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THE METHOD OF DATA ENVELOPMENT ANALYSIS
AS A TOOL FOR COMPILING A RATING LIST

The paper presents the method of Data Envelopment Analysis (DEA) as a tool for compiling rating lists in tasks of ordering elements of assemblages. We describe the differences that result from applying a one-dimensional model and multi-dimensional model, where in the latter a so-called partial order is created for incomparable elements in the assemblage being ordered. The example used to illustrate the issues discussed is the analysis of teaching results in a student group.

Key words: *DEA method, ordering of assemblages, partial order, rating classes*

1. Introduction

Assemblages of data are often used to describe reality. This involves making a decision on how to select appropriate data and analyse them. The purpose of data analysis should act as a decisive selection criterion. One of those purposes could be the process of ordering a data assemblage, i.e. creating an array of elements or compiling a rating list. Ordering can be followed by assessment – finding better and worse, larger and smaller, major and less important elements, those located higher or lower on the rating list.

Ordering is easy when dealing with an assemblage of homogeneous comparable data of measurable values, e.g. with a numerical assemblage. In this case, the order is called linear or total. This term originates in the fact that the elements of such an ordered assemblage can be arranged in a series or line.

If it is possible for the assemblage to contain incomparable elements, we are dealing with a partial order. This is the case for assemblages of inhomogeneous complex

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data, an example of which is an assemblage composed of matrix elements. In this case, order is to be created in multi-dimensional spaces, for which it is difficult to select a single, most appropriate method.

Sports results in some disciplines may provide an example of an assemblage of inhomogeneous and incomparable data. In biathlon races, the rating is based on cross-country ski racing (running time measured in seconds) and target shooting (performance measured by scoring). The task can be reduced to a linear order by adopting conventional weighting factors. Choosing other, equally good, weighting factors may result in another athlete becoming the winner.

For creating a partial order in an assemblage of incomparable data, it is advantageous to apply the method of Data Envelopment Analysis (DEA), also termed data boundary analysis or value weighting method. The scope of applications of this method has been continually increasing¹ [1], [3]–[14] ever since DEA was developed in 1978 [2]. Unfortunately, the method has not been very popular or widely used in Poland, while its applications have been restricted mainly to the analysis of processes and phenomena in economy.

DEA, like other mathematical instruments, makes it possible to analyse real issues and to precisely simulate phenomena at the abstract level, which explains why the method should be advocated and new applications sought.

2. The DEA method in a data ordering process

In DEA, object-related data assemblages are analysed. The object is understood to be an organisation, economic entity, action or process. A person or group of people can also be treated as an object. Order creation is based on the relations observed in objects between input signals, i.e. inputs (reasons, outlays, resources used, etc.), and output signals, i.e. outputs (effects, results, etc.). These relations delineate the efficiency of an object, whereas numerical measures of efficiency can act as a good instrument for creating an order. Analysis based on DEA modelling makes it possible to indicate which objects have the highest efficiency (those classified highest on the rating list) and then compare the remaining objects with the “best” ones, which constitutes a certain kind of benchmarking. In an order creating process, groups of objects, incomparable with one another, can be determined. These occupy the same place on the rating list, thus forming a rating class, which is specific to the DEA method. The concept of a rating list should therefore be understood in a special way.

¹ DEA, a mathematical programming method, has been described in many handbooks. The bulk of the literature demonstrates ways of applying the method. Only a few selected references have been quoted in the paper.

It is justifiable to apply the DEA method to a group of objects to which data matrixes have been ascribed by means of specifying multiple inputs and multiple outputs (figure 1).

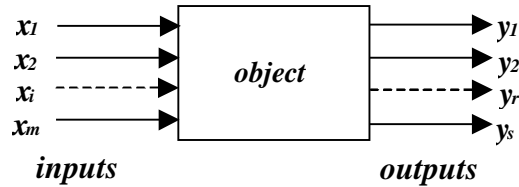


Fig. 1. Object with m inputs and s outputs
Source: the author

For each object, it is possible to compute many partial efficiencies as individual input to output ratios,

$$e_{r,i} = \frac{y_r}{x_i}, \tag{1}$$

where:

- m – number of inputs,
- s – number of outputs,
- $i = 1, \dots, m,$
- $r = 1, \dots, s.$

The DEA method determines the efficiency of an object with multiple inputs and outputs in the form of the ratio of a weighted sum of inputs to a weighted sum of outputs. Thus such a measure of the efficiency of an object is a function of weights μ and ν .

$$e = \frac{\sum_{r=1}^s \mu_r y_r}{\sum_{i=1}^m \nu_i x_i}. \tag{2}$$

As regards a group of objects, for the DEA method it is necessary to solve a problem of mathematical programming (non-linear). Solving this problem is equivalent to determining the relative efficiency of objects. Relative efficiency refers to the efficiency of an object compared with the object of highest efficiency in a group. Hence, the objects of highest efficiency have relative efficiency equal to 1. For each object ϕ in the assemblage $j = 1, \dots, n$, it is necessary to find the highest value of the ratio θ_ϕ which is a measure of the relative efficiency of successive objects. This is the objective function in the optimisation problem:

$$\theta_\phi \rightarrow \max, \quad (3)$$

where:

$$\theta_\phi = \frac{\sum_{r=1}^s \mu_r y_{r,\phi}}{\sum_{i=1}^m v_i x_{i,\phi}}, \quad (4)$$

under the following conditions:

$$\frac{\sum_{r=1}^s \mu_r y_{r,j}}{\sum_{i=1}^m v_i x_{i,j}} \leq 1, \text{ for } j = 1, \dots, n, \quad (5)$$

$$\mu_r \geq 0, \quad v_i \geq 0, \text{ for } r = 1, \dots, s, \quad i = 1, \dots, m. \quad (6)$$

Solving this optimisation problem is equivalent to determining the relative efficiency value, which makes it possible to order objects. Relative efficiency takes values in the interval $(0, 1]$ and can be given as a percentage $(0, 100\%]$. The objects with relative efficiency 1 are called efficient objects (regarded as the best in a given group of objects), the remaining objects are termed inefficient.

The optimisation problem presented above (3)–(6) can be converted into a linear form, which is advantageous for computing and makes it possible to solve the problem using linear programming. It becomes equivalent to solving, for each object ϕ of the assemblage $j = 1, \dots, n$, the optimisation problem (seeking the minimum) of the value θ_ϕ which acts as a measure of relative efficiency,

$$\theta_\phi \rightarrow \min, \quad (7)$$

while maintaining the reality conditions:

$$\sum_{j=1}^n x_{i,j} \lambda_j \leq \theta_\phi x_{i,\phi} \quad \text{for } i = 1, \dots, m, \quad (8)$$

$$\sum_{j=1}^n y_{r,i} \lambda_j \geq y_{r,\phi} \quad \text{for } r = 1, \dots, s, \quad (9)$$

$$\lambda_j \geq 0 \quad \text{for } j = 1, \dots, n, \quad (10)$$

where λ_j are weighted coefficients (linear combination coefficients).

The DEA method is non-parametric, therefore it is not necessary to know the functional dependence between the input and output signals for an object. Neither is it necessary to know the values of the weight coefficients, which are determined, for each object, in the optimisation process. In this way, it is possible to avoid subjectively ascribing values to these coefficients, which frequently occurs in other methods.

The weight coefficients determined, λ_j , make it possible to set so-called patterns for inefficient objects. These patterns are efficient objects, which an inefficient object can come close to, by the smallest possible changes in the input and output signals. If the inefficient object were to approach other efficient objects (those which are not patterns), greater changes in the signals would be required.

The relative efficiency of an object in the group analysed depends on its input and output signals. Under certain circumstances, however, changes in a signal do not result in changes in efficiency. Such admissible signal changes are determined by the so-called slack variables (or residues) and are equal to, for input signals:

$$\delta x_{i,\phi} = \frac{\sum_{j=1}^n x_{i,j} \lambda_j - \theta_\phi x_{i,\phi}}{\theta_\phi - \lambda_i}, \tag{11}$$

and output signals:

$$\delta y_{r,\phi} = \frac{\sum_{j=1}^n y_{r,i} \lambda_j - y_{r,\phi}}{1 - \lambda_j}. \tag{12}$$

For objects characterised by two partial efficiencies, the results can be presented in graphical form. Due to the interpretation of the Θ coefficient, it is advantageous to use inverse partial efficiencies as coordinates in the coordinate system.

Table 1 presents example data and the results of analysis for five objects² with two inputs (x_1, x_2) and one output (y_1).

Table 1. Example objects with two inputs and one output

	x_1	x_2	y_1	Θ	λ_1	λ_2	λ_3	λ_4	λ_5	δx_1
Object A	2	5	24	1	1	0	0	0	0	0
Object B	7	8	30	0.52	0.48	0	0	0.45	0	0
Object C	4	10	38	0.79	1.58	0	0	0	0	0
Object D	6	4	41	1	0	0	0	1	0	0
Object E	7	3	21	0.68	0	0	0	0.51	0	-2.5

Source: the author's work based on [8]

² This is an extended example based on [7].

The results of the analysis for the example discussed is presented graphically in figure 2.

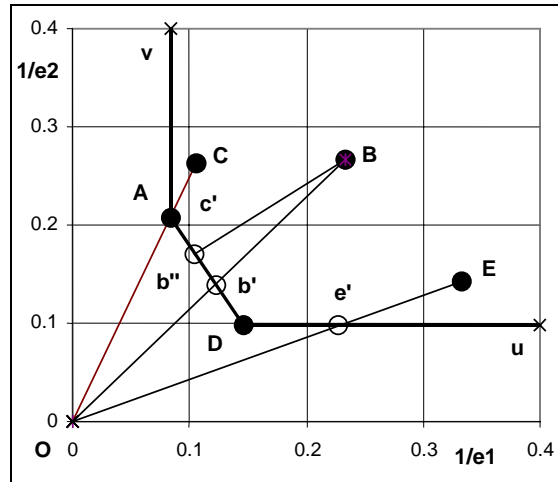


Fig. 2. Assemblage of objects with two inputs and one output
Source: the author's work based on [8]

In accordance with the computation results, objects A and D are efficient, whereas objects B, C, E are inefficient. The curve of efficient objects is a convex polygon v -A-D-u, of the least length (hence lines parallel to axes), the vertexes of which determine efficient objects. The inefficient objects (B, C, E) are located inside this polygon, their distance from the polygon (dependent on Θ) acts as an inefficiency measure.

For object C, efficiency can be increased by reducing the values of the input signals (both of them) so that they are 0.79 times their former values. The object will then be located at c' on the efficient object curve, just as the efficient object A. In the case of object B, it is necessary to reduce the values of the input signals to 0.52 times their former values to make the object efficient – point b' in figure 2. For object E, the values of the input signals have to be diminished to 0.68 times their former values. The object will then be located on the efficient object curve – point e' in figure 2, for which so-called weak efficiency is postulated. This means it is possible to reduce an input signal for this object while not increasing the efficiency. In the computation, this is manifested in the form of the slack variables. In this case, slack variable δ_{x_1} shows that, for object E, reducing signal x_1 by a value of 2.5 will not produce a change in efficiency.

The relative efficiency Θ indicates how to convert an inefficient object into an efficient one by means of proportionally reducing all input signals or increasing the output signals. Certainly, it is also possible to modify the input or output signals in such a way so that the object is located on the curve of efficient objects. In the problem under consideration, for point B this is possible, for example, by reducing signals

x_1 and x_2 to 0.45 and 0.64 times their original value, respectively. Consequently, object b will be positioned at b'' , thus becoming an effective object.

Non-zero values of the λ coefficients indicate patterns for inefficient objects. Here, object A is a pattern for object C, object D – for object E, and objects A and D are patterns for object B.

If more than two partial efficiencies describe an object, it is not possible to present a problem graphically in the plane. Such a problem is multi-dimensional, the analysis of which is performed in multi-dimensional space. The multi-dimensional surface of efficient objects is constructed as the surface of the convex polyhedron of least area, the vertices of which are determined by the points corresponding to efficient objects. A relative efficiency of 1 is ascribed to points on the faces of this polyhedron, not just the vertices.

In the simpler case of objects with one input and one output, object efficiencies can be presented on a single axis, where the curve of efficient objects is reduced to a single point. In such a case, the use of the DEA method is little justified, as efficiencies can simply be computed as the ratio of output to input signals.

Carrying out analysis using the DEA method, it is possible to determine so-called rating classes. The first rating class is constituted by efficient objects (coefficient $\theta = 1$). The second rating class is composed of objects determined to be efficient for the sub-group of objects excluding those belonging to the first rating class. Similarly, further classes, if such exist, can be established. It should be mentioned that objects in a single rating class of this kind are not comparable with one another, which is characteristic of a partial order.

3. Analysis of the efficiency of the teaching process

The same methods have been used for years in evaluating the results of the teaching process, especially the results of students in higher education institutions³. The arithmetic mean of grades obtained in a semester is computed and, on that basis, a rating list is compiled. Sometimes, some standardization measures are applied, in order to compensate for differences in the amount of work required by and the difficulty of particular subjects and semesters. Not denying in any way the importance of using such methodology, it is possible to look at the matter differently, i.e. to analyse the studying process in relation to the progress made by individual students⁴.

When studying is considered as a continuous process of gaining knowledge, it can be assumed that the grades in successive semesters depend on the knowledge gained in pre-

³ For many years, the author was a Dean in a department of Kielce University of Technology. This is why he is interested in this subject.

⁴ The DEA method has been applied in research into the efficiency of higher education institutions - in the Polish literature represented by Szuwarzyński [14].

vious ones. In this way, progress in studying in a given semester is defined by taking into account an initial state determined by the results from a previous semester. This is the essence of the methodology presented. A similar approach is used in the process of university admissions. Only high grades in one's high school exams can guarantee success in the first semester of studying and thus this is used as a criterion for admission.

3.1. Presentation of a tested group of students

A group of 24 students (St1, St2, ..., St24) was used for the research. They were students in a department of Kielce University of Technology in the academic year 2007/2008. The subjects of semesters 1 and 2 are presented in tables 2 and 4. The grades obtained in these semesters are given in tables 3 and 5. The traditional Polish grading scale (5.0; 4.5; 4.0; 3.5; 3.0; 2.0) is used, but the data presented in the table include the results of resit exams, hence other grades appear here (e.g. 1.7). Grades in English are not included in the analysis.

Table 2. Subjects in semester 1

Symbol	Subject	Type of class	Symbol	Subject	Type of class
Pa1	Linear Algebra	lecture	Pa8	Fundamentals of Computer Science	laboratory
Pa2	Linear Algebra	tutorial	Pa9	Fundamentals of Law	lecture
Pa3	Calculus	lecture	Pa10	Fundamentals of Law	tutorial
Pa4	Calculus	tutorial	Pa11	Mathematics Revision	lecture
Pa5	Physics	lecture	Pa12	Mathematics Revision	tutorial
Pa6	Microeconomics	lecture	Pa13	Statistics	lecture
Pa7	Microeconomics	tutorial	Pa14	Statistics	tutorial

Source: the author's study

Table 3. Grades in semester 1

	Pa1	Pa2	Pa3	Pa4	Pa5	Pa6	Pa7	Pa8	Pa9	Pa10	Pa11	Pa12	Pa13	Pa14
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
St1	3.0	3.0	3.5	3.5	3.0	5.0	5.0	3.0	3.5	3.5	3.5	3.5	3.5	3.5
St2	4.5	4.5	4.0	4.0	4.0	5.0	5.0	3.0	4.0	4.0	4.0	4.0	3.5	3.5
St3	4.0	4.0	4.5	4.5	3.5	5.0	5.0	3.5	4.5	4.5	5.0	5.0	4.0	4.0
St4	2.5	2.5	2.5	2.5	4.0	3.5	3.5	3.5	3.0	3.0	4.0	4.0	3.0	3.0
St5	3.5	3.5	3.0	3.0	3.5	5.0	5.0	3.5	3.5	3.5	4.0	4.0	4.0	4.0
St6	4.0	4.0	4.0	4.0	4.5	4.0	4.0	5.0	4.0	3.0	4.0	4.0	4.5	4.0
St7	3.0	3.0	2.5	2.5	5.0	4.0	4.0	3.0	3.0	3.0	4.5	4.5	3.5	4.0
St8	4.0	4.0	4.0	4.0	4.5	4.0	4.0	3.0	4.0	4.0	4.0	4.0	4.0	4.0
St9	5.0	5.0	5.0	5.0	3.5	4.0	4.0	4.0	4.5	4.5	5.0	5.0	4.0	4.0
St10	4.0	4.0	3.5	3.5	3.5	4.0	4.0	3.0	3.0	3.0	4.5	4.5	3.0	3.0

Table 3 continued

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
St11	4.0	4.0	4.5	4.5	3.5	4.0	4.0	4.0	3.0	3.0	5.0	5.0	4.0	4.0
St12	3.0	3.0	3.5	3.5	4.0	4.0	4.0	2.8	3.5	3.5	4.5	4.5	3.5	3.5
St13	4.0	4.0	3.0	3.0	4.5	3.5	3.5	2.5	3.5	3.5	4.5	4.5	3.5	3.5
St14	3.0	3.0	1.7	1.7	5.0	3.5	3.5	3.0	3.0	3.0	3.5	3.5	3.0	3.0
St15	4.0	4.0	5.0	5.0	4.0	4.5	4.5	3.0	5.0	5.0	5.0	5.0	4.0	4.0
St16	3.5	3.5	3.0	3.0	5.0	3.5	3.5	4.0	4.0	4.0	4.0	4.0	3.5	3.5
St17	3.0	3.0	3.0	3.0	4.0	3.5	3.0	5.0	3.0	3.0	3.0	3.0	4.0	3.5
St18	3.0	3.0	3.0	3.0	4.5	4.5	4.5	3.5	3.0	3.0	3.5	3.5	3.5	3.5
St19	5.0	5.0	5.0	5.0	3.0	5.0	5.0	4.0	5.0	5.0	5.0	5.0	4.5	4.5
St20	4.0	4.0	2.5	2.5	4.0	3.5	3.5	3.0	4.5	4.5	4.0	4.0	3.5	3.5
St21	5.0	5.0	5.0	5.0	3.5	4.0	4.0	3.0	3.5	3.5	5.0	5.0	4.0	4.0
St22	3.5	3.5	4.0	4.0	4.0	4.0	4.0	3.0	3.5	3.5	4.5	4.5	3.5	3.5
St23	1.8	1.8	2.5	2.5	3.0	3.5	3.0	3.0	3.0	3.0	4.0	4.0	3.0	3.0
St24	3.0	3.0	3.0	3.0	3.0	3.5	3.0	4.0	3.5	3.5	3.0	3.0	2.5	3.0

Source: the author's study

Table 4. Subjects in semester 2

Symbol	Subject	Type of class	Symbol	Subject	Type of class
Pb1	Calculus	lecture	Pb10	Financial Mathematics	lecture
Pb2	Calculus	tutorial	Pb11	Financial Mathematics	tutorial
Pb3	Physics	lecture	Pb12	Fundamentals of Computer Science	lecture
Pb4	Physics	tutorial	Pb13	Fundamentals of Computer Science	laboratory
Pb5	Physics	laboratory	Pb14	Economic Law	lecture
Pb6	Engineering Graphics	lecture	Pb15	Economic Law	tutorial
Pb7	Engineering Graphics	laboratory	Pb16	Intellectual Property	lecture
Pb8	Macroeconomics	lecture	Pb17	Materials Science	lecture
Pb9	Macroeconomics	tutorial	Pb18	Materials Science	laboratory

Source: the author's study

Table 5. Grades in semester 2

	Pb1	Pb2	Pb3	Pb4	Pb5	Pb6	Pb7	Pb8	Pb9	Pb10	Pb11	Pb12	Pb13	Pb14	Pb15	Pb16	Pb17	Pb18
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
St1	4.0	4.0	3.0	3.0	4.0	3.5	3.5	3.0	3.0	3.5	3.5	3.5	3.5	4.0	4.0	3.0	3.5	3.0
St2	4.0	4.0	3.5	4.0	4.5	4.5	4.5	3.0	4.0	3.5	3.5	3.5	5.0	5.0	4.5	3.0	4.0	4.0
St3	5.0	5.0	4.0	4.0	5.0	5.0	5.0	4.5	4.5	5.0	5.0	4.0	3.5	5.0	5.0	3.0	5.0	5.0
St4	2.5	1.5	4.0	3.0	4.0	4.0	4.0	3.0	3.0	2.0	0.0	3.0	3.0	4.0	4.0	3.0	1.5	1.5
St5	4.0	4.0	3.0	4.0	4.5	5.0	5.0	3.5	4.0	3.0	3.0	3.0	5.0	5.0	4.5	3.0	4.0	3.0
St6	4.0	5.0	4.0	4.0	5.0	5.0	5.0	4.5	4.0	4.5	4.5	4.0	2.5	5.0	5.0	4.5	5.0	5.0
St7	3.0	3.0	3.0	3.0	3.0	4.5	4.5	3.5	3.0	2.3	1.0	3.0	3.0	4.0	4.5	3.0	1.8	1.7
St8	4.5	4.5	3.0	4.0	4.5	4.0	4.0	3.5	3.0	3.0	3.0	3.5	3.0	4.5	4.5	3.0	4.0	3.5

Table 5 continued

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
St9	5.0	5.0	3.0	4.0	4.5	4.5	4.5	3.5	3.0	4.0	4.0	4.5	4.0	4.5	5.0	3.0	5.0	5.0
St10	4.0	4.0	3.5	4.0	3.5	4.0	4.0	3.5	3.5	3.0	3.0	3.0	3.0	5.0	5.0	3.0	4.0	3.0
St11	4.0	4.0	4.0	4.0	4.5	5.0	5.0	3.5	3.5	3.0	3.0	4.5	5.0	4.5	4.0	3.0	4.0	3.0
St12	4.0	4.0	3.5	3.0	4.5	4.0	4.0	3.5	1.3	3.5	3.5	3.5	3.5	4.0	4.0	3.0	3.5	3.5
St13	4.0	4.0	3.0	4.0	5.0	4.0	4.0	3.5	3.0	3.0	3.0	3.5	3.0	4.0	4.0	3.0	4.0	3.0
St14	2.0	0.0	3.0	4.0	4.5	3.5	3.5	3.0	3.0	2.0	0.0	3.0	3.0	4.0	3.5	3.0	3.0	1.0
St15	5.0	5.0	4.0	4.0	4.5	4.5	4.5	3.5	4.0	3.5	3.5	4.0	3.5	5.0	5.0	4.0	5.0	5.0
St16	3.0	1.5	4.0	4.0	4.0	3.5	3.5	3.5	1.0	3.0	3.0	5.0	5.0	4.0	4.0	3.0	1.5	1.8
St17	3.0	3.0	2.5	4.0	4.0	4.0	4.0	3.0	3.5	2.3	1.0	2.3	4.0	5.0	4.0	3.0	4.0	3.0
St18	3.0	3.0	3.5	3.5	4.5	3.5	3.5	3.0	3.0	3.0	3.0	3.5	3.5	4.0	4.5	3.0	4.0	3.0
St19	5.0	5.0	4.0	4.5	4.5	4.5	4.5	5.0	3.5	5.0	5.0	4.0	3.0	5.0	5.0	4.5	5.0	5.0
St20	3.0	1.5	3.0	3.5	4.0	4.0	4.0	3.0	3.5	2.0	0.0	3.5	3.5	4.0	4.0	3.0	4.0	4.0
St21	4.5	4.5	3.5	4.0	3.5	3.5	3.5	3.0	3.0	3.0	3.0	3.0	3.0	4.0	4.0	3.0	1.5	1.8
St22	4.5	4.5	3.0	4.5	4.0	5.0	5.0	3.0	4.0	3.0	3.0	3.5	3.0	4.0	4.5	3.0	4.0	3.0
St23	2.8	1.8	3.5	3.5	4.0	4.0	4.0	3.0	1.0	3.0	3.0	3.5	3.0	3.5	3.5	3.0	1.5	3.0
St24	4.0	4.0	3.0	3.5	4.0	4.5	4.5	4.0	3.0	2.7	1.3	3.0	3.0	4.0	4.0	3.0	4.0	1.5

Source: the author's study

Individual students (24 objects) are objects in the DEA analysis. The input signals are formed on the basis of the grades in semester 1 and the output signals - on the basis of the grades in semester 2. Efficiency, which is determined in this analysis, is a measure of progress in studying obtained by comparing the results in semester 2 to the inputs – the grades in semester 1.

3.2. One-dimensional analysis

The results of one-dimensional analysis are presented in table 6. Column 1 gives the identifier of an object-student as in tables 1 and 3. The average grades in the first semester are presented in column 2 (on the basis of table 3) and the average grades in the second semester are presented in column 3 (on the basis of table 5). Columns 4 and 5 present the ordered data arranged from columns 1 and 2, and columns 6 and 7 the ordered data from columns 1 and 3. As can be seen, student St19 got the highest average in the first semester, while student St23 obtained the lowest one. In the second semester, student St3 obtained the highest average whereas student St14 got the lowest one. This is a traditional way of analysing and compiling rating lists in successive semesters.

Table 6. One-dimensional analysis

1	2	3	4	5	6	7	8	9	10	11
St1	3.57	3.47	St19	4.71	St3	4.58	St6	1.098	1.000	1.000
St2	4.07	4.00	St9	4.46	St19	4.56	St24	1.078	0.982	1.019
St3	4.36	4.58	St15	4.43	St6	4.47	St3	1.052	0.958	1.044
St4	3.18	2.83	St3	4.36	St15	4.31	St5	1.035	0.942	1.062
St5	3.79	3.92	St21	4.25	St9	4.22	St23	1.033	0.941	1.063
St6	4.07	4.47	St2	4.07	St2	4.00	St10	1.017	0.925	1.081
St7	3.54	3.04	St6	4.07	St11	3.97	St22	1.005	0.915	1.093
St8	3.96	3.72	St11	4.04	St5	3.92	St13	0.991	0.902	1.108
St9	4.46	4.22	St8	3.96	St22	3.81	St17	0.986	0.898	1.114
St10	3.61	3.67	St5	3.79	St8	3.72	St11	0.984	0.896	1.116
St11	4.04	3.97	St22	3.79	St10	3.67	St18	0.984	0.896	1.116
St12	3.63	3.54	St16	3.71	St13	3.61	St2	0.982	0.894	1.118
St13	3.64	3.61	St13	3.64	St12	3.54	St12	0.977	0.889	1.125
St14	3.10	2.72	St20	3.64	St1	3.47	St1	0.972	0.885	1.130
St15	4.43	4.31	St12	3.63	St18	3.44	St15	0.972	0.885	1.130
St16	3.71	3.24	St10	3.61	St24	3.39	St19	0.966	0.880	1.137
St17	3.36	3.31	St1	3.57	St17	3.31	St9	0.946	0.861	1.161
St18	3.50	3.44	St7	3.54	St21	3.29	St8	0.939	0.855	1.170
St19	4.71	4.56	St18	3.50	St16	3.24	St4	0.891	0.812	1.232
St20	3.64	3.19	St17	3.36	St20	3.19	St14	0.878	0.799	1.251
St21	4.25	3.29	St4	3.18	St7	3.04	St20	0.877	0.798	1.253
St22	3.79	3.81	St24	3.14	St23	3.03	St16	0.872	0.794	1.260
St23	2.94	3.03	St14	3.10	St4	2.83	St7	0.861	0.784	1.276
St24	3.14	3.39	St23	2.94	St14	2.72	St21	0.775	0.706	1.417

Source: the author’s study

In the rightmost part of the table (columns 8–10), groups of students are arranged on the basis of efficiency. The averages in semester 1 are assumed to be the input signals (outlays) and the averages in semester 2 – the output signals (results). Efficiencies computed in this way are then ordered in column 9. The relative efficiencies are presented in column 10. They are obtained on the basis of data from column 9 by calculating the ratio to the greatest efficiency in the group. Student St6 showed the highest efficiency, while student St21- the lowest one. Column 11 presents the inverse efficiencies calculated from column 10 and illustrated in the following graph (figure 3).

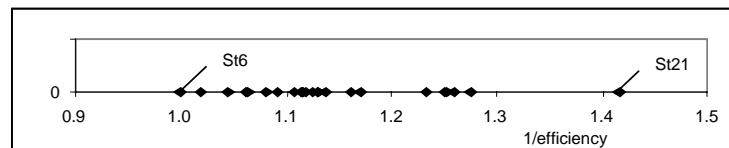


Fig. 3. The results of one-dimensional analysis presented geometrically.

Source: the author’s study

Due to the high density of points, only the two extreme points (objects) are highlighted on this graph. The distances between individual objects and the effective object (St6) indicate the degree of inefficiency. For example, the grades of St21 have to be increased by a factor of 1.417 to make object St21 as efficient as St6. This corresponds to an average in the second semester of 4.66. This object would also be effective if the average in the first semester (inputs) were 0.706 times the actual average, that is 3.001.

Optimisation procedures, resulting from the DEA method, are not used in the analysis presented. It is not necessary, because the objects only have one input and one output. Such procedures could obviously be applied here and the results would be the same.

3.3. Two-dimensional analysis

Three signals are ascribed to each object in the two-dimensional analysis presented here. This allows to determine two partial efficiencies (hence “two-dimensional analysis”). In the case considered, there is: one input signal (x_1) and two output signals (y_1 , y_2). These are:

x_1 – average grade for all subjects in the first semester

y_1 – average grade for basic and technical subjects in the second semester – table 5, subjects: Pb1, Pb2, Pb3, Pb4, Pb5, Pb6, Pb7, Pb12, Pb13, Pb17, Pb18.

y_2 – average grade for economic and law-related subjects in the second semester – table 5, subjects: Pb8, Pb9, Pb10, Pb11, Pb14, Pb15, Pb16.

In the DEA analysis, it is necessary to solve 24 optimisation problems for each ϕ of the assemblage $j = 1, \dots, 24$ in the form of:

$$\theta_\phi \rightarrow \min, \quad (13)$$

$$x_{1.1}\lambda_1 + x_{1.2}\lambda_2 + x_{1.3}\lambda_3 + \dots + x_{1.23}\lambda_{23} + x_{1.24}\lambda_{24} \leq \theta_\phi x_{1,\phi}, \quad (14)$$

$$y_{1.1}\lambda_1 + y_{1.2}\lambda_2 + y_{1.3}\lambda_3 + \dots + y_{1.23}\lambda_{23} + y_{1.24}\lambda_{24} \geq y_{1,\phi}, \quad (15)$$

$$y_{2.1}\lambda_1 + y_{2.2}\lambda_2 + y_{2.3}\lambda_3 + \dots + y_{2.23}\lambda_{23} + y_{2.24}\lambda_{24} \geq y_{2,\phi}, \quad (16)$$

$$\lambda_1, \dots, \lambda_{24} \geq 0. \quad (17)$$

The λ values can be different for each optimisation problem, but this is not indicated in the formulas above, since indexes are introduced (e.g, in the form of $\lambda_{j,\phi}$). In the computational results, we get 24 values for the coefficient Θ determining an object's relative efficiency, 576 values ($24 \cdot 24$) for the weighted coefficients λ , and 72

values (3*24) of coefficients determining the slacks ($\delta x_1, \delta y_1, \delta y_2$). Fortunately, a significant proportion of the coefficients determined take a value of zero, which makes the interpretation of the results easier. Data for the two-dimensional DEA analysis and the results obtained can be found in Table 7.

Table 7. Two-dimensional analysis

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
object	x_1	y_1	y_2	$e_{1,1}$	$e_{2,1}$	$1/e_{1,1}$	$1/e_{2,1}$	θ	λ_6	λ_{24}	δy_1	δy_2	object	θ
St1	3.57	3.50	3.43	0.980	0.960	1.020	1.042	0.891	0.49	0.38			St6	1
St2	4.07	4.14	3.79	1.016	0.930	0.984	1.075	0.908	0.18	0.94			St24	1
St3	4.36	4.59	4.57	1.054	1.049	0.949	0.953	0.962	0.76	0.35			St3	0.962
St4	3.18	2.91	2.71	0.915	0.854	1.093	1.171	0.822	0.2	0.57			St5	0.956
St5	3.79	4.05	3.71	1.069	0.981	0.936	1.019	0.956	0.19	0.9			St23	0.956
St6	4.07	4.41	4.57	1.083	1.123	0.923	0.891	1	1				St11	0.939
St7	3.54	3.05	3.04	0.861	0.861	1.161	1.162	0.787	0.52	0.22			St22	0.937
St8	3.96	3.86	3.50	0.975	0.883	1.026	1.133	0.869	0.11	0.95			St10	0.927
St9	4.46	4.45	3.86	0.998	0.864	1.002	1.157	0.885		1.26		0.092	St13	0.919
St10	3.61	3.64	3.71	1.008	1.030	0.992	0.971	0.927	0.74	0.1			St17	0.912
St11	4.04	4.27	3.50	1.059	0.867	0.945	1.153	0.939		1.21		0.288	St12	0.911
St12	3.63	3.73	3.26	1.027	0.898	0.974	1.114	0.911		1.05		0.047	St2	0.908
St13	3.64	3.77	3.36	1.036	0.922	0.966	1.085	0.919	0.02	1.04			St18	0.904
St14	3.10	2.77	2.64	0.894	0.853	1.118	1.173	0.808	0.28	0.44			St15	0.898
St15	4.43	4.45	4.07	1.006	0.919	0.994	1.088	0.898	0.19	1.03			St1	0.891
St16	3.71	3.35	3.07	0.901	0.827	1.110	1.209	0.805	0.16	0.75			St19	0.891
St17	3.36	3.44	3.11	1.024	0.928	0.977	1.078	0.912	0.1	0.84			St9	0.885
St18	3.50	3.50	3.36	1.000	0.959	1.000	1.043	0.904	0.38	0.51			St8	0.869
St19	4.71	4.45	4.71	0.945	1.000	1.058	1.000	0.891	1.03		0.092		St20	0.841
St20	3.64	3.45	2.79	0.948	0.765	1.055	1.308	0.841		0.97		0.277	St4	0.822
St21	4.25	3.30	3.29	0.776	0.773	1.288	1.293	0.709	0.54	0.25			St14	0.808
St22	3.79	4.00	3.50	1.057	0.925	0.946	1.082	0.937		1.13		0.046	St16	0.805
St23	2.94	3.15	2.86	1.071	0.973	0.933	1.028	0.956	0.1	0.76			St7	0.787
St24	3.14	3.55	3.14	1.128	1.000	0.886	1.000	1		1			St21	0.709

Source: the author's study

Columns 1–4 present the data to be analysed: column 1 – the identifiers of objects, column 2 – the input signals, columns 3 and 4 – the output signals. Columns 5–8 present the partial efficiencies and inverse partial efficiencies computed. These are illustrated in graphical form in figure 4. Columns 9–15 of table 7 contain the results of the DEA analysis.

Column 9 – presents the coefficients θ , which are a measure of the relative efficiency of individual objects. St6 and St24 are the efficient objects in the group tested. Object St21 has the lowest efficiency. To obtain the efficiency of object St21, the input signal has to be diminished to the level of 0.709 times the actual value or the out-

put signals have to be increased. This can be done by increasing both output signals to 1.41 times the actual value ($1/0.709$) or by another (non-proportional) change in the values of the output signals. Similarly, the necessary changes for the other objects to become efficient can be determined.

Column 10 and 11 present the weighted coefficients λ_6 and λ_{24} computed in the process of optimisation. The remaining weighted coefficients are not included in the table because they equal zero. The coefficients presented represent patterns for inefficient objects. St6 is a pattern for object St19 and object St24 – for objects St9, St11, St12, St20, St22. Both St6 and St24 are patterns for other inefficient objects.

Columns 12 and 13 contain the values of the slacks δy_1 , δy_2 . The Slacks δx_1 take the value zero (not included in the table). The slacks determine the ranges of changes in the output signals for which there are no changes in an object's efficiency. In the case analysed, non-zero slacks are ascribed to only six objects: St6, St9, St11, St12, St20, St22.

Columns 14 and 15 present objects arranged in descending order according to the coefficient Θ , presented in their original order in column 9.

The results of the two-dimensional analysis are presented in graphical form in figure 4, this shows the mutual location of the objects.

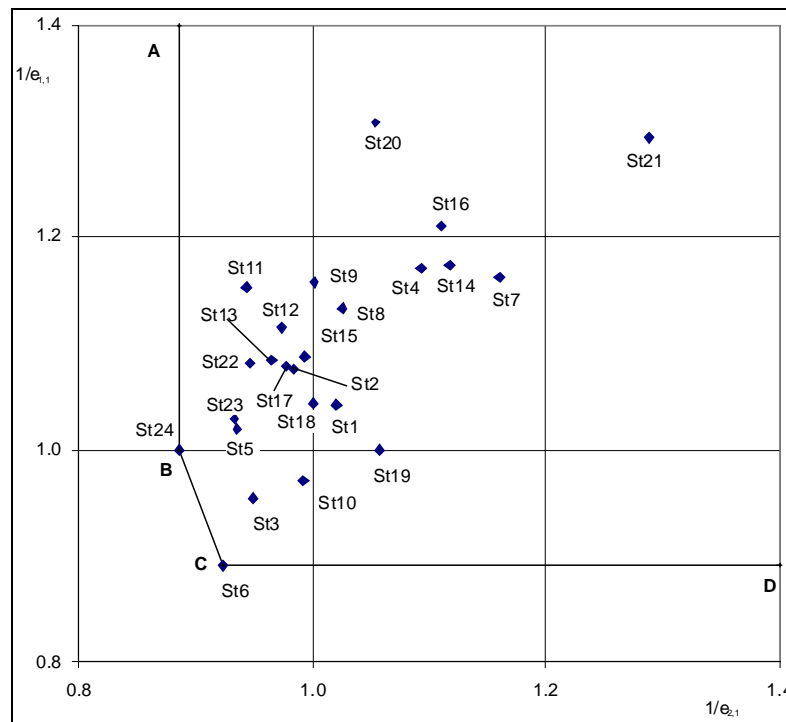


Fig. 4. The results of the two-dimensional analysis in graphical form
Source: the author's study

In this case, the curve of efficient objects is the polygon ABCD. The vertexes B and C are determined by the efficient objects St24 and St6. The sides AB and CD correspond to poor efficiency.

The next stage of the analysis is to distinguish rating classes. Four rating classes are determined in accordance with the procedure described above (table 8). The remaining elements are in the fifth rating class

Table 8. Rating classes

Class 1	Class 2	Class 3	Class 4	Class 5
St6 St24	St3 St23	St5 St10	St11 St19 St22	St1, St2, St4, St7, St8 St9, St12, St13, St14, St15 St16, St17, St18, St20, St21

Source: the author's study

The four graphs below (figures 5–8) present the curves of efficient objects and the location of individual objects in the rating classes distinguished.

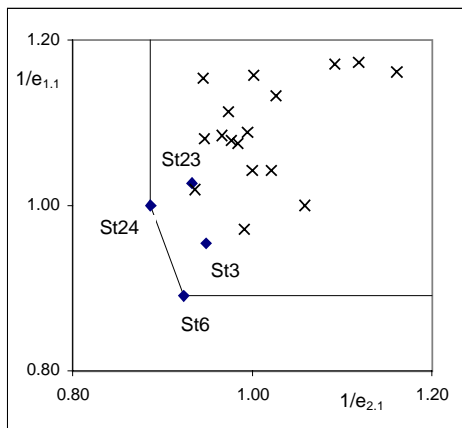


Fig. 5. The first rating class – St6, St24
Source: the author's study

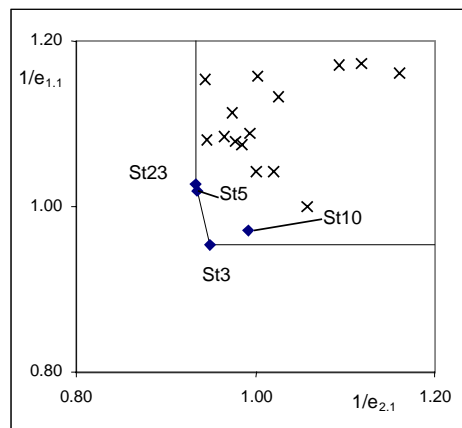


Fig. 6. The second rating class – St3, St23
Source: the author's study

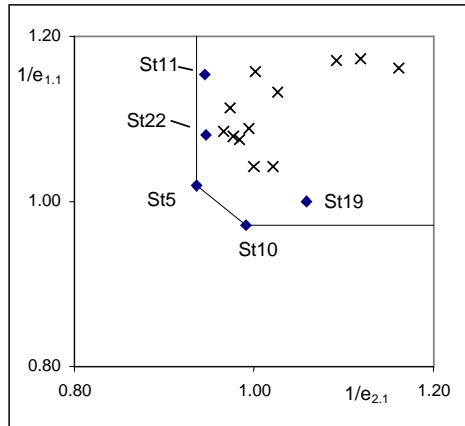


Fig. 7. The third rating class – St5, St10
Source: the author's study

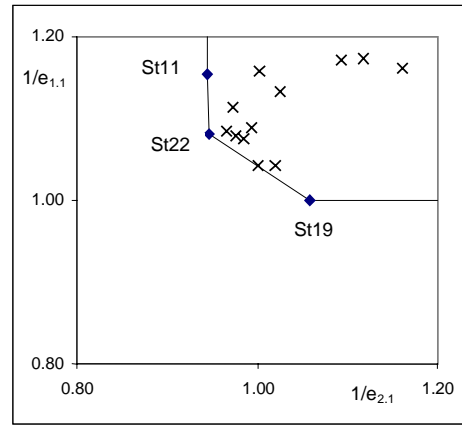


Fig. 8. The fourth rating class – St11, St19, St22
Source: the author's study

3.3. Multi-dimensional analysis

Each object has two input signals and three output signals in the example of multi-dimensional analysis presented here. This allows to determine six partial efficiencies (six-dimensional analysis). The following input signals (x_1, x_2) and output signals (y_1, y_2, y_3) were used in the problem under consideration:

x_1 – average grade for fundamental subjects in the first semester – table 3, subjects: Pa1, Pa2, Pb3, Pa4, Pa5, Pa8, Pba11, Pa12, Pb13, Pa14.

x_2 – average grade for economic and law-related subjects in the first semester – Table 3, subjects: Pa6, Pa7, Pa9, Pa10.

y_1 – average grade for basic subjects in the second semester – table 5, subjects: Pb1, Pb2, Pb3, Pb4, Pb5, Pb12, Pb13.

y_2 – average grade for economic and law-related subjects in the second semester – table 5, subjects: Pb8, Pb9, Pb10, Pb11, Pb14, Pb15, Pb16.

y_3 – average grade for technical subjects in the second semester – table 5, subjects: Pb6, Pb7, Pb17, Pb18.

The optimisation problems defined by the DEA method are solved and selected data are presented in table 9.

Columns 1–6 contain the data analyzed: column 1 – the identifiers of objects, columns 2 and 3 – the input signals, columns 4–6 – the output signals. The values of the coefficient θ determined are presented in column 7. Four objects: St3, St6, St11, St24 turned out to be efficient. Neither the values of the weighted coefficients λ , nor the slacks are included in the table. This solution has no graphical form which could be shown in three-dimensional space.

Table 9. Multi-dimensional analysis

1	2	3	4	5	6	7	8	9	10	11	12	13
object	x_1	x_2	y_1	y_2	y_3	θ	θ_2	θ_3	k_1	k_2	k_3	k_4
St2	3.90	4.50	4.07	3.79	4.25	0.930	0.95	1			St2	
St3	4.20	4.75	4.36	4.57	5.00	1			St3			
St4	3.15	3.25	3.00	2.71	2.75	0.863	0.89	0.943				St4
St5	3.60	4.25	3.93	3.71	4.25	0.993	1			St5		
St6	4.20	3.75	4.07	4.57	5.00	1			St6			
St7	3.55	3.50	3.00	3.04	3.13	0.807	0.84	0.936				St7
St8	3.95	4.00	3.86	3.50	3.88	0.895	0.93	0.982				St8
St9	4.55	4.25	4.29	3.86	4.75	0.915	0.97	1			St9	
St10	3.65	3.50	3.57	3.71	3.75	0.954	1			St10		
St11	4.25	3.50	4.29	3.50	4.25	1			St11			
St12	3.58	3.75	3.71	3.26	3.75	0.928	0.97	1			St12	
St13	3.70	3.50	3.79	3.36	3.75	0.966	1			St13		
St14	3.04	3.25	2.79	2.64	2.75	0.830	0.86	0.906				St14
St15	4.30	4.75	4.29	4.07	4.75	0.928	0.96	1			St15	
St16	3.70	3.75	3.79	3.07	2.58	0.929	0.97	1			St16	
St17	3.45	3.13	3.26	3.11	3.75	0.944	1			St17		
St18	3.40	3.75	3.50	3.36	3.50	0.937	0.97	1			St18	
St19	4.60	5.00	4.29	4.71	4.75	0.942	1	1			St19	
St20	3.50	4.00	3.14	2.79	4.00	0.960	0.98	1			St20	
St21	4.45	3.75	3.71	3.29	2.58	0.834	0.92	0.982				St21
St22	3.80	3.75	3.86	3.50	4.25	0.958	1			St22		
St23	2.86	3.13	3.16	2.86	3.13	0.970	1			St23		
St24	3.05	3.38	3.50	3.14	3.63	1			St24			

Source: the author's study

Rating classes have been distinguished in this example as previously. The first rating class consists of four objects determined to be efficient (column 10). After their removal, DEA analysis was repeated for the remaining 20 objects. The values of the coefficient θ determined can be found in column 8. Seven objects are efficient in this group. They belong to the second rating class (column 11). After removal, DEA analysis was carried out for the group of 13 objects remaining. The values of the coefficient θ obtained are located in column 9. The third rating class consists of 8 objects determined to be efficient (column 12). The remaining 5 objects are ascribed to the fourth (last) rating class (column 13).

4. Conclusions

The analyses carried out show the complexity of ordering assemblages. Depending on the approach used, different, though not contradictory, results can be obtained.

One-dimensional analysis leads to a linear ordering. The averaging of signals is required for such an order, which means that the differences arising from the specific character of different subjects are omitted. For example, physics and macroeconomics are treated similarly, without taking into consideration the differences between the distributions of grades in these subjects within a group of students. This is not the case for multi-dimensional analysis which, to some extent, takes into account the differences between subjects.

In the two-dimensional analysis presented, weighting has been performed by classifying subjects in the second semester into two groups. In the case of six-dimensional analysis, such a weighting was achieved by classifying two groups of subjects in the first semester and three groups of subjects in the second semester.

A greater number of effective students is the result of taking variety into account. Here, a greater number of students are considered to be “the best” or belong to one rating class in accordance with the measure used. It should also be added that students in the same rating class cannot be compared with each other in a direct way and they still occupy the same position on the rating list, despite having different partial efficiencies.

The analysis, to which the concept of efficiency and the DEA method are applied, enriches the information about a tested group of students. It indicates progress in studying and can be complementary to traditional rating lists.

It must also be mentioned that for the presented case of analysis we should take into account specific limitations related to the values of signals. They cannot increase freely and their highest possible value is 5, which results from the established grading scale.

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