Empirical Evidence of the Leverage Effect in a Stochastic Volatility Model: a Realized Volatility Approach

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Abstract

Increasing attention has been focused on the analysis of the realized volatility, which can be treated as a proxy for the true volatility. In this paper, we study the potential use of the realized volatility as a proxy in a stochastic volatility model estimation. We estimate the leveraged stochastic volatility model using the realized volatility computed from five popular methods across six sampling-frequency transaction data (from 1-min to 60-min). Availability of the realized volatility allows us to estimate the model parameters via the MLE and thus avoids computational challenge in the high dimensional integration. Six stock indices are considered in the empirical investigation. We discover some consistent findings and interesting patterns from the empirical results. In general, the significant leverage effect is consistently detected at each sampling frequency. The volatility persistence becomes weaker at the lower sampling frequency. We also find that the consistent-scaling and "optimal"-weighted realized volatility method proposed by Hansen and Lunde (2005) provide relatively better performances compared to other methods considered.

Keywords Realized Volatility; Stochastic Volatility Model; Leverage Effect; High Frequency Data; MLE.

JEL Classification C01, C51.

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1 Introduction

Rapid development in computer technology has made the financial transaction data visible at the highest granularity. High frequency and ultra high frequency data can be recorded transaction by transaction or at various short time intervals. Due to the availability of the high frequency transaction data, the daily return volatility, which is normally treated as a latent variable in various parametric models, now can be constructed using intra-day data. In particular, Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2001) propose using the sum of the squared intra-daily returns at different sampling frequencies as a proxy measure for the corresponding daily volatility. This measure provides a consistent estimator of the latent volatility under an ideal market condition. Barndorff-Nielsen and Shephard (2002), Andersen, Bollerslev, Diebold and Labys (2003), Meddahi (2002), among others, have established some theoretical foundations for realized volatility construction via high frequency data. Recently, increasing attention has been focused on the analysis of the realized volatility measures, see Zhang, Mykland and Ait-Sahalia (2005), van Dijk and Martens (2007), Andersen, Bollerslev and Diebold (2007), Maheu and McCurdy (2009) and etc. A recent survey paper by McAleer and Medeiros (2008) provides an excellent review of the rapidly expanding literature on realized volatility.

Given the theoretical foundations and empirical properties in the aforementioned papers, we can argue that realized volatility provides a venue in which the volatility is "observed" rather than being modeled latently. In this paper, we investigate this possibility by incorporating the high frequency realized volatility measures into a stochastic volatility framework. Based on this approach, we analyze the correlations between the return process and volatility process at various sampling frequency volatility measures.

Stochastic volatility (SV) model, proposed by Taylor (1986), is an alternative class of nonlinear financial models to the ARCH/GARCH specification which can capture the time-varying properties of the conditional volatility. Essentially, the SV model allows the conditional volatility to follow a certain latent stochastic process. Two innovations give the time-varying characteristics in the SV specification, while only one error process is specified in the ARCH/GARCH family. Nelson (1990) and Duan (1997) show that the ARCH/GARCH and SV models have strong similarities asymptotically. However, in the finite sample environment, the additional uncertainty in the volatility process from the SV model introduces more flexibilities. Consequently a SV model often provides a better in-sample fit than the ARCH/GARCH model, see Danielsson (1994) and Kim, Shephard and Chib (1998). The SV model has an intuitive appeal and a realistic modeling structure. However its empirical applications have been limited due to the intractability of its likelihood function. The problem essentially comes from the latent volatility dynamics: the latent conditional volatility at time t has to be integrated out in order to determine the likelihood function. As a result, the standard likelihood function for a SV model involves an integral with a dimension of sample sizes. As is well known, it is very difficult, if not impossible, to make any statistical inference directly under the SV model with latent volatility. By taking advantage of the availability of the high frequency data, we use the realized volatility as a proxy for the volatility in the SV model. This allows us to estimate the model parameters using the full information on the exact likelihood.

In this paper we investigate the feasibility and effectiveness of this approach based on both simulated and empirical data. The presentation of this paper is organized as follows. In Section 2 we provide details of the model specification and present its statistical properties. In Section 3 we illustrate the feasibility of the method using Monte Carlo simulations. In Section 4 we present the empirical data and describe various ways for the computation of the realized volatility measurements used in this paper. In addition we also discuss the applications and empirical results. Concluding remarks are provided in Section 5.

2 The SV Model and ML Estimation

There are two stochastic processes specified in a SV model. In this paper, we assume that

$$\begin{aligned} x_t &= \exp(h_t/2)e_t \\ h_{t+1} &= \lambda + \alpha h_t + v_{t+1} \end{aligned} \tag{1}$$

where x_t is the return time series, which is defined as the logarithmic closing price differences. The process h_t is the latent log-volatility. As in most of the literature, the latent volatility is assumed to follow a stationary AR(1) process, which requires the persistent parameter to be bounded by 1 in absolute value, i.e, $|\alpha| < 1$. In the standard SV model, e_t and v_{t+1} are assumed to be normally and independently distributed. However, under this assumption, the "leverage effect"[1] cannot be explained from the model. Consequently, in this paper, we follow the leveraged SV specification from Harvey and Shephard (1996) and Yu (2005) and assume the following bivariate structure:

$$\begin{pmatrix} e_t \\ v_{t+1} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho\sigma_v \\ \rho\sigma_v & \sigma_v^2 \end{pmatrix} \right) \tag{2}$$

where $\mathcal{N}(\mu, \Sigma)$ denotes a multivariate normal distribution with mean μ and covariance matrix Σ. Here the correlation coefficient, ρ, can be explained as the leverage effect if it is found to be negative.

To further examine the statistical properties of the model, we also derive the following general closed form cross moment conditions.

Proposition 1. For the SV model specified by (1) with the assumption (2), for m, n,

[1] The notion of the leverage effect for the financial returns was originally introduced by Black (1976) to capture the negative correlation between the innovation in the asset return process and the volatility process: negative returns increase financial leverage which extends the risk of the company and therefore its volatility.

$k \geq 0$, the closed form cross moments expression for x_t and x_{t+k} is given as follows,

$$\begin{aligned} E(x_t^m x_{t+k}^n) &= \exp\left(\frac{(m+n\alpha^k)^2\lambda}{2(1-\alpha)} + \frac{\sigma_v^2(m+n\alpha^k)^2}{8(1-\alpha^2)}\right) \\ &\times \exp\left(\frac{n\lambda}{2}\sum_{j=1}^{k}\alpha^{j-1}\right) + \exp\left(\frac{n^2\sigma_v^2}{8}\sum_{j=2}^{k}\alpha^{2k-2j}\right) \\ &\times \left.\frac{\partial M^{(n)}(r_1,r_2)}{\partial r_1^{(n)}}\right|_{r_1=0, r_2=0} \\ &\times \left.\frac{\partial M^{(m)}(r_1,r_2)}{\partial r_1^{(m)}}\right|_{r_1=0, r_2=\frac{n}{2}\alpha^{k-1}} \end{aligned} \quad (3)$$

where $M(r_1, r_2)$ is defined as the joint Moment Generating Function (MGF) of e_t and v_{t+1}. *Proof*: See the Appendix.

With the general cross moment formula in (3), it is straightforward to derive any order of the Autocorrelation Function (ACF) of the returns under the leveraged SV structure. For example, at the lag of 1 (when $k = 1$), the corresponding ACF of the squared returns can be expressed as the following closed form,

$$ACF_1 = \frac{(1+\rho^2\sigma_v^2)\exp(\frac{\alpha\sigma_v^2}{1-\alpha^2}) - 1}{3\exp(\frac{\sigma_v^2}{1-\alpha^2}) - 1} \quad (4)$$

To investigate some simple relationships among the parameters, based on (4), we fix the ACF value at its "true" value (a constant). The parameter values of (λ, α, ρ, σ_v) are set to be close to the typical values based on our empirical studies, e.g., (-0.2, 0.8, -0.1, 0.5).[2] We note that the ACF is not a function of λ. In other words, the value of λ does not affect this analysis. Firstly, with other parameters unchanged, one can see that the "leverage effect" parameter, ρ, is a decreasing function of the persistent parameter, α, over the domain of $\rho \in (-1, 0)$. In other words, when the detected leverage effect becomes larger, we expect that the volatility becomes more persistent in order to keep the ACF value constant. Another interesting aspect is that, with the same argument, we find that α is a decreasing function of σ_v^2 with other parameters being fixed. This is intuitive because if there are large variations in the volatilities, the volatility process should be less persistent. We will also illustrate these movements in the subsequent Monte Carlo and empirical sections.

In the above leveraged SV setting, there are four parameters, $\theta = (\lambda, \alpha, \sigma_v, \rho)$, to be estimated. In this paper, the return volatility is treated as observable in the model, which allows us to estimate the model parameters using the MLE based on the exact likelihood function. In particular, the exact likelihood function under the SV model (1) with the assumption (2) is as follows,

$$\begin{aligned} L(\theta; x, h) &= L_T(x_T, h_{T+1}|x_{T-1}, h_T, \theta) \times L_{T-1}(x_{T-1}, h_T|x_{T-2}, h_{T-1}, \theta) \times ... \\ &\times L_1(x_1, h_2|h_1, \theta) \times L_0(h_1|\theta) \end{aligned} \quad (5)$$

[2] We also experiment other sets of parameter values. Similar patterns are found.

The exact likelihood in (5) is the product of the serial conditional densities and the marginal density of the initial log volatility. From the AR process in (1), it is easy to get the asymptotic density for h_1 as the following,

$$L_0 = \sqrt{\frac{1-\alpha^2}{2\pi\sigma_v^2}} \exp\left(-\frac{1-\alpha^2}{2\sigma_v^2}(h_1 - \frac{\lambda}{1-\alpha})^2\right) \tag{6}$$

The conditional density can be obtained from the bivariate distributional assumption (2) and has the following form,

$$L_t(x_t, h_{t+1}|x_{t-1}, h_t, \theta) = \frac{1}{2\pi\sigma_v\sigma_{x_t}\sqrt{1-\rho^2}} e^{\left(-\frac{1}{2(1-\rho^2)}\left[\frac{x_t^2}{\sigma_{x_t}^2} + \frac{(h_{t+1}-\lambda-\alpha h_t)^2}{\sigma_v^2} - 2\rho\frac{x_t(h_{t+1}-\lambda-\alpha h_t)}{\sigma_{x_t}\sigma_v}\right]\right)} \tag{7}$$

where $\sigma_{x_t}^2 = \exp(h_t)$. The objective likelihood function (5) contains full information of both the return and volatility. We use the trust region method proposed in Coleman and Li (1996) for the bound constrained minimization to compute a maximizer for the likelihood function, as there are some bound constraints typically imposed on the model parameters.[3] The gradient of the maximum likelihood function is computed by automatic differentiation and the Hessian is approximated by finite difference. The global and superlinear convergence properties of the method ensures that a maximizer is computed with a high accuracy.

The standard error matrix of the estimates defined by the exact Hessian, H, is computed by automatic differentiation. The Hessian matrix is defined as the expectation of the second derivative of the log-likelihood function. Thus standard errors are given by the covariance matrix below:

$$\hat{\Theta} = -H^{-1}(\hat{\theta}) \tag{8}$$

3 Estimation Efficiency: Simulation Studies

In order to assess ML estimation efficiency for a SV model assuming volatility observable, several Monte Carlo simulation experiments are carried out under the assumed SV model in a controlled environment. Specifically, we conduct the following three groups of synthetic experiments. In Experiment # 1 we illustrates the sensitivity of estimation accuracy to the variation of model parameters and the sample sizes. Experiment # 2 analyzes consequence of the mis-specification on the leverage effect. In Experiment # 3 we demonstrate the effect of the microstructure in the realized volatility.

In the subsequent simulation studies, we assume the SV model specified in (1) with a specific set of (true) model parameters. We then estimate these parameters based on the MLE from a finite set of simulated observations. All computations are performed in MATLAB and each computation result is based on 1000 simulation replications. In particular, two standard measures, the bias (BIAS) and mean squared error (MSE) are computed for performance evaluations based on these 1000 replications.

[3]For example, the persistent parameter, α, and the leverage effect parameter, ρ, should be both bounded by 1 in the absolute terms.

We examine the performance of the MLE estimator under different parameter settings and different sample sizes. These settings are displayed in Table 1. Case 1a is set to be the benchmark case,[4], which is used for constructing comparisons with other cases. Case 1b only differs in the sample size. Case 1c has smaller persistent parameter (α) value. Case 1d investigates a larger leverage effect and Case 1e increases the variance of the h_t process.

Table 1. Parameter Settings for Monte Carlo Simulation Experiments

Case	λ	α	σ_v	ρ	n
1a.(*)	-0.2	0.9	0.5	-0.3	1000
1b.	-0.2	0.9	0.5	-0.3	200
1c.	-0.2	0.45	0.5	-0.3	1000
1d.	-0.2	0.9	0.5	-0.9	1000
1e.	-0.2	0.9	1.5	-0.3	1000

For Experiment # 1, we report the accuracy of the ML estimation under different parameter settings in Table 2. For all five cases, we find that the MLE estimates are close to the true values and have small MSEs for all the parameters. Specifically, for Case 1b, the sample size is reduced from 1000 to 200. As expected, due to a smaller sample size, the BIAS and MSE in Case 1b are both larger than those from the benchmark (Case 1a). Case 1c represents a situation where the volatility process is less persistent. We decrease the α value from 0.9 to 0.45. By comparisons with the results from the benchmark case, we find that the MSEs for λ and σ_v become smaller. Both the BIAS and MSE for α increase in the less-persistent case. Case 1d examines a scenario where the leverage effect is more dominant than the benchmark case, in which ρ is changed from -0.1 to -0.9. Interestingly, the MSEs all decrease except for the estimates of λ. Lastly, in Case 1e, we increase the variance of the residual from the latent volatility process, i.e., σ_v is increased to 1.5. Both the BIAS and MSE measures become uniformly larger than those from the benchmark case (except ρ). Overall, the magnitudes of the differences are quantitatively quite small for all the cases examined. This suggests that our proposed method is able to yield estimated parameter values which are consistently close to their true values.

The second group of the experiments are set up to investigate the implication of the mis-specification in the leverage effect. In particular, we first generate the observation data x and h following the SV process (1) assuming that there is no leverage effect (i.e., $\rho = 0$).[5] Then we estimate the parameters via MLE under both the leveraged SV model (with $\rho \in (-1, 1)$) and the model without leverage effect(with $\rho = 0$). The results are reported in Table 3. Under the assumed parameter settings, the biases and MSEs are found to be slightly smaller under the correct model assumption than under the mis-specified leveraged SV model. However, the magnitudes of both BIAS and MSE are very small across all the examined parameter settings. In addition, as noted from the table, the estimates of ρ are

[4]The true values of the parameters in the benchmark case are chosen to be close to the empirical estimates from the empirical application.

[5]Other parameter values in Experiment# 2 are set to be the same as those in Experiment # 1 (see Table 1).

Table 2. Monte Carlo Results for Experiment # 1.

	1a	1b	1c	1d	1e
BIAS:λ	-8.48e-003	-4.47e-002	3.07e-003	-9.64e-003	-2.41e-002
MSE:λ	4.71e-004	6.90e-003	1.53e-004	8.83e-004	5.20e-003
BIAS:α	-5.72e-003	-1.94e-002	-8.30e-003	-3.37e-003	-1.01e-002
MSE:α	2.01e-004	1.14e-003	4.24e-004	1.05e-004	3.67e-004
BIAS:ρ	1.25e-002	-9.39e-003	-3.58e-003	1.26e-002	1.01e-002
MSE:ρ	7.24e-004	5.36e-003	7.46e-004	8.49e-004	6.17e-004
BIAS:σ_v	-3.69e-003	-5.49e-003	-3.56e-003	-1.73e-003	-1.79e-002
MSE:σ_v	2.56e-004	3.16e-004	9.06e-005	1.80e-004	1.19e-003

Table 3. Monte Carlo Results for Experiment # 2.

	1a: $n, \lambda^*, \alpha^*, \sigma_v^*$		1b: $0.2n, \lambda^*, \alpha^*, \sigma_v^*$		1c: $n, \lambda^*, 0.5\alpha^*, \sigma_v^*$		1e: $n, \lambda^*, \alpha^*, 3\sigma_v^*$	
	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$
BIAS:λ	-6.74e-3	-6.73e-3	-4.24e-2	-4.26e-2	-1.82e-3	-1.77e-3	-1.08e-2	-1.08e-2
MSE:λ	1.13e-3	1.13e-3	8.42e-3	8.41e-003	3.64e-4	3.63e-4	3.49e-3	3.49e-3
BIAS:α	-3.53e-3	-3.53e-3	-2.06e-2	-2.06e-2	-3.19e-3	-3.14e-3	-3.82e-3	-3.83e-3
MSE:α	2.10e-4	2.09e-4	1.64e-3	1.64e-3	7.76e-4	7.72e-4	2.08e-4	2.08e-4
BIAS:ρ	-9.78e-4	–	-4.18e-3	–	1.36e-3	–	4.90e-4	–
MSE:ρ	9.94e-4	–	5.41e-3	–	9.65e-4	–	1.06e-3	–
BIAS:σ_v	-6.93e-4	-6.96e-4	-2.84e-3	-2.88e-3	-3.39e-4	-3.40e-4	-2.42e-3	-2.42e-3
MSE:σ_v	1.40e-4	1.40e-4	6.43e-4	6.43e-4	1.19e-4	1.19e-4	1.10e-3	1.10e-3

consistently around its true value of zero. These results indicate that even if there is no leverage effect in the true data generating process (DGP), the proposed estimation method under the leveraged SV model does not significantly overestimate or underestimate other parameters in the model.

Lastly, it is well known that the high frequency transaction data contains microstructure noise, which stems from market friction such as discreteness of price change, bid-ask bounce, and asymmetric information. Due to the presence of the market friction, the realized volatility measure can be a noisy estimator of the true volatility. In addition, there are several methods proposed to calculate the RV estimates in the literature. Different methods introduce different errors into the volatility process. To investigate these implications, we add a white noise into the generating process of the volatility and examine how it affects the estimation results. The data generating process (DGP) used in Experiment # 3 follows the benchmark parameter setting Case 1a in Table 1. Three levels of noises (10% , 25% and 50% of a standard normal) are then added into the volatility process; we compare the estimation accuracy with the case of no noise. The estimation results are provided in Table

Table 4. Monte Carlo Results for Experiment # 3.

	no noise	10% randn	25% randn	50% randn
BIAS:λ	-6.53e-003	-2.40e-002	-9.15e-002	-3.07e-001
MSE:λ	1.11e-003	1.80e-003	1.04e-002	9.97e-002
BIAS:α	-3.44e-003	-1.12e-002	-4.57e-002	-1.52e-001
MSE:α	2.07e-004	3.61e-004	2.49e-003	2.41e-002
BIAS:ρ	-1.05e-003	4.47e-003	1.94e-002	4.46e-002
MSE:ρ	9.69e-004	9.80e-004	1.37e-003	2.93e-003
BIAS:σ_v	-6.40e-004	1.67e-002	1.00e-001	3.16e-001
MSE:σ_v	1.39e-004	4.19e-004	1.02e-002	1.00e-001

4. As expected, both the biases and MSEs for all the parameter estimates become larger as the volatility process becomes more noisy. Negative biases for the persistent parameter (α) estimates are found at each level of the noises. In other words, as more noise is incorporated into the SV model, the volatility process becomes less persistent. Interestingly, we find that the bias and MSE for leverage effect parameter (ρ) are relatively small even in the 50% randn case. This suggests that, at least in the given parameter settings, the estimation of ρ is relatively robust to the presence of noises. Even when estimates of other parameters deteriorate as the noise level increases, our computational method is still able to produce relatively accurate estimates for the leverage effect parameter. This is consistent with our subsequent estimation results from the high frequency data in the equity market indices in Section 4.

4 Estimating SV Models for Equity Market Indices

Realized volatility can potentially improve our understanding of market volatility. We investigate this possibility here using intra-day transaction prices from six major equity market indices including Nasdaq Composite (COMPQ), Dow Jones Industrial Average (DJIA), Standard and Poor's 500 (SPX), France CAC 40 Index (PX1), Russell 2000 (RUA) and STOXX 600 index for Euro zone (SXXP). Roughly the data covers the period from 2003 to 2008; there are slight variations in sample sizes depending on the availability of the data across indices. The intra-daily transaction prices are available at various frequencies including 1-min, 5-min, 10-min, 15-min, 20-min, 30-min and hourly.

Realized volatility, defined as the sum of squared intra-day returns over a certain interval, has been proposed by Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2001). Under an ideal market condition, realized volatility has shown to provide a consistent estimator of the latent volatility. Here we consider five methods (RV1–RV5) to calculate the realized volatility. These methods are described as follows.

Define C_t as the closing price on the trading day t. The daily return x_t is calculated as the logarithmic closing price difference in the usual way, i.e.,

$$x_t = 100(\log C_t - \log C_{t-1}) \qquad t = 1, 2, ..., T \tag{9}$$

Let $p_{d,t}$ be the logarithmic price at the (intraday) sampling time d on day t. The continuously compounded returns with D observations on day t is then defined as, $r_{d,t} = 100(p_{d,t} - p_{d-1,t})$, where $d = 1, 2, ..., D$ and $t = 1, 2, ..., T$. When $D = 1$, the first subscript is ignored and r_t denotes the return series on a given day.

The first method RV1 simply computes the realized volatility by summing up the squared intra-day returns during the market open period, i.e.,

$$(\mathrm{RV1})_t = \sum_{d=1}^{D} r_{d,t}^2 \tag{10}$$

Unfortunately, this simple calculation method may encounter difficulties due to two potential issues associated with the measurement of the daily realized volatility from the high frequency data. One issue arises from the presence of the market microstructure noise in the transaction prices and the other stems from the presence of nontrading hours. Different modifications to RV1 have been proposed in the literature. Marten (2002) proposes a scale estimator to accommodate the "open-to-close" and "close-to-open" effects. In this paper, we refers to this estimator as RV2, which has the following form,

$$(\mathrm{RV2})_t = \xi_1 \cdot (\mathrm{RV1})_t \tag{11}$$

where $\xi_1 = (\sigma_{oc}^2 + \sigma_{co}^2)/\sigma_{oc}^2$. σ_{oc}^2 is the "open-to-close" variance, which can be computed as $\sigma_{oc}^2 = \sum_{t=1}^{T} 10000(p_{D,t} - p_{0,t})^2/T$. Similarly, σ_{co}^2 is the "close-to-open" variance, which can be computed as $\sigma_{co}^2 = \sum_{t=1}^{T} 10000(p_{0,t} - p_{D,t-1})^2/T$.

Alternatively, Hansen and Lunde (2005) propose a consistent scaling estimator for RV. They introduce a scale constant to the $(\mathrm{RV1})_t$ in order to producing a correct expected value. we denote this scaling estimator as (RV3), which is given below,

$$(\mathrm{RV3})_t = \xi_2 \cdot (\mathrm{RV1})_t \tag{12}$$

where $\xi_2 = \sum_{t=1}^{T}(r_t - \bar{r})^2 / \sum_{t=1}^{T}(\mathrm{RV1})_t$ and $\bar{r}$ is the sample mean of r_t.

Instead of scaling $RV1$, Hansen and Lunde (2005) propose an irregular RV estimator by incorporating the "overnight" information into the measurement. In this paper, we define it as RV4, which is given by,

$$(\mathrm{RV4})_t = r_{1,t}^2 + (\mathrm{RV1})_t \tag{13}$$

Finally, following Hansen and Lunde (2005), we assign different weights for the overnight component and the intra-day component. We use a linear combination of the overnight squared return, $r_{1,t}^2$, and $(\mathrm{RV1})_t$ to form a mean-square-error "optimal" realized volatility measure for the daily volatility. We denote this estimator as RV5, which is given by,

$$(\mathrm{RV5})_t = \omega_1 \cdot r_{1,t}^2 + \omega_2 \cdot (\mathrm{RV1})_t \tag{14}$$

where $\omega_1 = (1 - \phi)\mu_0/\mu_1$ and $\omega_2 = \phi\mu_0/\mu_2$ with $\phi = (\mu_2^2\eta_1^2 - \mu_1\mu_2\eta_{12})/(\mu_2^2\eta_1^2 + \mu_1^2\eta_2^2 - 2\mu_1\mu_2\eta_{12})$. $\eta_1^2 = var(r_{1,t}^2)$, $\eta_2^2 = \mathrm{var}((\mathrm{RV1})_t)$ and η_{12} is the covariance of $r_{1,t}^2$ and $(\mathrm{RV1})_t$.

Parameters μ_0, μ_1 and μ_2 are computed as the mean of $(r_{1,t}^2 + (\text{RV1})_t)$, $r_{1,t}$ and $(\text{RV1})_t$ respectively.

We report the summary statistics in Table 5 for the sample data as well as the RVs constructed using different methods across various frequencies. For brevity, we only present the descriptives for SPX.[6] The SPX data covers the period from 03/03/2003 to 24/09/2008 with 1401 daily observations. Standard statistics, such as mean (Mean), variance (Var), skewness (Skn), kurtosis (Kurt), minimum (Min), maximum (Max) and Jarque-Bera statistic (J-B Stats), are presented in Table 5. In general, the results are consistent with the stylized facts in the literature. For example, the daily returns are slightly skewed to the left (negative skewness) and exhibit heavy tail relative to the normal distribution. All the RVs exhibit significantly large J-B statistics indicating strong non-normal behavior. The logarithms of all the RVs are nearly Gaussian with the kurtosis values close to 3 and much smaller J-B statistics compared to the raw RV data. We also note the following interesting pattern from Table 5. As the sample frequency decreases (from 1-min to 60-min), the distribution of the logarithms of the RVs converge to a Gaussian distribution with the kurtosis values closer to 3. The J-B statistics also confirm this pattern with the evidence that at the 60-min level log-RVs, the normality cannot be rejected at a 5% significant level.

To further describe the data, we also provide several plots in Figure 1 and 2. Similarly, only plots for SPX and DJIA are provided here for illustration. Each figure consists of 4 subplots. Subplot (a) graphs the daily returns over the entire sample period. Subplot (b) provides the corresponding graphs for RV5 across different frequencies (from 1-min to 60-min). From these plots, we can observe the volatility clustering characteristics. In subplot (c), we graph the empirical distributions of the log-RV5 across different frequencies. We observe that these distributions are consistency with the stylized fact that, as the sampling frequency decreases, the distribution of the log-RVs converges to a Gaussian. Finally, following Andersen, Bollerslev, Diebold and Labys (2000), we produce a volatility signature plot for RV5, which graphs the average of the realized volatility against the sampling frequency. Interestingly, the RV5 fluctuates around a constant across the sampling frequency. This indicates that the microstructure effects are not pronounced in RV5 for both SPX and DJIA.

We now estimate the SV model using the empirical data, incorporating the RV measures (RV1 – RV5) constructed from different frequencies following the estimation strategy described in section 2. For each RV1–5, at each sampling frequency, we estimate the parameters under SV models both with the leverage effect and without leverage effect. We compare the estimation results results across six sampling frequencies (i.e., 1-min, 5-min, 10-min, 15-min, 30-min and 60-min), five RV constructions (i.e., RV1 to RV5), as well as with leverage effect models ($\rho \in (-1, 1)$) and without leverage-effect SV models ($\rho = 0$). We also make comparisons across all six stock indices. The estimation results are reported in the Table 6–11.

There are several interesting findings from these tables. Firstly, we observe that the

[6] The summary statistics for other indices are similar and are not reported in this paper. However, the descriptives for the entire sample data are available upon request.

estimated variance of the residual ($\hat{\sigma_v}^2$) from the volatility process increases as the sampling frequency decreases (from 1-min to 60-min), which is similar to the results established in Takahashi, Omori and Watanabe (2009). As expected, a decreasing persistency in the volatility process (with decreasing $\hat{\alpha}$ values) is observed as the sampling interval increases. This is also consistent with our Monte Carlo simulation results (Experiment #3) that if the volatility process has been affected by a larger variance of noise, negative bias is produced for the estimates of the persistence parameter. We note that that, as the sampling interval increases, the microstructure noise effect becomes less significant. However, the number of return samples available to calculate the daily realized volatility also decreases.

For all six stock indices, a significant leverage effect is detected at every sampling frequency level no matter which RV construction method is used. This finding strongly supports the negative relationship between the volatility and price/return. Interestingly, we also observe that the estimates of leverage parameter (ρ) do not vary much across different sampling frequencies (from 1-min to 60-min) over the five construction methods (RV1 to RV5). This result may indicate that the "true" underlying correlations between the return and future volatility, which can be stably and consistently estimated via the proposed methodology, are less distorted than other parameters in the model. This finding is consistent with our Monte Carlo results in Experiment #3, where as the volatility process becomes more noisy (even with a 50% of the standard normal noise distortion), the proposed method can still estimate the ρ parameter accurately with relatively small bias and MSE.

Finally, we discus some characteristics of the estimated models. We estimate the SV models both with and without the leverage parameter ρ. We find that the leveraged SV model has consistently smaller Bayesian Information Criteria (BIC) values. For most of the cases, the differences between the two BIC values are significantly greater than 10. This, according to Raftery (1995), indicates a strong model selection evidence in favor of the leveraged SV model. Another interesting finding across Table 6 – 11 is that the SV models with the leverage effect using RV3 and RV5 generally have relatively smaller BIC values. This consistently supports the analysis in Hansen and Lunde (2005), in which they demonstrate RV3 provides a bias-corrected RV measure and RV5 "optimally" combines the overnight squared returns and the conventional RV during the active period. Hence they can potentially produce better measures for the latent volatility.

5 Conclusion

Taking advantages of the availability of the high frequency (intra-daily) transaction data, we incorporate the RV measures into the well-known stochastic volatility model. Since the latent volatility becomes "visible", the MLE is feasible for the parameter estimation, avoiding the high-dimensional integration computation. Monte Carlo simulation studies are implemented to investigate the MLE performance under different parameter settings. We also examine the mis-specification and microstructure noise effect. In addition we apply our approach to study six stock indices based on the estimated SV models. We construct the RV measures across six different sampling frequencies using five different popular RV construction methods in the literature. From the empirical results, we observe some consistent findings and interesting patterns. In general, the leveraged SV model is preferred to the SV model without the leverage effect in terms of the BIC measures. The leverage effect is detected consistently and significantly at each sampling frequency level among all the RV construction methods. The volatility persistence becomes less prominent at the lower sampling frequency due to the noise. In addition, we find some empirical evidence to support RV measures proposed by Hansen and Lunde (2005).

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Appendix

Proof of Proposition 1

Since h_t follows an AR(1) process, we can write:

$$h_{t+k} = \alpha^k h_t + \lambda \sum_{j=1}^{k} \alpha^{j-1} + \sum_{j=1}^{k} \alpha^{k-j} v_{t+j}$$

Then,

$$\begin{aligned}
E(x_t^m x_{t+k}^n) &= E\,[\exp(\frac{mh_t}{2})e_t^m \exp(\frac{nh_{t+k}}{2})e_{t+k}^n] \\
&= E\,[\exp(\frac{m}{2}h_t + \frac{n}{2}\alpha^k h_t + \frac{n}{2}\lambda\sum_{j=1}^{k}\alpha^{j-1} + \frac{n}{2}\sum_{j=1}^{k}\alpha^{k-j}v_{t+j}) \times e_t^m \times e_{t+k}^n] \\
&= \frac{n}{2}\lambda\sum_{j=1}^{k}\alpha^{j-1} E\ \exp\left(\frac{m+n\alpha^k}{2}h_t\right) \times E\ \exp(\frac{n}{2}\sum_{j=2}^{k}\alpha^{k-j}v_{t+j}) \\
&\times\ E\,[\exp(\frac{n\alpha^{k-1}}{2}v_{t+1})e_t^m] \times E\ (e_{t+k}^n)
\end{aligned}$$

We then work out the above expectations term by term,

$$E\left(\exp(\frac{m+n\alpha^k}{2}h_t)\right) = \exp\left(\frac{(m+n\alpha^k)\lambda}{2(1-\alpha)} + \frac{\sigma_v^2(m+n\alpha^k)^2}{8(1-\alpha^2)}\right)$$

$$E\left(\exp(\frac{n}{2}\sum_{j=2}^{k}\alpha^{k-j}v_{t+j})\right) = \exp(\frac{n^2\sigma_v^2}{8}\sum_{j=2}^{k}\alpha^{2k-2j})$$

To work out the last two terms, we need to use the joint moment generating function of e_t and v_{t+1}.

By definition of the joint moment generating function (MGF) and under (2), we have,

$$\begin{aligned}
M_{e,v}(r_1, r_2) &= E\exp(r_1 e_t + r_2 v_{t+1}) \\
&= \exp(0.5r_1^2 + 0.5r_2^2\sigma_v^2 + \rho r_1 r_2 \sigma_v)
\end{aligned}$$

It is straightforward to get,

$$E\left(\exp(\frac{n\alpha^{k-1}}{2}v_{t+1})e_t^m\right) = \frac{\partial M^{(m)}(r_1, r_2)}{\partial r_1^{(m)}}\Bigg|_{r_1=0, r_2=\frac{n\alpha^{k-1}}{2}}$$

and,

$$E\left(\exp(e_{t+k}^n)\right) = \frac{\partial M^{(n)}(r_1, 0)}{\partial r_1^{(n)}}\Bigg|_{r_1=0}$$

Combining all the above expressions, we have the general cross moment conditions stated in Proposition 1.

Furthermore, in order to calculate the ACF at the lag of $k = 1$, we provide the following moment conditions generated from the general formula (3).

$$\begin{aligned} E\left(x_t^2\right) &= \exp\left(\frac{\lambda}{1-\alpha} + \frac{\sigma_v^2}{2(1-\alpha^2)}\right) \\ E\left(x_t^2 x_{t+1}^2\right) &= \exp\left(\frac{2\lambda}{1-\alpha} + \frac{\sigma_v^2}{(1-\alpha)}\right)\left(1+\rho^2\sigma_v^2\right) \\ E\left(x_t^4\right) &= 3\exp\left(\frac{2\lambda}{1-\alpha} + \frac{2\sigma_v^2}{1-\alpha^2}\right) \end{aligned}$$

With the first moment being zero, the first order ACF can be easily calculated from the above moment conditions following the standard formula of the ACF. Similarly, we can compute the ACF at any order of the lag k for x_t^2.

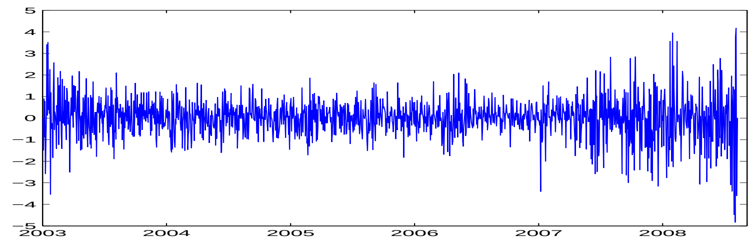
Table 5. Summary Statistics for SPX

	Mean	Var	Skn	Kurt	Min	Max	J-B Stats
DR	0.0252	0.8438	-0.2466	5.9098	-4.8354	4.1780	505.88 (0.00)
	1-min						
RV1	0.5249	0.5351	6.0666	58.6490	0.0331	10.0301	1.89e5 (0.00)
ln(RV1)	-1.0544	0.6734	0.6849	3.7114	-3.4083	2.3056	139.07 (0.00)
RV2	0.5373	0.5607	6.0666	58.6493	0.0339	10.2668	1.89e5 (0.00)
ln(RV2)	-1.0311	0.6734	0.6849	3.7114	-3.3850	2.3289	139.07 (0.00)
RV3	0.8441	1.3836	6.0666	58.6493	0.0532	16.1284	1.89e5 (0.00)
ln(RV3)	-0.5794	0.6734	0.6849	3.7114	-2.9333	2.7806	139.07 (0.00)
RV4	0.5434	0.5758	6.0771	58.7395	0.0334	10.6249	1.90e5 (0.00)
ln(RV4)	-1.0222	0.6786	0.6780	3.6816	-3.3992	2.3632	134.07 (0.00)
RV5	0.5434	0.5733	6.0678	58.6542	0.0341	10.4219	1.89e5 (0.00)
ln(RV5)	-1.0198	0.6734	0.6847	3.7093	-3.3778	2.3439	138.83 (0.00)
	5-min						
RV1	0.5529	0.6335	7.7570	100.4921	0.0281	14.4247	5.69e5 (0.00)
ln(RV1)	-1.0070	0.7013	0.5455	3.5035	-3.5709	2.6689	84.27 (0.00)
RV2	0.5660	0.6638	7.7570	100.4920	0.0288	14.7651	5.69e5 (0.00)
ln(RV2)	-0.9837	0.7013	0.5455	3.5035	-3.5476	2.6923	84.27 (0.00)
RV3	0.8441	1.4764	7.7570	100.4914	0.0429	22.0206	5.69e5 (0.00)
ln(RV3)	-0.5839	0.7013	0.5455	3.5035	-3.1479	3.0920	84.27 (0.00)
RV4	0.5714	0.6721	7.5920	95.3424	0.0292	14.4305	5.11e5 (0.00)
ln(RV4)	-0.9748	0.7036	0.5409	3.4876	-3.5325	2.6693	82.18 (0.00)
RV5	0.5714	0.6720	7.5754	94.8291	0.0292	14.3848	5.05e5 (0.00)
ln(RV5)	-0.9752	0.7043	0.5402	3.4849	-3.5320	2.6662	81.85 (0.00)
	10-min						
RV1	0.5456	0.6329	7.8468	105.3841	0.0320	14.9070	6.26e5 (0.00)
ln(RV1)	-1.0399	0.7502	0.4500	3.4404	-3.4408	2.7018	58.61 (0.00)
RV2	0.5585	0.6632	7.8468	105.3829	0.0328	15.2587	6.26e5 (0.00)
ln(RV2)	-1.0165	0.7502	0.4500	3.4404	-3.4175	2.7251	58.61 (0.00)
RV3	0.8441	1.5148	7.8468	105.3835	0.0496	23.0617	6.26e5(0.00)
ln(RV3)	-0.6035	0.7502	0.4500	3.4404	-3.0045	3.1382	58.61 (0.00)
RV4	0.5640	0.6717	7.6672	99.1448	0.0357	14.9127	6.26e5 (0.00)
ln(RV4)	-1.0062	0.7497	0.4523	3.4227	-3.3314	2.7022	58.19 (0.00)
RV5	0.5640	0.6716	7.6410	98.2402	0.0356	14.8417	5.43e5 (0.00)
ln(RV5)	-1.0066	0.7505	0.4520	3.4193	-3.3359	2.6974	57.97 (0.00)
	15-min						
RV1	0.5266	0.5689	6.9636	81.8789	0.0283	12.9876	3.74e5 (0.00)
ln(RV1)	-1.0854	0.7824	0.3794	3.3809	-3.5662	2.5640	42.07 (0.00)
RV2	0.5391	0.5961	6.9636	81.8791	0.0289	13.2941	3.74e5 (0.00)
ln(RV2)	-1.0620	0.7824	0.3794	3.3809	-3.5429	2.5873	42.07 (0.00)
RV3	0.8441	1.4616	6.9636	81.8782	0.0453	20.8166	3.74e5 (0.00)
ln(RV3)	-0.6136	0.7824	0.3794	3.3809	-3.0944	3.0358	42.07 (0.00)
RV4	0.5451	0.6016	6.7365	76.0894	0.0286	12.9934	3.22e5 (0.00)
ln(RV4)	-1.0503	0.7811	0.3838	3.3735	-3.5556	2.5644	42.53 (0.00)
RV5	0.5451	0.6005	6.6176	72.9545	0.0282	12.7572	2.95e5 (0.00)
ln(RV5)	-1.0520	0.7843	0.3843	3.3652	-3.5683	2.5461	42.27 (0.00)

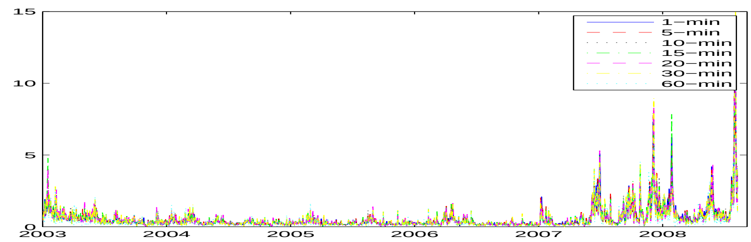
Table 5. Summary Statistics for SPX (Continued)

	Mean	Var	Skn	Kurt	Min	Max	J-B Stats
				20-min			
RV1	0.5157	0.5617	7.1015	84.1015	0.0228	12.7977	3.95e5 (0.00)
ln(RV1)	-1.1226	0.8236	0.3227	3.3113	-3.7799	2.5493	29.97 (0.00)
RV2	0.5279	0.5885	7.1015	84.1015	0.0234	13.0997	3.95e5 (0.00)
ln(RV2)	-1.0992	0.8236	0.3227	3.3113	-3.7566	2.5726	29.97 (0.00)
RV3	0.8442	1.5050	7.1015	84.1012	0.0374	20.9488	3.95e5 (0.00)
ln(RV3)	-0.6297	0.8236	0.3227	3.3113	-3.2871	3.0421	29.97 (0.00)
RV4	0.5342	0.5973	6.9474	79.2977	0.0231	12.8035	3.51e5 (0.00)
ln(RV4)	-1.0862	0.8214	0.3257	3.2998	-3.7668	2.5497	30.01 (0.00)
RV5	0.5342	0.5972	6.9171	78.3495	0.0230	12.7132	3.42e5 (0.00)
ln(RV5)	-1.0866	0.8222	0.3258	3.2970	-3.7713	2.5426	29.93 (0.00)
				30-min			
RV1	0.4986	0.6545	9.2002	137.6220	0.0093	16.0777	1.07e6 (0.00)
ln(RV1)	-1.2039	0.9248	0.2085	3.3741	-4.6818	2.7774	18.31 (0.00)
RV2	0.5104	0.6858	9.2002	137.6218	0.0095	16.4571	1.07e6 (0.00)
ln(RV2)	-1.1806	0.9248	0.2085	3.3741	-4.6585	2.8008	18.31 (0.00)
RV3	0.8442	1.8762	9.2002	137.6210	0.0157	27.2208	1.07e6 (0.00)
ln(RV3)	-0.6774	0.9248	0.2085	3.3741	-4.1552	3.3040	18.31 (0.00)
RV4	0.5170	0.6841	8.8803	128.8826	0.0093	16.0834	9.43e5 (0.00)
ln(RV4)	-1.1631	0.9149	0.2221	3.3633	-4.6725	2.7778	19.22 (0.00)
RV5	0.5170	0.6693	8.1527	108.5953	0.0089	14.9656	6.66e5 (0.00)
ln(RV5)	-1.1696	0.9260	0.2344	3.3412	-4.7259	2.7058	19.62 (0.00)
				60-min			
RV1	0.4218	0.4940	8.2595	115.6065	0.0056	13.4151	7.56e5 (0.00)
ln(RV1)	-1.4654	1.1698	0.0420	3.0857	-5.1767	2.5964	0.84 (0.65)
RV2	0.4317	0.5176	8.2595	115.6059	0.0058	13.7316	7.56e5 (0.00)
ln(RV2)	1.4420	1.1698	0.0420	3.0857	-5.1534	2.6197	0.84 (0.65)
RV3	0.8442	1.9792	8.2595	115.6066	0.0113	26.8509	7.56e5 (0.00)
ln(RV3)	-0.7714	1.1698	0.0420	3.0857	-4.4828	3.2903	0.84 (0.65)
RV4	0.4402	0.5172	7.9615	107.8172	0.0061	13.4209	6.56e5 (0.00)
ln(RV4)	-1.4103	1.1410	0.0610	3.0779	-5.0996	2.5968	1.22 (0.54)
RV5	0.4402	0.5007	7.2369	88.2674	0.0065	12.2817	4.36e5 (0.00)
ln(RV5)	-1.4125	1.1415	0.0818	3.0608	-5.0359	2.5081	1.77 (0.41)

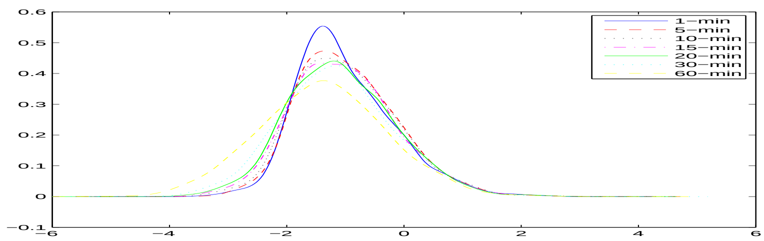
Figure 1. SPX Plots



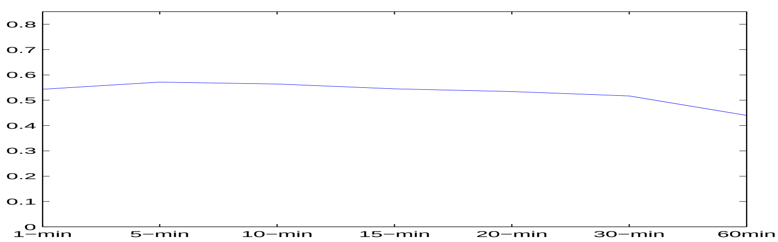
(a) Daily Return Plot



(b) Realized Volatility (RV5) across Different Frequencies

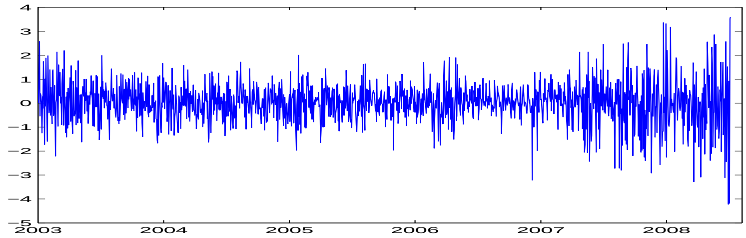


(c) Empirical Distributions of log-RV [ln(RV5)] across Different Frequencies

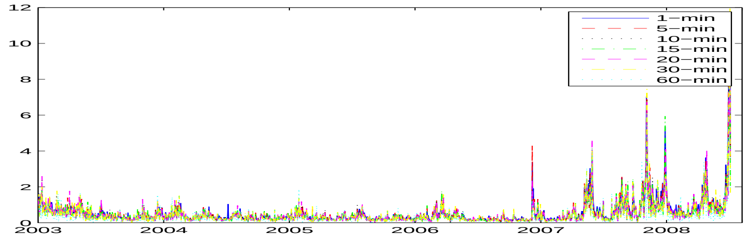


(d) Signature Plot for RV5

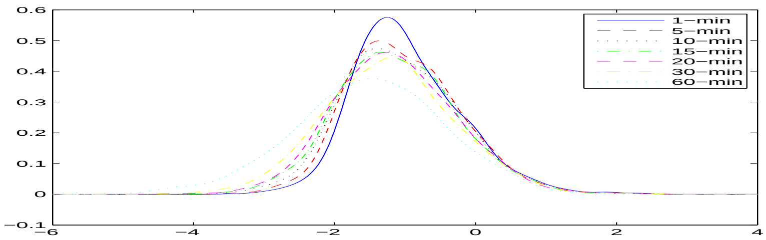
Figure 2. DJIA Plots



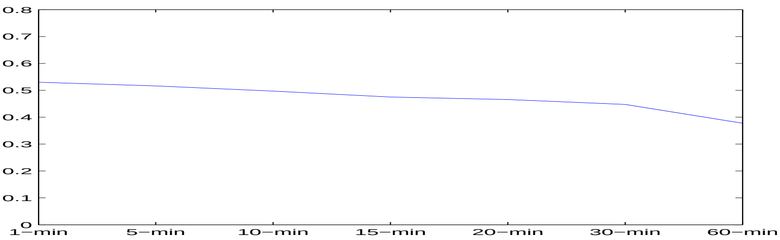
(a) Daily Return Plot



(b) Realized Volatility (RV5) across Different Frequencies



(c) Empirical Distributions of log-RV [ln(RV5)] across Different Frequencies



(d) Signature Plot for RV5

Table 6. Estimated Parameters and Standard Errors for COMPQ

	RV1		RV2		RV3		RV4		RV5	
	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$
1-min Frequency										
$\hat{\lambda}$	-0.393	-0.378	-0.303	-0.292	-0.144	-0.140	-0.355	-0.345	-0.258	-0.249
S.E.	(0.039)	(0.039)	(0.033)	(0.033)	(0.023)	(0.023)	(0.036)	(0.036)	(0.030)	(0.030)
$\hat{\alpha}$	0.654	0.670	0.654	0.670	0.654	0.670	0.579	0.596	0.658	0.675
S.E.	(0.030)	(0.029)	(0.030)	(0.029)	(0.030)	(0.029)	(0.032)	(0.032)	(0.030)	(0.029)
$\hat{\rho}$	-0.076	–	-0.087	–	-0.109	–	-0.153	–	-0.098	–
S.E.	(0.027)	–	(0.031)	–	(0.038)	–	(0.031)	–	(0.033)	–
$\hat{\sigma}_v$	0.484	0.485	0.484	0.485	0.485	0.485	0.595	0.598	0.482	0.483
S.E.	(0.014)	(0.014)	(0.014)	(0.014)	(0.014)	(0.014)	(0.017)	(0.017)	(0.013)	(0.014)
BIC	2738	2740	2583	2584	2478	2479	2761	2777	2521	2524
5-min Frequency										
$\hat{\lambda}$	-0.399	-0.383	-0.288	-0.278	-0.180	-0.175	-0.301	-0.293	-0.257	-0.248
S.E.	(0.037)	(0.037)	(0.031)	(0.031)	(0.025)	(0.026)	(0.032)	(0.033)	(0.029)	(0.029)
$\hat{\alpha}$	0.574	0.597	0.574	0.597	0.574	0.597	0.557	0.577	0.584	0.607
S.E.	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)
$\hat{\rho}$	-0.104	–	-0.118	–	-0.134	–	-0.175	–	-0.131	–
S.E.	(0.030)	–	(0.034)	–	(0.038)	–	(0.034)	–	(0.035)	–
$\hat{\sigma}_v$	0.540	0.542	0.540	0.542	0.542	0.542	0.601	0.603	0.535	0.536
S.E.	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.017)	(0.017)	(0.015)	(0.015)
BIC	2720	2725	2629	2634	2596	2602	2699	2718	2591	2598
10-min Frequency										
$\hat{\lambda}$	-0.446	-0.428	-0.322	-0.310	-0.209	-0.203	-0.323	-0.314	-0.285	-0.275
S.E.	(0.039)	(0.039)	(0.032)	(0.032)	(0.027)	(0.028)	(0.033)	(0.034)	(0.031)	(0.031)
$\hat{\alpha}$	0.522	0.548	0.522	0.548	0.522	0.548	0.522	0.544	0.539	0.565
S.E.	(0.034)	(0.033)	(0.034)	(0.033)	(0.034)	(0.033)	(0.033)	(0.033)	(0.033)	(0.033)
$\hat{\rho}$	-0.113	–	-0.129	–	-0.144	–	-0.176	–	-0.147	–
S.E.	(0.030)	–	(0.034)	–	(0.038)	–	(0.034)	–	(0.035)	–
$\hat{\sigma}_v$	0.581	0.583	0.582	0.583	0.583	0.583	0.630	0.633	0.574	0.575
S.E.	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.018)	(0.018)	(0.016)	(0.016)
BIC	2806	2814	2717	2724	2687	2694	2753	2772	2664	2674
15-min Frequency										
$\hat{\lambda}$	-0.512	-0.490	-0.372	-0.358	-0.244	-0.238	-0.361	-0.352	-0.332	-0.321
S.E.	(0.042)	(0.042)	(0.035)	(0.035)	(0.030)	(0.030)	(0.035)	(0.036)	(0.033)	(0.034)
$\hat{\alpha}$	0.463	0.494	0.463	0.494	0.463	0.494	0.477	0.500	0.478	0.508
S.E.	(0.035)	(0.034)	(0.035)	(0.034)	(0.035)	(0.034)	(0.034)	(0.034)	(0.034)	(0.034)
$\hat{\rho}$	-0.129	–	-0.147	–	-0.165	–	-0.189	–	-0.164	–
S.E.	(0.030)	–	(0.034)	–	(0.038)	–	(0.034)	–	(0.035)	–
$\hat{\sigma}_v$	0.636	0.640	0.638	0.640	0.640	0.640	0.674	0.677	0.629	0.631
S.E.	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.019)	(0.019)	(0.018)	(0.018)
BIC	2903	2914	2814	2826	2785	2796	2815	2837	2762	2776
30-min Frequency										
$\hat{\lambda}$	-0.644	-0.622	-0.486	-0.472	-0.318	-0.312	-0.457	-0.446	-0.428	-0.416
S.E.	(0.049)	(0.049)	(0.042)	(0.042)	(0.035)	(0.036)	(0.041)	(0.042)	(0.039)	(0.040)
$\hat{\alpha}$	0.392	0.422	0.392	0.422	0.392	0.422	0.408	0.431	0.413	0.441
S.E.	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)
$\hat{\rho}$	-0.107	–	-0.122	–	-0.140	–	-0.157	–	-0.144	–
S.E.	(0.029)	–	(0.033)	–	(0.037)	–	(0.034)	–	(0.034)	–
$\hat{\sigma}_v$	0.753	0.757	0.755	0.757	0.756	0.757	0.771	0.774	0.738	0.740
S.E.	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.022)	(0.022)	(0.021)	(0.021)
BIC	3169	3176	3054	3061	3002	3009	2987	3001	2954	2964
60-min Frequency										
$\hat{\lambda}$	-0.919	-0.890	-0.736	-0.716	-0.453	-0.446	-0.626	-0.616	-0.614	-0.601
S.E.	(0.061)	(0.062)	(0.054)	(0.055)	(0.044)	(0.045)	(0.050)	(0.051)	(0.049)	(0.050)
$\hat{\alpha}$	0.296	0.329	0.296	0.329	0.296	0.329	0.344	0.364	0.335	0.361
S.E.	(0.037)	(0.037)	(0.037)	(0.037)	(0.037)	(0.037)	(0.037)	(0.037)	(0.037)	(0.037)
$\hat{\rho}$	-0.117	–	-0.133	–	-0.162	–	-0.142	–	-0.146	–
S.E.	(0.025)	–	(0.028)	–	(0.034)	–	(0.030)	–	(0.031)	–
$\hat{\sigma}_v$	0.929	0.938	0.931	0.938	0.935	0.938	0.908	0.914	0.891	0.896
S.E.	(0.026)	(0.026)	(0.026)	(0.026)	(0.026)	(0.026)	(0.025)	(0.026)	(0.025)	(0.025)
BIC	3681	3696	3473	3488	3312	3327	3280	3294	3254	3269

Table 7. Estimated Parameters and Standard Errors for DJIA

	RV1		RV2		RV3		RV4		RV5	
	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$
1-min Frequency										
$\hat{\lambda}$	-0.193	-0.186	-0.191	-0.185	-0.124	-0.121	-0.191	-0.184	-0.193	-0.186
S.E.	(0.020)	(0.020)	(0.020)	(0.020)	(0.016)	(0.016)	(0.020)	(0.020)	(0.020)	(0.020)
$\hat{\alpha}$	0.801	0.810	0.801	0.810	0.801	0.810	0.801	0.811	0.799	0.808
S.E.	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)
$\hat{\rho}$	-0.089	–	-0.089	–	-0.106	–	-0.091	–	-0.088	–
S.E.	(0.022)	–	(0.022)	–	(0.026)	–	(0.022)	–	(0.023)	–
$\hat{\sigma}_v$	0.448	0.449	0.448	0.449	0.449	0.449	0.447	0.448	0.451	0.451
S.E.	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)
BIC	4859	4867	4856	4864	4753	4761	4847	4855	4871	4879
5-min Frequency										
$\hat{\lambda}$	-0.269	-0.260	-0.268	-0.259	-0.171	-0.167	-0.266	-0.257	-0.269	-0.260
S.E.	(0.024)	(0.024)	(0.024)	(0.024)	(0.019)	(0.019)	(0.024)	(0.024)	(0.024)	(0.024)
$\hat{\alpha}$	0.735	0.747	0.735	0.747	0.735	0.747	0.736	0.748	0.733	0.745
S.E.	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)
$\hat{\rho}$	-0.088	–	-0.088	–	-0.105	–	-0.089	–	-0.086	–
S.E.	(0.022)	–	(0.022)	–	(0.026)	–	(0.022)	–	(0.022)	–
$\hat{\sigma}_v$	0.529	0.530	0.529	0.530	0.530	0.530	0.527	0.528	0.532	0.533
S.E.	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
BIC	5344	5352	5340	5349	5210	5218	5330	5338	5356	5364
10-min Frequency										
$\hat{\lambda}$	-0.328	-0.317	-0.326	-0.315	-0.203	-0.198	-0.324	-0.313	-0.326	-0.315
S.E.	(0.026)	(0.026)	(0.026)	(0.026)	(0.021)	(0.021)	(0.026)	(0.026)	(0.026)	(0.026)
$\hat{\alpha}$	0.694	0.707	0.694	0.707	0.694	0.707	0.695	0.708	0.693	0.706
S.E.	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.020)	(0.019)
$\hat{\rho}$	-0.093	–	-0.093	–	-0.114	–	-0.094	–	-0.093	–
S.E.	(0.021)	–	(0.021)	–	(0.026)	–	(0.021)	–	(0.021)	–
$\hat{\sigma}_v$	0.583	0.585	0.583	0.585	0.585	0.585	0.581	0.582	0.584	0.586
S.E.	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)
BIC	5661	5672	5657	5668	5484	5496	5643	5655	5660	5671
15-min Frequency										
$\hat{\lambda}$	-0.400	-0.388	-0.398	-0.386	-0.239	-0.233	-0.396	-0.384	-0.397	-0.385
S.E.	(0.029)	(0.029)	(0.029)	(0.029)	(0.022)	(0.022)	(0.029)	(0.029)	(0.029)	(0.029)
$\hat{\alpha}$	0.643	0.658	0.643	0.658	0.643	0.658	0.644	0.659	0.643	0.658
S.E.	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)
$\hat{\rho}$	-0.093	–	-0.093	–	-0.117	–	-0.094	–	-0.093	–
S.E.	(0.021)	–	(0.021)	–	(0.026)	–	(0.021)	–	(0.021)	–
$\hat{\sigma}_v$	0.632	0.634	0.632	0.634	0.634	0.634	0.630	0.632	0.633	0.634
S.E.	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)
BIC	5893	5905	5889	5901	5685	5697	5875	5887	5887	5899
30-min Frequency										
$\hat{\lambda}$	-0.585	-0.571	-0.583	-0.568	-0.345	-0.338	-0.579	-0.565	-0.579	-0.565
S.E.	(0.035)	(0.036)	(0.035)	(0.035)	(0.027)	(0.027)	(0.035)	(0.035)	(0.035)	(0.035)
$\hat{\alpha}$	0.530	0.546	0.530	0.546	0.530	0.546	0.532	0.548	0.532	0.548
S.E.	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)
$\hat{\rho}$	-0.086	–	-0.087	–	-0.111	–	-0.087	–	-0.087	–
S.E.	(0.020)	–	(0.020)	–	(0.026)	–	(0.020)	–	(0.020)	–
$\hat{\sigma}_v$	0.766	0.768	0.766	0.768	0.768	0.768	0.763	0.765	0.763	0.765
S.E.	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.014)	(0.015)	(0.014)	(0.015)
BIC	6508	6519	6502	6513	6197	6208	6486	6497	6486	6497
60-min Frequency										
$\hat{\lambda}$	-0.961	-0.946	-0.958	-0.943	-0.533	-0.527	-0.946	-0.930	-0.942	-0.927
S.E.	(0.047)	(0.047)	(0.047)	(0.047)	(0.034)	(0.034)	(0.046)	(0.046)	(0.046)	(0.046)
$\hat{\alpha}$	0.371	0.386	0.371	0.386	0.371	0.386	0.376	0.391	0.378	0.393
S.E.	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)
$\hat{\rho}$	-0.072	–	-0.072	–	-0.101	–	-0.073	–	-0.073	–
S.E.	(0.017)	–	(0.017)	–	(0.024)	–	(0.017)	–	(0.017)	–
$\hat{\sigma}_v$	0.980	0.984	0.980	0.984	0.982	0.984	0.971	0.975	0.967	0.971
S.E.	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.018)	(0.019)	(0.018)	(0.019)
BIC	7815	7826	7804	7815	7000	7011	7742	7753	7722	7733

Table 8. Estimated Parameters and Standard Errors for SPX

	RV1		RV2		RV3		RV4		RV5	
	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$
1-min Frequency										
$\hat{\lambda}$	-0.188	-0.182	-0.184	-0.178	-0.102	-0.099	-0.186	-0.181	-0.182	-0.176
S.E.	(0.020)	(0.020)	(0.020)	(0.020)	(0.015)	(0.015)	(0.020)	(0.020)	(0.020)	(0.020)
$\hat{\alpha}$	0.818	0.826	0.818	0.826	0.818	0.826	0.814	0.822	0.818	0.826
S.E.	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)
$\hat{\rho}$	-0.086	–	-0.087	–	-0.109	–	-0.086	–	-0.088	–
S.E.	(0.021)	–	(0.021)	–	(0.026)	–	(0.021)	–	(0.021)	–
$\hat{\sigma}_v$	0.463	0.464	0.463	0.464	0.464	0.464	0.470	0.471	0.463	0.464
S.E.	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)
BIC	5273	5283	5251	5261	5049	5059	5270	5279	5237	5247
5-min Frequency										
$\hat{\lambda}$	-0.239	-0.232	-0.234	-0.226	-0.137	-0.134	-0.231	-0.224	-0.232	-0.225
S.E.	(0.022)	(0.022)	(0.022)	(0.022)	(0.017)	(0.017)	(0.022)	(0.022)	(0.022)	(0.022)
$\hat{\alpha}$	0.758	0.769	0.758	0.769	0.758	0.769	0.758	0.769	0.758	0.768
S.E.	(0.017)	(0.017)	(0.017)	(0.017)	(0.017)	(0.017)	(0.017)	(0.017)	(0.017)	(0.017)
$\hat{\rho}$	-0.088	–	-0.090	–	-0.109	–	-0.091	–	-0.091	–
S.E.	(0.021)	–	(0.021)	–	(0.026)	–	(0.022)	–	(0.022)	–
$\hat{\sigma}_v$	0.535	0.536	0.535	0.536	0.536	0.536	0.536	0.537	0.536	0.538
S.E.	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
BIC	5634	5644	5615	5625	5452	5462	5592	5602	5595	5605
10-min Frequency										
$\hat{\lambda}$	-0.287	-0.279	-0.280	-0.273	-0.164	-0.161	-0.276	-0.268	-0.277	-0.269
S.E.	(0.025)	(0.025)	(0.024)	(0.024)	(0.019)	(0.019)	(0.024)	(0.024)	(0.024)	(0.024)
$\hat{\alpha}$	0.719	0.731	0.719	0.731	0.719	0.731	0.721	0.732	0.720	0.732
S.E.	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)
$\hat{\rho}$	-0.085	–	-0.086	–	-0.106	–	-0.089	–	-0.089	–
S.E.	(0.021)	–	(0.021)	–	(0.026)	–	(0.021)	–	(0.021)	–
$\hat{\sigma}_v$	0.590	0.591	0.590	0.591	0.591	0.591	0.588	0.590	0.589	0.591
S.E.	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)
BIC	5925	5934	5905	5914	5721	5730	5860	5869	5862	5871
15-min Frequency										
$\hat{\lambda}$	-0.348	-0.339	-0.341	-0.332	-0.195	-0.191	-0.335	-0.327	-0.338	-0.330
S.E.	(0.027)	(0.027)	(0.027)	(0.027)	(0.021)	(0.021)	(0.027)	(0.027)	(0.027)	(0.027)
$\hat{\alpha}$	0.674	0.687	0.674	0.687	0.674	0.687	0.676	0.688	0.674	0.686
S.E.	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.019)	(0.020)	(0.020)
$\hat{\rho}$	-0.086	–	-0.087	–	-0.109	–	-0.088	–	-0.087	–
S.E.	(0.021)	–	(0.021)	–	(0.026)	–	(0.021)	–	(0.021)	–
$\hat{\sigma}_v$	0.641	0.643	0.641	0.643	0.643	0.643	0.640	0.641	0.643	0.644
S.E.	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)
BIC	6173	6183	6151	6161	5937	5947	6106	6116	6114	6124
30-min Frequency										
$\hat{\lambda}$	-0.515	-0.503	-0.505	-0.494	-0.287	-0.283	-0.488	-0.478	-0.495	-0.486
S.E.	(0.034)	(0.034)	(0.033)	(0.033)	(0.026)	(0.026)	(0.033)	(0.033)	(0.033)	(0.033)
$\hat{\alpha}$	0.567	0.581	0.567	0.581	0.567	0.581	0.575	0.589	0.572	0.584
S.E.	(0.022)	(0.022)	(0.022)	(0.022)	(0.022)	(0.022)	(0.022)	(0.022)	(0.022)	(0.022)
$\hat{\rho}$	-0.078	–	-0.079	–	-0.101	–	-0.079	–	-0.075	–
S.E.	(0.020)	–	(0.020)	–	(0.025)	–	(0.020)	–	(0.020)	–
$\hat{\sigma}_v$	0.780	0.782	0.781	0.782	0.782	0.782	0.772	0.773	0.779	0.781
S.E.	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)
BIC	6837	6845	6808	6817	6485	6493	6713	6721	6724	6730
60-min Frequency										
$\hat{\lambda}$	-0.824	-0.808	-0.811	-0.795	-0.430	-0.425	-0.766	-0.752	-0.756	-0.744
S.E.	(0.043)	(0.044)	(0.043)	(0.043)	(0.032)	(0.032)	(0.042)	(0.042)	(0.042)	(0.042)
$\hat{\alpha}$	0.432	0.449	0.432	0.449	0.432	0.449	0.452	0.466	0.460	0.473
S.E.	(0.024)	(0.024)	(0.024)	(0.024)	(0.024)	(0.024)	(0.024)	(0.024)	(0.024)	(0.024)
$\hat{\rho}$	-0.077	–	-0.078	–	-0.109	–	-0.075	–	-0.068	–
S.E.	(0.017)	–	(0.017)	–	(0.023)	–	(0.017)	–	(0.017)	–
$\hat{\sigma}_v$	0.962	0.966	0.962	0.966	0.965	0.966	0.941	0.945	0.939	0.942
S.E.	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)
BIC	8065	8079	8013	8027	7200	7214	7789	7800	7735	7743

Table 9. Estimated Parameters and Standard Errors for PX1

	RV1		RV2		RV3		RV4		RV5	
	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$
1-min Frequency										
$\hat{\lambda}$	-0.134	-0.134	-0.057	-0.057	-0.052	-0.052	-0.108	-0.109	-0.072	-0.073
S.E.	(0.017)	(0.017)	(0.014)	(0.014)	(0.013)	(0.013)	(0.018)	(0.018)	(0.015)	(0.015)
$\hat{\alpha}$	0.809	0.809	0.809	0.809	0.809	0.809	0.687	0.695	0.759	0.765
S.E.	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.019)	(0.019)	(0.017)	(0.017)
$\hat{\rho}$	-0.002	–	-0.002	–	-0.002	–	-0.103	–	-0.081	–
S.E.	(0.022)	–	(0.027)	–	(0.027)	–	(0.026)	–	(0.027)	–
$\hat{\sigma}_v$	0.475	0.475	0.475	0.475	0.475	0.475	0.635	0.635	0.540	0.540
S.E.	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)	(0.012)	(0.012)	(0.010)	(0.010)
BIC	5673	5666	5515	5508	5515	5507	6214	6222	5764	5766
5-min Frequency										
$\hat{\lambda}$	-0.152	-0.152	-0.062	-0.062	-0.068	-0.068	-0.103	-0.104	-0.078	-0.079
S.E.	(0.019)	(0.019)	(0.015)	(0.015)	(0.015)	(0.015)	(0.019)	(0.019)	(0.017)	(0.017)
$\hat{\alpha}$	0.779	0.779	0.779	0.779	0.779	0.779	0.684	0.693	0.734	0.741
S.E.	(0.017)	(0.017)	(0.017)	(0.017)	(0.017)	(0.017)	(0.019)	(0.019)	(0.018)	(0.018)
$\hat{\rho}$	0.004	–	0.005	–	0.005	–	-0.101	–	-0.082	–
S.E.	(0.022)	–	(0.027)	–	(0.027)	–	(0.027)	–	(0.027)	–
$\hat{\sigma}_v$	0.544	0.544	0.544	0.544	0.544	0.544	0.664	0.664	0.598	0.597
S.E.	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.013)	(0.013)	(0.011)	(0.011)
BIC	6009	6002	5875	5868	5875	5868	6317	6324	6027	6029
10-min Frequency										
$\hat{\lambda}$	-0.187	-0.187	-0.082	-0.082	-0.083	-0.083	-0.118	-0.119	-0.097	-0.098
S.E.	(0.021)	(0.021)	(0.017)	(0.017)	(0.017)	(0.017)	(0.020)	(0.020)	(0.018)	(0.018)
$\hat{\alpha}$	0.739	0.742	0.739	0.742	0.739	0.742	0.662	0.672	0.699	0.708
S.E.	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.020)	(0.020)	(0.019)	(0.019)
$\hat{\rho}$	-0.015	–	-0.018	–	-0.018	–	-0.107	–	-0.096	–
S.E.	(0.022)	–	(0.027)	–	(0.027)	–	(0.026)	–	(0.027)	–
$\hat{\sigma}_v$	0.590	0.590	0.590	0.590	0.590	0.590	0.689	0.689	0.639	0.639
S.E.	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.013)	(0.013)	(0.012)	(0.012)
BIC	6267	6260	6105	6099	6106	6099	6410	6419	6204	6209
15-min Frequency										
$\hat{\lambda}$	-0.230	-0.229	-0.110	-0.110	-0.100	-0.100	-0.138	-0.138	-0.129	-0.130
S.E.	(0.023)	(0.023)	(0.018)	(0.018)	(0.018)	(0.018)	(0.021)	(0.021)	(0.020)	(0.020)
$\hat{\alpha}$	0.703	0.706	0.703	0.706	0.703	0.706	0.641	0.650	0.654	0.664
S.E.	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.020)	(0.020)	(0.020)	(0.020)
$\hat{\rho}$	-0.020	–	-0.024	–	-0.024	–	-0.111	–	-0.108	–
S.E.	(0.021)	–	(0.026)	–	(0.026)	–	(0.026)	–	(0.026)	–
$\hat{\sigma}_v$	0.634	0.634	0.634	0.634	0.634	0.634	0.713	0.713	0.695	0.695
S.E.	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.013)	(0.013)	(0.013)	(0.013)
BIC	6547	6541	6334	6328	6330	6323	6494	6504	6420	6429
30-min Frequency										
$\hat{\lambda}$	-0.336	-0.333	-0.180	-0.180	-0.154	-0.154	-0.187	-0.187	-0.220	-0.221
S.E.	(0.027)	(0.027)	(0.022)	(0.022)	(0.022)	(0.022)	(0.023)	(0.023)	(0.025)	(0.025)
$\hat{\alpha}$	0.615	0.622	0.615	0.622	0.615	0.622	0.584	0.594	0.547	0.556
S.E.	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.022)	(0.022)	(0.022)
$\hat{\rho}$	-0.049	–	-0.060	–	-0.062	–	-0.118	–	-0.119	–
S.E.	(0.020)	–	(0.024)	–	(0.025)	–	(0.025)	–	(0.025)	–
$\hat{\sigma}_v$	0.745	0.745	0.745	0.745	0.745	0.745	0.787	0.787	0.841	0.842
S.E.	(0.014)	(0.014)	(0.014)	(0.014)	(0.014)	(0.014)	(0.015)	(0.015)	(0.016)	(0.016)
BIC	7113	7112	6813	6812	6795	6793	6754	6767	6964	6980
60-min Frequency										
$\hat{\lambda}$	-0.548	-0.545	-0.339	-0.338	-0.236	-0.236	-0.276	-0.276	-0.251	-0.251
S.E.	(0.035)	(0.035)	(0.029)	(0.029)	(0.027)	(0.027)	(0.027)	(0.027)	(0.026)	(0.026)
$\hat{\alpha}$	0.484	0.490	0.484	0.490	0.484	0.490	0.509	0.519	0.530	0.539
S.E.	(0.024)	(0.023)	(0.024)	(0.023)	(0.024)	(0.023)	(0.023)	(0.023)	(0.023)	(0.023)
$\hat{\rho}$	-0.030	–	-0.037	–	-0.041	–	-0.091	–	-0.080	–
S.E.	(0.018)	–	(0.022)	–	(0.024)	–	(0.024)	–	(0.025)	–
$\hat{\sigma}_v$	0.928	0.929	0.929	0.929	0.929	0.929	0.896	0.896	0.860	0.860
S.E.	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.017)	(0.017)	(0.016)	(0.016)
BIC	8167	8162	7636	7631	7518	7513	7146	7153	7030	7033

Table 10. Estimated Parameters and Standard Errors for RUA

	RV1		RV2		RV3		RV4		RV5	
	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$
1-min Frequency										
$\hat{\lambda}$	-0.746	-0.719	-0.737	-0.710	-0.382	-0.369	-0.779	-0.746	-0.759	-0.726
S.E.	(0.072)	(0.072)	(0.071)	(0.071)	(0.041)	(0.041)	(0.072)	(0.072)	(0.071)	(0.071)
$\hat{\alpha}$	0.524	0.542	0.524	0.542	0.524	0.542	0.489	0.512	0.502	0.525
S.E.	(0.044)	(0.044)	(0.044)	(0.044)	(0.044)	(0.044)	(0.045)	(0.045)	(0.044)	(0.044)
$\hat{\rho}$	-0.094	–	-0.095	–	-0.137	–	-0.110	–	-0.108	–
S.E.	(0.035)	–	(0.035)	–	(0.050)	–	(0.035)	–	(0.035)	–
$\hat{\sigma}_v$	0.410	0.413	0.410	0.413	0.412	0.413	0.409	0.412	0.404	0.407
S.E.	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)
BIC	1347	1349	1339	1340	1183	1184	1320	1324	1311	1314
5-min Frequency										
$\hat{\lambda}$	-0.732	-0.706	-0.722	-0.696	-0.453	-0.439	-0.753	-0.721	-0.761	-0.728
S.E.	(0.067)	(0.066)	(0.066)	(0.065)	(0.046)	(0.046)	(0.066)	(0.066)	(0.066)	(0.066)
$\hat{\alpha}$	0.454	0.475	0.454	0.475	0.454	0.475	0.423	0.450	0.417	0.445
S.E.	(0.046)	(0.046)	(0.046)	(0.046)	(0.046)	(0.046)	(0.047)	(0.046)	(0.047)	(0.046)
$\hat{\rho}$	-0.086	–	-0.087	–	-0.111	–	-0.109	–	-0.111	–
S.E.	(0.040)	–	(0.040)	–	(0.051)	–	(0.040)	–	(0.040)	–
$\hat{\sigma}_v$	0.476	0.477	0.476	0.477	0.477	0.477	0.471	0.472	0.472	0.474
S.E.	(0.017)	(0.018)	(0.017)	(0.018)	(0.018)	(0.018)	(0.017)	(0.017)	(0.017)	(0.017)
BIC	1360	1359	1355	1354	1287	1286	1336	1337	1339	1340
10-min Frequency										
$\hat{\lambda}$	-0.809	-0.782	-0.798	-0.771	-0.524	-0.508	-0.833	-0.798	-0.847	-0.811
S.E.	(0.069)	(0.069)	(0.068)	(0.068)	(0.050)	(0.050)	(0.068)	(0.068)	(0.068)	(0.068)
$\hat{\alpha}$	0.385	0.409	0.385	0.409	0.385	0.409	0.349	0.381	0.338	0.370
S.E.	(0.048)	(0.047)	(0.048)	(0.047)	(0.048)	(0.047)	(0.048)	(0.048)	(0.049)	(0.048)
$\hat{\rho}$	-0.100	–	-0.101	–	-0.125	–	-0.129	–	-0.133	–
S.E.	(0.040)	–	(0.040)	–	(0.050)	–	(0.041)	–	(0.041)	–
$\hat{\sigma}_v$	0.540	0.542	0.540	0.542	0.541	0.542	0.530	0.533	0.533	0.535
S.E.	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.019)	(0.020)	(0.019)	(0.020)
BIC	1440	1441	1435	1436	1376	1376	1409	1412	1411	1415
15-min Frequency										
$\hat{\lambda}$	-0.933	-0.907	-0.920	-0.894	-0.599	-0.584	-0.936	-0.903	-0.948	-0.913
S.E.	(0.074)	(0.073)	(0.073)	(0.072)	(0.053)	(0.052)	(0.072)	(0.071)	(0.072)	(0.071)
$\hat{\alpha}$	0.309	0.332	0.309	0.332	0.309	0.332	0.286	0.315	0.277	0.307
S.E.	(0.050)	(0.049)	(0.050)	(0.049)	(0.050)	(0.049)	(0.050)	(0.049)	(0.050)	(0.049)
$\hat{\rho}$	-0.077	–	-0.078	–	-0.098	–	-0.102	–	-0.106	–
S.E.	(0.040)	–	(0.040)	–	(0.050)	–	(0.041)	–	(0.040)	–
$\hat{\sigma}_v$	0.583	0.584	0.583	0.584	0.584	0.584	0.566	0.567	0.567	0.569
S.E.	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)	(0.021)
BIC	1512	1509	1506	1504	1437	1435	1470	1470	1471	1471
30-min Frequency										
$\hat{\lambda}$	-1.164	-1.146	-1.149	-1.131	-0.752	-0.742	-1.182	-1.158	-1.204	-1.178
S.E.	(0.083)	(0.082)	(0.082)	(0.081)	(0.061)	(0.060)	(0.081)	(0.080)	(0.081)	(0.081)
$\hat{\alpha}$	0.194	0.209	0.194	0.209	0.194	0.209	0.157	0.178	0.140	0.162
S.E.	(0.052)	(0.051)	(0.052)	(0.051)	(0.052)	(0.051)	(0.052)	(0.051)	(0.052)	(0.051)
$\hat{\rho}$	-0.052	–	-0.053	–	-0.067	–	-0.073	–	-0.080	–
S.E.	(0.039)	–	(0.039)	–	(0.050)	–	(0.040)	–	(0.040)	–
$\hat{\sigma}_v$	0.712	0.712	0.712	0.712	0.712	0.712	0.696	0.697	0.696	0.697
S.E.	(0.026)	(0.026)	(0.026)	(0.026)	(0.026)	(0.026)	(0.026)	(0.026)	(0.025)	(0.026)
BIC	1665	1661	1659	1654	1574	1570	1627	1624	1624	1622
60-min Frequency										
$\hat{\lambda}$	-1.503	-1.481	-1.486	-1.464	-0.934	-0.922	-1.474	-1.448	-1.475	-1.448
S.E.	(0.099)	(0.099)	(0.098)	(0.098)	(0.071)	(0.071)	(0.096)	(0.096)	(0.095)	(0.095)
$\hat{\alpha}$	0.111	0.128	0.111	0.128	0.111	0.128	0.099	0.119	0.093	0.114
S.E.	(0.052)	(0.051)	(0.052)	(0.051)	(0.052)	(0.051)	(0.052)	(0.052)	(0.052)	(0.052)
$\hat{\rho}$	-0.053	–	-0.053	–	-0.073	–	-0.066	–	-0.072	–
S.E.	(0.032)	–	(0.033)	–	(0.044)	–	(0.034)	–	(0.034)	–
$\hat{\sigma}_v$	0.879	0.881	0.879	0.881	0.880	0.881	0.855	0.858	0.844	0.847
S.E.	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)	(0.031)	(0.032)	(0.031)	(0.031)
BIC	2020	2017	2008	2005	1789	1786	1932	1929	1893	1892

Table 11. Estimated Parameters and Standard Errors for SXXP

	RV1		RV2		RV3		RV4		RV5	
	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$	$\rho \in (-1,1)$	$\rho = 0$
1-min Frequency										
$\hat{\lambda}$	-0.319	-0.319	-0.309	-0.309	-0.128	-0.128	-0.276	-0.273	-0.296	-0.291
S.E.	(0.031)	(0.031)	(0.030)	(0.030)	(0.019)	(0.019)	(0.029)	(0.028)	(0.030)	(0.030)
$\hat{\alpha}$	0.782	0.782	0.782	0.782	0.782	0.782	0.798	0.801	0.786	0.792
S.E.	(0.019)	(0.018)	(0.019)	(0.018)	(0.019)	(0.018)	(0.018)	(0.017)	(0.018)	(0.018)
$\hat{\rho}$	0.000	–	0.000	–	0.001	–	-0.017	–	-0.034	–
S.E.	(0.020)	–	(0.020)	–	(0.030)	–	(0.021)	–	(0.020)	–
$\hat{\sigma}_v$	0.538	0.538	0.538	0.538	0.538	0.538	0.523	0.523	0.548	0.549
S.E.	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)
BIC	5109	5102	5038	5031	4541	4534	4859	4853	4988	4984
5-min Frequency										
$\hat{\lambda}$	-0.314	-0.316	-0.304	-0.305	-0.155	-0.155	-0.268	-0.266	-0.319	-0.313
S.E.	(0.032)	(0.031)	(0.031)	(0.031)	(0.022)	(0.022)	(0.029)	(0.029)	(0.032)	(0.032)
$\hat{\alpha}$	0.775	0.773	0.775	0.773	0.775	0.773	0.795	0.797	0.763	0.770
S.E.	(0.019)	(0.018)	(0.019)	(0.018)	(0.019)	(0.018)	(0.018)	(0.018)	(0.019)	(0.019)
$\hat{\rho}$	0.008	–	0.008	–	0.011	–	-0.009	–	-0.044	–
S.E.	(0.020)	–	(0.021)	–	(0.029)	–	(0.021)	–	(0.021)	–
$\hat{\sigma}_v$	0.611	0.611	0.611	0.611	0.611	0.611	0.583	0.583	0.646	0.647
S.E.	(0.013)	(0.013)	(0.013)	(0.013)	(0.013)	(0.013)	(0.012)	(0.012)	(0.013)	(0.013)
BIC	5334	5327	5271	5264	4856	4849	5045	5038	5332	5329
10-min Frequency										
$\hat{\lambda}$	-0.338	-0.337	-0.327	-0.326	-0.171	-0.171	-0.291	-0.288	-0.331	-0.324
S.E.	(0.033)	(0.032)	(0.032)	(0.032)	(0.023)	(0.023)	(0.030)	(0.030)	(0.032)	(0.032)
$\hat{\alpha}$	0.754	0.755	0.754	0.755	0.754	0.755	0.774	0.778	0.749	0.758
S.E.	(0.020)	(0.019)	(0.020)	(0.019)	(0.020)	(0.019)	(0.019)	(0.018)	(0.019)	(0.019)
$\hat{\rho}$	-0.004	–	-0.004	–	-0.005	–	-0.019	–	-0.047	–
S.E.	(0.021)	–	(0.021)	–	(0.029)	–	(0.022)	–	(0.021)	–
$\hat{\sigma}_v$	0.643	0.643	0.643	0.643	0.643	0.643	0.617	0.617	0.668	0.669
S.E.	(0.013)	(0.013)	(0.013)	(0.013)	(0.013)	(0.013)	(0.013)	(0.013)	(0.014)	(0.014)
BIC	5451	5444	5388	5381	4987	4980	5179	5173	5402	5400
15-min Frequency										
$\hat{\lambda}$	-0.376	-0.376	-0.364	-0.363	-0.193	-0.192	-0.317	-0.315	-0.357	-0.351
S.E.	(0.035)	(0.034)	(0.034)	(0.033)	(0.024)	(0.024)	(0.031)	(0.031)	(0.034)	(0.033)
$\hat{\alpha}$	0.732	0.733	0.732	0.733	0.732	0.733	0.759	0.762	0.735	0.743
S.E.	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.019)	(0.019)	(0.020)	(0.019)
$\hat{\rho}$	-0.001	–	-0.001	–	-0.001	–	-0.015	–	-0.044	–
S.E.	(0.020)	–	(0.021)	–	(0.028)	–	(0.021)	–	(0.021)	–
$\hat{\sigma}_v$	0.678	0.678	0.678	0.678	0.678	0.678	0.644	0.644	0.697	0.697
S.E.	(0.014)	(0.014)	(0.014)	(0.014)	(0.014)	(0.014)	(0.013)	(0.013)	(0.014)	(0.014)
BIC	5656	5649	5589	5582	5137	5130	5324	5318	5540	5538
30-min Frequency										
$\hat{\lambda}$	-0.515	-0.509	-0.498	-0.493	-0.273	-0.272	-0.437	-0.431	-0.504	-0.495
S.E.	(0.040)	(0.040)	(0.039)	(0.039)	(0.029)	(0.029)	(0.037)	(0.037)	(0.040)	(0.040)
$\hat{\alpha}$	0.646	0.653	0.646	0.653	0.646	0.653	0.679	0.687	0.647	0.657
S.E.	(0.023)	(0.022)	(0.023)	(0.022)	(0.023)	(0.022)	(0.022)	(0.021)	(0.022)	(0.022)
$\hat{\rho}$	-0.024	–	-0.025	–	-0.034	–	-0.033	–	-0.053	–
S.E.	(0.019)	–	(0.020)	–	(0.027)	–	(0.021)	–	(0.020)	–
$\hat{\sigma}_v$	0.796	0.796	0.796	0.796	0.796	0.796	0.757	0.758	0.839	0.841
S.E.	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.017)	(0.017)
BIC	6133	6127	6057	6052	5529	5524	5760	5755	6109	6109
60-min Frequency										
$\hat{\lambda}$	-0.785	-0.779	-0.762	-0.756	-0.399	-0.398	-0.642	-0.635	-0.612	-0.604
S.E.	(0.050)	(0.050)	(0.049)	(0.049)	(0.035)	(0.035)	(0.045)	(0.045)	(0.044)	(0.044)
$\hat{\alpha}$	0.518	0.524	0.518	0.524	0.518	0.524	0.574	0.582	0.595	0.604
S.E.	(0.026)	(0.025)	(0.026)	(0.025)	(0.026)	(0.025)	(0.024)	(0.024)	(0.024)	(0.023)
$\hat{\rho}$	-0.016	–	-0.017	–	-0.024	–	-0.026	–	-0.036	–
S.E.	(0.017)	–	(0.018)	–	(0.026)	–	(0.019)	–	(0.019)	–
$\hat{\sigma}_v$	0.987	0.987	0.987	0.987	0.987	0.987	0.916	0.917	0.903	0.904
S.E.	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.020)	(0.019)	(0.019)	(0.019)	(0.019)
BIC	7158	7151	7050	7043	6141	6135	6486	6481	6395	6391