B A D A N I A   O P E R A C Y J N E   I   D E C Y Z J E

Nr 3–4 2005

Dorota KUCHTA*

FUZZY SOLUTION OF THE LINEAR PROGRAMMING PROBLEM WITH INTERVAL COEFFICIENTS IN THE CONSTRAINTS

A fuzzy concept of solving the linear programming problem with interval coefficients is proposed. For each optimism level of the decision maker (where the optimism concerns the certainty that no errors have been committed in the estimation of the interval coefficients and the belief that optimistic realisations of the interval coefficients will occur) another interval solution of the problem will be generated and the decision maker will be able to choose the final solution having a complete view of various possibilities.

Keywords: *interval linear programming, fuzzy solution*

1. Introduction

In many practical applications of linear programming the problem coefficients cannot be determined in a precise way. That is why quite a few researchers have been trying to propose a way to solve linear programming problems with imprecise coefficients. The difficulty lies in the fact that while dealing with such problems it is not clear what the optimal solution is. Its understanding depends strongly on the decision maker preferences and attitude. For this reason it is necessary to consider various concepts, so that each decision maker can choose one that suits him best.

In this paper, we propose a new concept of solving the linear programming problems with interval coefficients in the constraints. We consider the solution of such linear programming problems in a fuzzy set – thus, in fact, it will be a family of solutions for various membership degrees (family of λ-cuts). Such an approach will allow

* Institute of Industrial Engineering and Management, Wrocław University of Technology, ul. Smoluchowskiego 25, 50-370 Wrocław, Dorota.Kuchta@pwr.wroc.pl.

the decision maker to have a general view of the problem and to choose the final solution in a way that best suits his preferences. Our proposal is based on a known concept of an interval solution of the linear programming problem with interval coefficients.

2. Parameter-dependent relations between interval numbers

Two interval numbers $\overline{A} = [\underline{a}, \overline{a}]$ and $\overline{B} = [\underline{b}, \overline{b}]$ should often be compared with one another. The result of such a comparison, unlike in the case of crisp numbers, is not unique, that is why in the literature many ways of comparing interval numbers are considered (see, e.g., [2]–[4], [6]–[8]). Here, we will only propose one approach, discussed in more detail in [2]. We consider the "maximisation" case, i.e., the "greater" interval is preferred.

There are two basic extreme alternatives:

- the weak preference relation: $\overline{A} \geq_w \overline{B} \Leftrightarrow \overline{a} \geq \underline{b}$,
- the strong preference relation $\overline{A} \geq_s \overline{B} \Leftrightarrow \underline{a} \geq \overline{b}$.

Of course, if $\overline{A} \geq_s \overline{B}$ then $\overline{A} \geq_w \overline{B}$.

However, the decision maker may be interested in some relaxation or strengthening of the above relations. For any ε (positive, negative or 0, whereas the case $\varepsilon = 0$ would give us the above special cases) the following relations can be defined:

- the weak ε-preference relation: $\overline{A} \geq_{w(\varepsilon)} \overline{B} \Leftrightarrow \overline{a} \geq \underline{b} - \varepsilon(\overline{b} - \underline{b})$,
- the strong ε-preference relation $\overline{A} \geq_{s(\varepsilon)} \overline{B} \Leftrightarrow \underline{a} \geq \overline{b} - \varepsilon(\overline{b} - \underline{b})$.

If $\varepsilon > 0$, both ε-relations mean a relaxation of the corresponding original relations. If $\varepsilon < 0$, we deal with a strengthening of the original relations. It seems natural to choose ε as a number from interval [–1, 1].

The ε-preference relations give the decision maker more flexibility in deciding in each specific case which interval he prefers with respect to the case where he has only the two original preference relations at his disposal.

Example 1

Interval [2, 3.75] would not be definitely preferred to interval [4, 5] even according to the weak preference relation. And still there may be cases the decision maker would agree do choose [2, 3.75] – if it gave him a good compromise solution. Relation $\geq_{w(0,25)}$ would allow him to model this way of reasoning.

Interval [4.75, 7] would not be preferred to interval [4, 5] according to the strong preference relation, but it would be preferred according to relation $\geq_{s(0,25)}$.

Interval [2, 4] could be selected instead of interval [4, 5], if the weak preference relation is applied. The decision maker may want to avoid this and require more from an interval that should be preferred to interval [4,5]. Applying relation $\geq_{w(-0,25)}$ he would assure that it is "only" a shifted interval [2.25, 4.25] that could be preferred with respect to [4, 5].

Interval [5, 6] could be selected instead of interval [4, 5], if the strong preference relation is applied. Applying relation $\geq_{s(-0,25)}$ would allow the decision maker to require more – only a shifted interval [5.25, 6.25] could be preferred over [4,5].

3. Formulation of the problem

We consider the following problem:

$$\sum_{j=1}^{n} c_j x_j \to \min,$$

$$\sum_{j=1}^{n} [\underline{a}_j^i, \overline{a}_j^i]\, x_j \geq [\underline{b}^i, \overline{b}^i], \quad i = 1, \ldots, m, \tag{1}$$

$$x_j \geq 0 \ (j = 1, \ldots, n).$$

The upper and lower ends of the intervals can be equal, so that model (1) comprises also the case where some coefficients are precise. Arithmetical operations on interval numbers are understood in the standard way [5] – they can be performed on the lower and upper ends respectively, so that the left hand and the right hand sides of the expressions in (1) are also closed intervals.

The possible interpretation of the above problem can be as follows:

- x_j – the amount of product $j, j = 1, \ldots, n$,
- $[\underline{b}^i, \overline{b}^i]$ – the minimal requirements for the total amount of substance i ($i = 1, \ldots, m$) contained in the products,
- for each $j = 1, \ldots, n$: $[\underline{a}_j^i, \overline{a}_j^i]$ is the amount of substance $i\,(i = 1, \ldots, m)$ obtained during the manufacturing process of a unit of product j,
- c_j – the manufacturing unit cost of product $j, j = 1, \ldots, n$.

4. Solution concept proposed in the literature

In [1] and [9] the authors consider the problem in which additionally the coefficients in the objective function are intervals. Here, we refer to their concept limited to crisp objective function and interval constraints coefficients, which is a special case of the problem they consider. The authors propose to consider the following notions:

• the largest possible feasible region, determined in the case of positive decision variables by the following system of constraints:

$$\sum_{j=1}^{n} \overline{a}_j^i x_j \geq \underline{b}^i, \quad i = 1, \ldots, m, \quad x_j \geq 0 \, (j = 1, \ldots, n), \tag{2}$$

• the smallest possible feasible region, determined in the case of positive decision variables by the following system of constraints:

$$\sum_{j=1}^{n} \underline{a}_j^i x_j \geq \overline{b}^i, \quad i = 1, \ldots, m, \quad x_j \geq 0 \, (j = 1, \ldots, n). \tag{3}$$

Then they suggest to solve problem (1) by means of solving the following two crisp linear programming problems:

• the Best Optimum Problem: feasible region (2), objective function from (1),

• the Worst Optimum Problem: feasible region (3), objective function from (1).

Such a solution informs the decision maker about two extreme alternatives:

• what the optimum will be in the case the unknown coefficients (all at the same time) take on the most favourable values – let us denote this optimum by MIN,

• what the optimum will be in case the unknown coefficients (all at the same time) take on the least favourable values – let us denote this optimum by MAX.

Let us notice that the Best Optimum Problem concerns the optimistic approach: the requirements for the total amount of substance $i \, (i = 1, \ldots, m)$ turn out to be minimal and the amounts of the substances obtained in the manufacturing process of each product unit turn out to be maximal. The Best Optimum Problem assumes that in this most optimistic case no excess of the left hand sides over the minimal requirements is needed, but no shortage is allowed either.

The Worst Optimum Problem, on the other hand, represents the pessimistic case: the right hand sides turn out to be maximal and the manufacturing process leads to the smallest possible amount of the substances per product unit. The Worst Optimum Problem assumes that no excess is needed, but no shortage is allowed also in this pessimistic case.

In the next section, we will propose an approach in which the above pair of problems will be a special case. We think that the decision maker can be interested in some modifications (strengthening or relaxing) of the above two problems, in order to allow a greater flexibility: he may require some excess (to protect himself against the risk of having committed errors in the estimation of the interval coefficients) or allow some shortage (to obtain a better value of the objective function).

But before we pass to the fuzzy concept, let us reformulate this approach in terms of the preference relations discussed in the previous section:

- the Best Optimum Problem

$$\sum_{j=1}^{n} c_j x_j \to \min,$$

$$\sum_{j=1}^{n} [\underline{a}_j^i, \overline{a}_j^i]\, x_j \geq_w [\underline{b}^i, \overline{b}^i], \quad i = 1, \ldots, m, \tag{2}$$

$$x_j \geq 0\,(j = 1, \ldots, n)\,;$$

- the Worst Optimum Problem:

$$\sum_{j=1}^{n} c_j x_j \to \min,$$

$$\sum_{j=1}^{n} [\underline{a}_j^i, \overline{a}_j^i]\, x_j \geq_s [\underline{b}^i, \overline{b}^i], \quad i = 1, \ldots, m, \tag{3}$$

$$x_j \geq 0\,(j = 1, \ldots, n)\,.$$

5. Fuzzy solution concept

Let us denote by $\lambda \in [0, 1]$ a parameter. For each $\lambda \in [0,1]$ we propose solving a couple of problems:

- the λ-Best Optimum Problem

$$\sum_{j=1}^{n} c_j x_j \to \min,$$

$$\sum_{j=1}^{n} [\underline{a}_j^i, \overline{a}_j^i]\, x_j \geq_{w(-\lambda/2)} [\underline{b}^i, \overline{b}^i], \quad i = 1, \ldots, m, \tag{4}$$

$$x_j \geq 0 \;\; (j = 1, \ldots, n)\,.$$

The optimal objective function value in this problem will be denoted by $\mathrm{MIN}(\lambda)$.

• the λ-Worst Optimum Problem:

$$\sum_{j=1}^{n} c_j x_j \to \min,$$

$$\sum_{j=1}^{n} [\underline{a}_j^i, \overline{a}_j^i]\, x_j \geq_{s(\lambda/2)} [\underline{b}^i, \overline{b}^i], \quad i = 1, \ldots, m, \tag{5}$$

$$x_j \geq 0\,(j = 1, ..., n).$$

The optimal objective function value in this problem will be denoted by MAX(λ).

The following lemma assures an important property of the solution:

Lemma 1

For each $\lambda \in [0, 1]$ the following condition is fulfilled: $\mathrm{MIN}(\lambda) \leq \mathrm{MAX}(\lambda)$.

Proof. The lemma follows immediately from the fact that the feasible region of the λ-Worst Optimum Problem is included in the feasible region of the λ-Best Optimum Problem, and the proof of this fact is straightforward.

The following theorem also shows an important property of the solution:

Theorem 1

For each $\lambda_1, \lambda_2 \in [0, 1]$ $\lambda_1 \leq \lambda_2$ the following condition is fulfilled:

$$[\mathrm{MIN}(\lambda_2),\ \mathrm{MAX}(\lambda_2)\,] \subset [\mathrm{MIN}(\lambda_1),\ \mathrm{MAX}(\lambda_1)\,].$$

Proof. The thesis follows immediately from the easy to prove fact that the feasible region of the λ_2-Best Optimum Problem is included in the feasible region of the λ_1-Best Optimum Problem and that the feasible region of the λ_1-Worst Optimum Problem is included in the feasible region of the λ_2-Worst Optimum Problem.

Now, we are in a position to introduce the following definition:

Definition 1. The fuzzy solution of problem (1) is the fuzzy set whose α-levels ($\alpha \in [0, 1]$) are the intervals $[\mathrm{MIN}(\alpha), \mathrm{MAX}(\alpha)]$.

The interpretation of this solution is as follows: the parameter α is an indication of the optimism level of the decision maker. Here are the two extreme cases:

• $\alpha = 1$ corresponds to the case discussed in the previous section: in the optimistic case no shortage is allowed, in the pessimistic case no excess is required – an optimist thinks that the optimistic case will occur and he bothers more about this case, assuring that there will be no shortage. As far as the pessimistic case is concerned, the optimist does not require any excess, because even if the pessimistic case happens, the optimist is sure he has not made any error in the estimation of the interval coefficients, so that he thinks the pessimistic case from (1) is really the worst that can happen;

• α = 0 corresponds to the case where in the optimistic case a shortage of $(\overline{b}_i - \underline{b}_i)/2$ $(i = 1, ..., m)$ is allowed and in the pessimistic case an excess of $(\overline{b}_i - \underline{b}_i)/2$ $(i = 1, ..., m)$ is required. The pessimist does not think the optimistic case will ever happen, so that he does not bother a lot about it. He is more concerned with the pessimistic case and with his possible errors in the estimation of the interval parameters. That is why he wants to assure there will be an excess of $(\overline{b}_i - \underline{b}_i)/2$ $(i = 1, ..., m)$ in the total amount of each substance in the pessimistic version of (1).

The new concept will be illustrated with the following example:

Example 2

Let us consider the following problem

$x_1 + x_2 \to \min,$

$[4,5]\, x_1 + [1,2]\, x_2 \geq [2,3],$

$[1,2]\, x_1 + [1,7]\, x_2 \geq [4,5],$

$x_1, x_2 \geq 0.$

Here are some selected α-levels of the fuzzy solution:

• α = 0 (original approach): optimal objective function values comprised in the interval [0.71, 1.3],

• α = 0.25: optimal objective function values comprised in the interval [0.74, 1.26],

• α = 0.5: optimal objective function values comprised in the interval [0.77, 1.21],

• α = 0.25: optimal objective function values comprised in the interval [0.8, 1.19],

• α = 0: optimal objective function values comprised in the interval [0.84, 1.13].

Conclusions

The starting point of this paper is a concept of solving a linear programming problem with interval coefficients in the constraints, based on the idea of the so called best- and worst- optimum problem. The original approach does not give the decision maker any freedom in defining those notions, but imposes them. Our modification, proposing a fuzzy solution to the problem, makes these notions more flexible. The decision maker will obtain an overview of possible solutions (in the original sense) of the problem for various degrees of his optimism, i.e. for various degrees of readiness to accept risk of an estimation error of the interval constraints and for various degrees of certainty that a more optimistic or a more pessimistic version of problem (1) will occur. Our proposal comprises the original approach as its special case.

The author thanks the anonymous referee for his valuable remarks.

References

[1] CHINNECK J.W., RAMADAN K., *Linear Programming with Interval Coefficients*, Journal of the Operational Research Society, 2000, 51, s. 209–220.
[2] KUCHTA D., *User-tailored fuzzy relation between intervals*, submitted to Proceedings of EUROFUSE 2003.
[3] KUNDU S., *Min-transitivity of fuzzy leftness relationship and its application to decision making*, Fuzzy Sets and Systems, 1997, 86, s. 357–367.
[4] KUNDU S., *Preference relation on fuzzy utilities based on fuzzy leftness relation on intervals*, Fuzzy Sets and Systems, 1998, 97, s. 183–191.
[5] MOORE R.E., *Interval Analysis*, Prentice Hall, Englewood Cliffs, New Jersey, 1966.
[6] NAKAMURA K., *Preference relation on fuzzy utilities as a basis for decision making*, Fuzzy Sets and Systems, 1986, 20, s. 147–162.
[7] SENGUPTA A., PAL T.K., *On comparing interval numbers*, European Journal of Operations Research, 2000, 127, s. 28–43.
[8] SENGUPTA A., PAL T.K., CHAKRABORTY D., *Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming*, Fuzzy Sets and Systems, 2001, 119, s. 129–138.
[9] SHAOCHENG T., *Interval number and fuzzy number linear programming*, Fuzzy Sets and Systems, 1994, 66, s. 301–306.

Rozmyte rozwiązanie zadań programowania liniowego z przedziałowymi współczynnikami w ograniczeniach

Omówiono zadanie programowania liniowego z przedziałowymi współczynnikami w ograniczeniach – zarówno po prawej, jak i lewej stronie. Przedziałowy współczynnik po prawej stronie oznacza minimalne wymagania. Przewiduje się możliwość jego dokładniejszego określenia (zwężenia przedziału) – im dokładniej jest on określony, tym wyższy jest poziom zadowolenia decydenta. Dla każdego poziomu zadowolenia rozpatruje się przypadek pesymistyczny i optymistyczny, które dotyczą niezbędnych zasobów – danych w postaci przedziałowych współczynników lewej strony ograniczeń – i wyznacza się dwa rozwiązania. Stosuje się przy tym klasyczne metody rozwiązywania zadań programowania liniowego z precyzyjnie określonymi współczynnikami. Decydent, mając pełny przegląd rozwiązań dla różnych poziomów optymizmu, dokonuje wyboru ostatecznego rozwiązania, umiejąc ocenić ryzyko z nim związane. Zaproponowane podejście jest zilustrowane przykładem liczbowym.

Słowa kluczowe: *zadanie programowania interwałowego, rozwiązanie rozmyte*