Joint Tests for Zero Restrictions on Non-negative Regression Coefficients

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Joint tests for zero restrictions on nonnegative regression coefficients

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SUMMARY

Three tests for zero restrictions on regression coefficients that are known to be nonnegative are considered: the classical F test, the likelihood ratio test, and a one-sided t test in a particular direction. Critical values for the likelihood ratio test are given for the cases of two and three restrictions, and the power function is calculated for the case of two restrictions. The analysis is conducted in terms of a characterization of the clas . all similar tests for the problem, of which each of the above tests is a member. The likelihood ratio test emerges as the preferred test.

Some key words: Likelihood ratio test; One-sided alternative; Regression; Similar regions.

1. INTRODUCTION

There are numerous applications of the linear model in which the signs of at least some of the regression coefficients are known a priori. Without loss of generality we can assume that the coefficients of interest are known to be nonnegative. This paper is concerned with the problem of testing the joint null hypothesis that $k \geqslant 2$ such coefficients are zero, against the alternative that they are nonnegative, in the context of the classical normal linear model. Writing the model as

$$y = X\beta + Z\gamma + u, \quad u \sim N(0, \sigma^2 I_n), \tag{1}$$

with X an $n \times p$ matrix, Z an $n \times k$ matrix, and $W = (X, Z)$ of full column rank $(p+k)$, the problem of interest is that of testing H_0: $\gamma = 0$ against the one-sided alternative H_a^+: $\gamma > 0$, where $\gamma > 0$ means that $\gamma_i \geqslant 0$ for each $i = 1, \ldots, k$, with strict inequality for at least one i. Both the case of more general constraints on $\eta' = (\beta', \gamma')$, for example H_0: $R\eta = r$ against $R\eta \geqslant r$, R and r both known, and the case $u \sim N(0, \sigma^2\Omega)$, Ω known, are easily transformed into this form.

A number of authors, notably Bartholomew (1959a, b; 1961), Chacko (1963), Kudô (1963), Nüesch (1966), Perlman (1969), Oosterhoff (1969) and Shorack (1967), have considered closely related multivariate one-sided testing problems. Oosterhoff (1969, § 3.1) gives results for the case $\beta = 0$, while Gourieroux, Holly & Monfort (1982) and Yancey, Judge & Bock (1981) have recently considered the same problem under the simplifying assumptions that the covariance matrix is completely known, in the former paper, or that the regressors are orthonormal, in the latter. Kudô (1963, § 5) claims to have characterized the likelihood ratio test for the case where σ^2 in (1) is unknown, but we shall see later that Kudô's advice is incorrect.

In the present paper we assume that σ^2 is unknown, and impose no special restrictions on X and Z. The paper is primarily concerned with the likelihood ratio test but we also consider for comparison the traditional F test, which takes no account of the signs of the coefficients under the alternative, and a one-sided t test in a particular direction from the null. All three tests are similar tests for H_0, and we start with a characterization of the class of similar tests. The critical region for any similar test of H_0 must be defined in terms of a statistic that measures the direction of any departure from H_0, and a statistic that measures the extent of any such departure. This immediately suggests ways of improving the F test, which is based purely on the latter statistic, when the alternative hypothesis is restricted to H_a^+, and also provides a very simple way of describing the critical region for the likelihood ratio test.

2. SIMILAR REGIONS

For testing H_0: $\gamma = 0$ in (1) the parameters β and σ^2 are nuisance parameters, but under H_0 the statistics $\hat{\beta}_0 = (X'X)^{-1}X'y$ and $s_0^2 = y'M_x y$, where $M_x = I - X(X'X)^{-1}X'$, are jointly sufficient for (β, σ^2) and the distribution of $(\hat{\beta}_0, s_0^2)$ is complete. Hence, every size α critical region ω for testing H_0 consists of a fraction α of the surface content of the manifold in y-space defined by $(\hat{\beta}_0, s_0^2) = \text{const}$; see Cox & Hinkley (1974, pp. 134–6). If attention is confined to similar regions the relevant density for the problem becomes the conditional density of y given $\hat{\beta}_0$ and s_0^2, or the density of y on the manifold defined by $(\hat{\beta}_0, s_0^2) = \text{const}$.

We show in the Appendix that the manifold defined by $(\hat{\beta}_0, s_0^2) = \text{const}$ has three components: the surface of the unit m-sphere, S_m: $v'v = 1$, where $m = n - k - p$, the surface of the unit k-sphere, S_k: $h'h = 1$, and the line segment $0 \leq b \leq 1$, where b is related to the usual F statistic for testing H_0 by $b = (kF/m)/(1 + kF/m)$, and v and h are defined in the Appendix. Hence every similar region for testing H_0 must consist of some fraction of the surface $v'v = 1$, some fraction of the surface $h'h = 1$, and some fraction of the line segment $0 \leq b \leq 1$.

The statistic v is independent of h, b and s_0^2, and is uniformly distributed on S_m, whether or not H_0 is true. It follows from this that the most powerful critical regions must include the entire surface $v'v = 1$, so that attention may be confined to critical regions defined in terms of h and b alone.

Under H_0, h, b and s_0^2 are mutually independent, h is uniformly distributed on S_k, b has the beta distribution $B(\frac{1}{2}k, \frac{1}{2}m)$, and $s_0^2/\sigma^2 \sim \chi^2(m+k)$. Hence, as expected, the conditional distribution of y given $\hat{\beta}_0$ and s_0^2 is free of nuisance parameters when H_0 is true. Under the general alternative H_a: $\gamma \neq 0$ the conditional density of h and b given s_0^2 is

$$p(h, b \mid s_0^2) = K_1 b^{\frac{1}{2}k-1}(1-b)^{\frac{1}{2}m-1} \exp(s_0 b^{\frac{1}{2}} h'\bar{\gamma}/\sigma^2), \tag{2}$$

where $\bar{\gamma} = T\gamma$, with T as defined in the Appendix, and

$$K_1 = \Gamma(\tfrac{1}{2}k)[2\pi^{\frac{1}{2}k} B(\tfrac{1}{2}k, \tfrac{1}{2}m) \Sigma(\tfrac{1}{4}\lambda s_0^2/\sigma^2)^j \{j!(\tfrac{1}{2}m + \tfrac{1}{2}k)_j\}^{-1}]^{-1},$$

with $(a)_j = a(a+1)\ldots(a+j-1)$ and $\lambda = \bar{\gamma}'\bar{\gamma}/\sigma^2$. The statistic h may be interpreted as a measure of the direction of any departure from H_0, while b, or F, is a measure of the extent of any such departure.

3. THE F TEST AND DIRECTED t TESTS

Applying the Neyman–Pearson Lemma to the density (2) shows that the best similar region for testing H_0 against the specific alternative $\gamma = \gamma^* > 0$ consists of large values of $b^{\frac{1}{2}}\cos\theta^*$, where θ^* is the angle between h and the unit vector $\mu^* = \bar{\gamma}^*/(\bar{\gamma}^{*\prime}\bar{\gamma}^*)^{\frac{1}{2}}$. This has the following well-known consequences:

(i) if $k=1$ the one-sided t test that rejects H_0 for large positive values of $b^{\frac{1}{2}}h$ is uniformly most powerful similar;

(ii) if $k>1$ there is no uniformly most powerful similar test for H_0, whether or not the alternative is restricted to H_a^+;

(iii) if $k>1$ and, under H_a, $\gamma=\delta\gamma^*$, with $\gamma^*>0$ known and $\delta\geqslant 0$, the best similar region consists of large positive values of $b^{\frac{1}{2}}\cos\theta^*$.

This is equivalent to a one-sided t test of $\delta=0$ against $\delta>0$ in the equation $y=X\beta+\delta Z\gamma^*+u$, because the t statistic for this problem is

$$(m+k-1)^{\frac{1}{2}}b^{\frac{1}{2}}\cos\theta^*/(1-b\cos^2\theta^*)^{\frac{1}{2}}. \tag{3}$$

For testing H_0 against H_a^+, (iii) above suggests that the following strategy may yield a reasonable test: choose a particular vector γ^* in the region $\gamma>0$, and simply use a one-sided t-test for $\delta=0$ against $\delta>0$ in the equation $y=X\beta+\delta Z\gamma^*+u$. The resulting test is evidently locally most powerful similar in directions close to that of γ^*, and might be expected to have reasonable power over the whole region $\gamma>0$. We discuss this approach in more detail in § 5 below.

Let $\mu=\bar{\gamma}/(\bar{\gamma}'\bar{\gamma})^{\frac{1}{2}}$ be the point on S_k determined by the true vector $\bar{\gamma}$; μ indicates the direction in which the true vector $\bar{\gamma}$ lies, and μ and λ are the population analogues of h and F. From (2), the conditional power, given s_0^2, of a critical region ω is given by

$$P_\omega(s_0^2)=\int_\omega K_1 b^{\frac{1}{2}k-1}(1-b)^{\frac{1}{2}m-1}\exp\{(\lambda b s_0^2)^{\frac{1}{2}}h'\mu\}\,db(dh), \tag{4}$$

where (dh) denotes the invariant measure on the surface of the unit k-sphere (James, 1954). Using (4) it is easy to show that the power of the test depends only upon λ if and only if ω includes the entire surface of the S_k: $h'h=1$, a result due to Wolfowitz (1949). It is well known that the traditional F test has precisely this property. In fact, on integrating (4) over $h'h=1$, and then integrating out s_0^2, it follows that the F test is uniformly most powerful among similar tests whose power depends only upon λ (Hsu, 1941).

Now notice that $h'\mu=\cos\bar{\theta}$, where $\bar{\theta}$ is the angle between h and μ. For any fixed μ, that is, for $\bar{\gamma}$ in any fixed direction, $\cos\bar{\theta}$ ranges over the entire interval $-1\leqslant\cos\bar{\theta}\leqslant 1$ as h ranges over S_k, so that the power of the F test is in fact diminished by the inclusion in the critical region of that part of S_k for which $\cos\bar{\theta}$ is negative. Of course, if the direction μ of $\bar{\gamma}$ is arbitrary, this observation is nugatory, but for the problem of testing H_0 against H_a^+ it suggests that a test which excludes that part of the S_k: $h'h=1$ for which $\cos\bar{\theta}$ must be negative from the critical region is likely to improve upon the F test. We shall now show that the likelihood ratio test does exactly that, and does indeed improve upon the F test.

4. The likelihood ratio test

4·1. *Critical regions*

After maximizing the likehood with respect to the nuisance parameters β and σ^2 we find the profile likelihood to be proportional to

$$L=\{s_0^2+l^2-2ls\cos\bar{\theta}\}^{-\frac{1}{2}n}, \tag{5}$$

where $l=(\bar{\gamma}'\bar{\gamma})^{\frac{1}{2}}$ denotes the length of $\bar{\gamma}$ and the other notation is as in § 3 and the Appendix. Now, the region $\gamma>0$ for γ corresponds to a subset of the surface of the unit sphere S_k: $\mu'\mu=1$, say S_a. Provided the 'direction' $\mu=0$ is admitted as a possibility, the problem of maximizing (5) with respect to γ, subject to $\gamma>0$, is equivalent to that of maximizing (5) first with respect to the direction, μ, of $\bar{\gamma}$, subject to $\mu\in S_a$, then with respect to its length, l. But, if $h\in S_a$ it is clear at once that (5) is maximized by choosing $\mu=h$, so that $\cos\bar{\theta}=1$, and $l=s$, giving a maximum of $\{s_0^2(1-b)\}^{-\frac{1}{2}n}$.

Let $\bar{S}_a$ denote the subset of the S_k: $h'h=1$ for which $\cos\bar{\theta}=h'\mu$ must be negative when $\mu\in S_a$. That is, $\bar{S}_a$ is the set of points h on S_k that make an angle greater than $\frac{1}{2}\pi$ with every point $\mu\in S_a$. Clearly, if $h\in\bar{S}_a$, (5) is maximized by setting $\mu=0$, and hence $l=0$, giving a maximum of $(s_0^2)^{-\frac{1}{2}n}$.

If h is in neither S_a nor $\bar{S}_a$, so that μ cannot be chosen to give $\bar{\theta}=0$, but can be chosen so that $\bar{\theta}\leq\frac{1}{2}\pi$, then (5) is clearly maximized by choosing μ on the edge of S_a so as to minimize the angle between μ and the given point h. We denote such a choice for μ by $\tilde{\mu}_h$, and it is clear that (5) is then maximized by setting $l=sh'\tilde{\mu}_h$, giving a maximum of $[s_0^2\{1-b(h'\tilde{\mu}_h)^2\}]^{-\frac{1}{2}n}$.

Since the maximized value of the likelihood under H_0 is proportional to $\{s_0^2\}^{-\frac{1}{2}n}$, the critical region for the likelihood ratio test has the following form:

$$\omega_{\mathrm{LR}}:\begin{cases} b>c & (h\in S_a),\\ b(h'\tilde{\mu}_h)^2>c & (h\in(S_k-S_a-\bar{S}_a)),\end{cases} \tag{6}$$

where c is a suitably chosen critical value.

Notice that, because ω_{LR} is defined in terms of b and h, the likelihood ratio test is a member of the class of similar tests for H_0 against H_a^+. Also, since points $h\in\bar{S}_a$ are not included in the critical region, ω_{LR} excludes precisely those points h on S_k for which $\cos\bar{\theta}$ must be negative when $\mu\in S_a$.

The results above define the maximum likelihood estimates for γ, β and σ^2, subject to $\gamma>0$. The details are easily deduced from the results that follow for the cases $k=2$, 3 and are omitted.

4·2. *The case* $k=2$

Let $Z=(z_1, z_2)$, and write

$$Z'M_xZ=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho\\ \sigma_1\sigma_2\rho & \sigma_2^2\end{bmatrix},\quad T=\begin{bmatrix}\sigma_1 & \sigma_2\rho\\ 0 & \sigma_2(1-\rho^2)^{\frac{1}{2}}\end{bmatrix},$$

where $\sigma_i^2=z_i'M_xz_i$ $(i=1,2)$ and $\rho=(z_1'M_xz_2)/(\sigma_1\sigma_2)$. The region $\gamma>0$ corresponds to the region $\{\bar{\gamma}_1\geq\rho\bar{\gamma}_2/(1-\rho^2)^{\frac{1}{2}};\ \bar{\gamma}_2\geq0\}$, where $\bar{\gamma}'=(\bar{\gamma}_1, \bar{\gamma}_2)$. Writing $\mu'=(\mu_1, \mu_2)$, the region S_a for μ is an arc on the positive semicircle for μ_2 that subtends an angle $a=\cos^{-1}\rho$, measured clockwise from the vertical axis; see Fig. 1.

The region $(S_2-S_a-\bar{S}_a)$ has two components, S_a' and S_a'' in Fig. 1. For $h\in S_a'$, $\tilde{\mu}_h'=(1,0)$, while, for $h\in S_a''$, $\tilde{\mu}_h'=(\rho, (1-\rho^2)^{\frac{1}{2}})$. In terms of the elements of $\hat{\gamma}'=(\hat{\gamma}_1, \hat{\gamma}_2)$ the regions S_a, S_a', S_a'' are

$$S_a:\ \hat{\gamma}_1\geq0,\quad \hat{\gamma}_2\geq0;\quad S_a':\ \sigma_1\hat{\gamma}_1+\sigma_2\rho\hat{\gamma}_2\geq0,\quad \hat{\gamma}_2\leq0;$$

$$S_a'':\ \sigma_1\rho\hat{\gamma}_1+\sigma_2\hat{\gamma}_2\geq0,\quad \hat{\gamma}_1\leq0.$$

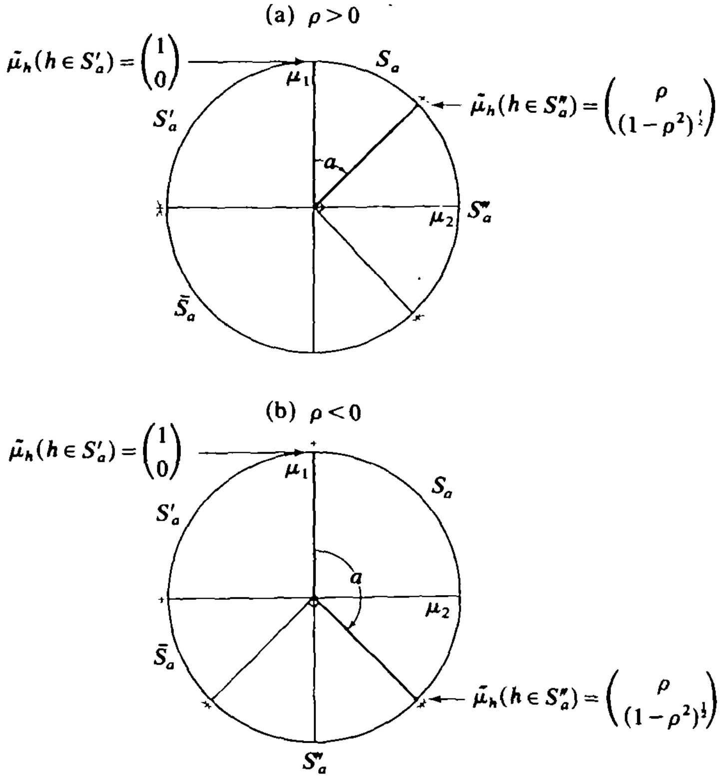

Fig. 1. Regions S_a, S'_a, S''_a and $\bar{S}_a$ when $\rho>0$ and $\rho<0$.

Using (3) and the fact that $b=(kF/m)(1+kF/m)^{-1}$, the critical region (6) has the form

$$\omega_{\mathrm{LR}}:\begin{cases} F>f & (\hat{\gamma}\in S_a),\\ t_1^2>2(m+1)f/m & (\hat{\gamma}\in S'_a),\\ t_2^2>2(m+1)f/m & (\hat{\gamma}\in S''_a),\end{cases}$$

where t_i is the t statistic for the coefficient of z_i in the regression of y on X and z_i alone, and the critical value c in (6) is related to f by $c=(2f/m)(1+2f/m)^{-1}$.

It is important that the boundary of S_a corresponds to points where, in the original parameter space, one or other of the elements of γ is zero, and the likelihood ratio test explicitly takes this into account in the denominators of t_1^2 and t_2^2. The tests of Kudô (1963, § 5) and Yancey, Judge & Bock (1981) do not incorporate this adjustment and hence are not the likelihood ratio test.

The critical value c, and hence f, is determined by the equation

$$\alpha=\int_{\omega_{\mathrm{LR}}}\frac{(1-b)^{\frac{1}{2}m-1}}{2\pi B(1,\frac{1}{2}m)}\,db(dh),$$

where α is the chosen level of significance. From the results above we see that this integral has three components, with regions of integration

$$\{b>c;\,h\in S_a\},\quad \{b(h'\tilde{\mu}_1)^2>c;\,h\in S'_a\},\quad \{b(h'\tilde{\mu}_2)^2>c;\,h\in S''_a\},$$

respectively, where $\tilde{\mu}'_1=(1,0)$ and $\tilde{\mu}'_2=(\rho,(1-\rho^2)^{\frac{1}{2}})$.

Now, the invariant differential form (dh) on, in this case, S_2, may be decomposed (James, 1954, pp. 57–8) into $(dh)=d\theta$, where θ is the angle, measured in a clockwise direction, between any conveniently chosen fixed point on S_2 and h. For the first of the above regions we measure θ from the vertical axis and we have at once

$$\int_{b>c}\int_0^a \frac{(1-b)^{\frac{1}{2}m-1}}{2\pi B(1,\frac{1}{2}m)}\,db\,d\theta=\frac{a}{2\pi}\int_{b>c}\frac{(1-b)^{\frac{1}{2}m-1}}{B(1,\frac{1}{2}m)}\,db$$

$$=\frac{a}{2\pi}\operatorname{pr}\{F(2,m)>f\},$$

where $f=\frac{1}{2}mc/(1-c)$.

For the third region we measure θ from the point $\tilde{\mu}_2$ and we have, on putting $b_1=b\cos^2\theta$, $b_2=b\sin^2\theta$,

$$\int_{b_1>c}\int_0^{1-b_1}\frac{b_1^{-\frac{1}{2}}b_2^{-\frac{1}{2}}(1-b_1-b_2)^{\frac{1}{2}m-1}}{4\pi B(1,\frac{1}{2}m)}\,db_2\,db_1=\frac{1}{4}\int_{b_1>c}\frac{b_1^{-\frac{1}{2}}(1-b_1)^{\frac{1}{2}m-\frac{1}{2}}}{B(\frac{1}{2},\frac{1}{2}m+\frac{1}{2})}\,db_1$$

$$=\tfrac{1}{4}\operatorname{pr}\{F(1,m+1)>f_1\},$$

where $f_1=(m+1)c/(1-c)=2(m+1)f/m$. It is easy to check that the contribution of the second region is identical to that of the third, so that the critical value f is the solution to the equation

$$\alpha=\frac{\cos^{-1}\rho}{2\pi}\operatorname{pr}\{F(2,m)>f\}+\tfrac{1}{2}\operatorname{pr}\{F(1,m+1)>2(m+1)f/m\}. \tag{7}$$

Note that (7) admits a solution for f only if $\alpha<\frac{1}{2}\{1+(\cos^{-1}\rho)/\pi\}$. This, of course, is unlikely to present any difficulty in practice; it is a consequence of the fact that ω_{LR} explicitly excludes part of the S_2: $h'h=1$. Table 1 gives selected critical values, f, calculated from (7) for various values of ρ and m, and for $\alpha=0{\cdot}05$ and $\alpha=0{\cdot}01$.

The last two lines in Table 1 give the comparable critical values for the F test, and it is striking that these can be much larger than those for the likelihood ratio test. Thus, in the case where it turns out that $\hat{\gamma}_1\geqslant 0$, $\hat{\gamma}_2\geqslant 0$, and it is known that $\gamma_1,\ \gamma_2\geqslant 0$, the acceptance region for the F test can be much too large. We shall see shortly that this is reflected in a comparison of the power of the two tests.

4·3. *The case* $k=3$

Let $Z=(z_1, z_2, z_3)$, and write

$$Z'M_xZ=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \sigma_1\sigma_3\rho_{13}\\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 & \sigma_2\sigma_3\rho_{23}\\ \sigma_1\sigma_3\rho_{13} & \sigma_2\sigma_3\rho_{23} & \sigma_3^2\end{bmatrix},$$

where $\sigma_i^2=z_i'M_xz_i$ $(i=1,2,3)$ and $\rho_{ij}=z_i'M_xz_j/(\sigma_i\sigma_j)$ $(i,j=1,2,3)$. The matrix T is given by

$$T=\begin{bmatrix}\sigma_1 & \sigma_2\rho_{12} & \sigma_3\rho_{13}\\ 0 & \sigma_2(1-\rho_{12}^2)^{\frac{1}{2}} & \sigma_3\rho_{23.1}(1-\rho_{13}^2)^{\frac{1}{2}}\\ 0 & 0 & \sigma_3(1-\rho_{13}^2)^{\frac{1}{2}}(1-\rho_{23.1}^2)^{\frac{1}{2}}\end{bmatrix},$$

and $\rho_{ij.k}=(\rho_{ij}-\rho_{ik}\rho_{jk})/\{(1-\rho_{ik}^2)(1-\rho_{jk}^2)\}^{\frac{1}{2}}$.

Table 1. 5% *and* 1% *critical values for the likelihood ratio test*: $k=2$

ρ		$m=10$	$m=15$	$m=20$	$m=30$	$m=50$
0·9	5%	1·83	1·76	1·72	1·69	1·66
	1%	4·03	3·68	3·51	3·36	3·24
0·7	5%	2·08	1·98	1·93	1·89	1·85
	1%	4·46	4·02	3·82	3·63	3·48
0·5	5%	2·25	2·13	2·07	2·02	1·97
	1%	4·73	4·23	4·01	3·80	3·64
0·3	5%	2·39	2·25	2·18	2·12	2·07
	1%	4·95	4·40	4·16	3·93	3·76
0·1	5%	2·51	2·35	2·28	2·21	2·16
	1%	5·14	4·55	4·29	4·04	3·86
0·0	5%	2·56	2·40	2·33	2·25	2·20
	1%	5·23	4·62	4·35	4·10	3·91
−0·1	5%	2·62	2·45	2·37	2·29	2·24
	1%	5·31	4·69	4·41	4·15	3·95
−0·3	5%	2·73	2·54	2·46	2·37	2·31
	1%	5·48	4·81	4·52	4·25	4·04
−0·5	5%	2·84	2·64	2·55	2·46	2·39
	1%	5·65	4·94	4·63	4·35	4·13
−0·7	5%	2·96	2·74	2·64	2·54	2·47
	1%	5·83	5·08	4·75	4·45	4·23
−0·9	5%	3·11	2·87	2·76	2·66	2·57
	1%	6·06	5·26	4·91	4·58	4·35
F		4·10	3·68	3·49	3·32	3·18
		7·56	6·36	5·85	5·39	5·06

Linear interpolation between successive ρ values for fixed m is extremely accurate.

The region $\gamma>0$ corresponds to the region $\{\bar\gamma - c_1\bar\gamma_2 - c_2\bar\gamma_3 \geq 0;\ \bar\gamma_2 - c_3\bar\gamma_3 \geq 0;\ \bar\gamma_3 \geq 0\}$ with $c_1=\rho_{12}/(1-\rho_{12}^2)^{\frac{1}{2}}$, $c_2=\rho_{13.2}/(1-R_{1.23}^2)^{\frac{1}{2}}$ and $c_3=\rho_{23.1}/(1-\rho_{23.1}^2)^{\frac{1}{2}}$, where $1-R_{1.23}^2 = (1-\rho_{12}^2)(1-\rho_{13.2}^2) = (1-\rho_{12}^2)(1-\rho_{13}^2)(1-\rho_{23.1}^2)/(1-\rho_{23}^2)$.

The regions S_a for μ is the region ABC in Fig. 2 which is drawn for the case $\rho_{12}>0$, $\rho_{23.1}>0$, $\rho_{13.2}>0$. The region $\bar S_a$ may be described as follows: draw three 'equators' EA, EB and EC with A, B and C respectively as 'north poles'. The southernmost unbroken line that can be drawn while remaining on one of these 'equators' is the northern boundary of $\bar S_a$.

The points A, B and C are

$$A=\{1,0,0\},\quad B=\{\rho_{13}, \rho_{23.1}(1-\rho_{13}^2)^{\frac{1}{2}}, (1-\rho_{13}^2)^{\frac{1}{2}}(1-\rho_{23.1}^2)^{\frac{1}{2}}\},$$
$$C=\{\rho_{12}, (1-\rho_{12}^2)^{\frac{1}{2}}, 0\}.$$

The coordinates of A', B' and C' are easily deduced from the obvious orthogonality relations with A, B and C. The line BB', for example, represents the locus of the points of tangency between latitudes drawn with B as 'north pole' and latitudes drawn with B' as 'north pole', so that points in $BB'C'$ are closest, in terms of angular distance, to B, while points in $AB'B$ are closest to points on the edge AB of S_a. Thus, for $h \in BB'C'$, $\tilde\mu_h$ is at B, while, for $h \in AB'B$, $\tilde\mu_h$ is a point on the edge AB of S_a. Similar remarks hold for the other four regions depicted in Fig. 2.

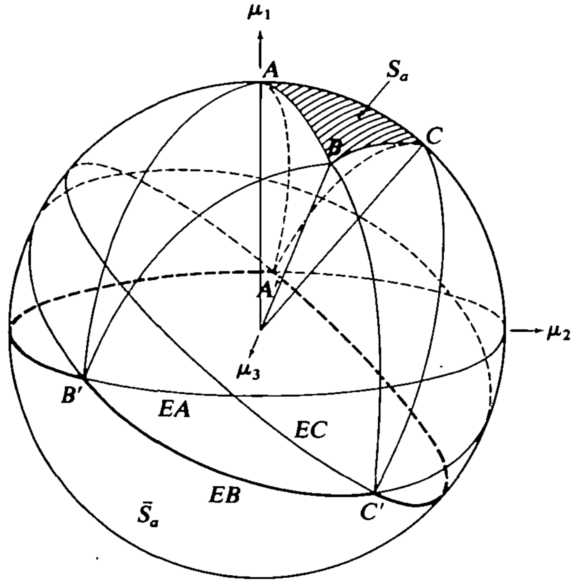

Fig. 2. Regions S_a and $\bar{S}_a$ when $\rho_{12} > 0$, $\rho_{13.2} > 0$, $\rho_{23.1} > 0$.

The six components of $(S_3 - S_a - \bar{S}_a)$ can be expressed in terms of the elements of $\hat{\gamma}' = (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3)$ by using the definition of h in terms of $\hat{\gamma}$. The results are summarized in Table 2, which also gives the appropriate test statistic and critical value for each section of $(S_3 - S_a - \bar{S}_a)$.

Table 2. *Critical region for the likelihood ratio test*: $k = 3$

	Region	Test statistic	Critical value
I	$\hat{\gamma}_1 \geq 0$, $\hat{\gamma}_2 \geq 0$, $\hat{\gamma}_3 \geq 0$	F	f
II	$\sigma_i(1-\rho_{ij}^2)^{\frac{1}{2}}\hat{\gamma}_i + \sigma_k\rho_{ik.j}(1-\rho_{jk}^2)^{\frac{1}{2}}\hat{\gamma}_k \geq 0$, $\sigma_j(1-\rho_{ij}^2)^{\frac{1}{2}}\hat{\gamma}_j + \sigma_k\rho_{jk.i}(1-\rho_{ik}^2)^{\frac{1}{2}}\hat{\gamma}_k \geq 0$, $\hat{\gamma}_k \leq 0$	F_{ij}	$\frac{3}{2}(m+1)f/m$
III	$\sigma_i\hat{\gamma}_i + \sigma_j\rho_{ij}\hat{\gamma}_j + \sigma_k\rho_{ik}\hat{\gamma}_k \geq 0$, $\sigma_j\rho_{jk.i}(1-\rho_{ij}^2)^{\frac{1}{2}}\hat{\gamma}_j + \sigma_k(1-\rho_{ik}^2)^{\frac{1}{2}}\hat{\gamma}_k \leq 0$, $\sigma_j(1-\rho_{ij}^2)^{\frac{1}{2}}\hat{\gamma}_j + \sigma_k\rho_{jk.i}(1-\rho_{ik}^2)^{\frac{1}{2}}\hat{\gamma}_k \leq 0$	t_i^2	$3(m+2)f/m$

II: $(i, j) = (1, 2), (1, 3), (2, 3)$; $k = 3, 2, 1$.
III: $i = 1, 2, 3$; $(j, k) = (2, 3), (1, 3), (1, 2)$.

In Table 2, F_{ij} is the F statistic that would be used to test the joint significance of z_i and z_j in the regression of y on X, z_i and z_j, with z_k excluded, and t_i is the t statistic for the coefficient of z_i in the regression of y on X and z_i alone. The critical value c in (6) is related to f in Table 2 by $c = (3f/m)(1+3f/m)^{-1}$. As before, the use of F_{ij} and t_i explicitly acknowledges that, on the boundary of S_a, one or more elements of γ are zero.

The integral that defines the critical value, c, for the test decomposes in this case into seven components, and the invariant measure (dh) on S_3 decomposes as $(dh) = \sin\theta_1\, d\theta_1\, d\theta_2$, where θ_1 is the angle between h and a fixed point μ_0, say, on S_3, and θ_2 is the angle between the unit vector lying along the orthogonal projection of h onto the S_2 orthogonal to μ_0 and a point $\bar{\mu}_0$, say, in that S_2. By judiciously choosing the points μ_0 and $\bar{\mu}_0$, the components of the integral can be calculated in exactly the same manner as was used in § 4·2 above for the case $k=2$. Thus we find that the critical value f in Table 2 is defined by

$$\alpha=\frac{1}{4\pi}[\{\cos^{-1}\rho_{12.3}+\cos^{-1}\rho_{13.2}+\cos^{-1}\rho_{23.1}-\pi\}\,\mathrm{pr}\,\{F(3,m)>f\}$$
$$+\{\cos^{-1}\rho_{12}+\cos^{-1}\rho_{13}+\cos^{-1}\rho_{23}\}\,\mathrm{pr}\,\{F(2,m+1)>\tfrac{3}{2}(m+1)f/m\}$$
$$+\{\cos^{-1}(-\rho_{12.3})+\cos^{-1}(-\rho_{13.2})+\cos^{-1}(-\rho_{23.1})\}$$
$$\times\,\mathrm{pr}\,\{F(1,m+2)>3(m+2)f/m\}]. \qquad (8)$$

As in the case $k=2$, equation (8) admits a solution for f only if $\alpha<\frac{1}{2}\{1+(\cos^{-1}\rho_{12}+\cos^{-1}\rho_{13}+\cos^{-1}\rho_{23})/(2\pi)\}$, but this constraint is unlikely to be of practical importance. Table 3 gives a small selection of critical values, f, calculated from (8).

Table 3. *Some 5% critical values for the likelihood ratio test*: $k=3$

ρ_{12}	ρ_{13}	ρ_{23}	$m=10$	$m=15$	$m=20$	$m=30$	$m=50$
0·9	0·9	0·9	1·23	1·21	1·21	1·20	1·19
0·9	0·6	0·3	1·50	1·46	1·44	1·42	1·41
0·9	0·1	−0·1	1·71	1·65	1·63	1·60	1·57
0·6	−0·5	−0·3	2·14	2·02	1·97	1·91	1·87
−0·1	−0·5	−0·3	2·52	2·35	2·26	2·18	2·12
−0·3	−0·5	−0·5	2·80	2·57	2·55	2·44	2·36
−0·9	−0·5	0·1	3·00	2·74	2·62	2·50	2·41
0·0	0·0	0·0	2·15	2·03	1·98	1·92	1·88
		F	3·71	3·29	3·10	2·92	2·79

Again, it is striking that the likelihood ratio critical values can be much smaller than those of the traditional F test, the differences being greatest when ρ_{12}, ρ_{13} and ρ_{23} are all large and positive.

4·4. *Power function*: $k=2$

As in the case of the size of the test, the conditional power function, given s_0^2,

$$P_{\mathrm{LR}}(s_0^2)=\int_{\omega_{\mathrm{LR}}} K_1(1-b)^{\frac{1}{2}m-1}\exp\{(\lambda b s_0^2)^{\frac{1}{2}}h'\mu/\sigma\}\,db(dh),$$

decomposes into three components, with regions of integration

$$\{b>c;\,h\in S_a\},\quad \{b(h'\tilde{\mu}_1)^2>c;\,h\in S'_a\},\quad \{b(h'\tilde{\mu}_2)^2>c;\,h\in S''_a\}.$$

Consider first the last of these regions. Setting $(dh)=d\theta$, as before, θ with the angle, measured in a clockwise direction, between $\tilde{\mu}_2$ and h, the contribution to the conditional power from this region is simply

$$\iint_R K_1(1-b)^{\frac{1}{2}m-1}\exp\{(\lambda b s_0^2)^{\frac{1}{2}}\cos(\theta+a-\theta_0)/\sigma\}\,db\,d\theta,$$

where $R=\{b\cos^2\theta>c;\,0\leq\theta\leq\frac{1}{2}\pi\}$, θ_0 is the angle between the vertical axis and μ, and $a=\cos^{-1}\rho$. Transforming to $b_1=b\cos^2\theta$, $b_2=b\sin^2\theta$ and evaluating the integral gives, after averaging with respect to the density of s_0^2, the contribution of this region to the unconditional power of the test. It is easy to see that the contribution of the second region is identical to that of the third except that $(a-\theta_0)$ is replaced by θ_0 itself. Hence we find that the contribution from these two regions is

$$\tfrac{1}{2}e^{-\lambda/2}\sum_{i,j=0}^{\infty}\frac{(\frac{1}{2}\lambda)^{\frac{1}{2}i+\frac{1}{2}j}(-1)^j}{\Gamma(\frac{1}{2}i+1)\Gamma(\frac{1}{2}j+1)}K_{ij}(\theta_0)I_{1-c}(\tfrac{1}{2}m+\tfrac{1}{2}j+\tfrac{1}{2},\tfrac{1}{2}i+\tfrac{1}{2}), \tag{9}$$

where

$$K_{ij}(\theta_0)=\tfrac{1}{2}\{\cos^i(a-\theta_0)\sin^j(a-\theta_0)+\cos^i\theta_0\sin^j\theta_0\} \tag{10}$$

and $I_u(a,b)$ denotes the incomplete beta integral with parameters a and b and upper limit u.

Turning next to the first region, we again put $(dh)=d\theta$ but in this case measure θ from the vertical axis, i.e. from $\tilde{\mu}_1$, so that $h'\mu=\cos(\theta-\theta_0)$ and $0\leqslant\theta\leqslant a$. Provided both θ_0 and $a-\theta_0$ are less than $\frac{1}{2}\pi$ the calculation is straightforward and gives, after averaging with respect to the density of s_0^2, the unconditional power deriving from the region $\{b>c;\ h\in S_a\}$ to be

$$\tfrac{1}{2}e^{-\frac{1}{2}\lambda}\sum_{j=0}^{\infty}\frac{(\frac{1}{2}\lambda)^{\frac{1}{2}j}}{\Gamma(\frac{1}{2}j+1)}J_j(\theta_0)I_{1-c}(\tfrac{1}{2}m,\tfrac{1}{2}j+1) \tag{11}$$

where

$$J_j(\theta_0)=\tfrac{1}{2}\{I_{\sin^2\theta_0}(\tfrac{1}{2},\tfrac{1}{2}j+\tfrac{1}{2})+I_{\sin^2(a-\theta_0)}(\tfrac{1}{2},\tfrac{1}{2}j+\tfrac{1}{2})\}. \tag{12}$$

Since both θ_0 and $(a-\theta_0)$ must be less than $\frac{1}{2}\pi$ when $\rho>0$, expressions (11) and (12) are valid when $\rho>0$. When $\rho<0$ either θ_0 or $(a-\theta_0)$ can exceed $\frac{1}{2}\pi$, and in that case $J_j(\theta_0)$ in (11) is replaced, for even values of j, by

$$J_j^*(\theta_0)=\begin{cases}\frac{1}{2}\{I_{\sin^2\theta_0}(\frac{1}{2},\frac{1}{2}j+\frac{1}{2})+2-I_{\sin^2(a-\theta_0)}(\frac{1}{2},\frac{1}{2}j+\frac{1}{2})\} & (a-\theta_0>\frac{1}{2}\pi),\\ \frac{1}{2}\{I_{\sin^2(a-\theta_0)}(\frac{1}{2},\frac{1}{2}j+\frac{1}{2})+2-I_{\sin^2\theta_0}(\frac{1}{2},\frac{1}{2}j+\frac{1}{2})\} & (\theta_0>\frac{1}{2}\pi).\end{cases} \tag{13}$$

The total power of the test is given by the sum of expressions (9) and (11), and is evidently a function of λ, m and θ_0, with θ_0 reflecting the direction in which the true vector $\bar{\gamma}$ lies. From (10), (12) and (13) it follows that the power function is symmetric with respect to θ_0 about the point $\theta_0=\frac{1}{2}a$, is maximized with respect to θ_0 when $\theta_0=\frac{1}{2}a$, and is minimized with respect to θ_0 when $\theta_0=a$. Some power calculations based on (9)–(13) are in Table 4. Comparison with the last two lines of Table 5 shows that the likelihood ratio test is superior to the F test over the whole range for ρ, and that the improvement in power in the likelihood ratio test is substantial when ρ is large and positive.

The power function for the case $k=3$ can be obtained by an obvious generalization of the argument above. We omit details and merely note that, since the source of the improvement in power is the same in $k\geqslant3$ dimensions as it is in the case $k=2$, there are good grounds for expecting that the above conclusions will remain valid in general.

5. Directed t tests

We consider next the test suggested in § 3, a one-sided t test based on a preselected vector, γ^* say, in the feasible region under H_a^+. From (3), this test rejects H_0 when $b^{\frac{1}{2}}h'\mu^*=b^{\frac{1}{2}}\cos\theta^*$ is large and positive. Hence, the critical region for the test is $\omega^*=\{b^{\frac{1}{2}}\cos\theta^*>c,\ h\in S^*\}$, where S^* is the set of points on the S_k: $h'h=1$ which make an angle less than $\frac{1}{2}\pi$ with μ^*. Notice that, since $\mu^*\in S_a$, S^* excludes all of $\bar{S}_a$, and also

Table 4. *Power of the likelihood ratio test*: $k=2$, $\alpha=0\cdot05$, $\lambda=2, 8$

ρ	a	λ	θ_0	$m=10$	$m=15$	$m=20$	$m=30$	$m=50$
0·9	0·4510	2	a	0·358	0·369	0·375	0·381	0·386
		2	$\frac{1}{2}a$	0·370	0·380	0·386	0·391	0·396
		8	a	0·820	0·836	0·844	0·852	0·859
		8	$\frac{1}{2}a$	0·834	0·848	0·856	0·863	0·868
0·5	1·0472	2	a	0·311	0·323	0·329	0·336	0·342
		2	$\frac{1}{2}a$	0·347	0·358	0·364	0·370	0·375
		8	a	0·766	0·790	0·802	0·813	0·822
		8	$\frac{1}{2}a$	0·803	0·822	0·831	0·840	0·847
0·0	1·5708	2	a	0·271	0·284	0·291	0·299	0·305
		2	$\frac{1}{2}a$	0·320	0·331	0·338	0·344	0·349
		8	a	0·722	0·752	0·767	0·781	0·792
		8	$\frac{1}{2}a$	0·768	0·791	0·803	0·814	0·823
−0·5	2·0944	2	a	0·239	0·253	0·261	0·269	0·275
		2	$\frac{1}{2}a$	0·292	0·304	0·310	0·317	0·323
		8	a	0·684	0·719	0·736	0·752	0·766
		8	$\frac{1}{2}a$	0·733	0·761	0·775	0·789	0·799
−0·9	2·6906	2	a	0·212	0·226	0·233	0·241	0·248
		2	$\frac{1}{2}a$	0·263	0·276	0·282	0·290	0·296
		8	a	0·646	0·685	0·705	0·724	0·739
		8	$\frac{1}{2}a$	0·698	0·730	0·746	0·762	0·776

Table 5. *Minimum power of the optimal directed t test*: $k=2$, $\alpha=0\cdot05$

ρ	λ	$m=10$	$m=15$	$m=20$	$m=30$	$m=50$
0·9	2	0·360	0·371	0·376	0·382	0·387
	8	0·817	0·834	0·843	0·850	0·857
0·5	2	0·298	0·310	0·316	0·322	0·328
	8	0·692	0·724	0·741	0·757	0·770
0·3	2	0·266	0·278	0·285	0·291	0·297
	8	0·614	0·654	0·675	0·695	0·712
0·1	2	0·235	0·247	0·253	0·260	0·266
	8	0·527	0·573	0·598	0·622	0·643
−0·2	2	0·187	0·199	0·205	0·211	0·217
	8	0·385	0·435	0·463	0·492	0·517
−0·5	2	0·140	0·149	0·155	0·160	0·165
	8	0·240	0·284	0·311	0·339	0·365
−0·9	2	0·070	0·076	0·080	0·083	0·086
	8	0·065	0·084	0·097	0·112	0·127
$\theta_0^*=0$	2	0·375	0·386	0·391	0·396	0·401
	8	0·842	0·856	0·862	0·869	0·874
Power of F:	2	0·178	0·192	0·200	0·207	0·216
	8	0·575	0·623	0·647	0·670	0·694

excludes part of $(S_k - S_a - \bar{S}_a)$. Integrating (2) over ω^*, and then averaging with respect to the density of s_0^2, gives the power function

$$P_t^* = \tfrac{1}{2}e^{-\frac{1}{2}\lambda} \sum_{i,j=0}^{\infty} \frac{(\frac{1}{2}\lambda)^{j+\frac{1}{2}i} \cos^i \theta_0^* \sin^{2j} \theta_0^*}{j!\,\Gamma(\frac{1}{2}i+1)} I_{1-c^2}(j+\tfrac{1}{2}m+\tfrac{1}{2}k-\tfrac{1}{2}, \tfrac{1}{2}i+\tfrac{1}{2}), \tag{14}$$

where θ_0^* is the angle between $\bar{\gamma}$ and $\bar{\gamma}^*$, and $1-c^2=\{1+t_c^2/(m+k-1)\}^{-1}$, with t_c^2 determined by $\mathrm{pr}\{F(1, m+k-1)>t_c^2\}=2\alpha$.

The power function (14) is evidently a function of λ, m, k and θ_0^*, the angle between the true vector $\bar{\gamma}$ and the preselected vector $\bar{\gamma}^*$. The power is a maximum at $\theta_0^*=0$ and is a decreasing function of θ_0^*, but it is, of course, impossible to choose γ^* so as to ensure that θ_0^* is small even when it is known that $\gamma>0$. Nevertheless, there is an optimal choice for γ^* when γ is restricted to the region $\gamma>0$ under H_a in the following sense. The test based on the vector γ^* for which the maximum value of θ_0^* over all $\mu \in S_a$ is smallest will maximize the minimum possible power of the test.

In the case $k=2$ the vector μ^* corresponding to γ^* should obviously bisect the angle a, see Fig. 1, which yields the vector $\gamma^{*\prime}=(\sigma_2, \sigma_1)$ as the optimal choice for γ^* in the above sense. For $k\geqslant 3$ the problem of finding the optimal γ^* is more complicated. When $k=3$, if the point μ_a, say, that makes the same angle with each of the extremities of S_a is within S_a, then $\mu^*=\mu_a$ and $\gamma^*=(Z'M_xZ)^{-1}d$, where $d'=(\sigma_1, \sigma_2, \sigma_3)$. However, the point μ_a may not lie within S_a, and in that case μ^* should be taken as the point on the longest edge of S_a that bisects the angle between its endpoints.

Table 5 gives some power calculations based on (14) for $k=2$, $\alpha=0{\cdot}05$, various values for λ and m and, in the body of the table, $\theta_0^*=\frac{1}{2}a$. Values in the main body of Table 5 represent the minimum power of the directed t test for which this minimum power is largest. Table 5 gives also the power of the directed t test when $\theta_0^*=0$, and the power of the F test. The entries for $\theta_0^*=0$ give the maximum possible power for any test of H_0 because the t test in the correct direction is uniformly most powerful similar.

The minimum power of the best directed t test exceeds that of F for $\rho\geqslant-0{\cdot}2$ when $\lambda=2$, for $\rho\geqslant0{\cdot}1$ when $\lambda=5$, and for $\rho\geqslant0{\cdot}3$ when $\lambda=8$. Thus, when ρ is reasonably large this test is better than the F test for $k=2$. Tables 4 and 5 reveal that the likelihood ratio test dominates the directed t test, in terms of minimum power, for almost all values of ρ. Both tests are excellent when ρ is large, but the power of the directed t test falls off much more sharply than that of the likelihood ratio test as ρ declines. Hence, at least for the case $k=2$, the likelihood ratio test is superior.

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APPENDIX

The manifold defined by $(\hat{\beta}_0, s_0^2)$ constant

Let $m=n-k-p$, and let C be an $n\times m$ matrix such that $CC'=I_n-W(W'W)^{-1}W'$ and $C'C=I_m$. Put $w=C'y$, $\hat{\beta}_0=(X'X)^{-1}X'y$ and $\hat{\gamma}=(Z'M_xZ)^{-1}Z'M_xy$. The vectors w, $\hat{\beta}_0$ and $\hat{\gamma}$ are mutually independent, $w\sim N(0, \sigma^2 I_m)$, $\hat{\beta}_0\sim N(\beta+(X'X)^{-1}X'Z\gamma, \sigma^2(X'X)^{-1})$ and $\hat{\gamma}\sim N(\gamma, \sigma^2(Z'M_xZ)^{-1})$. Let T be a $k\times k$ upper triangular matrix such that $T'T=Z'M_xZ$, and put $\tilde{\gamma}=T\hat{\gamma}$, $\bar{\gamma}=T\gamma$, so that $\tilde{\gamma}\sim N(\bar{\gamma}, \sigma^2 I_k)$. Note that $s_0^2=w'w+\tilde{\gamma}'\tilde{\gamma}$.

Now make the transformations $w\to vr$, $\tilde{\gamma}\to hs$, with $v=w/(w'w)^{\frac{1}{2}}$, $h=\tilde{\gamma}/(\tilde{\gamma}'\tilde{\gamma})^{\frac{1}{2}}$, $r^2=w'w$ and $s^2=\tilde{\gamma}'\tilde{\gamma}=\hat{\gamma}'Z'M_xZ\hat{\gamma}$. By construction, $v'v=1$ and $h'h=1$, and the volume elements dw and $d\tilde{\gamma}$ are transformed as

$$dw=\tfrac{1}{2}(r^2)^{\frac{1}{2}m-1}\,dr^2(dv), \quad d\tilde{\gamma}=\tfrac{1}{2}(s^2)^{\frac{1}{2}k-1}\,ds^2(dh),$$

where (dv) and (dh) denote the invariant differential forms on the surfaces of the unit m-sphere and the unit k-sphere respectively (James, 1954). Because $w \sim N(0, \sigma^2 I_m)$, v and r^2 are independent, v is uniformly distributed on the m-sphere S_m: $v'v = 1$, and $r^2/\sigma^2 \sim \chi^2(m)$. Note that at this point we have resolved $p(y)$ into $p(\hat{\beta}_0) \times p(v) \times p(r^2) \times p(h, s^2)$.

Now make the transformations $(r^2, s^2) \to (s_0^2, b)$, with $s_0^2 = r^2 + s^2$ and $b = s^2/(r^2+s^2)$, and note that $b = (kF/m)(1+kF/m)^{-1}$, where $F = ms^2/kr^2$ is the usual F statistic for testing H_0. We then have the resolution of $p(y)$ into $p(\hat{\beta}_0) \times p(s_0^2) \times p(v) \times p(h, b \mid s_0^2)$, with $p(h, b \mid s_0^2)$ given by (2) in the text, and it is straightforward to check that $s_0^2/\sigma^2 \sim \chi'^2(m+k, \lambda)$, with $\lambda = \bar{\gamma}'\bar{\gamma}/\sigma^2 = \gamma' Z' M_x Z \gamma/\sigma^2$. The manifold defined by $(\hat{\beta}_0, s_0^2) = \text{const}$ thus has three components: the surface of the unit m-sphere S_m: $v'v = 1$, the surface of the unit k-sphere S_k: $h'h = 1$, and the line segment $0 \leq b \leq 1$.

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