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Consistency and Monotonicity in One-Sided Assignment Problems

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Consistency and Monotonicity in One-Sided Assignment Problems*

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Abstract

One-sided assignment problems combine important features of two well-known matching models. First, as in roommate problems, any two agents can be matched and second, as in two-sided assignment problems, the payoffs of a matching can be divided between the agents. We take a similar approach to one-sided assignment problems as Sasaki (1995) for two-sided assignment problems and we analyze various desirable properties of solutions including consistency and weak pairwise-monotonicity. We show that for the class of solvable one-sided assignment problems (i.e., the subset of one-sided assignment problems with a non-empty core), if a subsolution of the core satisfies [indifference with respect to dummy agents, continuity, and consistency] or [Pareto indifference and consistency], then it coincides with the core (Theorems 1 and 2). However, we also prove that on the class of all one-sided assignment problems (solvable or not), no solution satisfies consistency and coincides with the core whenever the core is non-empty (Theorem 3). Finally, we comment on the difficulty in obtaining further positive results for the class of solvable one-sided assignment problems in line with Sasaki's (1995) characterizations of the core for two-sided assignment problems.

JEL classification: C71, C78, D63.

Keywords: (One-sided) assignment problems, consistency, core, matching.

1 Introduction

Most racket sports (tennis, squash, badminton, etc.) have established top level doubles competitions. At the start of each season there is a predefined time frame in which players have to organize themselves into pairs. Once pairs are formed, partners cannot be changed during the season. If a player fails to form a pair she cannot participate in the doubles competition. The players, with

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very few exceptions, are professionals who are driven by their passion for the sport but also by pecuniary interests. The latter plays an important role as the prizes at stake in the tournaments throughout the year represent the players most significant source of revenue. This situation leads to a problem, where (rational) players have to simultaneously decide upon how to form pairs and how to distribute payoffs. Since reshuffling of the pairs throughout the seasons is minor, we assume that there are many instances where the problem faced by the players has a stable solution. But what properties would this solution to the problem satisfy? What would happen to the solution if some pairs dropped out of the competition with their gains? Considering that the changes in the doubles hierarchy are relatively small from one season to another, players estimations with respect to the potential gains of each pair in the competition are likely to be adjusted from one season to another by small amounts. Does the adjustment affect the solution? Finally, would the solution be affected if one focuses on pecuniary driven players who are indifferent with respect to the identity of their partners as long as their revenue is kept invariant? We call the situation described in this example a one-sided assignment problem and we call the properties of the solutions described above consistency, continuity, respectively Pareto indifference.

A one-sided assignment problem has many similarities with two well known models in matching: roommate problems (Gale and Shapley, 1962) and assignment problems (Shapley and Shubik, 1972). In roommate problems, agents have preferences over other agents and being alone (or consuming an outside option) and any two agents can either be matched as pairs or remain single. If the value a single agent creates can be consumed by herself and if the value a pair of agents creates is divided between them according to a fixed division, then a roommate problem as described above is the result. With a one-sided assignment problem we model a roommate problem where the assumption that the payoffs to agents are fixed ex-ante is relaxed. In two-sided assignment problems, the set of agents is partitioned into two sets and only agents from different sets can be paired. Then, based on how agents are matched in such a two-sided market, the division of payoffs to agents is flexible as part of the solution.

Here, we generalize both models by allowing for one-sided matching as in roommate problems and for flexible division of payoffs as in two-sided assignment problems. Eriksson and Karlander (2001) and Sotomayor (2005) modeled and analyzed one-sided assignment problems. A one-sided assignment problem consists of a set of agents and a value function that specifies the worth of trade gain or the payoff of working together for each pair of agents. A feasible outcome for a one-sided assignment problem is a matching that partitions the set of agents in pairs and singletons and a payoff vector that divides the total value of the matching between the agents. A solution assigns to any one-sided assignment problem a non-empty subset of feasible outcomes. As in many other economies, a concept of special interest is the core. Eriksson and Karlander (2001) give a characterization of the core by a forbidden minors criterion while Sotomayor (2005) shows that there are one-sided assignment problems with an empty core and identifies necessary and sufficient conditions for the non-emptiness of the core. Hence, strictly speaking, the core is not a solution for the class of all one-sided assignment problems.

We call a one-sided assignment problem with a non-empty core solvable. First, with Example 2, we show that a solvable one-sided assignment problem is not essentially equal to a two-sided assignment problem, that is, a solvable one-sided assignment problem cannot always be mapped onto a core-isomorphic two-sided assignment problem. Then, we aim to extend insights from the normative analysis of two-sided assignment problems to solvable one-sided assignment problems.

For two-sided assignment problems, there are several characterizations of the core using consistency as a central property. Consistency is an invariance requirement of the solutions if some couples and singles decide to leave with their payoffs. To understand this property, suppose that after agents are matched and payoffs are divided according to the solution, some of the agents decide to leave with their payoffs and with their match, and the remaining agents decide to apply the same solution to the restricted one-sided assignment problem. A solution would be considered to be "inconsistent" if it solves such a restricted one-sided assignment problem differently than before. For a comprehensive survey on consistency, see Thomson (2009).

In the first characterization, Sasaki (1995, Theorem 2) considers consistency in conjunction with individual rationality, couple rationality, Pareto optimality, continuity, and weak pairwise-monotonicity. In the second characterization, Sasaki (1995, Theorem 4) replaces continuity by Pareto indifference. Sasaki (1995) proves both characterizations by showing that (*Step 1*) the core satisfies all properties used in both characterizations, (*Step 2*) a solution that satisfies all properties as stated in each of the characterizations is a subsolution of the core, and (*Step 3*) a solution that is a subsolution of the core and satisfies all properties as stated in each of the characterizations equals the core. For a two-sided assignment problem closely related to the one investigated by Sasaki (1995), Toda (2005) also obtains two characterizations of the core (see Toda, 2005, Theorems 3.1 and 3.2, and the discussion of how his results relate to Sasaki's results on page 249).

We adopt the same properties as considered by Sasaki (1995) to see in how far his results for two-sided assignment problems can be extended to one-sided assignment problems. Since Sasaki (1995) characterized the core, we start by restricting attention to the class of solvable one-sided assignment problems. First, and corresponding to Step 1 of Sasaki's analysis, we prove that on the class of solvable one-sided assignment problems, the core satisfies all properties as considered by Sasaki (1995) (Proposition 1). Second, and corresponding to Step 3 of Sasaki's analysis, we show that for the class of solvable one-sided assignment problems, if a subsolution of the core satisfies [indifference with respect to dummy agents, continuity, and consistency] or [Pareto indifference and consistency], then it coincides with the core (Theorems 1 and 2). Note that indifference with respect to dummy agents is a new property that we introduce, and that without this property neither our Theorem 1 nor Sasaki's corresponding Theorem 1 would be correct (see Remark 3 and the counterexample therein). Adapting Sasaki's Step 2 directly to solvable one-sided assignment problems turns out to be impossible because certain steps in the proof would transform a solvable one-sided assignment problem into one with an empty core. We discuss this issue with Step 2 for solvable one-sided assignment problems in Subsection 4.2 in more detail. It is currently an open problem if Sasaki's (1995, Theorems 2 and 4) characterizations of the core can be extended to the class of solvable one-sided assignment problems. Finally, we prove that on the class of all one-sided assignment problems (solvable or not), no solution satisfies consistency and coincides with the core whenever the core is non-empty (Theorem 3).

2 Model and Definitions

We first introduce the one-sided version of Shapley and Shubik's (1972) well-known (two-sided) assignment problems. Our model extends Sasaki (1995) and it coincides with the roommate model with transferable utility introduced by Eriksson and Karlander (2001) and with the one-sided assignment model introduced by Sotomayor (2005).

Let $\mathbb{N}$ be the set of potential agents and $\mathcal{N}$ be the set of all non-empty finite subsets of $\mathbb{N}$, i.e., $\mathcal{N} = \{N \subseteq \mathbb{N} \mid \infty > |N| > 0\}$. For any $N \in \mathcal{N}$ we denote the set of distinct pairs that agents in N can form (including the degenerate case where only one agent $i \in N$ forms a "pair" (i,i)) by $P(N) = \{(i,j) \in N \times N \mid i \leq j\}$. For any $N \in \mathcal{N}$ a function $\pi : P(N) \to \mathbb{R}_+$ such that for each $i \in N$, $\pi(i,i) = 0$, is a *characteristic function for* N. For each pair $(i,j) \in P(N)$, $\pi(i,j) \geq 0$ is the monetary benefit or *value* that i and j can jointly obtain; in particular, $\pi(i,i) = 0$ denotes the (fixed) *reservation value* of agent i. Let $\Pi(N)$ be the set of all characteristic functions on $P(N)$.

A *one-sided assignment problem* γ is a pair $(N,\pi) \in \mathcal{N} \times \Pi(N)$. A *two-sided assignment problem* is a one-sided assignment problem where the set of agents N can be partitioned in two subsets M and W, i.e., $N = M \cup W$ and $M \cap W = \emptyset$, and all coalitions from the same side of the market fail to generate additional value, i.e., for each $(i,j) \in M \times M$ and $(i,j) \in W \times W$, $\pi(i,j) = 0$. We denote the set of all one-sided (two-sided) assignment problems for $N \in \mathcal{N}$ by Γ^N ($\tilde{\Gamma}^N$) and the set of all one-sided (two-sided) assignment problems by $\Gamma = \cup_{N \in \mathcal{N}} \Gamma^N$ ($\tilde{\Gamma} = \cup_{N \in \mathcal{N}} \tilde{\Gamma}^N$).

A *matching* μ for $\gamma \in \Gamma^N$ is a function $\mu : N \to N$ of order two, i.e., for each $i \in N$, $\mu(\mu(i)) = i$. Two agents $i,j \in N$ *are matched* if $\mu(i) = j$ (or equivalently $\mu(j) = i$); for notational convenience we also use the notation $(i,j) \in \mu$. If $i \neq j$, then we say that agents i and j *are paired* and they *form a couple*. If $i = j$, we say that agent i *is paired to herself* and she *remains single*. Thus, at any matching μ the set of agents is partitioned into a set of couples $C(\mu) = \{(i,j) \in P(N) \mid \mu(i) = j, i \neq j\}$ and a set of singles $S(\mu) = \{i \in N \mid \mu(i) = i\}$, with $|N| = 2|C(\mu)| + |S(\mu)|$. For $N \in \mathcal{N}$, let $\mathcal{M}(N)$ denote the *set of matchings*.

A matching that generates maximal value is *(socially) optimal*. That is, for $\gamma \in \Gamma^N$, a matching $\mu \in \mathcal{M}(N)$ is optimal if for each $\mu' \in \mathcal{M}(N)$, $\sum_{(i,j)\in\mu} \pi(i,j) \geq \sum_{(i,j)\in\mu'} \pi(i,j)$. For $\gamma \in \Gamma^N$, let $\mathcal{OM}(\gamma)$ denote the *set of optimal matchings*. Note that for any $\gamma \in \Gamma^N$, $\mathcal{OM}(\gamma) \neq \emptyset$.

For $\gamma \in \Gamma^N$, a *feasible outcome* is a pair (μ, u) where $\mu \in \mathcal{M}(N)$ is a matching and $u \in \mathbb{R}^{|N|}$ is a *payoff vector* such that $\sum_{i \in N} u_i = \sum_{(i,j)\in\mu} \pi(i,j) = \sum_{(i,j)\in C(\mu)} \pi(i,j)$. For $\gamma \in \Gamma^N$, let $\mathcal{F}(\gamma)$ denote the *set of feasible outcomes*.

For $\gamma \in \Gamma^N$, a feasible outcome (μ, u) is *Pareto optimal* if for each $\mu' \in \mathcal{M}(N)$, $\sum_{i\in N} u_i = \sum_{(i,j)\in\mu} \pi(i,j) \geq \sum_{(i,j)\in\mu'} \pi(i,j)$. That is, for $\gamma \in \Gamma^N$ a feasible outcome (μ, u) is Pareto optimal, if $\mu \in \mathcal{OM}(\gamma)$. For $\gamma \in \Gamma^N$, let $\mathcal{PO}(\gamma)$ denote the *set of Pareto optimal outcomes*.

Individual rationality is a voluntary participation condition based on the idea that agents cannot be forced to enter in agreements that yield negative payoffs. For $\gamma \in \Gamma^N$, a feasible outcome (μ, u) is *individually rational* if for each $i \in N$, $u_i \geq 0$. For $\gamma \in \Gamma^N$, let $\mathcal{IR}(\gamma)$ denote the *set of individually rational outcomes*.

Couple rationality ensures that paired agents receive a payoff greater or equal to their own value. For $\gamma \in \Gamma^N$, a feasible outcome (μ, u) is *couple rational* if for each $(i,j) \in C(\mu)$, $u_i + u_j \geq \pi(i,j)$. Note that the non-negativity of values and feasibility imply that if $(\mu, u) \in \mathcal{F}(\gamma)$ is couple rational, then for each $(i,j) \in C(\mu)$, $u_i + u_j = \pi(i,j)$ and for each $i \in S(\mu)$, $u_i = \pi(i,i)$. For $\gamma \in \Gamma^N$, let $\mathcal{CR}(\gamma)$ denote the *set of couple rational outcomes.*[1]

Let $\gamma \in \Gamma^N$ and $(\mu, u) \in \mathcal{F}(\gamma)$. If there are two agents $(i,j) \in P(N)$ such that $i \neq j$ and $u_i + u_j < \pi(i,j)$, then i and j have an incentive to form a couple in order to obtain a higher payoff. In this case, $\{i,j\}$ is a *blocking pair* for the outcome (μ, u).

For $\gamma \in \Gamma^N$, a feasible outcome (μ, u) is *stable* if it is individually rational and no blocking pairs exist, i.e., $(\mu, u) \in \mathcal{IR}(\gamma)$ and for each $(i,j) \in P(N)$ such that $i \neq j$, $u_i + u_j \geq \pi(i,j)$. For $\gamma \in \Gamma^N$, let $\mathcal{S}(\gamma)$ denote the *set of stable outcomes*. Let $\gamma \in \Gamma^N$ be *solvable* if $\mathcal{S}(\gamma) \neq \emptyset$. The following example shows that the set of stable outcomes may be empty.

Example 1. ***A one-sided assignment problem that is not solvable.***
Let $N = \{1,2,3\}$, π such that $\pi(1,2) = \pi(2,3) = \pi(1,3) = 1$, and $\gamma = (N, \pi)$. Then, for each $(\mu, u) \in \mathcal{F}(\gamma)$, $u_1 + u_2 + u_3 \leq 1$ and there exist two agents i and j, $i \neq j$, such that $u_i + u_j < \pi(i,j) = 1$. Thus, $\mathcal{S}(\gamma) = \emptyset$. $\diamond$

Remark 1. On the class of solvable one-sided assignment problems, the set of stable outcomes coincides with the core (Sotomayor, 2005, Proposition 1), i.e., for each $\gamma \in \Gamma$, $\mathcal{S}(\gamma) = \mathcal{C}(\gamma)$, where $\mathcal{C}(\gamma)$ denotes the core for problem γ.[2] $\diamond$

Since for the class of solvable one-sided assignment problems the set of stable outcomes and the core coincide, from now on we will use the two notions interchangeably.

[1]Our definition of couple rationality is identical to the one in Sasaki (1995) and it implies (pairwise) feasibility in Toda (2005) and Sotomayor (2005).

[2]Alternatively, we could model a one-sided assignment problem $\gamma \in \Gamma^N$ as the following cooperative game with transferable utility (TU). Let v be the associated *TU characteristic function* that assigns to each coalition S, $v(S) \equiv \max_{\mu\in\mathcal{M}(S)}\{\sum_{(i,j)\in\mu} \pi(i,j)\}$ with $v(\emptyset) = 0$. The *core* of $\gamma \in \Gamma^N$ equals $\mathcal{C}(\gamma) = \{(\mu, u) \in \mathcal{F}(\gamma) \mid \text{for all } S \subseteq N, \sum_{i\in S} u_i \geq v(S)\}$. Thus, a feasible outcome is in the core if no coalition of agents $S \subseteq N$ can improve their payoffs by rematching among themselves.

Investigating the precise conditions which guarantee the solvability of a one-sided assignment problem and a full comparison of solvable one-sided assignment problems to two-sided assignment problems is beyond the purpose of this paper. However, we note that Eriksson and Karlander (2001) find that there are many similarities between solvable one-sided assignment problems and two-sided assignment problems. The impression one might get is that in the core of solvable one-sided assignment problems there always exists a "two-sided" partition of the agents. Then, it could be possible that for each solvable one-sided assignment problem there exists a core-isomorphic two-sided assignment problem, i.e., by choosing the appropriate partition of agents (setting the values of now incompatible pairs equal to zero) one can convert the one-sided assignment problem into a two-sided assignment problem without changing the set of core outcomes. The following example shows that a solvable one-sided assignment problem cannot always be mapped onto a core-isomorphic two-sided assignment problem.

Example 2. ***Two-sided and solvable one-sided problems are not core-isomorphic.***
Let $N = \{1, 2, 3\}$, π such that $\pi(1, 2) = 2$, $\pi(2, 3) = \pi(1, 3) = 1$, and $\gamma = (N, \pi)$. Then, γ is solvable because $\mathcal{S}(\gamma) = \{(\mu, u) \mid \mu = (2, 1, 3) \text{ and } u = (1, 1, 0)\}$. The unique stable matching μ induces a natural partition of the set of agents $N = M \cup W$ where agents 1 and 2 have different genders. Assume that this solvable one-sided assignment problem can be mapped onto a core-isomorphic two-sided assignment problem and, without loss of generality, $M = \{1, 3\}$ and $W = \{2\}$. Then, formally we can associate with γ a two-sided assignment problem $\gamma' \equiv (M \cup W, \pi')$, where we define π' as the restriction of π to feasible (man-woman) pairs, i.e., $\pi'(1, 2) = 2$, $\pi'(2, 3) = 1$, and in difference to π, $\pi'(1, 3) = 0$ (agents 1 and 3 are both male and now do not create any positive surplus). The problem γ' is still solvable since $\mathcal{S}(\gamma') = \{(\mu, u') \mid \mu = (2, 1, 3) \text{ and } u' = (\alpha, 2 - \alpha, 0) \text{ for } \alpha \in [0, 1]\}$. Observe that $u'_1 \leq u_1$, $u'_2 \geq u_2$, and $u'_3 = u_3$, and that the same matching μ is part of both $\mathcal{S}(\gamma)$ and $\mathcal{S}(\gamma')$. Thus, $\mathcal{S}(\gamma) \subsetneq \mathcal{S}(\gamma')$. Therefore, γ' has additional stable outcomes and it is not, as suspected, core-isomorphic to γ. $\diamond$

A solution specifies how to form couples and how to distribute payoffs among the agents. Formally, a *solution* φ is a correspondence that associates with each $\gamma \in \Gamma$ a non-empty subset of feasible outcomes, i.e., for each $\gamma \in \Gamma$, $\varphi(\gamma) \subseteq \mathcal{F}(\gamma)$ and $\varphi(\gamma) \neq \emptyset$. A solution φ' is a *subsolution* of solution φ if for each $\gamma \in \Gamma$, $\varphi'(\gamma) \subseteq \varphi(\gamma)$.

3 Properties of Solutions

In this section we introduce desirable properties of solutions.

Definition 1. ***Individual rationality***
For each $\gamma \in \Gamma$, $\varphi(\gamma) \subseteq \mathcal{IR}(\gamma)$.

Definition 2. ***Couple rationality***
For each $\gamma \in \Gamma$, $\varphi(\gamma) \subseteq \mathcal{CR}(\gamma)$.

Definition 3. ***Pareto optimality***
For each $\gamma \in \Gamma$, $\varphi(\gamma) \subseteq \mathcal{PO}(\gamma)$.

Pareto indifference requires that if an outcome is chosen by the solution, then all feasible outcomes containing the same payoff vector have to be part of the solution.

Definition 4. ***Pareto indifference***
For each $\gamma \in \Gamma$ and each $(\mu, u) \in \varphi(\gamma)$, if $(\mu', u) \in \mathcal{F}(\gamma)$, then $(\mu', u) \in \varphi(\gamma)$.

For our next property we introduce the notion of dummy agents: agents who when paired at a given matching do not create any positive surplus. For $\gamma \in \Gamma^N$, $\mu \in \mathcal{M}(N)$, i and j are *dummy agents* for μ if $\pi(i,j) = 0$ and $(i,j) \in C(\mu)$. For $\gamma \in \Gamma^N$, $\mu \in \mathcal{M}(N)$, let $DA(\gamma, \mu)$ denote the *set of dummy agents* for μ.

Indifference with respect to dummy agents requires that if an outcome for which the matching pairs some dummy agents is chosen by the solution, then all feasible outcomes obtained by "unmatching" some of the dummy agents have to be part of the solution.

Definition 5. ***Indifference with respect to dummy agents***
For each $\gamma \in \Gamma$, each $(\mu, u) \in \varphi(\gamma)$, and all $(\mu', u) \in \mathcal{F}(\gamma)$ such that $C(\mu') \subseteq C(\mu)$ and $S(\mu') \backslash S(\mu) = DA(\gamma, \mu)$, $(\mu', u) \in \varphi(\gamma)$.[3]

Note that indifference with respect to dummy agents has no bite if $DA(\gamma, \mu) = \emptyset$. Furthermore, Pareto indifference implies indifference with respect to dummy agents.

Continuity, loosely speaking requires that small changes in the value function induce small changes in solution outcomes.

Definition 6. ***Continuity***
For each $N \in \mathcal{N}$, for any natural number k, and each $\gamma, \gamma^k \in \Gamma$, where $\gamma = (N, \pi)$ and $\gamma^k = (N, \pi^k)$ such that $(\mu, u^k) \in \varphi(N, \pi^k)$, if for each $(i,j) \in P(N)$, $\pi^k(i,j) \xrightarrow[k \to \infty]{} \pi(i,j)$ and for each $i \in N$, $u_i^k \xrightarrow[k \to \infty]{} u_i$, then $(\mu, u) \in \varphi(N, \pi)$.

To define the next property we first introduce the notion of a subproblem. Let $\gamma = (N, \pi)$, $N' \subseteq N$, and $P(N') = \{(i,j) \in N' \times N' \mid i \leq j\}$. We denote by $\pi_{|N'}$ the restriction of value function π to $P(N')$, i.e., $\pi_{|N'} : P(N') \to \mathbb{R}_+^{|N'|}$ such that for each $(i,j) \in P(N')$, $\pi_{|N'}(i,j) = \pi(i,j)$. Then, $\gamma_{|N'} = (N', \pi_{|N'}) \in \Gamma^{N'}$ is a *subproblem* of γ.

Let $N \in \mathcal{N}$ and $N' \subseteq N$. For any matching $\mu \in \mathcal{M}(N)$, we denote by $\mu(N')$ the set of agents that are matched to agents in N', i.e., $\mu(N') = \{i \in N \mid \mu^{-1}(i) \in N'\}$. Furthermore, for any

[3]Note that $C(\mu') \subseteq C(\mu)$ implies $S(\mu') \supseteq S(\mu)$.

$u \in \mathbb{R}^{|N|}$, let $u_{|N'}$ denote the restriction of vector u to N', i.e., $u_{|N'} \equiv u' \in \mathbb{R}^{|N'|}$ such that for each $i \in N'$, $u'_i = u_i$.

Consistency is an invariance requirement of the solutions if some couples and singles decide to leave with their payoffs. For a comprehensive survey on consistency, see Thomson (2009).

Definition 7. ***Consistency***
For each $N \in \mathcal{N}$, each $\gamma \in \Gamma^N$, each $(\mu, u) \in \varphi(\gamma)$, and each $N' \subseteq N$ such that $\mu(N') = N'$, $(\mu_{|N'}, u_{|N'}) \in \varphi(\gamma_{|N'})$.

Weak pairwise-monotonicity requires that if the value of a couple is increased, then the total payoff of the couple should not decrease.

Definition 8. ***Weak pairwise-monotonicity***
For each $\gamma \in \Gamma$, where $\gamma = (N, \pi)$, each $(i, j) \in P(N)$, $i \neq j$, and all $\gamma^* = (N, \pi^*)$ such that

$$\pi^*(i, j) \geq \pi(i, j) \qquad \text{and} \tag{1}$$

$$\pi^*(i', j') = \pi(i', j'), \qquad \text{otherwise,} \tag{2}$$

if $(\mu, u) \in \varphi(\gamma)$, then there exists $(\mu^*, u^*) \in \varphi(\gamma^*)$ such that $u^*_i + u^*_j \geq u_i + u_j$.

4 Results

4.1 Positive results on the class of solvable one-sided assignment problems

Our first positive result shows that the core of solvable one-sided assignment problems satisfies all of the above properties, which extends a similar result by Sasaki (1995, Propositions 3 and 4) for two-sided assignment problems to solvable one-sided assignment problems.[4]

Proposition 1. *On the class of solvable problems, the core satisfies individual rationality, couple rationality, Pareto optimality, Pareto indifference, indifference with respect to dummy agents, continuity, consistency, and weak pairwise-monotonicity.*

Proof.
Individual rationality, couple rationality, and Pareto optimality: For each solvable $\gamma \in \Gamma$, if $(\mu, u) \in \mathcal{S}(\gamma)$, then it is immediate that $(\mu, u) \in \mathcal{PO}(\gamma) \cap \mathcal{IR}(\gamma) \cap \mathcal{CR}(\gamma)$.

Pareto indifference and indifference with respect to dummy agents: For each solvable $\gamma \in \Gamma$, assume $(\mu, u) \in \mathcal{S}(\gamma)$. Let $(\mu', u) \in \mathcal{F}(\gamma)$. Then, since any blocking pair for (μ', u) would also be a blocking pair for (μ, u), $(\mu', u) \in \mathcal{S}(\gamma)$. Furthermore, Pareto indifference implies indifference with respect to dummy agents.

[4]It is also easy to show that the core of solvable one-sided assignment problems satisfies converse consistency. However, since we do not use converse consistency in the sequel, we do not include it in this paper.

Continuity: Let $\gamma, \gamma^k \in \Gamma$ be solvable, $(\mu, u^k) \in \mathcal{S}(\gamma^k)$ as in the definition of continuity, and $u_i^k \xrightarrow[k\to\infty]{} u_i$, $\pi^k(i,j) \xrightarrow[k\to\infty]{} \pi(i,j)$. Since the correspondence of feasible outcomes is continuous, (μ, u) is feasible. By stability, for each $i \in N$, $u_i^k \geq 0$ and for each $(i,j) \in P(N)$, $u_i^k + u_j^k \geq \pi^k(i,j)$. Letting $k \to \infty$, for each $i \in N$, $u_i \geq 0$ and for each $(i,j) \in P(N)$, $u_i + u_j \geq \pi(i,j)$. Hence, $(\mu, u) \in \mathcal{S}(\gamma)$.

Consistency: For each solvable $\gamma \in \Gamma$, if $(\mu, u) \in \mathcal{S}(\gamma)$, then there do not exist any blocking pairs for (μ, u). Hence, no blocking pairs exist in any of the (smaller) reduced problems and all reduced problems are solvable.

Weak pairwise-monotonicity: Let $\gamma, \gamma^* \in \Gamma$ be solvable and $(i,j) \in P(N)$ as in the definition of weak pairwise-monotonicity. We show that $(\mu, u) \in \mathcal{S}(\gamma)$ implies that there exists $(\mu^*, u^*) \in \mathcal{S}(\gamma^*)$ such that $u_i^* + u_j^* \geq u_i + u_j$.

Let $(\mu, u) \in \mathcal{S}(\gamma)$. Then, $\mu \in \mathcal{OM}(\gamma)$ and for each $(i', j') \in P(N), i' \neq j', u_{i'} + u_{j'} \geq \pi(i', j')$.

Case 1: Assume $\mu(i) = j$.

Let $\mu^* = \mu$ and define u^* as follows: $u_i^* = u_i + [\pi^*(i,j) - \pi(i,j)]/2$, $u_j^* = u_j + [\pi^*(i,j) - \pi(i,j)]/2$ and for each $i' \in N \setminus \{i,j\}$, $u_{i'}^* = u_{i'}$. By (1), $\pi^*(i,j) \geq \pi(i,j)$ and consequently u_i^*, $u_j^* \geq 0$. Thus, $(\mu^*, u^*) \in \mathcal{IR}(\gamma^*)$. Since $(i,j) \in C(\mu^*)$, $u_i^* + u_j^* = \pi^*(i,j)$. By definition of u^*, for each $(i',j') \neq (i,j)$, $u_{i'}^* + u_{j'}^* = u_{i'} + u_{j'}$. Since $(\mu, u) \in \mathcal{S}(\gamma)$, $u_{i'} + u_{j'} \geq \pi(i',j')$ and by (2), $\pi(i',j') = \pi^*(i',j')$. Thus, for each $(i',j') \in P(N)$ such that $i' \neq j'$ we have $u_{i'}^* + u_{j'}^* \geq \pi^*(i',j')$, i.e., $(\mu^*, u^*) \in \mathcal{S}(\gamma^*)$. Note that $u_i^* + u_j^* = u_i + u_j + [\pi^*(i,j) - \pi(i,j)]$ which by (1) yields $u_i^* + u_j^* \geq u_i + u_j$.

Case 2: Assume $\mu(i) \neq j$.

Let $(\mu^*, u^*) \in \mathcal{S}(\gamma^*)$. Then, $\mu^* \in \mathcal{OM}(\gamma^*)$ and consequently, $\sum_{(i',j')\in\mu^*} \pi^*(i',j') \geq \sum_{(i',j')\in\mu} \pi^*(i',j')$. By (2) and since $(i,j) \notin C(\mu)$, $\sum_{(i',j')\in\mu} \pi^*(i',j') = \sum_{(i',j')\in\mu} \pi(i',j')$. Thus,

$$\sum_{(i',j')\in\mu^*} \pi^*(i',j') \geq \sum_{(i',j')\in\mu} \pi(i',j'). \tag{3}$$

Case 2.1: Assume $\mu^*(i) = j$.

By (3) and feasibility, $\sum_{i'\in N} u_{i'}^* = \sum_{(i',j')\in\mu^*} \pi^*(i',j') \geq \sum_{(i',j')\in\mu} \pi(i',j') = \sum_{i'\in N} u_{i'}$. Suppose $u_i^* + u_j^* < u_i + u_j$. Then, $\sum_{i'\in N\setminus\{i,j\}} u_{i'}^* = \sum_{(i',j')\in\mu^*\setminus\{(i,j)\}} \pi^*(i',j') > \sum_{i'\in N\setminus\{i,j\}} u_{i'}$. Thus, there exists $(i'',j'') \neq (i,j)$ such that $(i'',j'') \in C(\mu^*)$ and $\pi^*(i'',j'') > u_{i''} + u_{j''}$. But by (2), $\pi^*(i'',j'') = \pi(i'',j'')$. Hence, $\pi(i'',j'') > u_{i''} + u_{j''}$, which is a contradiction to $(\mu, u) \in \mathcal{S}(\gamma)$. Therefore, $u_i^* + u_j^* \geq u_i + u_j$.

Case 2.2: Assume $\mu^*(i) \neq j$.

Further, assume $\mu \notin \mathcal{OM}(\gamma^*)$. Then, (3) is strict and by (2), $\sum_{(i',j')\in\mu^*} \pi^*(i',j') = \sum_{(i',j')\in\mu^*} \pi(i',j')$. Consequently, $\sum_{(i,j)\in\mu^*} \pi(i,j) > \sum_{(i,j)\in\mu} \pi(i,j)$, which is a contradiction to $\mu \in \mathcal{OM}(\gamma)$. Alternatively, now assume $\mu \in \mathcal{OM}(\gamma^*)$. Then, we could have chosen $u^* = u$, which gives $u_i^* + u_j^* = u_i + u_j$. □

Remark 2. ***The class of solvable one-sided assignment problems is closed***
An immediate consequence of the continuity of the core (Proposition 1) is that the class of solvable one-sided assignment problems is *closed*, i.e., if for all $k \in \mathbb{N}$, $\gamma^k \in \Gamma^N$ is solvable and $\gamma^k \xrightarrow[k\to\infty]{} \gamma \in \Gamma^N$, then γ is also solvable.

We next prove that if a consistent subsolution of the core satisfies indifference with respect to dummy agents and continuity, then it coincides with the core. Note that Sasaki (1995, Theorem 1) states this result without using indifference with respect to dummy agents for two-sided assignment problems. We show in Remark 3 that adding indifference with respect to dummy agents is indeed necessary (also for two-sided assignment problems).

Theorem 1. *On the class of solvable problems, if φ is a subsolution of the core satisfying indifference with respect to dummy agents, continuity, and consistency, then φ coincides with the core.*

Proof. Let $\gamma \in \Gamma^N$ be a solvable problem. Then, since φ is a subsolution of the core, $\varphi(\gamma) \subseteq \mathcal{S}(\gamma)$. We prove that $\varphi(\gamma) \supseteq \mathcal{S}(\gamma)$, i.e., we show that $(\mu, u) \in \mathcal{S}(\gamma)$ implies $(\mu, u) \in \varphi(\gamma)$.

Step 1: Intuitively, we start from γ and a stable outcome (with a maximal number of dummy agents) by adding a new agent n. By construction, we preserve the original set of optimal matchings and we extend each optimal matching with the single new agent n. Further, we require that any paired agent can maintain the same utility as the one obtained within her current partnership when pairing with n.

Formally, let $(\mu, u) \in \mathcal{S}(\gamma)$. We will not work with matching μ directly, but with a matching μ^{DA} (possibly $\mu^{DA} = \mu$) that has a maximal number of dummy agents such that the only possible difference between μ^{DA} and μ is that some dummy agents for μ^{DA} are single at matching μ, i.e., $C(\mu^{DA}) \supseteq C(\mu)$ and $S(\mu) \setminus S(\mu^{DA}) = DA(\gamma, \mu^{DA})$ and there exists no matching $\hat{\mu}$ such that $C(\hat{\mu}) \supseteq C(\mu)$, $S(\mu) \setminus S(\hat{\mu}) = DA(\gamma, \hat{\mu})$, and $|DA(\gamma, \hat{\mu})| > |DA(\gamma, \mu^{DA})|$. Note that $(\mu^{DA}, u) \in \mathcal{S}(\gamma)$ and that at μ^{DA} at most one agent is single: if more than one agent is single, then the matching μ^{DA} is not optimal (two single agents have a positive value) or it does not match the maximal number of dummy agents (two single agents have value 0). In the remainder of the proof we distinguish the two Cases **(a)** $S(\mu^{DA}) = \{d\}$ and **(b)** $S(\mu^{DA}) = \emptyset$.

Let $n \in \mathbb{N} \setminus N$ and $N^* = N \cup \{n\}$. For Case **(a)** we define μ^* such that for each $i \in N \setminus \{d\}$, $\mu^*(i) = \mu^{DA}(i)$, $\mu^*(d) = d$, and $\mu^*(n) = n$ and $\bar{\mu}^*$ such that for each $i \in N \setminus \{d\}$, $\bar{\mu}^*(i) = \mu^{DA}(i)$ and $\bar{\mu}^*(d) = n$. For Case **(b)** we define μ^* such that for each $i \in N$, $\mu^*(i) = \mu^{DA}(i)$ and $\mu^*(n) = n$. Hence, **(a)** $S(\mu^*) = \{d, n\}$ and $S(\bar{\mu}^*) = \emptyset$ and **(b)** $S(\mu^*) = \{n\}$.

For each $i \in N$, let $\pi^*(i, n) = u_i = u_i^*$ and for each $(i, j) \in P(N)$, $\pi^*(i, j) = \pi(i, j)$. Let $\gamma^* = (N^*, \pi^*)$. Because the entrant n does not create any new blocking pairs, it follows that **(a)** $(\mu^*, u^*), (\bar{\mu}^*, u^*) \in \mathcal{S}(\gamma^*)$ and **(b)** $(\mu^*, u^*) \in \mathcal{S}(\gamma^*)$. Note that **(a)** $DA(\gamma^*, \mu^*) = \emptyset$ and $DA(\gamma^*, \bar{\mu}^*) = \{d, n\}$ and **(b)** $DA(\gamma^*, \mu^*) = \emptyset$.

Step 2: Next, starting from γ^* we construct γ^ε as follows: ceteris paribus, we increase the value of all couples at μ^* that do not include agent n by ε and we symmetrically distribute the benefits within these couples.

Formally, for Case **(a)** for each $(i,j) \in C(\mu^{DA})$,[5] $[\pi^\varepsilon(i,j) = \pi^*(i,j) + \varepsilon$, $u_i^\varepsilon = u_i^* + \frac{\varepsilon}{2}$, and $u_j^\varepsilon = u_j^* + \frac{\varepsilon}{2}]$, $\pi^\varepsilon(d,n) = \pi^*(d,n)$, and $u_d^\varepsilon = u_n^\varepsilon = 0$. For each $(i,j) \in P(N) \setminus C(\mu^{DA})$, $\pi^\varepsilon(i,j) = \pi^*(i,j)$. For Case **(b)** for each $(i,j) \in C(\mu^*) = C(\mu^{DA})$, $[\pi^\varepsilon(i,j) = \pi^*(i,j)+\varepsilon$, $u_i^\varepsilon = u_i^* + \frac{\varepsilon}{2}$, and $u_j^\varepsilon = u_j^* + \frac{\varepsilon}{2}]$ and $u_n^\varepsilon = 0$. For each $(i,j) \in P(N) \setminus C(\mu^*)$, $\pi^\varepsilon(i,j) = \pi^*(i,j)$.

Let $\gamma^\varepsilon = (N^*, \pi^\varepsilon)$. Because the change in couples' values does not create any new blocking pairs, it follows that **(a)** $(\mu^*, u^\varepsilon), (\bar{\mu}^*, u^\varepsilon) \in \mathcal{S}(\gamma^\varepsilon)$ and **(b)** $(\mu^*, u^\varepsilon) \in \mathcal{S}(\gamma^\varepsilon)$. Hence, **(a)** $\{\mu^*, \bar{\mu}^*\} \subseteq \mathcal{OM}(\gamma^\varepsilon)$ and **(b)** $\{\mu^*\} \subseteq \mathcal{OM}(\gamma^\varepsilon)$. Note that **(a)** $DA(\gamma^\varepsilon, \mu^*) = \emptyset$ and $DA(\gamma^\varepsilon, \bar{\mu}^*) = \{d, n\}$ and **(b)** $DA(\gamma^\varepsilon, \mu^*) = \emptyset$.

Claim 1: **(a)** $\mathcal{OM}(\gamma^\varepsilon) = \{\mu^*, \bar{\mu}^*\}$ and **(b)** $\mathcal{OM}(\gamma^\varepsilon) = \{\mu^*\}$.

Let $\mu' \in \mathcal{M}(N^*)$ with **(a)** $\mu' \neq \mu^*, \bar{\mu}^*$ or **(b)** $\mu' \neq \mu^*$. Since for **(a)** and **(b)** $(\mu^*, u^*) \in \mathcal{S}(\gamma^*)$, $\mu^* \in \mathcal{OM}(\gamma^*)$. By construction,

$$\textstyle\sum_{(i,j)\in\mu^*} \pi^\varepsilon(i,j) = \sum_{(i,j)\in\mu^*} \pi^*(i,j) + |C(\mu^{DA})|\ \varepsilon \qquad \text{and} \qquad (4)$$

$$\textstyle\sum_{(i,j)\in\mu'} \pi^\varepsilon(i,j) = \sum_{(i,j)\in\mu'} \pi^*(i,j) + |C(\mu^{DA}) \cap C(\mu')|\ \varepsilon. \qquad (5)$$

Observe that

$$|C(\mu^{DA}) \cap C(\mu')| \leq |C(\mu^{DA})|. \qquad (6)$$

By construction of μ^{DA}, **(a)** $|C(\mu^{DA})| = \frac{|N^*|-2}{2}$ or **(b)** $|C(\mu^{DA})| = \frac{|N^*|-1}{2}$. Hence, if $|C(\mu^{DA}) \cap C(\mu')| = |C(\mu^{DA})|$,[6] then $C(\mu') = C(\mu^{DA}) = C(\mu^*)$ and $S(\mu') = S(\mu^*)$. Consequently, $\mu' = \mu^*$, a contradiction. Thus, (6) is strict, which taken together with (4) and (5) yields $\sum_{(i,j)\in\mu^*} \pi^\varepsilon(i,j) > \sum_{(i,j)\in\mu'} \pi^\varepsilon(i,j)$. Hence, **(a)** $\mathcal{OM}(\gamma^\varepsilon) = \{\mu^*, \bar{\mu}^*\}$ and **(b)** $\mathcal{OM}(\gamma^\varepsilon) = \{\mu^*\}$.

By Claim 1, **(a)** μ^* and $\bar{\mu}^*$ are and **(b)** μ^* is the only optimal matching(s) for γ^ε. However, there might be infinitely many payoff vectors associated with these optimal matching(s) (Sotomayor, 2003, Theorem 1).

Claim 2: *Let $(\mu^*, \tilde{u}) \in \mathcal{S}(\gamma^\varepsilon)$. Then, for each $i \in N$, $|\tilde{u}_i - u_i^\varepsilon| \leq \frac{\varepsilon}{2}$.*

For Case **(a)** $S(\mu^*) = \{d, n\}$, $\tilde{u}_d = u_d^\varepsilon = 0$, and $\tilde{u}_n = u_n^\varepsilon = 0$. Hence, $|\tilde{u}_d - u_d^\varepsilon| = |\tilde{u}_n - u_n^\varepsilon| = 0$. For Case **(b)** $S(\mu^*) = \{n\}$ and $\tilde{u}_n = u_n^\varepsilon = 0$. Hence, $|\tilde{u}_n - u_n^\varepsilon| = 0$. Now consider couples' payoffs. Recall that compared to γ^*, in γ^ε the value of each couple is increased by ε. Intuitively, we show that any payoff renegotiations within each pair should be limited to ε, as any attempt of a paired agent to negotiate a payoff in excess of ε would induce her partner to leave the couple in favor of a partnership with agent n. Formally, let $(i,j) \in C(\mu^*)$.

[5]Note that for Case (a) $C(\mu^{DA}) = C(\mu^*) = C(\bar{\mu}^*) \setminus \{(d,n)\}$.

[6]Recall that in Case **(a)** we also assume $\mu' \neq \bar{\mu}^*$.

Case 1: $\tilde{u}_i - u_i^\varepsilon < -\frac{\varepsilon}{2}$

Then $\tilde{u}_i < u_i^\varepsilon - \frac{\varepsilon}{2} = u_i^* = u_i$. By construction, for each $i \in N$, $\pi^*(i,n) = u_i$. Thus, (i,n) forms a blocking pair, and $(\mu^*, \tilde{u}) \notin \mathcal{S}(\gamma^\varepsilon)$.

Case 2: $\tilde{u}_i - u_i^\varepsilon > \frac{\varepsilon}{2}$

Since $\mu^* \in \mathcal{OM}(\gamma^\varepsilon)$, for any $(i,j) \in C(\mu^*)$, $\tilde{u}_i + \tilde{u}_j = u_i^\varepsilon + u_j^\varepsilon$. Then $\tilde{u}_i - u_i^\varepsilon = u_j^\varepsilon - \tilde{u}_j > \frac{\varepsilon}{2}$. Thus, $\tilde{u}_j - u_j^\varepsilon < -\frac{\varepsilon}{2}$ and similarly as in Case 1, it follows that $(\mu^*, \tilde{u}) \notin \mathcal{S}(\gamma^\varepsilon)$.

In Case **(a)** we in addition have $(\mu^*, \tilde{u}) \in \mathcal{S}(\gamma^\varepsilon)$ if and only if $(\bar{\mu}^*, \tilde{u}) \in \mathcal{S}(\gamma^\varepsilon)$.

Step 3: By assumption, $\varphi(\gamma^\varepsilon) \subseteq \mathcal{S}(\gamma^\varepsilon)$. Thus, $\mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon) \neq \emptyset$. For Case **(a)**, assume that $(\bar{\mu}^*, \bar{u}) \in \mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon)$. Then, by indifference with respect to dummy agents, $(\mu^*, \bar{u}) \in \mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon)$. Hence, for both cases, there exists $(\mu^*, \bar{u}) \in \mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon)$. By Claim 2, for each $i \in N$, $|\bar{u}_i - u_i^\varepsilon| \leq \frac{\varepsilon}{2}$. Letting $\varepsilon \to 0$, for each $i \in N$, $|\bar{u}_i - u_i^\varepsilon| \to 0$ and $u_i^\varepsilon \to u_i^*$. By continuity, $(\mu^*, u^*) \in \varphi(\gamma^*)$. Note that $N \subseteq N^*$ such that $\mu^*(N) = N$, $\gamma^*_{|N} = \gamma$, and $(\mu^*_{|N}, u^*_{|N}) = (\mu^{DA}, u)$. By consistency, $(\mu^*_{|N}, u^*_{|N}) \in \varphi(\gamma^*_{|N})$. Thus, $(\mu^{DA}, u) \in \varphi(\gamma)$. Since $(\mu, u) \in \mathcal{F}(\gamma)$, by indifference with respect to dummy agents, $(\mu, u) \in \varphi(\gamma)$. □

Remark 3. ***Indifference with respect to dummy agents and Sasaki's (1995) Theorem 1***
For Theorem 1 to hold, the requirement that the subsolution of the core φ satisfies indifference with respect to dummy agents is necessary.

Proof. For each solvable $\gamma \in \Gamma$, define $\tilde{\varphi}$ as the subsolution of the core with a maximum number of matched (dummy) agents. To illustrate the way $\tilde{\varphi}$ selects from the core, consider the construction of (μ^{DA}, u) starting from a stable outcome (μ, u) in the proof of Theorem 1: if μ has fewer matched agents than μ^{DA}, then $(\mu^{DA}, u) \in \tilde{\varphi}(\gamma)$ and $(\mu, u) \notin \tilde{\varphi}(\gamma)$. For $N = \{1,2\}$, π such that $\pi(1,2) = 0$, and $\gamma = (N, \pi)$ define $\mu = (2,1)$ and $\mu' = (1,2)$. Then, $\tilde{\varphi}(\gamma) = \{(\mu, u)\} \subsetneq \{(\mu, u), (\mu', u)\} = \mathcal{S}(\gamma)$, where $u = (0,0)$. It is easy to see that $\tilde{\varphi}$ is a subsolution of the core that satisfies continuity and consistency, but not indifference with respect to dummy agents. □

Note that for the *two-sided* assignment model Sasaki (1995, Theorem 1) states that "If φ is a subsolution of the core satisfying consistency and continuity, then $\varphi = \mathcal{S}$." However, solution $\tilde{\varphi}$ establishes a counterexample to Sasaki's result where only $\tilde{\varphi} \subsetneq \mathcal{S}$, but not $\varphi = \mathcal{S}$ holds (in problem γ above we can assume that agent 1 is a man and agent 2 is a woman). For *two-sided* and solvable *one-sided* assignment problems where reservation values are not fixed but are allowed to vary (see, for instance, Toda, 2005), our Theorem 1 holds without requiring indifference with respect to dummy agents.[7] Hence, fixing reservation values in our one-sided (or in classic two-sided) assignment model is *not* without loss of generality. ◇

The next theorem extends a result by Sasaki (1995, Theorem 3) for two-sided assignment problems to solvable one-sided assignment problems.

[7]For completeness, we include the proof of this result in Appendix A.

Theorem 2. *On the class of solvable problems, if φ is a subsolution of the core satisfying consistency and Pareto indifference, then φ coincides with the core.*

Proof. Let $\gamma \in \Gamma^N$ be a solvable problem. Then, since φ is a subsolution of the core, $\varphi(\gamma) \subseteq \mathcal{S}(\gamma)$. We prove that $\varphi(\gamma) \supseteq \mathcal{S}(\gamma)$, i.e., we show that $(\mu, u) \in \mathcal{S}(\gamma)$ implies $(\mu, u) \in \varphi(\gamma)$.

Let $(\mu, u) \in \mathcal{S}(\gamma)$. Let $n \in \mathbb{N} \setminus N$ and $N^* = N \cup \{n\}$. Let $\mu^*(n) = n$ and for each $i \in N$, $\mu^*(i) = \mu(i)$. For each $i \in N$, let $\pi^*(i, n) = u_i = u_i^*$ and for each $(i, j) \in P(N)$, $\pi^*(i, j) = \pi(i, j)$. Let $\gamma^* = (N^*, \pi^*)$. Because the entrant n does not create any new blocking pairs, it follows that $(\mu^*, u^*) \in \mathcal{S}(\gamma^*)$ and $\mu^* \in \mathcal{OM}(\gamma^*)$. From the definition of π^* and u^*, observe that every agent $i \in N$ can maintain his utility level u_i^* by matching with the new agent n. Let $(\tilde{\mu}, \tilde{u}) \in \mathcal{S}(\gamma^*)$. It is well-known that if an agent is single at a stable outcome than at any stable outcome she will get exactly her reservation value (e.g., Roth and Sotomayor, 1990, Lemma 8.5). Thus, since $\mu^*(n) = n$, $u_n^* = \tilde{u}_n = 0$. Since $(\tilde{\mu}, \tilde{u}) \in \mathcal{S}(\gamma^*)$, for each $i \in N$, $\tilde{u}_i = \tilde{u}_i + \tilde{u}_n \geq \pi^*(i, n) = u_i^*$. Note that the inequality cannot be strict as it would contradict $\mu^* \in \mathcal{OM}(\gamma^*)$. Thus, we have shown that $\tilde{u} = u^*$, i.e., $(\tilde{\mu}, u^*) \in \mathcal{S}(\gamma^*)$.

By assumption, $\varphi(\gamma^*) \subseteq \mathcal{S}(\gamma^*)$. Thus, $\mathcal{S}(\gamma^*) \cap \varphi(\gamma^*) \neq \emptyset$, i.e., there exists $(\tilde{\mu}, u^*) \in \mathcal{S}(\gamma^*)$ such that $(\tilde{\mu}, u^*) \in \varphi(\gamma^*)$. Since $(\mu^*, u^*) \in \mathcal{F}(\gamma^*)$, by Pareto indifference $(\mu^*, u^*) \in \varphi(\gamma^*)$. Note that $N \subseteq N^*$ such that $\mu^*(N) = N$, $\gamma^*_{|N} = \gamma$, and $(\mu^*_{|N}, u^*_{|N}) = (\mu, u)$. By consistency, $(\mu^*_{|N}, u^*_{|N}) \in \varphi(\gamma^*_{|N})$. Thus, $(\mu, u) \in \varphi(\gamma)$. □

4.2 Impossibilities and limitations

Recall that for two-sided assignment problems, the core is always non-empty. However, for one-sided assignment problems, this need not be the case (see Example 1). The positive results centered around consistency in the previous subsection were obtained when restricting attention to the class of solvable one-sided assignment problems. Next, it is natural to ask if it is possible to obtain consistency and nice "core properties" whenever possible for the entire class of one-sided assignment problems. The following theorem and corollary shows some impossibilities.

Theorem 3. *There exists no solution φ that coincides with the core whenever the core is nonempty and that satisfies consistency.*

Proof. Let φ be a consistent solution such that for each solvable one-sided assignment problem $\gamma \in \Gamma$, $\varphi(\gamma) = \mathcal{S}(\gamma)$. Let $N = \{1, 2, 3, 4, 5\}$, π such that $\pi(1, 2) = \pi(2, 3) = \pi(3, 4) = \pi(4, 5) = \pi(1, 5) = 1$, for all $(i, j) \in P(N) \setminus \{(1, 2), (2, 3), (3, 4), (4, 5), (1, 5)\}$, $\pi(i, j) = 0$, and $\gamma = (N, \pi)$. Note that for any $\mu \in \mathcal{M}(\gamma)$, $2|C(\mu)| + |S(\mu)| = |N| = 5$. One can easily show that $\mathcal{S}(\gamma) = \emptyset$ (the proof is similar to the arguments used in Example 1).

Case 1: Let $(\mu, u) \in \varphi(\gamma)$ such that $[|C(\mu)| = 0$ and $|S(\mu)| = 5]$ or $[|C(\mu)| = 1$ and $|S(\mu)| = 3]$.

Hence, there exists $\{i, j\} \in S(\mu)$ such that $j = i + 1$ (modulo 5). Thus, $\pi(i, j) = 1$. Let $N' = \{i, j\}$ and consider the subproblem $\gamma_{|N'}$. Then, $\mathcal{S}(\gamma_{|N'}) = \{(\mu', u') \mid \mu'(i) = j$ and $u' = (\alpha, 1 - \alpha)$ for $\alpha \in [0, 1]\}$ and $(\mu_{|N'}, u_{|N'}) \notin \mathcal{S}(\gamma_{|N'})$ (because $u_i = u_j = 0$). Since $\mathcal{S}(\gamma_{|N'}) \neq \emptyset$, $\varphi(\gamma_{|N'}) = \mathcal{S}(\gamma_{|N'})$. Hence, in contradiction to φ being consistent, $(\mu_{|N'}, u_{|N'}) \notin \varphi(\gamma_{|N'})$.

Case 2: Let $(\mu, u) \in \varphi(\gamma)$ such that $|C(\mu)| = 2$ and $|S(\mu)| = 1$.

Without loss of generality, assume $C(\mu) = \{(1, 2), (3, 4))\}$ and $S(\mu) = \{5\}$.

Step 1: Let $N' = \{1, 2, 5\}$ and consider the subproblem $\gamma_{|N'}$. Then, $\mathcal{S}(\gamma_{|N'}) = \{(\mu', u'), (\tilde{\mu}, u') \mid \mu' = (2, 1, 5),\ \tilde{\mu} = (5, 2, 1),$ and $u' = (1, 0, 0)\}$. Since $\mathcal{S}(\gamma_{|N'}) \neq \emptyset$, $\varphi(\gamma_{|N'}) = \mathcal{S}(\gamma_{|N'})$.

If $u_1 \neq 1$, then in contradiction to φ being consistent, $(\mu_{|N'}, u_{|N'}) \notin \varphi(\gamma_{|N'})$. Hence, $u_1 = 1$.

Step 2: Let $N'' = \{3, 4, 5\}$ and consider the subproblem $\gamma_{|N''}$. Then, $\mathcal{S}(\gamma_{|N''}) = \{(\mu'', u''), (\bar{\mu}, u'') \mid \mu'' = (4, 3, 5),\ \bar{\mu} = (3, 5, 4)$ and $u'' = (0, 1, 0)\}$. Since $\mathcal{S}(\gamma_{|N''}) \neq \emptyset$, $\varphi(\gamma_{|N''}) = \mathcal{S}(\gamma_{|N''})$.

If $u_4 \neq 1$, then in contradiction to φ being consistent, $(\mu_{|N''}, u_{|N''}) \notin \varphi(\gamma_{|N''})$. Hence, $u_4 = 1$.

Step 3: Let $N^* = \{1, 2, 3, 4\}$ and consider the subproblem $\gamma_{|N^*}$. Note that $\mathcal{S}(\gamma_{|N^*}) \neq \emptyset$ (e.g., $(\mu_{|N^*}, (0, 1, 1, 0)) \in \mathcal{S}(\gamma_{|N^*})$). Hence, $\varphi(\gamma_{|N^*}) = \mathcal{S}(\gamma_{|N^*})$.

By consistency, $(\mu_{|N^*}, u_{|N^*}) \in \varphi(\gamma_{|N^*})$. Recall that $\mu_{|N^*} = (2, 1, 4, 3)$, $\pi(1, 2) + \pi(3, 4) = 2$, and by *Steps 1 and 2*, $u_1 = u_4 = 1$. But then, $u_2 = u_3 = 0$ and $\pi(2, 3) = 1$ imply that $(2, 3)$ is a blocking pair for $(\mu_{|N^*}, u_{|N^*})$; contradicting $\varphi(\gamma_{|N^*}) = \mathcal{S}(\gamma_{|N^*})$. □

Theorem 3 together with Theorems 1 and 2 implies the following two impossibility results.

Corollary 1.
(a) *There exists no solution φ that is subsolution of the core whenever the core is nonempty and that satisfies indifference with respect to dummy agents, continuity, and consistency.*
(b) *There exists no solution φ that is subsolution of the core whenever the core is nonempty and that satisfies Pareto indifference and consistency.*

Sasaki (1995) provides the following two characterizations of the core for two-sided assignment problems (similarly as for Sasaki's, 1995, Theorem 1, we have corrected his Theorem 2 by adding indifference with respect to dummy agents.[8]).

Sasaki's (1995), Theorem 2. *On the class of two-sided assignment problems, the core is the unique solution satisfying individual rationality, couple rationality, Pareto optimality, indifference with respect to dummy agents, continuity, consistency, and weak pairwise-monotonicity.*

Sasaki's (1995) Theorem 4. *On the class of two-sided assignment problems, the core is the unique solution satisfying individual rationality, couple rationality, Pareto optimality, Pareto indifference, consistency, and weak pairwise-monotonicity.*

[8]Recall that solution $\tilde{\varphi} \subsetneq \mathcal{S}$ as defined in Remark 3 satisfies all properties stated in Sasaki's (1995) Theorem 2.

On the class of two-sided assignment problems, Sasaki (1995) proves his characterizations of the core as follows.

Step 1. Sasaki (1995, Propositions 3 and 4) proves that the core satisfies all properties used in both characterizations

Step 2. Sasaki (1995, Proof of Theorems 2 and 4) shows that a solution that satisfies all properties as stated in each of the characterizations is a subsolution of the core.

Step 3. Finally, using Sasaki (1995, Theorems 1 and 3), the characterizations follow.

Note that with Proposition 1 and Theorems 1 and 2 we have established Sasaki's Steps 1 and 3 for solvable one-sided assignment problems. Observe, however, that in these steps weak pairwise-monotonicity has not been actively used. A close look at the proofs related to Sasaki's proof Step 2 reveals that this is where weak pairwise-monotonicity is heavily used. Even though the core is a weakly pairwise-monotonic solution on the class of solvable one-sided assignment problems, the strength of weak pairwise-monotonicity as a property on that class of assignment problems turns out to be quite different from its strength on the class of two-sided assignment problems. The main difference is that in Sasaki's two-sided assignment model any pairwise-monotonic transformation of a characteristic function (see Definition 8) leads to another two-sided assignment problem and therefore to solvability by default. However, for solvable one-sided assignment problems, a small pairwise-monotonic transformation can transform a solvable one-sided assignment problem into a one-sided assignment problem with an empty core. In other words, for problems on the boundary of the class of solvable one-sided assignment problems, we are not able to use weak pairwise-monotonicity in the same way as Sasaki (1995) does because it would lead us outside the class of solvable one-sided assignment problems. The validity of Sasaki's (1995) characterizations of the core on the class of solvable one-sided assignment problems is currently an **open problem**.

We conclude with an illustrative example that shows that for the class of solvable one-sided assignment problems, increasing the value of a couple (i, j) might change the position of a one-sided assignment problem within the class of one-sided assignment problems. In particular, the solvable one-sided assignment problem γ' is one on the boundary of the class of solvable one-sided assignment problems for which a direct adaptation of Sasaki's proof Step 2 would not work (for instance, the pairwise-monotonic transformation from γ' to γ^* transforms a solvable one-sided assignment problem in one with an empty core).

Example 3. ***Changes of the core when the value of a couple changes.***
Let $N = \{1, 2, 3\}$ and $\varepsilon \in (0, 1)$. Consider the following characteristic functions: π such that $\pi(1, 2) = 2$, $\pi(1, 3) = 1$, $\pi(2, 3) = 1 - \varepsilon$, π' such that $\pi'(1, 2) = 2$, $\pi'(1, 3) = 1$, $\pi'(2, 3) = 1$, and π^* such that $\pi^*(1, 2) = 2$, $\pi^*(1, 3) = 1$, $\pi^*(2, 3) = 1 + \varepsilon$. Then for the corresponding one-sided assignment problems $\gamma = (N, \pi)$, $\gamma' = (N, \pi')$, and $\gamma^* = (N, \pi^*)$, we have $\mathcal{S}(\gamma) = \{(\mu, u) \mid \mu = (2, 1, 3) \text{ and } u = (1 + \alpha, 1 - \alpha, 0) \text{ for } \alpha \in [0, \varepsilon]\}$; $\mathcal{S}(\gamma') = \{(\mu, u) \mid \mu = (2, 1, 3) \text{ and } u = (1, 1, 0)\}$; and $\mathcal{S}(\gamma^*) = \emptyset$. ◇

For the solvable one-sided assignment problem γ in Example 3 there is a unique optimal matching and an infinite number of payoff vectors associated with it; so $|\mathcal{S}(\gamma)| = \infty$. Furthermore, these properties are maintained for small changes of the characteristic function. Without introducing the formal definitions here, we state that the one-sided assignment problem γ is in the *interior* of the set of solvable one-sided assignment problems. More generally, one can show that a one-sided assignment problem is in the interior of the set of solvable one-sided assignment problems whenever the core exhibits an infinite set of payoff vectors.

For the solvable one-sided assignment problem γ' in Example 3 there is a unique optimal matching and a unique payoff vector; so $|\mathcal{S}(\gamma')| = 1$. Furthermore, small changes of the characteristic function (e.g., as represented by γ or γ^* for small ε), completely change the core: either from a finite set to an infinite set (if γ' is changed to γ) or from a finite set to an empty set (if γ' is changed to γ^*). More generally, one can show that a one-sided assignment problem is on the *boundary* of the set of solvable one-sided assignment problems whenever the core exhibits a unique payoff vector (and thus, a finite outcome set).

Any one-sided assignment problem that is not solvable, e.g., γ^* in Example 3, is clearly *outside* of the set of solvable one-sided assignment problems.

We conjecture, that if Sasaki's (1995, Theorems 2 and 4) characterizations hold on the class of solvable one-sided assignment problems, the proof techniques for the interior and the boundary of the class differ.

A Appendix

We slightly modify the model as introduced in Section 2 by extending the definition of a characteristic function π to allow for variable reservation values, i.e., for any $N \in \mathcal{N}$ a function $\pi : P(N) \to \mathbb{R}_+$ is a *characteristic function for* N. In particular, we now do not require that for each agent $i \in N$, the reservation value $\pi(i,i)$ is fixed to equal 0.

Theorem 4. *On the class of solvable one-sided problems with nonnegative reservation values that are allowed to vary, if φ is a subsolution of the core satisfying continuity and consistency, then φ coincides with the core.*

Proof. Let $\gamma \in \Gamma^N$ be a solvable problem. Then, since φ is a subsolution of the core, $\varphi(\gamma) \subseteq \mathcal{S}(\gamma)$. We prove that $\varphi(\gamma) \supseteq \mathcal{S}(\gamma)$, i.e., we show that $(\mu, u) \in \mathcal{S}(\gamma)$ implies $(\mu, u) \in \varphi(\gamma)$. The proof strategy is similar to Theorem 1.

Step 1: Let $(\mu, u) \in \mathcal{S}(\gamma)$. Let $n \in \mathbb{N} \setminus N$ and $N^* = N \cup \{n\}$. We define μ^* such that for each $i \in N$, $\mu^*(i) = \mu(i)$ and $\mu^*(n) = n$. For each $i \in N$, let $\pi^*(i,n) = u_i = u_i^*$ and for each $(i,j) \in P(N)$, $\pi^*(i,j) = \pi(i,j)$. Let $\gamma^* = (N^*, \pi^*)$. Because the entrant n does not create any new blocking pairs, it follows that $(\mu^*, u^*) \in \mathcal{S}(\gamma^*)$.

Step 2: For each $(i,j) \in C(\mu^*)$, $\pi^\varepsilon(i,j) = \pi^*(i,j) + \varepsilon$, $u_i^\varepsilon = u_i^* + \frac{\varepsilon}{2}$, and $u_j^\varepsilon = u_j^* + \frac{\varepsilon}{2}$, and for each $i \in S(\mu^*)$, $\pi^\varepsilon(i,j) = u_i^\varepsilon = \frac{\varepsilon}{2}$.[9] For each $(i,j) \in P(N) \setminus C(\mu^*)$, $\pi^\varepsilon(i,j) = \pi^*(i,j)$. Let $\gamma^\varepsilon = (N^*, \pi^\varepsilon)$. Because the change in agents' values does not create any new blocking pairs, it follows that $(\mu^*, u^\varepsilon) \in \mathcal{S}(\gamma^\varepsilon)$.

Claim 1: *μ^* is the unique optimal matching for γ^ε, i.e, $\mathcal{OM}(\gamma^\varepsilon) = \{\mu^*\}$.*

Let $\mu' \in \mathcal{M}(N^*)$ with $\mu' \neq \mu^*$. Since $(\mu^*, u^*) \in \mathcal{S}(\gamma^*)$, $\mu^* \in \mathcal{OM}(\gamma^*)$. By construction,

$$\textstyle\sum_{(i,j)\in\mu^*} \pi^\varepsilon(i,j) = \sum_{(i,j)\in\mu^*} \pi^*(i,j) + |C(\mu^*)|\, \varepsilon + |S(\mu^*)|\, \frac{\varepsilon}{2} \quad \text{and} \tag{7}$$

$$\textstyle\sum_{(i,j)\in\mu'} \pi^\varepsilon(i,j) = \sum_{(i,j)\in\mu'} \pi^*(i,j) + |C(\mu^*)\cap C(\mu')|\, \varepsilon + |S(\mu^*)\cap S(\mu')|\, \frac{\varepsilon}{2}. \tag{8}$$

Observe that

$$|C(\mu^*)\cap C(\mu')| \leq |C(\mu^*)| \text{ and } |S(\mu^*)\cap S(\mu')| \leq |S(\mu^*)|. \tag{9}$$

Since the number of agents $|N^*|$ at both μ' and μ^* is invariant, if $|S(\mu^*)\cap S(\mu')| = |S(\mu^*)|$ and $|C(\mu^*)\cap C(\mu')| = |C(\mu^*)|$, then $S(\mu') = S(\mu^*)$ and $C(\mu') = C(\mu^*)$. Consequently, $\mu' = \mu^*$, a contradiction. Thus, at least one of the inequalities in (9) is strict, which taken together with (7) and (8) yields $\sum_{(i,j)\in\mu^*} \pi^\varepsilon(i,j) > \sum_{(i,j)\in\mu'} \pi^\varepsilon(i,j)$. Hence, $\mathcal{OM}(\gamma^\varepsilon) = \{\mu^*\}$.

By Claim 1, μ^* is the unique optimal matching for γ^ε.

Claim 2: *Let $(\mu^*, \tilde{u}) \in \mathcal{S}(\gamma^\varepsilon)$. Then, for each $i \in N$, $|\tilde{u}_i - u_i^\varepsilon| \leq \frac{\varepsilon}{2}$.*

For each $i \in S(\mu^*)$, $\tilde{u}_i = u_i^\varepsilon = 0$. Hence, $|\tilde{u}_i - u_i^\varepsilon| = 0$. Let $(i,j) \in C(\mu^*)$.

Case 1: $\tilde{u}_i - u_i^\varepsilon < -\frac{\varepsilon}{2}$

Then $\tilde{u}_i < u_i^\varepsilon - \frac{\varepsilon}{2} = u_i^* = u_i$. By construction, for each $i \in N$, $\pi^*(i,n) = u_i$. Thus, (i,n) forms a blocking pair, and $(\mu^*, \tilde{u}) \notin \mathcal{S}(\gamma^\varepsilon)$.

Case 2: $\tilde{u}_i - u_i^\varepsilon > \frac{\varepsilon}{2}$

Since $\mu^* \in \mathcal{OM}(\gamma^\varepsilon)$, for any $(i,j) \in C(\mu^*)$, $\tilde{u}_i + \tilde{u}_j = u_i^\varepsilon + u_j^\varepsilon$. Then $\tilde{u}_i - u_i^\varepsilon = u_j^\varepsilon - \tilde{u}_j > \frac{\varepsilon}{2}$. Thus, $\tilde{u}_j - u_j^\varepsilon < -\frac{\varepsilon}{2}$ and similarly as in Case 1, it follows that $(\mu^*, \tilde{u}) \notin \mathcal{S}(\gamma^\varepsilon)$.

Step 3: By assumption, $\varphi(\gamma^\varepsilon) \subseteq \mathcal{S}(\gamma^\varepsilon)$. Thus, $\mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon) \neq \emptyset$, i.e., there exists $(\mu^*, \bar{u}) \in \mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon)$. By Claim 2, for each $i \in N$, $|\bar{u}_i - u_i^\varepsilon| \leq \frac{\varepsilon}{2}$. Letting $\varepsilon \to 0$, for each $i \in N$, $|\bar{u}_i - u_i^\varepsilon| \to 0$ and $u_i^\varepsilon \to u_i^*$. By continuity, $(\mu^*, u^*) \in \varphi(\gamma^*)$. Note that $N \subseteq N^*$ such that $\mu^*(N) = N$, $\gamma^*_{|N} = \gamma$, and $(\mu^*_{|N}, u^*_{|N}) = (\mu, u)$. By consistency, $(\mu^*_{|N}, u^*_{|N}) \in \varphi(\gamma^*_{|N})$. Hence, $(\mu, u) \in \varphi(\gamma)$. □

[9] Note that we increase the reservation values of single agents at μ^* – this is only possible if we change the assignment model to allow reservation values to vary.

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