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Mariusz MAZURKIEWICZ*
Jacek W. MERCIK*

MODIFIED SHAPLEY–SHUBIK POWER INDEX FOR PARLIAMENTARY COALITIONS

Classical power analysis does not involve preferences of players (parties). Classical power indices are constructed under assumption of equal probability of occurrence for each coalition. The paper contains a proposition of relaxation of this assumption, based on extended Shapley–Shubik power index approach.

Keywords: *power index, ideological preferences, ideological distance*

Introduction

Forming a coalition in a decision making body is a process deeply complicated and depends on various determinants, among which probably the most important is conformity of interests. Analysing a parliament as a decision making body one can observe that converging the programs of political parties is one of the most apparent determinants. How this convergence of parties manifesto should be measured and how one can use it in power indices? In the last case we somehow have to determine, in a quantitative way of analysis, factors which are usually hard to describe.

In classical concept of power indices the assumption about equal probability of all coalitions is essential. In any power index formula there is no place for any preference structure or analysis. The first relaxation of equal probability assumption was introduced by Owen (1977, 1982). Owen modified Shapley–Shubik and Penrose-Banzhaf index formulas to involve a special structure – a priori unions. A priori unions structure is a partition of the set of committee members, a collection of disjoint voting configuration whose union is the set of all members. A priori unions are configura-

* Institute of Production Engineering and Management, ul. Smoluchowskiego 25, 50-370 Wrocław, Mariusz.Mazurkiewicz@pwr.wroc.pl, Jacek.Mercik@pwr.wroc.pl

tions that made a priori commitment to cooperate. Another innovation relaxing equal probability assumption was introduced by Edelman (1997). A simple model of positions in one dimensional (ideological) space was applied to modify Shapley–Shubik formulas. Perlinger (2000) extends Edelman's modified index to a class of voting power indices based on a Markov-Polya model of coalition formation. It is possible to call these approaches fundamental and basic. There are certain interesting definitions, mathematical theorems, but in our opinion, there still exist some problems of connections between real coalition formation process and theory of coalition formation. For example, Berg and Perlinger (2001) use ideology scale. Each player (voter) poses position from left to right on the ideology scale. Neighbours on the scale are more likely to form coalitions. But in their paper there is no way how to reach ideology scale using observed data – general election results, voting results, number of members, roll call voting, etc[1].

Mazurkiewicz and Mercik's papers (Mazukiewicz and Mercik, 1996, Mazurkiewicz, 2002) present less sophisticated approach to the problem of coalition formation. The main goal of this work is to define and compute distance between parties in an ideological space spanned on socio-economic factors. Socio-economic factors determine behaviour of voters and indirectly determine preferences of voters in ideological sense. Based on this property it is possible to formulate definition of distance in ideological space. Distances are computed for each pair of parties. The distance between party A and party B is reciprocally proportional to probability of occurrence coalition (A, B). Distances were used as weights in the formula of Shapley–Shubik index (Mazurkiewicz, 1997).

Ideological Scale

Let us come back to Berg and Perlinger methodology. They use in the power index concept positions along an ideological dimension. This requires a definition of one dimensional ideological space and also knowledge about positions of all voters along this dimension.

Let us consider a voting game on the set of players $N = \{1, 2, \ldots, n\}$, in which certain coalitions are excluded *a priori*. Assume that there are n positions on ideological scale occupied by one player each, from left to right (whatever "left" or "right" means). Let $[q, v_1, v_2, \ldots, v_n]$ denote the weighted voting game.

A chain is a subset of N of type $\{i_1, i_2, \ldots, i_n\}$ where the subscript indicates the order in which the players join the coalition (Berg, Perlinger, 2001). A maximal chain is a chain of length n.

[1] For an approach to this problem in the case of Polish Parliament, see Mazurkiewicz et al. (2001).

Let Π denote a sub-class of permutations on the players set N such that:

$\Pi = \{\pi\colon \pi = \{i_1, i_2, \ldots, i_n\}\ i_{k+1} \in \{\min[i_1, i_2, \ldots, i_n] - 1, \max[i_1, i_2, \ldots, i_n] + 1\}, k = 1, 2, \ldots, n - 1\}$

Hasse's diagram of allowable coalitions (ordered by inclusion) for six players is shown in Figure 1.

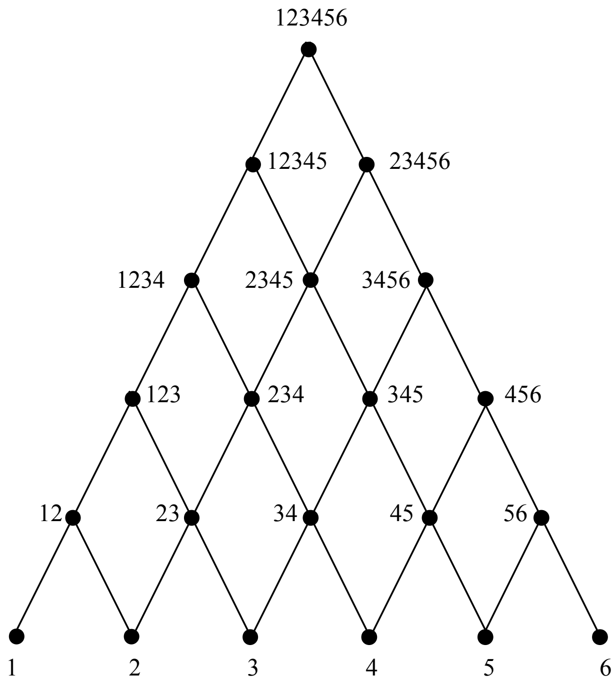

Fig. 1. Hasse's diagram for six players

Let n denote the number of voters. Maximal chain corresponds to the path of maximal length on the graph. The length of maximal path (maximal chain) is equal to $n - 1$.

Let x_k denote binary variables defined by

$$x_k = \begin{cases} 1 & \text{if} \quad i_{k+1} = \max(i_1, i_2, \ldots, i_k) + 1, \\ -1 & \text{if} \quad i_{k+1} = \min(i_1, i_2, \ldots, i_k) - 1 \end{cases}$$

for $k = 1, 2, \ldots, n - 1$. The value 1 in Hasse's diagram may be interpreted as the right step, the value -1 as the left step.

Consider now $\pi = \{j_1, j_2, \ldots, j_n\}$ as an outcome of a random experiment. In this approach, π is the realization of a sequence of binary variables $X_1, X_2, \ldots, X_{n-1}$. Now, for each maximal chain there is a unique element that turns a losing coalition into a winning one. This is the pivotal element. Let us define next random variable as the indicator of pivot in a given permutation (maximal chain, maximal path).

$$\xi(\pi) = i \Leftrightarrow \pi = \{j_1, j_2, \ldots j_n\},\ j_{k+1} = i$$

$$\text{and } \left(\sum_{l=1}^{k} v_{j_l} < q\right) \wedge \left(\sum_{l=1}^{k+1} v_{j_l} \geq q\right) \text{ for given } k = 1,2,\ldots,n-1.$$

This random variable indicates the following event: the player is at pivotal position of the randomly chosen chain. It is now possible to define Shapley–Shubik extended voting power index using definition of $\xi(\pi)$ random variable.

Definition 1 (Berg, Perlinger 2001)

The extended Shapley–Shubik index for player iϵ*N is given by probability*

$$\varphi_i^P(n) = P(\xi(\pi) = i)$$

where permutations $\pi \in \Pi$.

The above definition was originally introduced for a simple game $\left[\frac{n}{2}+1; 1, 2, \ldots, n\right]$. In this situation, the probability of move is described by Polya distribution. It is tacitly assumed that neighbours on an ideological scale form a coalition with the same probability. Introducing a definition of distance between voters (political parties, respectively) we make the assumption weaker by describing the probability of two neighbouring voters' coalition as reciprocal to the distance between them.

The ordering introduced along ideological dimension determines a set of admissible permutations. Every ordering on ideological dimension is connected with different set of admissible coalitions. Therefore, every order generates a new power index. Hence, without some acceptable technique of ordering voters along ideological dimension it is not possible to use in practice extended Shapley–Shubik power index.

Distances between parties

In the process of constructing extended power index for political decision-making body such as parliament we can use distances between voters – political parties. It is quite a natural assumption that the distance is reciprocal to probability of coalition appearance. A matrix of all distances may be obtained via, for example, econometrical analysis of real results of voting and thus via electorate's preference profile.

Let $N = \{1, 2, 3, \ldots, n\}$ denote a set of actors corresponding to n political parties. A given political party may be analysed as a player in political competition. Deputies belonging to this party may be seen as one player under assumption of sufficient homogeneity. In this case we have a weighted voting game $[q, v_1, v_2, \ldots, v_n]$, where v_i denotes the number of seats belonging to party i with decision rule q (usually simple majority).

Analysis of coalition formation process is based on rational choice models of coalition behaviour. There are two main approaches: minimal winning coalition – minimal in the sense of the number of members (Riker, 1962) and minimal in the sense of ideological diversity (Axelrod, 1970). Both conditions are significant in the coalition formation process. We try to build a common approach based on these two original concepts.

Distances and Hasse's diagram

The diversity of voters (parties) is described by distances between them. Enlargement of a certain coalition C by another voter should lead to a decrease of the probability of such coalition establishing. In this case, both criterions of minimal winning coalition in the sense of the numbers of members and in the sense of ideological conformity are fulfilled. A minimal winning coalition in the sense of probability is a coalition for which the probability is minimal.

In original Berg and Perlinger's concept of restricted set of all possible permutations, set Π, which denotes a sub-class of permutations on the player set N, some coalitions are rejected by definition. For them the probability of being established equals 0. In our concept, the assumption about the probability of being established as reciprocal to distance between voters implicates positive value of probability for all coalitions. This attempt enables us to form any coalition and classify them to more or less probable only. In the first round, one can establish $\binom{n}{2}$ coalitions consisting of two voters, in the next round $\binom{n}{3}$ coalitions with three voters and so on.

Using Hasse's diagram one can evaluate the probability for every coalition. Let $D = \{d_{ij}\}_{i,\, j = 1, 2, \ldots, n}$ denote the matrix of distances between all voters from the set N, where

$$d_{ij} = \text{distance}(i, j)$$

denotes the distance between voters i and j.

The matrix of distances does not allow the voters to be distributed along any ideological dimension. Distances were obtained for points in non-determined space, not in the sense of the number of dimensions, and neither in the sense of orientation of this space.

Let us assume that the probability of establishing a coalition comprising voters i and j is reciprocal to the distance d_{ij}:

$$P(\text{coalition}(i, j)) \sim \frac{1}{d_{ij}}.$$

Three-voter coalition can be seen as an additional voter incorporated in the already existing two-voter coalition. The probability of establishing such a coalition of voters i, j, k is proportional to

$$P(\text{coalition}(i, j, k)) \sim \frac{1}{\frac{v_i}{v_i + v_j} d_{ik} + \frac{v_j}{v_i + v_j} d_{jk}}.$$

The distance between voter k and coalition (i, j) is the weighted average of distances between i, k and i, j. Weights used here depend on individual weights of voters v_i, v_j forming a coalition.

Generally, the distance between voter p_k and coalition c_{k-1} (voter p_k accesses coalition c_{k-1}) of voters $c_{k-1} = \{p_1, p_2, ..., p_{k-1}\}$ may be described as

$$d(p_k, c_{k-1}) = \frac{1}{v_{c_{k-1}}} \sum_{i=1}^{k-1} v_{p_i} d_{ik},$$

where $v_{c_{k-1}} = \sum_{i=1}^{k-1} v_{p_i}$ denotes the weight of coalition c_{k-1} for $k = 3, 4, ..., n$.

Generalized Hasse's diagram for all possible coalitions has $2^n - 1$ nodes. Let us fix temporarily a node enumerated as p. Assume additionally that this node represents a single voter coalition. From node p there are $n - 1$ edges to nodes: 1, 2, ..., $p - 1$, $p + 1$, …, n. Let us define random variable X_p, with values equal to the edge number leading to a certain node. The probability distribution of this random variable is described by:

$$P(X_p = k) = \frac{w_p}{d_{pk}},$$

where $k = 1, 2, ..., p - 1, p + 1, …, n$; $w_p = \left(\sum_{i \neq p} \frac{1}{d_{pi}} \right)^{-1}$; d_{pk} – distance between p and k.

More generally, assume that the given node c_l represents a coalition consisting of voters $\{p_1, p_2, ..., p_l\}$. From this node $n - l$ edges are emanating for $l = 1, 2, ..., n - 1$.

Distribution of random variable connected with a given edge is defined by the probability function:

$$P(X_{c_l} = k) = \frac{w_{c_l}}{d_{c_l p_k}},$$

where k denotes the number of player p_k such that $p_k \notin c_l$; $w_{c_l} = \left(\sum_{i \notin c_l} \frac{1}{d_{c_l i}} \right)^{-1}$.

The above probabilistic distribution is conditional one – first the coalition c_l must be established. Hence, the probability of maximal chain is a product of certain probabilities of every subchain entering chain $\pi = \{j_1, j_2, \ldots, j_n\}$:

$$P(\pi) = P(X_1 = j_1) \cdot P(X_2 = j_2) \cdot P(X_3 = j_3) \cdot \ldots \cdot P(X_n = j_n),$$

where random variable X_k describes the choice of edge from node equivalent to coalition $j_1, j_2, \ldots, j_{k-1}$ (random choice of one from $n - k$ edges) for $k = 1, 2, 3, \ldots, n$.

The above random distribution makes it possible to determine extended Shapley–Shubik power index via definition 1. This extension for known ideological distances may be used in practice.

Empirical Example

This empirical example is based on the results of general election in Poland in 2001. In this election, deputies of six parties entered into the lower chamber of Polish Parliament. Table 1 shows distances between the winning parties. In computing the distance socio-economic data were used, which explained behaviour of voters in 2001 (Mazurkiewicz, 2002).

Table 1

Party distances for parties in Polish Parliament of 2001

	SLD-UP	SRP	PiS	PSL	PO	LPR
SLD-UP	–	0.097	0.030	0.185	**0.027**	0.060
SRP	0.097	–	0.089	0.131	0.099	0.051
PiS	0.030	0.089	–	0.176	0.029	0.046
PSL	0.185	0.131	0.176	–	0.180	0.158
PO	**0.027**	0.099	0.029	0.180	–	0.064
LPR	0.060	0.051	0.046	0.158	0.064	–

Consider a cooperative n-person game on the set of players $N = \{1, 2, 3, 4, 5, 6\}$. It is a weighted voting game, where weights are defined by numbers of seats, a decision rule is a simple majority $q = 231$.

Table 2

Number of seats[2] for Polish Parliament of 2001

Party	Number of seats (v_i)
SLD-UP	216
SRP	53
PiS	44
PSL	42
PO	65
LPR	38
Total	458

The minimal distance is a distance equal to 0.027. This means that the probability of occurrence of a coalition containing SLD-UP and PO as members: is the highest. This coalition is the minimal winning coalition in the sense of the number of members (there are also 3 other two-player coalitions) and this coalition is also the minimal winning one in the sense of ideological closeness.

Table 3

Classical Shapley–Shubik index of power

Party	Index
SLD-UP	0.6667
SRP	0.0667
PiS	0.0667
PSL	0.0667
PO	0.0667
LPR	0.0667
Total	1

Values of classical Shapley–Shubik power index are presented in table 3. It is obvious that SLD-UP dominates this parliament, however it has no majority.

The values of extended Shapley–Shubik power index are presented in table 4. It seems that the position of SLD-UP is even stronger than it was evaluated before and moreover, all other parties are more differentiable.

[2] Total number of MPs is 460. We exclude 2 members of German Minority, which in fact is not a political party.

Table 4

Extended Shapley–Shubik index of power

Party	Index
SLD-UP	0.73327
SRP	0.03323
PiS	0.07474
PSL	0.02992
PO	0.08519
LPR	0.04365
Total	1

Conclusion

All original definitions of voting power are based on a simple model of a weighted voting game, described by distribution of votes and quota. In classical approaches there is nothing about preferences, restrictions, voting configuration etc. All coalitions or voting configuration are equally probable. *A priori* unions, extended Shapley–Shubik index are some techniques of relaxation of equal probability assumption. In our opinion general problem relevant to application side of these techniques is transformation of real available data about voting process to theoretical assumption of a priori union or ideological scale for example. This paper contains proposition of solution of the problem of relation between observed data and theoretical assumption. Certainly advantage of described method is possibility of applications just using statistical computation, without expert's opinion for example. Disadvantage is connected to theoretical properties of our approach. The extended Shapley–Shubik index with respect to our modification loose (probably, we are not able to reconstruct proofs – not yet maybe) transparency of theoretical properties proved by Berg and Perlinger, but that is a price (we believe not too big) paid for practical applications.

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Modyfikacja indeksu siły Shapleya–Shubika z uwzględnieniem koalicji

Obserwacja procesu formowania koalicji, zwłaszcza koalicji parlamentarnych, pozwala zidentyfikować dwa główne czynniki determinujące ten proces. Jednym z nich jest dążenie do uzyskania większości – zawarcia koalicji wygrywającej. Należy również uwzględniać czynnik związany z podobieństwem partii na płaszczyźnie ideologicznej. Wydaje się, że ten czynnik odgrywa dużą rolę podczas formowania koalicji. Klasyczna koncepcja indeksów siły nie uwzględnia preferencji ideologicznych. Indeksy siły wyznacza się przy założeniu, że każda koalicja jest tak samo prawdopodobna. Artykuł zawiera propozycję modyfikacji klasycznego podejścia Shapleya–Shubika, uwzględniającą preferencje ideologiczne. Wyznaczenie zmodyfikowanej wartości indeksu siły jest możliwe, co potwierdzają obliczenia przeprowadzone dla wyborów parlamentarnych z 2001 roku w Polsce.

Słowa kluczowe: *indeks siły, preferencje ideologiczne, odległość ideologiczna*