $x \in A$. Thus $\nu(B \cap f_x^{-1}(E_2)) = \nu(B)\nu(f_x^{-1}(E_2))$ for almost all $x \in A$. Finally, note that from (P2) we have $\nu(f_x^{-1}(E_2)) = \mu(E_2)$ for all $x \in A$.

Putting all these together, we may conclude that, for any $B \in \Sigma$,

$$\begin{aligned}
\int_B \mu(E_1 \cap f_y^{-1}(E_2))\, d\nu(y) &= \int_{E_1} \nu(B \cap f_x^{-1}(E_2))\, d\mu(x) \\
&= \int_{E_1} \nu(B)\nu(f_x^{-1}(E_2))\, d\mu(x) \\
&= \int_{E_1} \nu(B)\mu(E_2)\, d\mu(x) \\
&= \nu(B)\mu(E_2)\mu(E_1).
\end{aligned}$$

By the Radon-Nikodym theorem it follows that $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for almost all $y \in \Sigma$. Thus, as E_1, $E_2 \in \mathcal{A}$ are arbitrary, f satisfies (P3). □

[7] A warning may be in order here. Conditions (i) and (ii) cannot be satisfied simultaneously when $(A, \mathcal{A}, \mu)$ is taken to be $[0, 1]$ with Lebesgue measure. Indeed, with this choice of the space of agents, there can be no random matching satisfying (P3); see Section 4 for this.

As noted in Remark 1, if a random matching satisfies (P3) then it satisfies (P7) for every type assignment. Thus we have the following corollary of Theorem 3.

Corollary 1. *Let $(A, \mathcal{A}, \mu)$ be an atomless probability space of agents, (Ω, Σ, ν) a sample probability space, and $f\colon A \times \Omega \to A$ a random matching. Let λ be the product probability measure on $A \times \Omega$ defined from μ and ν. Suppose:*

(i) *There is a Fubini extension $\bar{\lambda}$ of λ such that f is $(\bar{\Lambda}, \mathcal{A})$-measurable, writing $\bar{\Lambda}$ for the domain of $\bar{\lambda}$.*

(ii) *f satisfies* (P2) *and* (P4)*.*

Then f satisfies (P7) *for every type assignment.*

Remark 5. We note here that even when a random matching is jointly measurable (or satisfies (i) of Proposition 3), mixing properties do not follow from proportionality properties if no independence properties are satisfied. This is illustrated in the following example where (P6) holds but (P7) does not.

Example 1. Take the probability space $(A, \mathcal{A}, \mu)$ of agents to be $([0,1], \mathcal{B}, \lambda)$, where λ is Lebesgue measure, and $\mathcal{B}$ the Borel $\sigma-$algebra of $[0,1]$. Partition $[0,1]$ into eight measurable subsets $A_1, \ldots, A_8$, each with measure $1/8$. Let $\left(A_i, A_j\right)$ denote "the agents in A_i are matched with the agents in A_j." Recall that given any $C, C' \in \mathcal{B}$ of the same measure, there is an inverse measure preserving bijection from C onto C'. Using this fact, we can construct four matchings $f_1, \ldots, f_4$ on $[0,1]$ such that each f_i is inverse measure-preserving and such that

f_1 satisfies	(A_1, A_2)	(A_3, A_7)	(A_4, A_8)	(A_5, A_6)
f_2 satisfies	(A_1, A_5)	(A_2, A_6)	(A_3, A_4)	(A_7, A_8)
f_3 satisfies	(A_1, A_2)	(A_3, A_4)	(A_5, A_6)	(A_7, A_8)
f_4 satisfies	(A_1, A_6)	(A_2, A_5)	(A_3, A_8)	(A_4, A_7)

Let the sample probability space (Ω, Σ, ν) be the set $\{1, 2, 3, 4\}$ with normalized counting measure and let a random matching $f\colon [0,1] \times \Omega \to [0,1]$ be given by $f(x, i) = f_i(x)$ for $x \in [0,1]$ and $i \in \Omega$. Assume that there are just two types 0 and 1, and that the type assignment $\theta : [0,1] \to \{0,1\}$ is given by $\theta(x) = 0$ for $x \in \bigcup_{j=1}^{4} A_j$, and $\theta(x) = 1$ for $x \in \bigcup_{j=5}^{8} A_j$. Then, since in state 3 there is no match between any agents of different types, f fails to satisfy (P7). On the other hand, since each of the four matchings is equally likely, it is easy to check that f satisfies (P6). Moreover, f is $\mathcal{B}\otimes\Sigma-$measurable. However, since there are only finitely many sub-σ-algebras of Σ, f cannot satisfy independence conditions as in (P4) or (P8).

4 The main existence result

In this section we state our main theorem on existence of random matching. It is important to note that in this theorem the space of agents cannot be $[0,1]$

with Lebesgue measure (see Remark 8 for more on this). As already remarked in the introduction, by not taking the space of agents to be $[0,1]$ with Lebesgue measure, we depart from the approach in Alós-Ferrer (1999). The following notation applies in the sequel. For any set X, $\#(X)$ denotes its cardinal; $\mathfrak{c}$ denotes the cardinal of the continuum.

Theorem 1. *There exists an atomless probability space $(A, \mathcal{A}, \mu)$ of agents, a sample probability space (Ω, Σ, ν), and a random matching $f\colon A \times \Omega \to A$ such that the following hold.*

(a) *f satisfies* (P1) *to* (P5)*.*

(b) *Given any type space $(T, \mathcal{T})$ and type assignment $\theta\colon A \to T$, f satisfies* (P6) *to* (P8)*.*

(c) *Let λ be the product measure on $A \times \Omega$ defined from μ and ν. There is a Fubini extension $\bar{\lambda}$ of λ such that f is $(\bar{\Lambda}, \mathcal{A})$-measurable, writing $\bar{\Lambda}$ for the domain of $\bar{\lambda}$; in particular, given any type space $(T, \mathcal{T})$ and type assignment $\theta\colon A \to T$, the type process $\theta \circ f$ is $(\bar{\Lambda}, \mathcal{T})$-measurable.*

(d) *The probability space $(A, \mathcal{A}, \mu)$ of agents can be constructed with $\#(A) = \mathfrak{c}$.*

The proof is in Section 6.1. The following remarks comment on the theorem.

Remark 6. The theorem will be proved by showing that there is a random matching on an atomless probability space $(A, \mathcal{A}, \mu)$ of agents with $\#(A) = \mathfrak{c}$ such that (P1) to (P4) are satisfied. By Remark 1 and Proposition 2, this establishes the entire theorem.

Remark 7. Note that Theorem 1 gives a random matching which is independent of type spaces and type assignments. In particular, it gives a random matching such that the important types mixing property is satisfied for every possible type assignment.

Moreover, our result applies to the case of infinitely many types. Indeed, recall that given any atomless probability space $(A, \mathcal{A}, \mu)$ and any Borel probability measure γ on a Polish space Z, there is a mapping $\theta\colon A \to Z$ which is inverse-measure-preserving for μ and γ; in other words, every such γ is the distribution of some measurable mapping from A to Z. Consequently, Theorem 1 allows for any Borel probability measure on a Polish space to be taken as type distribution.

In both of these aspects, our existence result for random matching improves on that in Alós-Ferrer (1999). Another difference from Alós-Ferrer (1999) is that, in our result, the random matching satisfies general independence. As pointed out in Section 2, in a random matching model with a continuum population it may be natural to require this property.

In regard to independence properties, Theorem 1 also differs from the existence result for random matching in Duffie and Sun (2007, Theorem 2.4) where only pairwise independence in types is required, i.e., property (P8) of our paper, weakened to pairwise independence.[8] However, given that one wants random matchings to satisfy independence properties, it seems more natural to require independence directly for the matching process, as with property (P4), and to require stochastic independence in the usual sense, rather than only pairwise independence.

Furthermore, unlike Duffie and Sun (2007), our result is not based on nonstandard analysis. In particular, it does not depend on Loeb space constructions.[9]

Remark 8. Theorem 1(d) says that the probability space $(A, \mathcal{A}, \mu)$ of agents can be constructed so that A may be identified as a set with the unit interval $[0, 1]$ via a bijection; that is, if one likes, one can take $A = [0, 1]$ in Theorem 1. The point is that the measure μ cannot be Lebesgue measure on $[0, 1]$. In fact, as noted in Alós-Ferrer (1999, Proposition 3.1 and Corollary 3.2), if the space of agents is taken to be $[0, 1]$ with Lebesgue measure, then a random matching satisfying (P1) must fail (P3) and in particular cannot satisfy (P7) for every possible type assignment.

We note here that one does not need to invoke (P1) to reach this negative conclusion. In Remark 9 below we show that, in fact, (P3) alone cannot be satisfied by any random matching when the space of agents is taken to be $[0, 1]$ with Lebesgue measure. By Proposition 1, this in turn implies that, with this choice of the space of agents, no random matching can satisfy (P7) for every type assignment.

Similarly, by Proposition 3, if one would like to have a random matching that satisfies independence and proportionality properties, as well as some joint measurability property with respect to agents and sample points, then $[0, 1]$ with Lebesgue measure is also not the appropriate choice of the probability space of agents.

Of course, there are economic contexts where it has a specific meaning that the space of agents is taken to be $[0, 1]$ with Lebesgue measure, e.g., contexts where geographical location of agents matters. In such contexts our result does not apply.

Frequently, however, as for instance in standard general equilibrium models, it is of no economic significance whether or not the space of agents is $[0, 1]$ with Lebesgue measure. Indeed, if a large set of negligible agents is modeled as an atomless probability space just to establish that any single agent has strictly no influence on aggregate levels, then, to quote Hildenbrand (1974, p.

[8] Actually, Duffie and Sun (2007) speak of *essential* pairwise independence in types, which in our notation means that for almost all $x \in A$, $\theta \circ f_x$ is stochastically independent of $\theta \circ f_{x'}$ for almost all $x' \in A$.

[9] Built on their 2007 result, Duffie and Sun (2010) recently have constructed a random matching satisfying pairwise independence in types for some type assignments where the space of agents is $[0, 1]$ with an *extension* of Lebesgue measure. However, in that result, the random matching is not universal with respect to type assignments. Also, the measure preservation property fails.

113), the σ-algebra should be considered as "only been introduced for technical reasons" and, conceptually, "be considered ... as the set of all subsets" of the set of agents.[10] Under this view, any atomless probability measure on $[0,1]$ is as good as any other in modeling a large set of negligible agents, and a particular choice, e.g. according to our Theorem 1, of a σ-algebra, or probability measure, on the set of agents should not be discussed in terms of economic meaning, but should be seen as a technical device having to do some job.

Remark 9. That (P3) cannot be satisfied by any random matching if the space of agents is $[0,1]$ with Lebesgue measure (regardless of whether or not (P1) is satisfied) may be seen as follows. Let $\mathcal{B}$ be the Borel σ-algebra of $[0,1]$, let μ be Lebesgue measure on $[0,1]$, and let $C \subset \mathcal{B}$ be a countable algebra generating $\mathcal{B}$. Suppose there would be a random matching $f\colon [0,1] \times \Omega \to [0,1]$ such that (P3) is satisfied with respect to μ. Pick any $E_2 \in \mathcal{B}$ with $\mu(E_2) = 1/2$. Then (P3) implies that there is a $\bar{y} \in \Omega$ such that $\mu(E_1 \cap f_{\bar{y}}^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for all $E_1 \in C$. Note that for any $y \in \Omega$, $\{E_1 \in \mathcal{B}\colon \mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)\}$ is a monotone class. It follows that $\mu(E_1 \cap f_{\bar{y}}^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for all $E_1 \in \mathcal{B}$. But this is impossible. To see this, take E_1 to be any member of $\mathcal{B}$ which differs from $f_{\bar{y}}^{-1}(E_2)$ by a null set if $\mu(f_{\bar{y}}^{-1}(E_2)) > 0$, and take $E_1 = [0,1]$ otherwise.

5 A uniqueness result

In the introduction we mentioned the observation in Molzon and Puzzello (2010) that a random matching is not uniquely determined by measure preservation, proportionality, and mixing properties. In this section we will address this issue. It will turn out that the crucial properties to get uniqueness of random matching are general proportionality and general independence. Some additional notation is needed.

Notation. Given a probability space $(A, \mathcal{A}, \mu)$ of agents, $M_A \subset A^A$ denotes the set of all matchings on A, i.e., the set of all fixed point free involutions on A; further, writing $\bar{\gamma}$ for the product probability measure on A^A defined from μ, γ denotes the restriction of $\bar{\gamma}$ to the σ-algebra generated by the measurable cylinders in A^A, γ_A the subspace measure on M_A induced from γ, and Γ_A the domain of γ_A. Given in addition a sample probability space (Ω, Σ, ν) and a random matching $f\colon A \times \Omega \to A$, $\phi\colon \Omega \to M_A$ denotes the mapping defined by setting $\phi(y) = f_y$ for $y \in \Omega$.

Now, given a probability space $(A, \mathcal{A}, \mu)$ of agents, the following theorem shows that if a random matching exists, then, in terms of distributions on (M_A, Γ_A), it is unique subject to (P2) and (P4).

[10] In particular, then, it should not be considered an essential point whether or not the σ-algebra is countably generated. Further, the σ-algebra need not be derived from any topological structure on the set of agents, and hence, in case of $[0,1]$ as set of agents, it should also not be considered an essential point whether or not the sub-intervals of $[0,1]$ belong to the σ-algebra.

Theorem 2. *Let $(A, \mathcal{A}, \mu)$ be a probability space of agents. Then if (Ω, Σ, ν) is any sample probability space and $f: A \times \Omega \to A$ is a random matching, the mapping ϕ is (Σ, Γ_A)-measurable, and if f satisfies* (P2) *and* (P4)*, the distribution of ϕ on (M_A, Γ_A) is γ_A.*

Proof. It suffices to show that ϕ, viewed as a mapping from Ω to A^A, has the property that $\phi^{-1}(Z) \in \Sigma$ whenever Z is a measurable cylinder in A^A, and that if f satisfies (P2) and (P4) then $\nu(\phi^{-1}(Z)) = \gamma(Z)$ for any such Z. Thus let Z be a measurable cylinder in A^A. Then for some finite collection $x_1, \ldots, x_n$ of distinct members of A, together with members $B_1, \ldots, B_n$ of $\mathcal{A}$, we have $Z = E_{B_1}^{x_1} \cap \cdots \cap E_{B_n}^{x_n}$ where $E_{B_i}^{x_i} = \{z \in A^A : z(x_i) \in B_i\}$, $i = 1, \ldots, n$. Note that for each $i = 1, \ldots, n$, $\phi^{-1}(E_{B_i}^{x_i}) = f_{x_i}^{-1}(B_i)$, because for any $y \in \Omega$,

$$\phi(y) \in E_{B_i}^{x_i} \Leftrightarrow \phi(y)(x_i) \in B_i \Leftrightarrow f_y(x_i) \in B_i \Leftrightarrow f_{x_i}(y) \in B_i.$$

Now by definition of random matching, f_x is $(\Sigma, \mathcal{A})$-measurable for any $x \in A$. It follows that $\phi^{-1}(E_{B_i}^{x_i}) \in \Sigma$ for each $i = 1, \ldots, n$, and hence that $\phi^{-1}(Z) \in \Sigma$. Moreover, if f satisfies (P2) and (P4), then

$$\begin{aligned}
\nu(\phi^{-1}(Z)) &= \nu(\phi^{-1}(E_{B_1}^{x_1}) \cap \cdots \cap \phi^{-1}(E_{B_n}^{x_n})) \\
&= \nu(f_{x_1}^{-1}(B_1) \cap \cdots \cap f_{x_n}^{-1}(B_n)) \\
&= \prod_{i=1}^{n} \nu(f_{x_i}^{-1}(B_i)) \quad \text{by (P4)} \\
&= \prod_{i=1}^{n} \mu(B_i) \quad \text{by (P2)} \\
&= \gamma(E_{B_1}^{x_1} \cap \cdots \cap E_{B_n}^{x_n}) = \gamma(Z),
\end{aligned}$$

the first equality in the previous line by the definition of product measure since the elements $x_1, \ldots, x_n$ of A are distinct. This completes the proof. □

As noted in Proposition 1, if a random matching satisfies (P6) and (P8) for any type assignment with a finite type space, then it satisfies (P2) and (P4). Therefore the above uniqueness result can equivalently be stated in the following way in terms of type assignments.

Corollary 2. *Let $(A, \mathcal{A}, \mu)$ be a probability space of agents. Then if (Ω, Σ, ν) is any sample probability space and $f: A \times \Omega \to A$ is a random matching satisfying* (P6) *and* (P8) *for every type assignment with a finite type space, the distribution of ϕ on (M_A, Γ_A) is γ_A.*

6 Remaining proofs

6.1 Proof of Theorem 1

Let ω_1 be the least uncountable ordinal. For each $\xi < \omega_1$, choose a subset $K_\xi \subset \omega_1$ with $\#(K_\xi) = \#(\xi)$ such that $\eta > \xi$ for each $\eta \in K_\xi$, and then choose a bijection $\rho_\xi \colon \xi \to K_\xi$. Define $h_\xi \colon \omega_1 \to \omega_1$ by setting

$$h_\xi(\eta) = \begin{cases} \rho_\xi(\eta) & \text{for } \eta < \xi \\ \rho_\xi^{-1}(\eta) & \text{for } \eta \in K_\xi \\ \eta & \text{for } \eta \notin \xi \cup K_\xi. \end{cases}$$

Then for each $\xi < \omega_1$, h_ξ is an involution on ω_1.

Consider the product space $\{0,1\}^{\omega_1}$. Let λ be the usual measure on $\{0,1\}^{\omega_1}$, and let Λ denote the domain of λ. Recall that λ is complete. For each $\xi < \omega_1$, define a mapping $\hat{\phi}_\xi \colon \{0,1\}^{\omega_1} \to \{0,1\}^{\omega_1}$ by setting, for each $x \in \{0,1\}^{\omega_1}$,

$$\hat{\phi}_\xi(x) = x \circ h_\xi\,.$$

(Thus $\hat{\phi}_\xi(x)$ is the element in $\{0,1\}^{\omega_1}$ that is given by $\hat{\phi}_\xi(x)(\eta) = x(h_\xi(\eta))$ for $\eta < \omega_1$.) Then for each $\xi < \omega_1$, $\hat{\phi}_\xi$ is inverse-measure-preserving for λ, and since h_ξ is an involution, $\hat{\phi}_\xi$ is an involution, too. (To see that $\hat{\phi}_\xi$ is an involution, observe that for each $x \in \{0,1\}^{\omega_1}$,

$$\hat{\phi}_\xi(\hat{\phi}_\xi(x)) = \hat{\phi}_\xi(x \circ h_\xi) = (x \circ h_\xi) \circ h_\xi = x \circ (h_\xi \circ h_\xi) = x.$$

To see that $\hat{\phi}_\xi$ is inverse-measure-preserving for λ, observe that whenever I is a finite subset of ω_1, we have

$$\lambda(\{x \in \{0,1\}^{\omega_1} \colon x(h_\xi(\eta)) = 1 \text{ for every } \eta \in I\}) = 2^{-\#(I)},$$

because h_ξ is an injection.)

We claim that given any E_1, $E_2 \in \Lambda$, for all but countably many $\xi < \omega_1$ the sets E_1 and $\hat{\phi}_\xi^{-1}(E_2)$ are stochastically independent, i.e., $\lambda(E_1 \cap \hat{\phi}_\xi^{-1}(E_2)) = \lambda(E_1)\lambda(\hat{\phi}_\xi^{-1}(E_2))$. To see this, pick any E_1, $E_2 \in \Lambda$. There is an $E_1' \in \Lambda$ which differs from E_1 by a null set and is determined by coordinates in a countable subset of ω_1, say D_1, and there is an $E_2' \in \Lambda$ which differs from E_2 by a null set and is determined by coordinates in a countable subset of ω_1, say D_2. Then by choice of $\hat{\phi}_\xi$, for each $\xi < \omega_1$ the set $\hat{\phi}_\xi^{-1}(E_2')$ is determined by coordinates in $h_\xi(D_2)$. As ω_1 has uncountable cofinality, we can find a $\beta < \omega_1$ such that $\eta < \beta$ for every $\eta \in D_1 \cup D_2$. Then by choice of h_ξ, for each $\xi < \omega_1$ with $\xi > \beta$, we have $\eta > \beta$ for every $\eta \in h_\xi(D_2)$. Hence for each $\xi < \omega_1$ with $\xi > \beta$, $D_1 \cap h_\xi(D_2) = \emptyset$, which implies that the sets E_1' and $\hat{\phi}_\xi^{-1}(E_2')$ are stochastically independent, E_1' being determined by coordinates in D_1, and $\hat{\phi}_\xi^{-1}(E_2')$ by coordinates in $h_\xi(D_2)$. Since $\hat{\phi}_\xi$ is inverse-measure-preserving for λ, the fact that E_1' and E_2' differ by null sets from E_1, E_2, respectively, implies that $\hat{\phi}_\xi^{-1}(E_2')$ differs by a null set from $\hat{\phi}_\xi^{-1}(E_2)$, and $E_1' \cap \hat{\phi}_\xi^{-1}(E_2')$ by a null set from $E_1 \cap \hat{\phi}_\xi^{-1}(E_2)$. Consequently E_1 and $\hat{\phi}_\xi^{-1}(E_2)$ are stochastically independent for each $\xi < \omega_1$ with $\xi > \beta$, and thus the claim above is established.

Because each $\hat{\phi}_\xi$ is inverse-measure-preserving for λ, it follows that given any E_1, $E_2 \in \Lambda$ we have $\lambda(E_1 \cap \hat{\phi}_\xi^{-1}(E_2)) = \lambda(E_1)\lambda(E_2)$ for all but countably many $\xi < \omega_1$.

Let

$$A = \{x \in \{0,1\}^{\omega_1} \colon \text{ for some } \alpha < \omega_1,\ x(\xi) = 1 \text{ for all } \xi < \omega_1 \text{ with } \xi > \alpha\}.$$

Evidently A is expressible as the union of ω_1 sets of cardinal $\mathfrak{c}$, so $\#(A) = \mathfrak{c}$. Also, A has full outer measure for λ, by the fact that every non-negligible member of Λ includes a non-empty set that is determined by coordinates in some countable subset of ω_1, together with the fact that ω_1 has uncountable cofinality.

Let μ be the subspace measure on A induced from λ, and let $\mathcal{A}$ denote its domain. Then, as A has full outer measure for λ, $(A, \mathcal{A}, \mu)$ is a probability space. Clearly, as λ is complete and atomless, so is μ.

For each $\xi < \omega_1$ let $\tilde{\phi}_\xi$ be the restriction of $\hat{\phi}_\xi$ to A. Note that by construction, for each $\xi < \omega_1$ and each $x \in \{0,1\}^{\omega_1}$, $\hat{\phi}_\xi(x)$ and x agree in all but countably many coordinates in ω_1. Consequently, for each $\xi < \omega_1$, whenever $x \in A$ then $\tilde{\phi}_\xi(x) \in A$, again using the fact that ω_1 has uncountable cofinality. Thus since $\hat{\phi}_\xi$ is an involution on $\{0,1\}^{\omega_1}$, $\tilde{\phi}_\xi$ is an involution on A. By the fact that A has full outer measure for λ, the properties of the functions $\hat{\phi}_\xi$, $\xi < \omega_1$, also imply that, for each $\xi < \omega_1$, $\tilde{\phi}_\xi$ is inverse-measure-preserving for μ, and that, given any E_1, $E_2 \in \mathcal{A}$, for all but countably many $\xi < \omega_1$ we have $\mu(E_1 \cap \tilde{\phi}_\xi^{-1}(E_2)) = \mu(E_1)\mu(E_2)$.

We will now modify the mappings $\tilde{\phi}_\xi$ so as to make them fixed point free. Pick any $\xi < \omega_1$ with $\xi \geq \omega$. Let

$$\Delta_\xi = \{x \in \{0,1\}^{\omega_1} \colon x(\eta) = x(h_\xi(\eta)) \text{ for each } \eta < \omega_1\}$$

and let $\Delta_\xi^A = \Delta_\xi \cap A$. Then by the definitions of $\hat{\phi}_\xi$ and $\tilde{\phi}_\xi$, Δ_ξ^A is exactly the set of fixed points of $\tilde{\phi}_\xi$. Now by the definition of h_ξ,

$$\{\eta < \omega_1 \colon \eta < \xi\} \cap h_\xi(\{\eta < \omega_1 \colon \eta < \xi\}) = \emptyset.$$

Hence since $\xi \geq \omega$, Δ_ξ is a λ-null set in $\{0,1\}^{\omega_1}$ (directly from the definition of λ to be the usual measure on $\{0,1\}^{\omega_1}$), and thus Δ_ξ^A is a μ-null set in A. Finally, Δ_ξ^A is an infinite subset of A. (To see this, note that by definition of h_ξ, for some countable $D \subset \omega_1$ we have $h_\xi(\eta) = \eta$ for all $\eta < \omega_1$ with $\eta \notin D$, and let B be the set of those x in A for which $x(\eta) = 1$ for all $\eta < \omega_1$ with the exception of exactly one $\eta < \omega_1$ with $\eta \notin D$. Then B is an infinite subset of A, and since h_ξ is a bijection we must have $B \subset \Delta_\xi^A$.)

Now by the fact that any infinite set can be partitioned into two sets of the same cardinality, we can choose a fixed point free involution $\kappa_\xi \colon \Delta_\xi^A \to \Delta_\xi^A$. As Δ_ξ^A is the set of fixed points of $\tilde{\phi}_\xi$, the restriction of $\tilde{\phi}_\xi$ to $A \backslash \Delta_\xi^A$ is an involution on $A \backslash \Delta_\xi^A$. Therefore, defining $\phi_\xi \colon A \to A$ by

$$\phi_\xi(x) = \begin{cases} \kappa_\xi(x) & \text{if } x \in \Delta_\xi^A \\ \tilde{\phi}_\xi(x) & \text{if } x \in A \backslash \Delta_\xi^A, \end{cases}$$

ϕ_ξ is a fixed point free involution on A. As ϕ_ξ agrees with $\tilde{\phi}_\xi$ on the complement of a μ-null set, ϕ_ξ is inverse-measure-preserving for μ.

Doing this construction for all $\xi < \omega_1$ with $\xi \geq \omega$, and then letting $\phi_\xi = \phi_\omega$ for $\xi < \omega$, we get a family $\langle \phi_\xi \rangle_{\xi<\omega_1}$ of fixed point free involutions on A, each of them inverse-measure-preserving for μ. Moreover, given any $E_1, E_2 \in \mathcal{A}$, for all but countably many $\xi < \omega_1$ we have $\mu(E_1 \cap \phi_\xi^{-1}(E_2)) = \mu(E_1)\mu(E_2)$, by the corresponding property of the family $\langle \tilde{\phi}_\xi \rangle_{\xi<\omega_1}$, because ϕ_ξ agrees with $\tilde{\phi}_\xi$ on the complement of a μ-null set for $\omega \leq \xi < \omega_1$.

Now choose a family $\langle x_\xi \rangle_{\xi<\omega_1}$ of elements of A so that given any countable $D \subset A$, for some $\xi < \omega_1$ we have both $x_\xi \notin D$ and $\phi_\xi(x_\xi) \notin D$. Such a choice is possible. Indeed, by transfinite recursion on ω_1 choose a family $\langle x_\xi \rangle_{\xi<\omega_1}$ as follows. Let x_0 be an arbitrary point of A. Given that $\langle x_\eta \rangle_{\eta<\xi}$ has been chosen, where $\xi < \omega_1$, consider the set $A_\xi = \{x_\eta, \phi_\eta(x_\eta) : \eta < \xi\}$. Then A_ξ is countable, so, because A is uncountable and ϕ_ξ is a bijection, we can choose an x_ξ in A such that both $x_\xi \notin A_\xi$ and $\phi_\xi(x_\xi) \notin A_\xi$. This completes the recursion. The result is a family $\langle x_\xi \rangle_{\xi<\omega_1}$ of distinct elements of A such that the family $\langle \phi_\xi(x_\xi) \rangle_{\xi<\omega_1}$ also consists of distinct elements. Thus $\langle x_\xi \rangle_{\xi<\omega_1}$ is a family as desired.

Let $\bar{\nu}$ be the complete product probability measure on A^A defined from μ, and let $\bar{\Sigma}$ denote the domain of $\bar{\nu}$. For each $\xi < \omega_1$ let

$$
\begin{aligned}
N_\xi = \Big\{ y \in A^A \colon & \text{(a) } y \text{ is a fixed point free involution on } A, \\
& \text{(b) } y(x_\xi) = \phi_\xi(x_\xi), \\
& \text{(c) } y \restriction A \setminus N = \phi_\xi \restriction A \setminus N \ \text{ for some } \mu\text{-null set } N \subset A \Big\},
\end{aligned}
$$

and then let $\Omega = \bigcup_{\xi<\omega_1} N_\xi$.

From (c) in the definition of N_ξ, each $y \in \Omega$ is inverse-measure-preserving for μ. From (b) in that definition, each N_ξ is a $\bar{\nu}$-null set in A^A because, μ being atomless, singletons in A are μ-null sets.

On the other hand, Ω has full outer measure for $\bar{\nu}$. To see this, note first that it suffices to show that Ω intersects every non-negligible subset of A^A that is determined by coordinates in some countable subset of A (since every non-negligible element of $\bar{\Sigma}$ includes such a set). Thus let E be a non-negligible subset of A^A, determined by coordinates in a countable subset of A, say D.

As D is countable and $(A, \mathcal{A}, \mu)$ is atomless, the set of all y in A^A such that $y \restriction D$ is injective is an element of $\bar{\Sigma}$ with $\bar{\nu}$-measure 1 (see Fremlin, 2001, 254V). Also, since a countable subset of A is a μ-null set in A, for each $x \in A$ the set of all y in A^A such that $y(x) \in D$ is a $\bar{\nu}$-null set in A^A, and hence (using again the fact that D is countable) the set of all y in A^A such that $D \cap y(D) = \emptyset$ belongs to $\bar{\Sigma}$ and has $\bar{\nu}$-measure 1. Consequently, because E is non-negligible, there is an element of E, say y_0, such that $y_0 \restriction D$ is a bijection onto $y_0(D)$ and such that $D \cap y_0(D) = \emptyset$.

Set $D' = y_0(D)$. Then $D \cup D'$ is countable, so we can choose a countably infinite subset H of A with $H \cap (D \cup D') = \emptyset$. Set $C = H \cup D \cup D'$. Then C is again countable, so by choice of the family $\langle x_\xi \rangle_{\xi<\omega_1}$, there is a $\xi < \omega_1$ such that $x_\xi \notin C$ as well as $\phi_\xi(x_\xi) \notin C$. Fix such a ξ and set $C' = C \cup \phi_\xi(C)$. Using the fact that ϕ_ξ is an involution, we may see that $x_\xi \notin C'$.

Also by the fact that ϕ_ξ is an involution, we have $\phi_\xi(C') = C'$ and therefore $\phi_\xi(A \setminus C') = A \setminus \phi_\xi(C') = A \setminus C'$. Thus $\phi_\xi \restriction A \setminus C'$ is a fixed point free involution on $A \setminus C'$.

Note that by choice of C, the set $C' \setminus (D \cup D')$ is infinite. Hence, since an infinite set can be partitioned into two sets of the same cardinality, we can choose a fixed point free involution $\zeta\colon C' \setminus (D \cup D') \to C' \setminus (D \cup D')$.

Now as $y_0 \restriction D$ is a bijection onto D', and $D \cap D' = \emptyset$, we get a fixed point free involution $y_1\colon A \to A$ by setting, for $x \in A$,

$$y_1(x) = \begin{cases} y_0(x) & \text{if } x \in D \\ y_0^{-1}(x) & \text{if } x \in D' \\ \zeta(x) & \text{if } x \in C' \setminus (D \cup D') \\ \phi_\xi(x) & \text{if } x \in A \setminus C'. \end{cases}$$

In particular, then, since $x_\xi \notin C'$, we have $y_1(x_\xi) = \phi_\xi(x_\xi)$. Thus $y_1 \in \Omega$, because the countable set C' is a μ-null set in A. On the other hand, y_1 agrees with y_0 on D, and since $y_0 \in E$ and E is determined by coordinates in D, we have $y_1 \in E$. Thus $\Omega \cap E \neq \emptyset$, proving that Ω has full outer measure for $\bar{\nu}$.

Let ν be the subspace measure on Ω induced from $\bar{\nu}$, and let Σ denote its domain. Then, as Ω has full outer measure for $\bar{\nu}$, (Ω, Σ, ν) is a probability space. Note that N_ξ is a ν-null set in Ω for each $\xi < \omega_1$.

Now let $f\colon A \times \Omega \to A$ be defined by setting

$$f(x, y) = y(x), \; x \in A, \; y \in \Omega.$$

Further, for each $x \in A$, let π_x be the coordinate projection $y \mapsto y(x)\colon A^A \to A$. Then, by definition of product measure, for each $x \in A$, π_x is inverse-measure-preserving for $\bar{\nu}$ and μ, and the family $\langle \pi_x \rangle_{x \in A}$ is stochastically independent. Evidently $f(x, \cdot)$ agrees with π_x on Ω for each $x \in A$, and since Ω has full outer measure for $\bar{\nu}$, it follows that for each $x \in A$, $f_x \equiv f(x, \cdot)$ is inverse-measure-preserving for ν and μ, and that the family $\langle f_x \rangle_{x \in A}$ is stochastically independent. On the other hand, for each $y \in \Omega$, f_y is the same as y. Hence, for each $y \in \Omega$, f_y is a fixed point free involution on A, and by what was noted following the definition of the sets N_ξ above, f_y is inverse-measure-preserving for μ. As was also noted above, given any $E_1, E_2 \in \mathcal{A}$, we have $\mu(E_1 \cap \phi_\xi^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for all but countably many $\xi < \omega_1$. By (c) in the definition of the sets N_ξ, this means that, given any $E_1, E_2 \in \mathcal{A}$, there is a countable $D \subset \omega_1$ such that whenever $y \in \Omega \setminus \bigcup_{\xi \in D} N_\xi$ then $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$. As each N_ξ is a null set in Ω, it follows that, given any $E_1, E_2 \in \mathcal{A}$, we have $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for almost all $y \in \Omega$.

Taken together, these properties of f mean that f is a random matching satisfying (P1) to (P4). By Remark 1 in Section 2, (P5) is also satisfied and, given any type space $(T, \mathcal{T})$ and type assignment $\theta\colon A \to T$, (P6) to (P8) are satisfied as well. Thus (a) and (b) of the theorem hold. By Proposition 2, (c) of the theorem holds, and the choice of A shows that (d) is true. This completes the proof. $\square$

6.2 Proof of Proposition 2

The following notation will be used in the sequel.

Notation. If H is a subset of $A \times \Omega$ then for $x \in A$, H_x denotes the x-section of H, and for $y \in \Omega$, H_y denotes the y-section of H. Thus, if $x \in A$, then $H_x = \{y \in \Omega\colon (x, y) \in H\}$; similarly, for $y \in \Omega$, $H_y = \{x \in A\colon (x, y) \in H\}$.

For convenience, we first establish a lemma.

Lemma. *Let $(A, \mathcal{A}, \mu)$ and (Ω, Σ, ν) be probability spaces, and $(A \times \Omega, \Lambda, \lambda)$ the corresponding product probability space. Let $\mathcal{M}$ be the set of all $M \subset A \times \Omega$ for which M_x is a null set in Ω for almost all $x \in A$, and M_y a null set in A for almost all $y \in \Omega$. Further, let $\langle J_i \rangle_{i \in I}$ be a family of sets, and $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$ a family of subsets of $A \times \Omega$. Suppose:*

(a) *For all $x \in A$ and all $y \in \Omega$, $H_x^{i,j} \in \Sigma$ and $H_y^{i,j} \in \mathcal{A}$ for each $i \in I$ and $j \in J_i$.*

(b) *For each $i \in I$ there is a real number $\alpha_i > 0$ such that whenever $j_1, \ldots, j_n$ are finitely many distinct members of J_i, then given $B \in \mathcal{A}$,*

$$\mu(B \cap H_y^{i,j_1} \cap \cdots \cap H_y^{i,j_n}) = \mu(B)\alpha_i 2^{-n}$$

for almost all $y \in \Omega$, and given $C \in \Sigma$,

$$\nu(C \cap H_x^{i,j_1} \cap \cdots \cap H_x^{i,j_n}) = \nu(C)\alpha_i 2^{-n}$$

for almost all $x \in A$.

(c) *$H^{i,j} \cap H^{i',j'} = \varnothing$ whenever $i \neq i'$.*

Then λ has a Fubini extension $\bar{\lambda}$ such that $\mathcal{M} \cup \{H^{i,j}\colon i \in I, j \in J_i\} \subset \bar{\Lambda}$, writing $\bar{\Lambda}$ for the domain of $\bar{\lambda}$.

Proof. Let $\mathcal{F}$ be the set of all those subsets F of $A \times \Omega$ for which the integrals $\int_A \bar{\nu}(F_x)\, d\mu(x)$ and $\int_\Omega \bar{\mu}(F_y)\, d\nu(y)$ are well-defined and equal, writing $\bar{\mu}$ and $\bar{\nu}$ for the completions of μ and ν respectively. Then $\mathcal{F}$ is a Dynkin class (i.e., $\varnothing \in \mathcal{F}$ and $\mathcal{F}$ is closed under complements and countable disjoint unions) as may easily be checked. In addition, (a) to (c) imply that whenever $B_1 \times C_1, \ldots, B_n \times C_n$ are finitely many measurable rectangles in $A \times \Omega$ and $F_1, \ldots, F_m$ are finitely many elements of $\mathcal{M} \cup \{H^{i,j}\colon i \in I, j \in J_i\}$, then the intersection

$$(B_1 \times C_1) \cap \cdots \cap (B_n \times C_n) \cap F_1 \cap \cdots \cap F_m$$

belongs to $\mathcal{F}$. Therefore, by the monotone class theorem, there is a σ-algebra $\bar{\Lambda} \subset \mathcal{F}$ which contains all measurable rectangles in $A \times \Omega$ and all members of $\mathcal{M} \cup \{H^{i,j}\colon i \in I, j \in J_i\}$. In particular, $\Lambda \subset \bar{\Lambda}$. Define $\bar{\lambda}\colon \bar{\Lambda} \to \mathbb{R}$ by setting $\bar{\lambda}(F) = \int_A \bar{\nu}(F_x)\, d\mu(x)$ for $F \in \bar{\Lambda}$. Using the monotone convergence theorem, we may see that $\bar{\lambda}$ is a probability measure on $A \times \Omega$. This completes the proof of the lemma. $\square$

Proof of Proposition 2. Using Maharam's theorem, we can choose a countable partition $\langle A_i \rangle_{i \in I}$ of A into non-negligible measurable sets so that for each $i \in I$ there is a family $\langle F^{i,j} \rangle_{j \in J_i}$ of measurable subsets of A, with $F^{i,j} \subset A_i$ for all $j \in J_i$, such that all of (i)–(iii) below hold, writing μ_i for the probability measure on A_i obtained by normalizing the subspace measure induced by μ on A_i:

(i) For each $i \in I$, $\mu_i(F^{i,j}) = 1/2$ for all $j \in J_i$.

(ii) For each $i \in I$, the family $\langle F^{i,j} \rangle_{j \in J_i}$ is stochastically independent for μ_i.

(iii) Denoting by $\mathcal{A}'$ the sub-σ-algebra of $\mathcal{A}$ generated by $\{F^{i,j} \colon i \in I, j \in J_i\}$, for any $B \in \mathcal{A}$ there is a $B' \in \mathcal{A}'$ such that B' differs from B by a μ-null set.

For each $i \in I$ and $j \in J_i$, let $H^{i,j} = f^{-1}(F^{i,j})$. We will show that the family $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$ satisfies the conditions of the lemma above.

Clearly $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$ satisfies (c) of these conditions. As earlier, write f_x for $f(x, \cdot)$ and f_y for $f(\cdot, y)$. Note that for each $i \in I$ and $j \in J_i$, the sections $H_x^{i,j}$ and $H_y^{i,j}$ satisfy

$$H_x^{i,j} = f_x^{-1}(F^{i,j}) \text{ and } H_y^{i,j} = f_y^{-1}(F^{i,j})$$

for all $x \in A$ and $y \in \Omega$ respectively. Thus, in particular, (a) of the above lemma is satisfied by the family $\langle H^{i,j} \rangle_{i \in I, j \in J_i}$.

For each $i \in I$ set $\alpha_i = \mu(A_i)$. Fix any $i \in I$, and let $j_1, \ldots, j_n$ be distinct members of J_i. Note that (i) and (ii) imply:

$$(*) \qquad \mu(F^{i,j_1} \cap \cdots \cap F^{i,j_n}) = \alpha_i 2^{-n}.$$

Consider any $B \in \mathcal{A}$. As f satisfies (P3) by hypothesis, for almost all $y \in \Omega$ we have

$$\mu(B \cap f_y^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})) = \mu(B)\mu(F^{i,j_1} \cap \cdots \cap F^{i,j_n}).$$

Using this fact together with $(*)$, we may see that for almost every $y \in \Omega$,

$$\begin{aligned} \mu(B \cap H_y^{i,j_1} \cap \cdots \cap H_y^{i,j_n}) &= \mu(B \cap f_y^{-1}(F^{i,j_1}) \cap \cdots \cap f_y^{-1}(F^{i,j_n})) \\ &= \mu(B \cap f_y^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})) \\ &= \mu(B)\mu(F^{i,j_1} \cap \cdots \cap F^{i,j_n}) \\ &= \mu(B)\alpha_i 2^{-n}. \end{aligned}$$

Now consider any $C \in \Sigma$. For each $x \in A$ let Σ_x be the sub-σ-algebra of Σ generated by f_x, and let Σ_C be the sub-σ-algebra of Σ generated by C. By hypothesis, f satisfies (P4), i.e., the family $\langle \Sigma_x \rangle_{x \in A}$ is stochastically independent. By Fremlin (2008, 5A6-272W), it follows that there is a countable $D \subset A$ such that for each $x \in A \backslash D$, Σ_C and Σ_x are stochastically independent. Since $(A, \mathcal{A}, \nu)$ is atomless by hypothesis, this means that Σ_C and Σ_x are stochastically independent for almost every $x \in A$. Now for each $x \in A$, we have $f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n}) \in \Sigma_x$, and it follows that for almost all $x \in A$,

$$\nu(C \cap f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})) = \nu(C)\nu\big(f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n})\big).$$

Using this fact together with $(*)$ and the hypothesis that f satisfies (P2), i.e., that for each $x \in A$, f_x is inverse-measure-preserving for μ and ν, we may conclude that for almost all $x \in A$,

$$\begin{aligned}\nu(C \cap H_x^{i,j_1} \cap \cdots \cap H_x^{i,j_n}) =&\nu(C \cap f_x^{-1}(F^{i,j_1}) \cap \cdots \cap f_x^{-1}(F^{i,j_n}))\\ =&\nu(C \cap f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n}))\\ =&\nu(C)\nu(f_x^{-1}(F^{i,j_1} \cap \cdots \cap F^{i,j_n}))\\ =&\nu(C)\mu(F^{i,j_1} \cap \cdots \cap F^{i,j_n})\\ =&\nu(C)\alpha_i 2^{-n}.\end{aligned}$$

Thus (b) of the above lemma is also satisfied by the family $\langle H^{i,j}\rangle_{i\in I, j\in J_i}$.

Now let

$$\mathcal{G} = \{f^{-1}(N) \colon N \text{ is a } \mu\text{-null set in } A\}.$$

Then for each $M \in \mathcal{G}$, and each $x \in A$, the section M_x is a ν-null set in Ω, by the facts that $M = f^{-1}(N)$ implies $M_x = f_x^{-1}(N)$ and f_x is inverse-measure-preserving. Also, by (P3) with $E_1 = A$, for each $M \in \mathcal{G}$, M_y is a μ-null set in A for almost all $y \in \Omega$.

We may now appeal to the lemma above to find a Fubini extension of λ such that, denoting by $\bar{\Lambda}$ its domain, $\bar{\Lambda}$ contains every member of $\mathcal{G}$ and every member of $\langle H^{i,j}\rangle_{i\in I, j\in J_i}$. In view of (iii) above, it follows that f is $(\bar{\Lambda}, \mathcal{A})$-measurable. This completes the proof. □

6.3 Proof of Proposition 1

(a) We have to show that whenever $x_1, \ldots, x_n$ are distinct members of A and $E_1, \ldots, E_n$ are members of $\mathcal{A}$, then

$$\nu(f_{x_1}^{-1}(E_1) \cap \cdots \cap f_{x_n}^{-1}(E_n)) = \prod_{i=1}^{n} \nu(f_{x_i}^{-1}(E_i)).$$

Thus let such $x_1, \ldots, x_n$ and $E_1, \ldots, E_n$ be given. There is a finite partition $\mathcal{P}$ of A into measurable subsets such that for each $i = 1, \ldots, n$, E_i is the union of members of $\mathcal{P}$. Let the finite type space $(T, \mathcal{T})$ be given by setting $T = \mathcal{P}$ and $\mathcal{T} = 2^{\mathcal{P}}$, and let the type assignment $\theta \colon A \to T$ be the mapping that takes an $x \in A$ to that element of $\mathcal{P}$ which contains x. Evidently θ is $(\mathcal{A}, \mathcal{T})$-measurable and we have $\theta^{-1}(\theta(E_i)) = E_i$ for each $i = 1, \ldots, n$. Now the hypothesis implies that

$$\nu(f_{x_1}^{-1}(\theta^{-1}(\theta(E_1))) \cap \cdots \cap f_{x_n}^{-1}(\theta^{-1}(\theta(E_n)))) = \prod_{i=1}^{n} \nu(f_{x_i}^{-1}(\theta^{-1}(\theta(E_i)))),$$

and since $\theta^{-1}(\theta(E_i)) = E_i$ for each $i = 1, \ldots, n$, we have the desired conclusion.

(b) Consider any E_1, $E_2 \in \mathcal{A}$. Let the type space $(T, \mathcal{T})$ be given by setting $T = \{0, 1, 2, 3\}$ and $\mathcal{T} = 2^T$, and let the type assignment $\theta \colon A \to T$ be given by setting $\theta(x) = 0$ for $x \in E_1 \backslash E_2$, $\theta(x) = 1$ for $x \in E_1 \cap E_2$, $\theta(x) = 2$ for $x \in E_2 \backslash E_1$, and $\theta(x) = 3$ for $x \in A \backslash (E_1 \cup E_2)$. Then the hypothesis implies that there is a ν-null set $N \subset \Omega$ such that for each $y \in \Omega \backslash N$,

$$\mu(\theta^{-1}(\{0,1\}) \cap f_y^{-1}(\theta^{-1}(\{1,2\}))) = \mu(\theta^{-1}(\{0,1\}))\mu(\theta^{-1}(\{1,2\})).$$

By the choice of θ, this means $\mu(E_1 \cap f_y^{-1}(E_2)) = \mu(E_1)\mu(E_2)$ for each $y \in \Omega \backslash N$.

(c) Fix any $E \in \mathcal{A}$. Let $(T, \mathcal{T}) = (\{0,1\}, 2^{\{0,1\}})$ and let $\theta \colon A \to T$ be given by $\theta(x) = 1$ if $x \in E$ and $\theta(x) = 0$ if $x \in A \backslash E$. Then, for every $x \in A$, the hypothesis implies $\nu(f_x^{-1}(\theta^{-1}(\{1\}))) = \mu(\theta^{-1}(\{1\}))$ and thus $\nu(f_x^{-1}(E)) = \mu(E)$. □

6.4 Proof of the claim in Remark 2

Suppose $C \subset \mathcal{T}$ is a countable algebra generating $\mathcal{T}$ and, for any $y \in \Omega$ and any $B_1, B_2 \in \mathcal{T}$, let $\mathrm{P}_y(B_1, B_2)$ stand for the statement

$$\text{“}\mu\big(\theta^{-1}(B_1) \cap (\theta \circ f_y)^{-1}(B_2)\big) = \tau(B_1)\tau(B_2).\text{"}$$

Since C is countable, (P7) implies that there is a null set $N \subset Y$ such that for any $y \in \Omega \backslash N$, $\mathrm{P}_y(B_1, B_2)$ is true for all $B_1, B_2 \in C$. Fix any $y \in \Omega \backslash N$ and any $B_2 \in C$. The set $\{B_1 \in \mathcal{T} \colon \mathrm{P}_y(B_1, B_2) \text{ is true}\}$ is readily seen to be a monotone class, and it follows that this set is $\mathcal{T}$. As B_2 was an arbitrary member of C, this means that $\mathrm{P}_y(B_1, B_2)$ is true for all $B_1 \in \mathcal{T}$ and all $B_2 \in C$. Now fix any $B_1 \in \mathcal{T}$. The set $\{B_2 \in \mathcal{T} \colon \mathrm{P}_y(B_1, B_2) \text{ is true}\}$ is again a monotone class, and it follows that this set is $\mathcal{T}$. As B_1 was an arbitrary member of $\mathcal{T}$, it follows that $\mathrm{P}_y(B_1, B_2)$ is true for all B_1, $B_2 \in \mathcal{T}$. Thus, as y was an arbitrary point in $\Omega \backslash N$, (P7') holds. □

7 Examples

Our paper provides foundations also to random matching models with infinitely many types. This section provides examples that show this is very important. Models with infinitely many types are not uncommon in economics. Examples 2 and 3 describe random matching models that require a continuum of types. Example 4 shows what could go wrong if the notion of type is not appropriately defined. It also clarifies that our existence result allows for a correct definition of types also for models with infinitely many types.[11]

Example 2. *Evolutionary Game Theory*
In economics, most work of evolutionary game theory focuses on populations of agents who are randomly matched to play a game with repeated rounds. In these environments, types are identified with strategies. Thus, games with continuous strategy spaces involve random matching with a continuum of types. Examples can be found in Sandholm (2001), Oechssler and Riedel (2002), Hofbauer et al. (2008). In these games, the distribution of strategies in the population is given by a probability distribution on the strategy space S, written as

[11] Examples 3 and 4 are taken from Molzon and Puzzello (2009).

τ. Let $R(s, s')$ denote the payoff function to a player selecting strategy s when his partner/opponent chooses strategy s'. Then, the expected payoff to a player selecting strategy s is written as

$$E(s, \tau) = \int_S R(s, s') d\tau(s').$$

This expression makes implicit use of the types proportional law (P6) with a continuum of types.

Example 3. *Monetary Theory*
We start by describing the aspects of the model of Molico (2006) (see also Zhu (2005)) that are relevant to random matching. Time is discrete and the population $A = [0, 1]$ consists of a continuum of infinitely lived agents whose discount factor is $\beta \in (0, 1)$. Let $\tau_t(E)$ the measure of agents whose money holdings are in $E \subset [0, \infty)$ at the beginning of period t. In this model, the agent's type is given by his money holdings, and thus there may be a continuum of types. In every period agents are randomly and bilaterally matched. An agent is the buyer in his match with probability α, the seller with probability α, and neither with probability $(1 - 2\alpha)$.

The trading rule is determined by means of Nash bargaining. We follow Molico (2006) and denote by $q_t(m_b, m_s)$ and $d_t(m_b, m_s)$ the amount of output and the amount of money determined by bargaining in a match where the buyer has m_b money holdings and the seller has m_s money holdings. Note that the payoff only depends on types.

The expected lifetime utility of an agent who enters period t with m money holdings is given by

$$\begin{aligned} V_t(m) = & \alpha \int_0^\infty \{u[q_t(m, m_s)] + \beta V_{t+1}[m - d_t(m, m_s)]\} d\tau_t(m_s) \\ & + \alpha \int_0^\infty \{-c[q_t(m_b, m)] + \beta V_{t+1}[m + d_t(m_b, m)]\} d\tau_t(m_b) \\ & + (1 - 2\alpha) \beta V_{t+1}(m). \end{aligned}$$

The state of the system at any time is defined by the distribution τ_t, whose law of motion depends on the proportion of sellers and the proportion of buyers. With x denoting the proportion of buyers and sellers during a period, the law of motion for the distribution of money in Molico (2006) can be written as

$$\begin{aligned} \tau_{t+1}(B) = & \alpha \int \int_{m_b - d_t(m_b, m_s) \in B} d\tau_t(m_b) d\tau_t(m_s) \\ & + \alpha \int \int_{m_s + d_t(m_b, m_s) \in B} d\tau_t(m_b) d\tau_t(m_s) + (1 - 2\alpha)\tau_t(B). \end{aligned}$$

where the first and second terms are the measure of consumers and producers whose post-trade money holdings are in B. The last term accounts for those agents who do not trade and thus their money holdings remain in B.

The expressions above suggest that the expected payoff and the law of motion equations implicitly postulate a matching process that satisfies properties (P6) and (P7) with a continuum of types.

Example 4. *On the notion of type in economics*
It is intuitive that if the notion of type in a random matching model does not capture all payoff relevant characteristics of agents, then the model may fail to give proper predictions on aggregate outcomes. We make this intuition precise by providing a simple example with finitely many agents.

Suppose there is an even number of agents, say 8, of two types, "a" and "b." Denote the set of agents by

$$A = \{a_1, ..., a_4, b_1, ..., b_4\}.$$

Let M_A denote the set of all possible matchings on this set of agents, and let elements of M_A be denoted by φ. The randomness of matching will be modeled by placing a probability distribution on the set M_A.

Each agent $x \in A$ is endowed with a non-negative amount k_x of some input. Suppose that production of a certain good occurs only when agents of opposite type meet, and that in this case the production of agent x depends on his input and the input of the agent with whom agent x is matched. A simple specification capturing such a complementarity of inputs is, denoting by $F_x(\varphi)$ the production amount of agent x given φ,

$$F_x(\varphi) = \begin{cases} f(\min\{k_x, k_{\varphi(x)}\}) & \text{if } \ x \text{ and } \varphi(x) \text{ have different types} \\ 0 & \text{if } \ x \text{ and } \varphi(x) \text{ have the same type,} \end{cases}$$

where $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $f(0) = 0$.

We now consider two distinct probability distributions on M_A. The two distributions are described in the tables below, listing the matchings and corresponding probabilities. Matchings that do not appear are assigned probability 0. The notation (x, x') is used to denote that agent x is paired with agent x'.

Distribution I

Matching	Probability
$(a_1,a_2)(a_3,b_3)(a_4,b_4)(b_1,b_2)$	.5
$(a_1,b_1)(a_2,b_2)(a_3,a_4)(b_3,b_4)$	.5

Distribution II

Matching	Probability
$(a_1,a_2)(a_3,b_3)(a_4,b_4)(b_1,b_2)$	.25
$(a_1,a_2)(a_3,b_1)(a_4,b_2)(b_3,b_4)$	.25
$(a_1,b_1)(a_2,b_2)(a_3,a_4)(b_3,b_4)$	.25
$(a_1,b_3)(a_2,b_4)(a_3,a_4)(b_1,b_2)$	.25

Note that both distributions satisfy property (P6) (types proportional law) since any individual agent has probability .5 of being matched with a type "a" agent and probability .5 of being matched with a type "b" agent. Both distributions also satisfy the types mixing property (P7) since for each listed matching, exactly one-half of the type "a" agents are matched with type "a" agents and one-half are matched with type "b" agents. Now, suppose that agents are given initial endowments as described in the following table:

Input endowments

Agent	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
Input	1	1	0	0	0	0	1	1

In the case of Distribution I, nothing can be produced. For both matchings, either two agents of the same type are paired or a pair involves one agent with 0 resource. In the case of Distribution II, if one of the first three matchings is realized, no production takes place because two agents of the same type meet or agents of opposite type meet but one of them has 0. However if the fourth matching is realized (and this occurs with probability .25) then agents a_1, a_2, b_3, and b_4 all produce an amount $f(\min\{1,1\}) = f(1)$. Thus, if payoffs depend on production output, these distributions could give rise to very different predictions about expected aggregate outcomes.

Now as shown in Molzon and Puzzello (2010), if individual payoff functions depend only on types, then the information contained in the types proportionality property is all one needs to know about the matching process to make predictions about expected aggregate payoff outcomes; in particular, these outcomes do not depend on the actual choice of the random matching, i.e., the distribution on the set of matchings (see Theorems 4.2 and 4.3 in Molzon and Puzzello (2010)). The point in the example is that this is actually true only if the notion of type includes all payoff relevant attributes of the agents; otherwise, in addition to types proportionality, the choice of the random matching could indeed be relevant. Now in models with a continuum of agents, infinitely types could matter just because different agents could have different payoff relevant attributes.[12] If this is the case, then it is necessary to formulate the model in terms of an infinite type space to avoid the problem that aggregate outcomes could depend on the actual choice of the random matching. Our results provide mathematical foundations to such models.

8 Concluding remarks

This paper provides existence and uniqueness results for random matchings on continuum populations with infinitely many types. Our results suggest that there is a trade-off between the choice of the measure space of agents and the strength of the random matching properties one could hope for. In particular,

[12] See the introduction for references to such models.

if one needs to model the population as a continuum endowed with Lebesgue measure, then one should be ready to face impossibility results regarding desirable properties of random matching (see Alós-Ferrer (1999)). However, if the Lebesgue structure does not have substantive economic implications for the model at hand, then our results can be interpreted as providing solid foundations to a large class of random matching models, including those with infinitely many types.

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