# The War of Information[†]

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## Abstract

We analyze political contests (campaigns) between two parties with opposing interests. Parties provide costly information to voters who choose a policy. The information flow is continuous and stops when both parties quit. Parties' actions are strategic substitutes: increasing one party's cost makes that party provide more and its opponent provide less information. For voters, parties' actions are complements and hence raising the advantaged party's cost may be beneficial. Asymmetric information adds a signaling component resulting in a belief-threshold at which the informed party's decision to continue campaigning offsets other unfavorable information.

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# 1. Introduction

A political party proposes a new policy, for example, a health care plan. Interest groups favoring or opposing the plan gather information to convince voters of their respective positions. This process continues until polling data suggest that voters decisively favor or oppose the new policy and Congress responds accordingly. Recent health care debates and the social security debate during the Bush administration are prominent examples of this pattern.

A key question is how asymmetric access to funds affects the outcome of such campaigns. For example, health care reform proponents often cite their opponents' superior funding as the main reason for the failure of health care reform during the Clinton administration. Hence, the question is to what degree superior funding can determine the outcome of a political campaign and whether asymmetric access to funds can reduce voter welfare. We formulate a model of competitive advocacy to address this and related questions.

We assume that parties cannot distort information; rather, they trade-off the cost of information provision and the probability of convincing the (median) voter.[1] The underlying uncertainty is about the voter's utility of the proposed policy. There are two states; the voter prefers party 1's policy in one and party 2's policy in the other. We first study the symmetric information case in which neither the parties nor the voter know the state and information is revealed gradually.

Information flows continuously as long as one of the parties is willing to incur its cost. All players observe the signal, a Brownian motion with a state-dependent drift. The game ends when no party is willing to pay the information cost. At that point, the voter picks his preferred policy based on his beliefs. We call this game *the war of information.*

The war of information has a unique subgame perfect equilibrium. In that equilibrium, each party chooses a threshold and stops providing information once the voter's belief is less favorable than that threshold. The lower a party's cost, the more aggressive is its equilibrium threshold and the higher is its probability of winning. Viewed as a game between the two parties, the war of information is a game of strategic substitutes: a more

[1] In Gul and Pesendorfer (2009), we consider a variant of the war of information that allows for information distortions.

aggressive opponent threshold implies a less aggressive best response. Hence, a party's easy access to resources will stifle its opponent. If the signal is very informative, the effect of asymmetric costs is small. In that case, the war of information is resolved quickly with nearly full information revelation. If the signal is very uninformative, a party with a large cost advantage captures nearly all the surplus.

For the voter, the parties' thresholds are complements. Raising one party's threshold increases the marginal benefit of raising the other's. This complementarity implies that the voter's payoff is highest when the campaigns are "balanced," that is, when they feature two parties with similar costs of providing information. If the parties have sufficiently asymmetric costs, the voter benefits from regulation that raises the cost of the advantaged (i.e., low-cost) party and may even benefit from regulation that raises both parties' costs equally. Such regulation makes the advantaged party provide less and the disadvantaged party provide more information. If costs are sufficiently asymmetric, the latter affect dominates and increases voter welfare. We also show that, to benefit the voter, regulation must increase total campaign expenditures and hence reduce the combined payoff of parties.

US political campaigns devote substantial effort to fundraising while US election laws hinder these efforts by limiting the amount of money an individual donor can give. Such regulation disproportionately affects the advantaged party. Our results show that the median voter may benefit from this type of regulation.

In section 4, we consider two extensions of our model. First, to allow for the possibility that fundraising becomes more difficult as public opinion turns against a party, we assume that information costs depend on the voter's belief. In the second extension, parties are impatient and discount future payoffs. Both extensions yield unique equilibria similar to the equilibrium of our original game. We show that discounting magnifies the deterrent effect of a cost advantage. Specifically, holding all other parameters fixed, a party's payoff converges to the total surplus as its cost converges to zero.

In section 5, we incorporate asymmetric information by assuming that one party knows the true state. Hence, the party advocating the new policy knows its merit and provides noisy information. Communication may be noisy either because parties cannot communicate directly with voters and rely on intermediaries or because voters require time to fully understand and evaluate the policy. In either case, parties cannot simply "disclose" their information. Instead, they convey information through a costly and noisy campaign.[2]

For example, suppose a type 1 party advocates banning an unsafe technology that would hurt the (median) voter while a type 0 party advocates banning a safe technology that would benefit him. The voter's prior does not warrant a ban and therefore the party must convince him. As before, the party provides hard information through a Brownian motion with a type-dependent drift. However, the voter now takes the party's private information into account and draws the appropriate conclusions from its decision to quit or continue. The natural inference is to interpret quitting as weakness and persistence as strength; that is, assume that the party is more likely to continue if it knows that the technology is unsafe. We call an equilibrium that satisfies this restriction a *monotone equilibrium* and show that it is unique.[3]

In a monotone equilibrium, type 1 never quits and type 0 provides information as long as the voter's belief that the technology is unsafe remains above a threshold $p$. Once the belief reaches $p$, type 0 randomizes between quitting and not quitting. The randomization is calibrated to balance unfavorable evidence so that the voter's belief never drops below $p$. Asymmetric information therefore leads to a *signaling barrier*, i.e., a lower bound that cannot be crossed as long as the party provides information. Once the party quits, its type is revealed and the voter knows that the technology is safe.

As long as the party does not quit, the voter remains unconvinced of the technology's safety. An observer who ignores the signaling component might incorrectly conclude that the voter is biased in the informed party's favor. Unfavorable information is discounted – offset by the party's decision not to quit – while favorable information is not.

The probability of an incorrect choice (banning a safe technology) depends on the voter's prior but not on the party's cost. Changing this cost changes the signaling barrier's location and the expected duration of the game but not the probability of an incorrect choice. Increasing the cost has two offsetting effects: first, not quitting becomes more

[2] The literature on strategic transmission of verifiable information (Milgrom and Roberts (1996), Austen-Smith (1992)) has focused on the incentive to disclose a known signal. This literature assumes that disclosure is costless.

[3] There are also non-monotone equilibria. We discuss non-monotone equilibria at the end of Section 5.

costly (hence there is less incentive to provide information). Second, not-quitting becomes a more informative signal.

### 1.1 Related Literature

The war of information resembles the war of attrition. However, there are two key differences: first, in a war of attrition both players bear costs as long as the game continues while in a war of information only one player incurs a cost at each moment. Second, the resources spent during a war of information generate a payoff relevant signal. If the signal were uninformative and both players incurred costs throughout the game, the war of information would become a war of attrition with a public randomization device. The war of information is similar to models of contests (Rosenthal and Rubinstein (1984), Dixit (1987), and rent-seeking games (Tullock (1980)). The key difference is that in a war of information, the two sides generate useful information.

Austen-Smith and Wright (1992) examine strategic information transmission between two competing lobbies and a legislator. They consider a static setup in which lobbies may provide a single binary signal and analyze whether and when lobbies provide useful information to the legislator. A problem in Austen-Smith and Wright (1992) and in Austen-Smith (1994) is ensuring that the informed party has incentive to disclose the information. In our model, this incentive problem is absent. Our model fits situations in which the informed party cannot simply disclose information but must convey it through a costly and noisy campaign. The Austen-Smith and Wright setting is appropriate when an informed lobby interacts with a sophisticated policy maker to whom information can be conveyed at no cost and without noise.

The literature on strategic experimentation (Harris and Bolton (1999, 2000), Cripps, Keller and Rady (2005)) analyzes the free rider problem that arises when agents incur costs to learn the true state but can also learn from the behavior of others. Our information structure is similar to that of Harris and Bolton (1999); the signal is a Brownian motion with unknown drift.[4] However, the war of information provides different incentives: a party would like to deter its opponent from providing information and therefore benefits from

[4] See also Moscarini and Smith (2001) for an analysis of the optimal level of experimentation in a decision problem.

a cost advantage beyond the direct cost saving. In a model of strategic experimentation, agents have an incentive to free-ride on other players and therefore would like to encourage opponents to provide information.

Our model is related to work on campaign advertising, most notably, Prat (2002) who assumes that campaign expenditures are not inherently informative but may signal private information about the candidate's ability.[5] Our asymmetric information game is a hybrid of Prat's model and one of informative advertising. Prat provides a different argument for restricting political advertising: a party caters to a privately informed campaign donor at the voters' expense. Prohibiting political advertising may decrease the resulting policy bias and hence yield higher welfare. In our model, campaign spending may hurt voters by restricting their information. Clearly, both effects play a role in public policy debates about campaign finance regulation.

Yilankaya (2002) analyzes the optimal burden of proof. He assume an informed defendant, an uninformed prosecutor and an uninformed judge. This setting is similar to our asymmetric information model. However, Yilankaya's model is static; that is, parties commit to a fixed expenditure at the outset. Yilankaya explores the trade-off between increasing the burden of proof and increasing penalties for convicted defendants. He shows that higher penalties may lead to larger errors, i.e., a larger probability of convicting innocent defendants or acquitting guilty defendants. A higher penalty his model is like a lower cost in ours. Hence, our analysis shows that in a dynamic setting, if the defendant is informed, increasing penalties have no effect on the probability of convicting an innocent defendant or acquitting a guilty one.

[5] See also Potters, Soof and Van Winden (1997).

## 2. The War of Information

The *War of Information* is a three-person, continuous-time game. Players 1 and 2 are *parties* and player 3 is the *voter*. Nature endows one party with the correct (voter-preferred) position. Then, both parties decide whether or not to provide information. Once the flow of information stops, the voter chooses a party (or its policy). The voter's payoff is 1 if he chooses the party with the correct position and 0 otherwise. Party $i$ incurs flow cost $k_i/2$ while providing information but earns an additional payoff of 1 if it is chosen.

Players are symmetrically informed.[6] Let $p_t$ denote the probability that the voter (and parties) assigns at time $t$ to party $i$ having the correct position and let $T$ be the time at which the flow of information stops. It is optimal for the voter to choose party 1 if and only if $p_T \geq 1/2$. We say that party 1 (2) is *trailing* at time $t$ if $p_t < 1/2$ ($p_t \geq 1/2$).

We assume that only the trailing party may provide information. Hence, the game stops whenever the trailing party quits. The equilibrium below remains an equilibrium when this assumption is relaxed and parties are allowed to provide information while they are ahead. We discuss the more general case at the end of this section.

We say that the game is running at time $t$ if, at no $\tau \leq t$, a trailing player has quit. As long as the game is running, all three players observe the process $X$ where

$$X_t = \mu t + Z_t \tag{2}$$

and $Z$ is a Wiener process. Hence, $X$ is a Brownian motion with uncertain drift $\mu$ and variance 1. The realization $\mu = 1/2$ ($\mu = -1/2$) means that party 1 (party 2) holds the correct position. The prior probability that party $i$ holds the correct position is $1/2$ for $i = 1, 2$. Let $p$ be the logistic function; that is,

$$p(x) = \frac{1}{1 + e^{-x}} \tag{3}$$

for all $x \in I\!R$. We set $p(-\infty) = 0$ and $p(\infty) = 1$. A straightforward application of Bayes' law yields

$$p_t := \Pr\{\mu = 1/2 \,|\, X_t\} = p(X_t)$$

[6] See Section 5 for the case of asymmetric information.

and therefore, $i$ is trailing if and only if

$$(-1)^{i-1} X_t < 0 \tag{4}$$

In this section, we restrict both parties to stationary, pure strategies. In Appendix B, we show that this restriction is without loss of generality. Specifically, we show that the war of information has a unique subgame perfect equilibrium and this equilibrium is in stationary strategies.

A stationary pure strategy for player 1 is a number $y_1 < 0$ ($y_1 = -\infty$ is allowed) such that player 1 quits providing information as soon as $X$ reaches $y_1$. That is, player 1 provides information when $y_1 < X_t < 0$ and quits as soon as $X_t = y_1$. Similarly, a stationary pure strategy for player 2 is an extended real number $y_2 > 0$ such that player 2 provides information when $0 \leq X_t < y_2$ and quits as soon as $X_t = y_2$. Let

$$T = \inf\{t > 0 \,|\, X_t - y_i = 0 \text{ for some } i = 1, 2\} \tag{5}$$

if $\{t \,|\, X_t = y_i \text{ for some } i = 1, 2\} \neq \emptyset$ and $T = \infty$ otherwise. Observe that the game runs until time $T$. At time $T < \infty$, player 3 rules in favor of player $i$ if and only if $X_T = y_j$ for $j \neq i$. If $T = \infty$, we let $p_T = 1/2$ and assume that both players win.[7] Let $y = (y_1, y_2)$ and let $v_1(y)$ denote the probability that player 1 wins given the strategy profile $y$; that is, $v_1(y) = \Pr\{p_T > 1/2\}$. The probability that 2 wins is $v_2(y) = 1 - v_1(y)$.

To compute the parties' expenditures given the strategy profile $y$, define $C : [0, 1] \to \{0, 1\}$ such that

$$C(s) = \begin{cases} 1 & \text{if } s < 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Let $C_1 = C$ and $C_2 = 1 - C$. Then, party $i$'s (expected) expenditure given the strategy profile $y$ is

$$c_i(y) = \frac{k_1}{2} E \int_0^T C_i(p_t) dt \tag{7}$$

The parties' utilities are

$$U_i(y) = v_i(y) - c_i(y) \tag{8}$$

[7] The specification of payoffs for $T = \infty$ has no effect on the equilibrium outcome since staying in the game forever is never a best response under any specification. We chose this particular specification to simplify the notation and exposition.

while the voter's utility is

$$U_3(y) = E[\max\{p_T, 1 - p_T\}] \tag{9}$$

When the belief $p(X_t)$ is in the range $(p(y_1), 1/2]$, party 1 provides information while $[1/2, p(y_2))$ is the corresponding range for party 2. It is convenient to describe strategies as a function of $p$. Let

$$\alpha_i := (-1)^{i-1}\big(1 - 2p(y_i)\big)$$

Hence, $\alpha_1 = 1 - 2p(y_1) \in (0,1]$ and $\alpha_2 = 2p(y_2) - 1 \in (0,1]$. For both players, higher values of $\alpha_i$ indicate a greater willingness to bear the cost of information provision. If $\alpha_i$ is close to 0, then $i$ is not willing to provide much information and quits at $y_i$ close to zero. Conversely, if $\alpha_i = 1$, $i$ provides information no matter how far behind he is (i.e., $y_1 = -\infty$ or $y_2 = \infty$). Without risk of confusion, we write $U_i(\alpha)$, where $\alpha = (\alpha_1, \alpha_2)$ in place of $U_i(y)$. We let $W^k$ denote the war of information with costs $k = (k_1, k_2)$ and strategy sets $(0,1]^2$; that is, $W^k$ restricts players to stationary strategies. Lemma 1 below derives a simple expression for the players' payoffs given the strategy profile $\alpha$.

**Lemma 1:** *Player $i = 1,2$ wins with probability $\alpha_i/(\alpha_1+\alpha_2)$ and*

$$\begin{aligned} U_i(\alpha) &= \frac{\alpha_i}{\alpha_1+\alpha_2}\left(1 - k_i\alpha_j \ln\frac{1+\alpha_i}{1-\alpha_i}\right) \text{ for } j \neq i = 1,2 \\ U_3(\alpha) &= \frac{1}{2} + \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} \end{aligned}$$

*If $\alpha_i = 1$, then $U_i(\alpha) = -\infty$.*

The win-probabilities in Lemma 1 follow from the fact that $p(X_t)$ is a martingale and therefore

$$\Pr(1 \text{ wins})p(y_2) + \Pr(2 \text{ wins})p(y_1) = 1/2 \tag{10}$$

where the right hand side of the above equation is the prior. Substituting $(1-\alpha_1)/2$ for $p(y_1)$ and $(1+\alpha_2)/2$ for $p(y_2)$ yields the desired win-probabilities.

Lemma 2 below uses Lemma 1 to establish that player $i$'s best response to $\alpha_j$ is well-defined, single valued and differentiable. The lemma also shows that the war of information is dominance solvable. In Appendix B, we use this last fact to show that the war of information has a unique subgame perfect Nash equilibrium even if nonstationary strategies are permitted.

The function $B_i : (0,1] \to (0,1]$ is party 1's best response function if

$$U_1(B_1(\alpha_2), \alpha_2) > U_1(\alpha_1, \alpha_2)$$

for all $\alpha_2 \in (0,1]$ and $\alpha_1 \neq B_1(\alpha_2)$. Party 2's best response function is defined in an analogous manner. Then, $\alpha_1$ is a Nash equilibrium strategy for party 1 if and only if it is a fixed-point of the mapping $\phi$ defined by $\phi(\alpha_1) = B_1(B_2(\alpha_1))$. Lemma 2 below ensures that $\phi$ has a unique fixed-point.

**Lemma 2:** *There exist differentiable, strictly decreasing best response functions for both parties. Furthermore, if $\alpha_1 \in (0,1)$ is a fixed-point of $\phi$, then $0 < \phi'(\alpha_1) < 1$.*

Using Lemma 2, Proposition 1(i) below establishes that the war of information has a unique equilibrium. Proposition 1(ii) shows that a player becomes more aggressive if his cost decreases or his opponent's cost increases. Player $i$'s equilibrium strategy converges to 0 as his cost goes to infinity and converges to 1 as it goes to zero. It follows that any strategy profile $\alpha \in (0,1)^2$ is the equilibrium for some pair of costs.

**Proposition 1:** *(i) $W^k$ has a unique Nash equilibrium $\alpha^k$. (ii) The function $\alpha_i^k$ is strictly decreasing in $k_i$, strictly increasing in $k_j$ and has range $(0,1)^2$.*

**Proof:** Appendix A.

We have assumed that the states have equal prior probability. To model situations with an arbitrary prior $\pi$, we can choose the initial state $X_0 = x$ so that $p(x) = \pi$. The initial state does not affect the equilibrium; that is, if $(\alpha_1, \alpha_2)$ is the equilibrium for $X_0 = 0$ then $(\alpha_1, \alpha_2)$ is also an equilibrium for $X_0 = x$.

However, the prior does affect equilibrium payoffs and win probabilities. For example, if $\pi \neq 1/2$, then one of the parties may quit at time 0. If $(\alpha_1, \alpha_2)$ are the equilibrium strategies, then for

$$\pi \leq \frac{1 - \alpha_1}{2}$$

party 1 quits at time 0 while for

$$\pi \geq \frac{1+\alpha_2}{2}$$

party 2 quits at time 0. In those cases, the prior is so lopsided that the trailing party does not find the campaign worthwhile. The game ends in period 0 and the voter chooses the policy that the prior favors. For $\pi \in \left(\frac{1-\alpha_1}{2}, \frac{1+\alpha_2}{2}\right)$, the win probabilities satisfy the following version of Equation (10):

$$\Pr(1 \text{ wins})p(y_2) + \Pr(2 \text{ wins})p(y_1) = \pi$$

Recall that $p(y_1) = (1-\alpha_1)/2$ and $p(y_2) = (1+\alpha_2)/2$ and therefore

$$\Pr(1 \text{ wins}) = \frac{2\pi - 1 + \alpha_1}{\alpha_1 + \alpha_2}.$$

We have assumed that the drift of $X_t$ is $\mu \in \{-1/2, 1/2\}$ and its variance is 1. We can show that these assumptions are normalizations and entail no loss of generality. Let $\mu_1 > \mu_2$ be the drift parameters and let $\sigma^2$ be the variance. Define,

$$\delta = \frac{\sigma^2}{(\mu_1 - \mu_2)^2}$$

As we show in Appendix A (section 7.4), the parties' payoffs with arbitrary $\sigma^2, \mu_1, \mu_2$ are

$$U_i(\alpha) = \frac{\alpha_i}{\alpha_1 + \alpha_2}\left(1 - \delta k_i \alpha_j \ln \frac{1+\alpha_i}{1-\alpha_i}\right) \tag{11}$$

while the voter's payoff is unchanged. Hence, in equation (11), $\delta k_i$ replaces the $k_i$ of Lemma 1. After this modification, the analysis above extends immediately to the general $\mu_1, \mu_2$, and $\sigma^2$ case. The parameter $1/\delta$ measures the signal's informativeness and therefore increasing $\delta$ is like increasing both $k_1$ and $k_2$.

## 2.1 Both Parties Provide Information

Throughout, we have assumed that only the trailing party can provide information. Consider a simple extension in which both parties may incur costs if they choose but the second party's efforts generate no additional information. Then, if actions are unobservable, the leading party will never provide information. It can be shown that even if players can observe information provision efforts, the equilibrium of the war of information remains the unique subgame perfect equilibrium.

A more natural alternative extension is the Moscarini and Smith (2001) formulation. These authors assume the following signal process:

$$dX_t = \mu dt + \sigma(n_t)dZ_t$$

where $n_t$ is the number of parties that provide information at time $t$ and $\sigma(2) \leq \sigma(1)$.[8] Thus, the signal variance is reduced if both parties provide information. With this formulation, the equilibrium of Proposition 1 remains an equilibrium. To see why, note that if a party's strategy is stationary, its opponent has a strict incentive *not* to provide information when leading: $p_t$ is a martingale and therefore a player cannot increase the probability of winning (at most he may change the *speed* of learning) by providing additional information. Since information provision is costly, such a deviation would lower the party's payoff. Therefore, our equilibrium is also an equilibrium with the Moscarini-Smith formulation. However, the new game may admit other equilibria.[9]

## 3. Resources, Outcomes and Welfare

The parameters $k_1$ and $k_2$ quantify the effort a party or a candidate must exert to raise funds. A small $k_i$ means that the party has easy access to funds while a large $k_i$ indicates that the party finds it difficult to raise money. Proposition 1 implies that party 1's chance of winning is decreasing in $k_1$ and increasing in $k_2$. Hence, the advantaged party is more likely to win.

However, the effect of superior resources is limited: let $\pi = 1/2$ and suppose that party 1 has unlimited access to resources (i.e., $k_1$ is arbitrarily close to zero). For any fixed $k_2$, the probability that party 2 wins remains bounded away from zero. To see this, note

[8] Moscarini and Smith (2001) use the model $dX_t = \mu dt + \frac{\sigma}{\sqrt{n_t}}dZ_t$ to analyze the optimal level of experimentation in a decision problem with unknown drift. In their case, $n_t$ represents the number of signals the agent acquires and $dX_t$ represents the running sample mean of $n_t$ signals.

[9] To see how one might construct other equilibria, assume that $\sigma(2)/\sigma(1)$ is small so that the signal is much more informative when both parties provide information. We conjecture that there are equilibria in which parties "cooperate" by simultaneously providing information over some range of voter beliefs. This behavior reduces expenditures and can be sustained with the threat of reverting to the (less efficient) equilibrium in which only the trailing party provides information.

that $B_2(1)$ depends on $k_2$ but not $k_1$. Also, $\alpha_2 \geq B_2(1) > 0$ and $\alpha_1 \leq 1$ and therefore, party 2's win-probability $v_2$ satisfies

$$v_2 = \alpha_2/(\alpha_1 + \alpha_2) \geq B_2(1)/(1 + B_2(1)).$$

Next, we examine how the signal's informativeness (i.e., $\delta = \frac{\sigma^2}{(\mu_1-\mu_2)^2}$) affects the parties' win-probabilities and payoffs. The following proposition shows that if the signal is very informative ($\delta \to 0$), then both parties' payoffs converge to 1/2. In that case, all information is revealed and both parties win with equal probability. If the signal is very uninformative ($\delta \to \infty$), then the parties' payoffs depend on the cost ratio $k_2/k_1$. Define $h : I\!R_+ \to [0, 1]$ as follows:

$$h(s) = \frac{1}{3s}\left(s + 2\sqrt{1 - s + s^2} - 2\right)$$

and note that $h$ is increasing, $h(0) = 0, h(1) = 1/3$ and $\lim_{s\to\infty} h(s) = 1$. The following proposition shows that party 1's payoff in an uninformative war of information is $h(k_2/k_1)$ and while party 2's is $h(k_1/k_2)$. Let $W^{\delta k}$ be the war of information with cost $k$ and informativeness $\delta$, let $\alpha^{\delta k} = (\alpha_1^{\delta k}, \alpha_2^{\delta k})$ be the unique Nash equilibrium of $W^{\delta k}$ and let $V_i(\delta k)$ be player $i$'s payoff in that equilibrium.

**Proposition 2:** *(i)* $\lim_{\delta\to 0} V_i(\delta k) = 1/2$ *and* $\lim_{\delta\to\infty} V_i(\delta k) = h(k_j/k_i)$ *for* $j \neq i = 1, 2$. *(ii)* $\lim_{\delta\to 0} V_3(\delta k) = 1$, $\lim_{\delta\to\infty} V_3(\delta k) = 1/2$ *and* $V_3(\delta k)$ *is decreasing in* $\delta$.

**Proof:** Appendix A.

Since $h(0) = 0$, Proposition 2(ii) reveals that if the signal is uninformative, a party's win-probability converges to one as its cost goes to zero (and the opponent's cost stays fixed). If the two parties are evenly matched, then both prefer a very informative to a very uninformative signal; that is, if $k_1 = k_2$, $\lim_{\delta\to 0} V_i = 1/3$ while $\lim_{\delta\to\infty} V_i = 1/2$ for $i = 1, 2$. An informative signal leads to a quick resolution and therefore information expenditures are a vanishing fraction of the surplus. By contrast, a third of the surplus is spent providing information if the signal is uninformative and $k_1 = k_2$.

To determine campaign's value for the voter, note that without it the voter's payoff is 1/2. Hence the *value of the campaign* $w$ is

$$\begin{aligned} w &= U_3 - 1/2 \\ &= \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \end{aligned}$$

The above expression reveals that parties' actions are complements for the voter. If one party does not provide information ($\alpha_i = 0$), then $w = 0$. This complementarity suggests that the voter is best served by "balanced" campaigns; that is, campaigns in which costs are comparable. Our next results confirm this intuition.

Let $\delta = 1$; hence, $V_3(k)$ is the voter's equilibrium payoff. Let $c(k)$ be the sum of the parties' equilibrium expenditures given costs $k$. We say that $f : (0, \infty) \to \mathbb{R}_+$ is a threshold function if it satisfies the following properties:

(i) $f(s) < s$ and there is $z < \infty$ such that $f(s) = 0$ if and only if $s \leq z$.

(ii) $f$ is strictly increasing for $s \geq z$ and unbounded;

**Proposition 3:** *There is a threshold function $f$ such that*

*(i) $V_3(k)$ is increasing in $k_1$ at $k_1 < f(k_2)$ and decreasing in $k_1$ at $k_1 > f(k_2)$.*

*(ii) $c(k)$ is increasing in $k_1$ if $k_1 < f(k_2)$.*

**Proof:** Appendix A.

Proposition 3(i) shows that when parties' costs are sufficiently asymmetric, regulation that raises the advantaged party's cost increases voter welfare. Since $f(s) < s$, only the advantaged party can be below the threshold and hence raising the disadvantaged party's cost never benefits the voter. Moreover, if the disadvantaged party has costs below $z$, the threshold is zero. In that case, regulation that raises campaign costs always harms the voter. Figure 1 below illustrates the relation between costs and voter utility.

Regulation that increases the advantaged party's cost lowers its threshold and increases the disadvantaged party's (by Proposition 1) threshold. As a result, the disadvantaged party's payoff increases while the advantaged party's payoff decreases. Proposition 3(ii) implies that the sum of parties' payoffs decreases as a consequence of any regulation that benefits the voter.

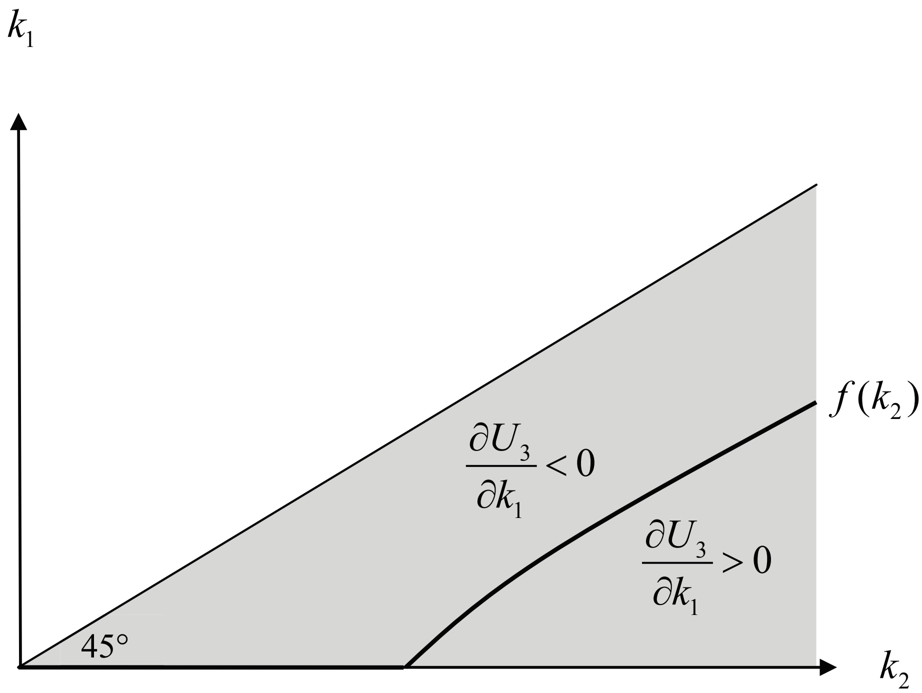


Figure 1

In some situations, regulation cannot target the advantaged party but affects both parties. Our next result shows regulation that increases both parties' costs equally will benefit the voter if the disadvantaged party's cost is sufficiently large.

**Proposition 4:** *For every $k_1$, there is $\bar{k}_2$ such that $k_2 > \bar{k}_2$ implies*

$$\frac{\partial V_3}{\partial k_1} + \frac{\partial V_3}{\partial k_2} > 0.$$

*at $k = (k_1, k_2)$.*

**Proof:** Appendix A.

Propositions 3(i) and 4 consider the welfare of the median voter who is indifferent between the two parties when the states are equally likely. Suppose every voter has a threshold $\gamma$ such that at $p_t = \gamma$ he is indifferent between the parties. At $p_t = 1/2$, voters with thresholds below 1/2 prefer party 1 while voters with thresholds above 1/2 prefer party 2. If party 1 is the advantaged party, any regulation that increases the median voter's utility also increases the utility of all voters in the latter group. Thus, a majority of voters benefit from the regulation but voters who have a sufficiently strong preference for the advantaged party's policy do not. Therefore, with a diverse population of voters, Propositions 3(i) and 4 imply only that the majority benefits from the regulation under the stated conditions.

Together, Propositions 3 and 4 provide a rationale for political campaigns. The key insight is that the war of information is a game of strategic substitutes between parties. Raising the advantaged party's cost will raise the disadvantaged party's threshold. For the median voter, the parties' actions are complements and, as a result, he prefers balanced campaigns. However, as we show in Proposition 3(ii), regulation that raises the median voter's utility also raises the resources spent during the campaign.

## 4. Extensions

So far, we have assumed that information costs are constant. If we interpret a party's cost as its fund-raising ability, then it seems plausible that this cost might depend on the party's standing in the polls. We can model this dependence by letting $k_i$ be a function of the voter's belief $p_t$. In section 4.1 below, we assume these cost functions are log-linear, compute the resulting payoff functions and establish that our earlier results are robust to this modification.

In Section 4.2, we investigate the effect of impatience. The difference between discounted and undiscounted cases is significant if one of the parties has unlimited resources (near-zero cost). As we show below, a party with near-zero cost captures all the surplus in the discounted case and therefore wins with near certainty. As we have shown in section 3, this is not true in the undiscounted case.

### 4.1 Variable Costs

In this subsection, we assume that party 1's information cost is decreasing while party 2's cost is increasing in $p_t$. To get a closed form expression similar to the one in Lemma 1, we assume that costs are linear functions of the log-likelihood ratio $\ln \frac{p_t}{1-p_t}$. Since

$X_t = \ln \frac{p_t}{1-p_t}$ this implies that costs are linear functions of $X_t$. Hence, party $i$ incurs flow-cost $(-1)^i k_i X_t$ while it provides information. Then, the expenditure functions (equations (6) and (7)) must be modified as follows:

$$c_1(y) = k_1 \, E \int_0^T \ln \frac{1-p_t}{p_t} C(p_t) dt$$

and

$$c_2(y) = k_2 \, E \int_0^T \ln \frac{p_t}{1-p_t} (1 - C(p_t)) dt$$

The game is unchanged in all other respects. Lemma 3 below shows how player's payoffs change with this simple formulation of belief-dependent costs.

**Lemma 3:** *Player $i = 1, 2$ wins with probability $\alpha_i/(\alpha_1 + \alpha_2)$ and*

$$U_i(\alpha) = \frac{\alpha_i}{\alpha_1 + \alpha_2} \left( 1 - k_i \alpha_j \left( \left( \ln \frac{1+\alpha_i}{1-\alpha_i} \right)^2 + \frac{2}{\alpha_i} \ln \frac{1+\alpha_i}{1-\alpha_i} - 4 \right) \right) \quad \textit{for } j \neq i \in \{1, 2\}.$$

$$U_3(\alpha) = \frac{1}{2} + \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}$$

*If $\alpha_i = 1$, then $U_i(\alpha) = -\infty$.*

The payoffs in Lemma 3 are similar to those in Lemma 1. The game is still dominance solvable and therefore has a unique equilibrium. Finally, the comparative statics of Proposition 1 continue to hold.

### 4.2 Discounting

In this subsection, we modify the war of information so that payoffs are discounted. Otherwise, the game is as described in section 2. Let $r > 0$ be the common discount rate and let $y_1 < 0 < y_2$ be stationary strategies for players 1 and 2 respectively. As before, let $T$ be the random time at which the game ends and let $p_t$ be the probability of the high-drift state. Let $C : [0, 1] \to [0, 1]$ be as defined in section 2 and set $C_1 = C$ and $C_2 = 1 - C$. Then, note that party $i$'s (expected discounted) expenditure is

$$c_i(y) = \frac{k_i}{2} \, E \int_0^T e^{-rt} C_i(p_t) dt \tag{5c}$$

Party $i$'s overall payoff is

$$U_i(y) = E\left[e^{-rT}C_j(p_T)\right] - c_j(y)$$

for $j \neq i$. In appendix C, we provide closed form expressions the players payoffs. The following result extends Proposition 1 to the discounted war of information $W_r^k$.

**Proposition 5:** *$W_r^k$ has a unique Nash equilibrium $y^k = (y_1^k, y_2^k)$. Furthermore, $|y_i^k|$ is strictly decreasing in $k_i$ and strictly increasing in $k_j$ for $j \neq i = 1, 2$.*

The next result describes the key difference between the discounted and the undiscounted cases. Fix $k_2$ and let $k_1$ converge to zero. Then, as in the undiscounted case, player 1's equilibrium strategy converges to $-\infty$, i.e., player 1 never gives up. However, unlike the undiscounted case, player 2's equilibrium strategy converges to zero, i.e., player 2 gives up immediately. Hence, with discounting, if a player has zero cost but his opponent does not he is almost sure to win. In that case, in equilibrium, the campaign provides no information and therefore has no value for the voter.

**Proposition 6:** *Let $k_* = (0, z)$ for some $z > 0$. Then, $\lim_{k \to k_*} y^k = (-\infty, 0)$.*

To see the intuition for Proposition 6, note that $y_1^k$ must converge to $-\infty$ as $k_1$ converges to zero because the marginal benefit of extending the threshold is always positive while the cost is going to zero. Since $y_1^k$ is going to $-\infty$, the random time at which player 2 can win is converging to $\infty$ (almost surely). Since player 2 discounts future payoffs, the value of winning goes to zero. However, expenditure stays bounded away from zero for any strictly positive threshold and hence quitting immediately is optimal for player 2.

## 5. Asymmetric Information

In this section, we analyze the war of information with asymmetric information. Specifically, we assume that party 1 knows the state and the voter is uninformed. To simplify the analysis, we consider a one-sided war of information in which only party 2 provides information. For the remainder of this section, we call party 1 *the party* and let $k = k_1$. Extending the analysis to include an uninformed[10] party 2 is straightforward and, since the analysis of the uninformed party would be the same as in the symmetric information case, we omit this extension.

The party is either type 1 or type 0. Type 1 is campaigning in the voter's interest while type 0 is advocating a policy that is bad for the voter. As in the symmetric information model, the party provides information at flow cost $k/2$. Information provision stops when the party quits or when the voter's belief that the party holds the correct position reaches 1/2. As long as information flow continues, the type $i$ party and the voters observe the process $X^i$:

$$X_t^i = \mu_i t + Z_t$$

where $\mu_i = i - {}^1\!/_2$ and $Z$ is a Wiener process. The key difference between this and the symmetric information setting is that now "not quitting" is itself a signal. As a result, the voter's beliefs depend not only on the current public signal $X_t^i$ but also on its history.

**Mixed Strategies:**

The analysis of asymmetric information requires that we introduce mixed strategies. Recall that a (stationary) pure strategy for the party is a number $x$ such that the party quits whenever $X_t$ reaches $x$. Thus, if the party chooses strategy $x$ then it quits by time $t$ if $x \geq \min\{X_\tau^i \,|\, \tau \leq t\}$. A *mixed strategy* is a cumulative distribution function (cdf), $G$, on the extended reals. The value $G(z)$ is the probability the party plays a pure strategy $x \geq -z$, that is; $G(z)$ is the probability that the party chooses a threshold that is less aggressive than or equal to $-z$. The party's strategy is a pair of cdfs $\alpha = (G^0, G^1)$ where $G^i$ is the strategy of type $i$. Let

$$Y_t^i = \inf_{\tau < t} X_\tau^i$$

[10] The uninformed party would have the same information as the voter.

be the stochastic process that keeps track of the lowest realization of $X_t^i$ during the interval $[0,t]$. Given the strategy $\alpha = (G^0, G^1)$, $G^i(-Y_t^i)$ is the probability that type $i$ quits by time $t$ as a function of the realized sample path.

**Beliefs:**

For a given strategy profile $\alpha$, the stochastic process $L_t^{i\alpha}$ is type-$i$'s prediction[11] of the voter's belief at time $t$; that is, the probability that the voter assigns to the party being type 1. If the probability that the party quits by time $t$ is less than one, i.e., $G^0(-Y_t^i) \cdot G^1(-Y_t^i) < 1$, then, $L_t^{i\alpha}$ is determined by Bayes' Law. In that case, we have

$$\begin{aligned} L_t^{i\alpha} &= \frac{(1 - G^1(-Y_t^i))f_1(X_t^i - X_0)\pi}{(1 - G^1(-Y_t^i))f_1(X_t^i - X_0)\pi + (1 - G^0(-Y_t^i))f_0(X_t^i - X_0)(1-\pi)} \\ &= \frac{1 - G^1(-Y_t^i)}{1 - G^1(-Y_t^i) + (1 - G^0(-Y_t^i))e^{-X_t^i}} \end{aligned}$$

where $f_i$ be the normal density with mean $\mu_i$ and variance 1. When $G^0(-Y_t^i) \cdot G^1(-Y_t^i) = 1$ Bayes' Law does not apply. In this case, we set

$$L_t^{i\alpha} = 1$$

That is, if a party deviates and does not quit after a history at which both party types were supposed to quit with probability 1, the voter interprets this as a sign of strength and assigns probability 1 to type 1. This notion of equilibrium incorporates a signalling refinement similar to those in Banks and Sobel (1987) and Cho and Kreps (1987). It is a stronger requirement than necessary for our result. Our results continue to hold as long as voters do not interpret not-quitting off the equilibrium path as evidence of type 0.[12] We provide a more detailed discussion of equilibrium refinements below. There we also identify other equilibria that emerge without any restrictions on beliefs.

**Payoffs and Equilibrium:**

For a type $i$ party, the game ends if $L^{i\alpha}$ reaches 1/2 or if the party quits. The voter chooses policy 1 in the former case and policy 0 in the latter. For any belief process $L$, let

[11] Since the party has more information than the voter, it's estimate of the current voter belief is correct.
[12] We use the stronger requirement purely for expositional reasons.

$\tau(L) = \inf_t\{L_t \geq 1/2\}$. The probability that type $i$ wins, given any $L$ and strategy $G$, is $1 - G(-Y_{\tau(L)})$. Hence, the ex ante winning probability is

$$v^i(G, L) = E[1 - G(-Y^i_{\tau(L)})]$$

where the expectation is taken over the possible realizations of $X^i$.[13]

Over the finite time interval $[0, T]$, the party's expenditure is

$$c^i_T(G, L) = \frac{k}{2} E \int_{t=0}^{\tau(L)\wedge T} t dG(-Y^i_t)$$

where $\tau(L)$ and the expectation are as defined above. Note that $c^i_T(G, L)$ is increasing in $T$ and may have an infinite limit. The party's payoff given $\alpha = (G^0, G^1)$ is $U^i(G^i, L^{i\alpha}) = v^i(G^i, L^{i\alpha}) - c^i(G^i, L^{i\alpha})$. When we wish to be explicit about the initial state $x = X_0$, we write $U^i_x$ instead of $U^i$.

Let $W^k_*$ denote the game defined in this section. The strategy $\alpha$ is a *monotone equilibrium* for game $W^k_*$ if no type has an incentive to deviate given the beliefs $L^{i\alpha}$. That is, $U^i(L^{i\alpha}, G^i) \geq U^i(L^{i\alpha}, G)$ for all $G$ and $i = 0, 1$.[14] We refer to the equilibrium as *monotone* since it builds in the restriction on out of equilibrium beliefs described above. We discuss non-monotone equilibria below.

The voter is not a player in the game $W^k_*$; his behavior is an exogenously specified function of $L$. The following alternative formulation with a strategic voter would yield exactly the same results: at time $t$, the voter specifies what he will do if the party quits during the interval $(t, t + \Delta]$. If the party quits, this decision is implemented; otherwise, the voter revises his decision and the game continues.[15]

[13] Thus, we are assuming that the party wins if it never quits. This convention does not affect our results.

[14] Note that deviations do not affect $L^{i\alpha}$.

[15] If the voter observes the quit decision before choosing a policy, then equilibria in which the party provides information beyond the belief threshold 1/2 can be sustained: the voter may infer from an off-equilibrium path quit decision that the party is type 0 and this inference may deter the party from quitting. Such equilibria are not robust and are ruled out by a perturbation in which information provision stops exogenously with some small type-independent probability. Hence, our equilibria are also the robust equilibria of the game with a strategic voter who moves after the quit decision.

**Results:**

Proposition 7, below, shows that in a monotone equilibrium, type 1 never quits. Thus, the unique equilibrium strategy of type 1 is $G^1 = 0$. Next, we identify a class of strategies that contains the equilibrium strategy of type 0. For any real number $z \in I\!R$, let $F_z$ be the following cdf:

$$F_z(-x) = \begin{cases} 0 & \text{if } x > z \\ 1 - e^{x-z} & \text{if } x \leq z \end{cases}$$

Let $Y_t^{iz} = \min\{z, Y_t^i\}$ and note that $F_z(-Y_t^i) = F_z(-Y_t^{iz})$.

Next, we compute the beliefs $L_t^{i\alpha}$ for the party strategy $\alpha = (F_z, 0)$ and show that these beliefs are bounded below by $p(z) = 1/(1 + e^{-z})$.

$$\begin{aligned} L_t^{iz} := L_t^{i\alpha} &= \frac{1}{1 + (1 - F_z(-Y_t^{iz}))e^{-X_t^i}} \\ &= \frac{1}{1 + e^{-(X_t^i - Y_t^{iz} + z)}} \\ &\geq \frac{1}{1 + e^{-z}} \end{aligned} \tag{$*$}$$

where the inequality follows from $X_t^i - Y_t^{iz} \geq 0$. The voter's belief would be $p(X_t^i)$ if both types never quit. By adding $z - Y_t^{iz}$ to the signal, the voter incorporates the information that the party reveals by not quitting until $t$. Since $X_t^i - Y_t^{iz} \geq 0$, the belief can never drop below $p(z)$ and hence we call $p(z)$ the *signaling barrier*. To sustain this reflecting barrier, type 0 quits with a probability that exactly offsets any negative $X_t^i$-information once the barrier is reached. If the initial belief is below the signaling barrier (i.e., $\pi < p(z)$ or equivalently $X_0 < z$), then $F_z(X_0) > 0$. In this case, type 0 quits with strictly positive probability $F_z(X_0)$ at $t = 0$ so that $L_0^{0z} = p(z)$.

Let $z_*$ be the unique negative solution to the equation

$$e^{-z_*} + z_* = \frac{k+1}{k} \tag{$**$}$$

**Proposition 7:** *The strategy $(F_{z_*}, 0)$ is the unique monotone equilibrium of $W_*^k$.*

**Proof:** Appendix D.

As long as $L_t^{iz_*} > p(z_*)$, the above equilibrium of $W_*^k$ is like the equilibrium of the war of information $W^k$; the current signal $X_t^i$ determines beliefs. However, once $L^{iz_*}$ reaches $p(z_*)$, the quit decision also affects beliefs. In fact, type 0 quits at a rate that exactly offsets any negative information revealed by $X_t^i$. If the party has not quit and $X_t^i < z_*$, the voter concludes that either he is facing type 1 or he is facing type 0 but by chance the random quitting strategy had the party continue until time $t$. The probability that type 0 quits by time $t$ is $1 - e^{X_t^i - z_*}$. Hence, when $X_t^i$ is "very negative," the party counters the public information with its private information.

An observer who ignores this signaling component might incorrectly conclude that the voter chooses the wrong position. Evidence that in a nonstrategic environment would indicate that the party holds the incorrect position (i.e., $X_T^i < 0$) may nonetheless result in the voter adopting the party's favored position. Hence, ignoring the signaling component creates the appearance of bias in favor of the party conducting the campaign.

When the belief depends only on $X^i$ (as in the case of symmetric information), it is a function of the current signal $X_t^i$ and independent of the path $(X_\tau^i)_{\tau<t}$. By contrast, the belief process is path-dependent in $W_*^k$. In particular, conditional on the party not having quit, recent (positive) public information is given greater weight than past negative information. To see this, note that for a given $X_t^i$, the belief is decreasing in $Y_t^i = \inf_{\tau \le t} X_t^i$. Thus, if the signaling component is ignored, the voter appears to put too much weight on recent information.

The signaling barrier's location depends on $k$ but not $\pi$ while the probability that the party wins depends on $\pi$ but not $k$. In particular, equation (**) reveals that $z_*$ is increasing in $k$. If $\pi < p(z_*)$, then type 0 quits with strictly positive probability at time 0 so that conditional on not quitting the voter's belief jumps to $p(z_*)$.

Type 1 wins for sure because he never quits and $X^1$ has strictly positive drift. To compute type 0's win-probability, note that once the game terminates, the voter assigns either probability 1/2 (in case the party wins) or 0 (in case the party quits) to type 1. Therefore,

$$\pi = \frac{1}{2} \cdot \left(\pi \Pr(\text{type 1 wins}) + (1-\pi) \Pr(\text{type 0 wins})\right)$$

Since Pr(type 1 wins) = 1, we have

$$\Pr(\text{type 0 wins}) = \frac{\pi}{1-\pi}$$

A higher $k$ makes information provision more costly but also makes not-quitting a stronger signal. These two effects cancel leaving type 0's win-probability unchanged. Thus, we have demonstrated the following corollary:

**Corollary 1:** *The probability that type* 0 *wins the game* $W_*^k$ *is* $\frac{\pi}{1-\pi}$ *irrespective of* $k$.

Our analysis of $W_*^k$ incorporates a simple and strong restriction on off-equilibrium-path beliefs: we require that if the party does not quit when the candidate equilibrium strategy specifies quitting, the voter should interpret this as strength and assume that he is dealing with a type-1 party. Since $\mu_1 > \mu_0$, the type-1 party does indeed get a higher payoff from continuing while both types get 0 if they quit. Hence, our refinement is in the same spirit as those of Banks and Sobel (1987) and Cho and Kreps (1987).[16] Our off-equilibrium-path beliefs can be rationalized with perturbations that put infinitely less weight on type-0 not quitting than on type-1 not quitting.

The same result would obtain if we used the following weaker refinement. After every history, type 1's deviation (to not-quitting) is deemed at least as likely as type 0's deviation. This refinement would also identify the equilibrium in Proposition 7 as the unique equilibrium. The reason we used the stronger requirement is expositional. The weaker refinement would necessitate a cumbersome specification of how off equilibrium path beliefs respond to $X_t^i$. Our stronger restriction facilitates the simpler exposition above.

As in the case of symmetric information, we have restricted the party to (probability distributions over) stationary strategies. However, it is not too difficult to see that even if we allowed non-stationary strategies the equilibrium of Proposition 7 would remain the unique monotone equilibrium. Thus, as in the symmetric information case, the stationarity restriction is without loss of generality.

[16] Since $W_*^k$ is an infinite horizon continuous-time game, we cannot literally apply the Banks-Sobel or Cho-Kreps refinements.

As is typical of signaling games, without any restriction on off equilibrium path beliefs, the war of information with asymmetric information has many stationary and nonstationary equilibria. For example, take any $z \in [z_*, 0]$ and consider the pure strategy profile $\alpha = (z, z)$, (i.e., both parties quit the first time $X_t^i$ reaches $z$). Now, suppose that if the game does not end when $X_t^i$ reaches $z$, the voter assumes that the party is the weak type. Formally, define $\hat{L}^{i\alpha}$ as follows: $\hat{L}_t^{i\alpha} = \frac{1}{1+e^{-X_t^i}}$ whenever $Y_t^i > z$ and $\hat{L}_t = 0$ otherwise.[17] Hence, $\hat{L}_t^{i\alpha}$ is derived from $\alpha$ whenever Bayes' Law applies and is equal to 0 if it does not.

The strategy profile $\alpha$ is an equilibrium if we replace $L^{i\alpha}$ with $\hat{L}^{i\alpha}$ in the above definition: since the voter believes that deviating by continuing to provide information when $X_t^i < z$ is proof of weakness, the party has no incentive to do so. In this equilibrium, the voter's out of equilibrium beliefs punish the party for not quitting and therefore *less* information is revealed than would have been revealed if the party were uninformed.

## 6. Conclusion

We have analyzed political campaigns with a model in which two parties provide information to convince a voter. A key feature of our model is that information is conveyed to voters through a continuous process. This feature adds tractability but also has substantive implications.

If only one party can provide information (as in our asymmetric information model), it would stop as soon as the voter is convinced that its policy is as good as the alternative, i.e., when the voter is just indifferent. Because information arrives continuously, the voter can indeed be made just indifferent, and, as a result, receives no surplus: the policy that was optimal given the prior remains optimal at the end of the campaign. To benefit from the campaign, the voter needs competition between parties. We show that the voter benefits most when parties are equally matched - providing a rationale for regulating political campaigns.

When a party knows the state, the indirect inference from its campaign spending will interact with the direct information it provides. If the strategic interaction between the party and voter is ignored, the latter seems biased in favor of the party conducting the campaign. In particular, we show that no matter how much unfavorable direct information is revealed, the voter's belief cannot drop below a threshold we call the signaling barrier.

[17] Recall that $Y_t^i = \min_{\tau \leq t} X_t^i$.

# 7. Appendix A

## 7.1 Proof of Lemma 1

Let $c_1(y \mid \mu)$ be player 1's expenditure given $y = (y_1, y_2)$ and drift $\mu$. First, we will show that

$$c_1(y \mid \mu) = \frac{k_1}{\mu^2} \left( \frac{1 - e^{-2\mu y_2}}{1 - e^{-2\mu(y_2 - y_1)}} \right) \left(1 - e^{2\mu y_1}(1 - 2\mu y_1)\right) \tag{A1}$$

For $z_1 < 0 < z_2$, let $P(z_1, z_2)$ be the probability that $X_t$ hits $z_2$ before it hits $z_1$ and $T(z_1, z_2)$ be the expected time $X_t$ spends until it hits either $z_1$ or $z_2$ given $X_0 = 0$ and drift $\mu$. Harrison (1985 p. 43 and 52) shows that

$$\begin{aligned} P(z_1, z_2) &= \frac{1 - e^{2\mu z_1}}{1 - e^{-2\mu(z_2 - z_1)}} \\ T(z_1, z_2) &= \frac{(z_2 - z_1)P(z_1, z_2) + z_1}{\mu} \end{aligned} \tag{A2}$$

To compute $c_1(y \mid \mu)$, let $\epsilon \in (0, y_2]$ and assume that player 1 bears the cost until $X_t \in \{y_1, \epsilon\}$. Then, player 2 bears the cost until $X_{t+\tau} \in \{0, y_2\}$ if $X_t = \epsilon$; otherwise (i.e., if $X_{t+\tau} = 0$), the process repeats with player 1 again bearing the cost until $X_{t+\tau+\tau'} \in \{y_1, \epsilon\}$ and so on. This procedure yields an upper bound for $c_1(y \mid \mu)$. Let $(k_1/2)T^\epsilon$ denote that upper bound and note that

$$T^\epsilon = T(y_1, \epsilon) + P(y_1, \epsilon)(1 - P(-\epsilon, y_2 - \epsilon))T^\epsilon$$

Substituting for $T(y_1, \epsilon)$ and $P(y_1, \epsilon)$ from $(A2)$, we get

$$\mu T^\epsilon = \left( \frac{(\epsilon - y_1)(1 - e^{2\mu y_1})}{1 - e^{-2\mu(\epsilon - y_1)}} + y_1 \right) \left( 1 - \frac{(1 - e^{2\mu y_1})(e^{-2\mu\epsilon} - e^{-2\mu y_2})}{(1 - e^{-2\mu(\epsilon - y_1)})(1 - e^{-2\mu y_2})} \right)^{-1}$$

and therefore

$$c_1(y \mid \mu) \leq (k_1/2) \lim_{\epsilon \to 0} T^\epsilon = \frac{k_1}{\mu^2} \left( \frac{1 - e^{-2\mu y_2}}{1 - e^{-2\mu(y_2 - y_1)}} \right) \left(1 - e^{2\mu y_1}(1 - 2\mu y_1)\right)$$

An analogous lower bound converges to the right hand side of $(A1)$ as $\epsilon \to 0$ from below proving $(A1)$.

Since $c_1(y)$ is the average of the two $c_1(y,\mu)$'s, $(A1)$ and the definition $\alpha_i$ yield

$$c_1(y) = \frac{k_1\alpha_1 \cdot \alpha_2}{\alpha_1 + \alpha_2} \ln \frac{1+\alpha_1}{1-\alpha_1}$$

Let $v$ be the probability that player 1 wins. Since $p_T$ is a martingale and $T < \infty$,

$$vp(y_2) + (1-v)p(y_1) = E(p_T) = 1/2$$

and hence, $v = \frac{\alpha_1}{\alpha_1+\alpha_2}$ and

$$U_1(\alpha) = \frac{\alpha_1}{\alpha_1+\alpha_2}\left(1 - k_1\alpha_2 \ln \frac{1+\alpha_1}{1-\alpha_1}\right)$$

A symmetric argument establishes the desired result of $U_2$. □

### 7.2 Proof of Lemma 2

By Lemma 1, party $i$'s utility is strictly positive if and only if

$$\alpha_i \in \left(0, \frac{e^{\frac{1}{k_i\alpha_j}} - 1}{e^{\frac{1}{k_i\alpha_j}} + 1}\right).$$

Furthermore, throughout this range, $U_i(\cdot, \alpha_j)$ is twice continuously differentiable and strictly concave in $\alpha_i$. To verify strict concavity, note that $U_i$ is the product of a strictly increasing concave function $f \geq 0$ and a strictly decreasing concave function $g \geq 0$. Hence, $(f \cdot g)^{''} = f^{''}g + 2f^{'}g^{'} + fg^{''} < 0$. Therefore, the first order condition characterizes the unique best response to $\alpha_j$. Player $i$'s first order condition is:

$$U_i = \frac{2\alpha_i^2 k_i}{1-\alpha_i^2} \qquad (A3)$$

Note that $(A3)$ implicitly defines the best response functions $B_i$. Equation $(A3)$ together with the implicit function and the envelope theorems yield

$$\frac{dB_i}{d\alpha_j} = \frac{\partial U_i}{\partial \alpha_j} \cdot \frac{(1-\alpha_i^2)^2}{4\alpha_i k_i} \qquad (A4)$$

Equation $(A3)$ also implies

$$\frac{\partial U_i}{\partial \alpha_j} = -\frac{1}{\alpha_1+\alpha_2}\left(U_i + \alpha_i k_i \ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right)\right) \qquad (A5)$$

Note that $(A5)$ implies $\frac{\partial U_i}{\partial \alpha_j} < 0$. The three equations $(A3)$, $(A4)$, and $(A5)$ yield

$$\frac{dB_i}{d\alpha_j} = -\frac{\alpha_i(1-\alpha_i^2)}{2(\alpha_1+\alpha_2)} \cdot \left(1 + \frac{1-\alpha_i^2}{2\alpha_i} \ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right)\right). \tag{A6}$$

Then, since $\ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right) \leq \frac{2\alpha_i}{1-\alpha_i}$, we have

$$\frac{dB_i}{d\alpha_j} \geq -\frac{\alpha_i(1-\alpha_i^2)(2+\alpha_i)}{2(\alpha_1+\alpha_2)}.$$

Hence, since $\phi' = \frac{dB_1}{d\alpha_2}\frac{dB_2}{d\alpha_1}$, we conclude

$$0 < \phi'(\alpha_1) \leq \frac{\alpha_1(1-\alpha_1^2)(2+\alpha_1)\alpha_2(1-\alpha_2^2)(2+\alpha_2)}{4(\alpha_1+\alpha_2)^2}$$

Note that the $\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2} \leq 1/2$ and, hence, $\phi'(\alpha_1) < 1$ if

$$(1-\alpha_i^2)(2+\alpha_i) < 2\sqrt{2}.$$

The left-hand side of the equation above reaches its maximum at $\alpha_i < 1/2$ and at such $\alpha_i$ is no greater than $5/2 < 2\sqrt{2}$, proving that $0 < \phi'(\alpha_1) < 1$. $\square$

### 7.3 Proof of Proposition 1

*Part (i):* By Lemma 2, $B_i$'s are decreasing, continuous functions. It is easy to see that $B_i(1) > 0$ and $\lim_{s\to 0} B_i(s) = \sqrt{\frac{1}{1+2k_i}}$. Hence, we can continuously extend $B_i$ and $\phi$ to the compact interval $[0,1]$ and the extended $\phi$ must have a fixed-point. Since $B_i$ is strictly decreasing, $B_i(0) < 1$ implies that this fixed-point is not 1. Since $B_i(1) > 0$, every fixed-point must be in the interior of $[0,1]$. Let $s$ be the infimum of all fixed-points. Clearly, $s$ itself is a fixed-point and hence $s \in (0,1)$. Since $\phi'(s) < 1$, there exists $\varepsilon > 0$ such that $\phi(s') < s'$ for all $s' \in (s, s+\varepsilon)$. Let $s^* = \inf\{s' \in (s,1) \mid \phi(s') = s'\}$. If the latter set in nonempty, $s^*$ is well-defined, a fixed-point and not equal to $s$. Since $\phi(s') < s'$ for all $s' \in (s, s^*)$, we must have $\phi'(s^*) \geq 1$, contradicting Lemma 2. Hence, $\{s' \in (s,1) \mid \phi(s') = s'\} = \emptyset$ proving that $s$ is the unique fixed-point of $\phi$ and hence the unique equilibrium of the war of information.

*Part (ii):* View party 1's best response as a function of both $\alpha_2$ and $k_1$. Then, the unique equilibrium $\alpha_1$ satisfies

$$B_1(B_2(\alpha_1), k_1) = \alpha_1.$$

With the arguments of Lemma 2, it is straightforward to show that $B_1(\cdot,\cdot)$ is a differentiable function. Taking the total derivative of the equation above and rearranging terms yields

$$\frac{d\alpha_1}{dk_1} = \frac{\frac{\partial B_1}{\partial k_1}}{1 - \frac{d\phi}{d\alpha_1}}$$

where $\frac{d\phi}{d\alpha_1} = \frac{\partial B_1}{\partial \alpha_2} \cdot \frac{dB_2}{d\alpha_1}$. By Lemma 1, $\phi' < 1$. Taking the total derivative of $(A3)$ (for fixed $\alpha_2$) establishes that $\frac{\partial B_1}{\partial k_1} < 0$ and hence $\frac{d\alpha_1}{dk_1} < 0$ as desired. Then, note that $k_1$ does not appear in $(A3)$ for player 2. Hence, a change in $k_1$ affects $\alpha_2$ only through its effect on $\alpha_1$ and therefore has the same sign as

$$\frac{dB_2}{dk_1} = \frac{dB_2}{d\alpha_1} \cdot \frac{d\alpha_1}{dk_1} > 0 \qquad (A7)$$

By symmetry, we also have $\frac{d\alpha_2}{dk_2} < 0$ and $\frac{d\alpha_1}{dk_2} > 0$.

As $k_i$ goes to 0, the left-hand side of $(A3)$ is bounded away from 0. Hence, $\frac{2\alpha_i^2}{1-\alpha_i^2}$ must go to infinity and therefore $\alpha_i$ must go to 1. Since $U_i \leq 1$, it follows from $(A3)$ that $k_i \to \infty$ implies $\alpha_i$ goes to 0. Fix $(\alpha_1, \alpha_2)$ and note that $B_i(\alpha_j, \cdot)$ is a continuous function and hence by the above argument there is $k_i$ such that $B_i(\alpha_j, k_i) = \alpha_i$. □

### 7.4 Arbitrary $\mu_1$, $\mu_2$ and $\sigma$

Let $X_t$ be a signal state-dependent drift ($\mu_1 > \mu_2$) and arbitrary variance $\sigma^2$. We can rescale time so that each new unit corresponds to $1/\delta = \frac{(\mu_1-\mu_2)^2}{\sigma^2}$ old units. The flow-costs with the new time units is $\hat{k}_i = \delta k_i$, where $k_i$ is party $i$'s in the old time units. Let $\hat{X}_i$ be the signal process in the new time unit and note that the state-dependent drift is $\hat{\mu}_i = \delta\mu_i$ and the variance is $\hat{\sigma}^2 = \delta\sigma^2$. Observe that $(\hat{\mu}_1 - \hat{\mu}_2)/\hat{\sigma} = 1$.

Let

$$Z_i = \frac{\mu_1 - \mu_2}{\sigma^2}\left(\hat{X}_i - \frac{\hat{\mu}_1 + \hat{\mu}_2}{2} t\right)$$

A simple calculation shows that $Z_1$ has drift $1/2$ and variance 1 and $Z_2$ has drift $-1/2$ and variance 1. Since $Z_i$ is a deterministic function of $\hat{X}_i$ the equilibrium with signal $\hat{X}_i$ must be the same as the equilibrium with signal $Z_i$. Hence, the game with time renormalized corresponds to the simple war of information analyzed above.

### 7.5 Proof of Proposition 2

Let $\alpha_i = 1 - \epsilon$. Then, for $\delta$ small we have $U_i \geq \frac{1-\epsilon}{2-\epsilon} - \epsilon$ for $i = 1, 2$. Since $\epsilon$ can be chosen arbitrarily small, it follows that $U_i \to 1/2$ as $\delta \to 0$. Equation $(A3)$ implies that $\alpha_i \to 1$ which in turn implies that $U_3 \to 1$.

We suppress the superscript $\delta k$ and note that $\alpha_i \to 0$ and hence $U_3 \to 1/2$ as $\delta \to \infty$. Let $s = k_2/k_1$ and define $a = \alpha_2/\alpha_1$ and $z = \alpha_1^2 \delta k_1$. Then, $(A3)$ can be re-written as

$$\frac{1}{1+a}\left(1 - az\frac{\ln\left(\frac{1+\alpha_1}{1-\alpha_1}\right)}{\alpha_1}\right) = \frac{2z}{1-\alpha_1^2}$$
$$\frac{1}{1+a}\left(1 - azs\frac{\ln\left(\frac{1+\alpha_2}{1-\alpha_2}\right)}{\alpha_2}\right) = \frac{2azs}{1-\alpha_2^2}$$

These two equations imply that $z, a$ are bounded away from zero and infinity for large $\delta$. Moreover, as $\delta \to \infty$ it must be that $\alpha_i \to 0$ for $i = 1, 2$ and therefore, $\frac{1}{\alpha_i}\ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right) \to 2$. Hence, the limit solution to the above equations satisfies

$$\frac{1}{1+a}(1 - 2az) = 2z$$
$$\frac{1}{1+a}(1 - 2azs) = 2azs.$$

Solving the two equations for $a, z$ and substituting the solutions into $(A3)$ yields $U_1 = 2z = \frac{1}{3s}\left(s + 2\sqrt{1 - s + s^2} - 2\right)$ and $U_2 = 2a^2zs = \frac{1}{3}\left(1 + 2\sqrt{1 - s + s^2} - 2s\right)$.

Note that $U_3$ is decreasing in $\alpha_i$'s. Therefore, it is sufficient to show that both $\alpha_i$'s are decreasing in $\delta$. Substituting for $U_i$ from Lemma 1 into $(A3)$ and yields

$$\frac{1}{\delta k_i} = \alpha_j \ln\frac{1+\alpha_i}{1-\alpha_i} + 2\alpha_i\frac{\alpha_1 + \alpha_2}{1-\alpha_i^2}$$

Taking a derivative with respect to $\delta$ and evaluating at $\delta = 1$ yields

$$-\frac{1}{k_1}d\delta = \left(4\frac{\alpha_1 + \alpha_2}{(1-\alpha_1)^2(1+\alpha_1)^2}\right)d\alpha_1 + \left(\ln\left(\frac{1+\alpha_1}{1-\alpha_1}\right) + \frac{2\alpha_1}{1-\alpha_1^2}\right)d\alpha_2$$

and an analogous equation for player 2.

Let $D_i = \ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right)$. Then the four equations above yield

$$-\left(\alpha_2 D_1 + \frac{2\alpha_1(\alpha_1+\alpha_2)}{1-\alpha_1^2}\right) d\delta = 4\frac{\alpha_1+\alpha_2}{(1-\alpha_1)^2(\alpha_1+1)^2} d\alpha_1 + \left(D_1 + \frac{2\alpha_1}{1-\alpha_1^2}\right) d\alpha_2$$

and

$$-\left(\alpha_1 D_2 + \frac{2\alpha_1(\alpha_1+\alpha_2)}{1-\alpha_2^2}\right) d\delta = 4\frac{\alpha_1+\alpha_2}{(1-\alpha_2)^2(\alpha_2+1)^2} d\alpha_2 + \left(D_2 + \frac{2\alpha_2}{1-\alpha_2^2}\right) d\alpha_2$$

Solving the two equations above for $\frac{d\alpha_1}{d\delta}$ yields

$$\frac{\left(\alpha_1 D_2 + \frac{2\alpha_2(\alpha_1+\alpha_2)}{1-\alpha_2^2}\right)\left(D_1 + \frac{2\alpha_1}{1-\alpha_1^2}\right) - \left(\alpha_2 D_1 + \frac{2\alpha_1(\alpha_1+\alpha_2)}{1-\alpha_1^2}\right)\frac{4(\alpha_1+\alpha_2)}{(1-\alpha_2)^2(1+\alpha_2)^2}}{\frac{4(\alpha_1+\alpha_2)}{(1-\alpha_2)^2(1+\alpha_2)^2}\frac{4(\alpha_1+\alpha_2)}{(1-\alpha_1)^2(1+\alpha_1)^2} - \left(D_1 + \frac{2\alpha_1}{1-\alpha_1^2}\right)\left(D_2 + \frac{2\alpha_2}{1-\alpha_2^2}\right)}$$

Next, we will show that the above expression is always negative. We will verify that the numerator is always negative; analogous calculations for that the denominator reveal that it is always positive. Using the bound $D_i \leq \frac{2\alpha_i}{1-\alpha_i}$, the numerator is less than

$$\left(\frac{2\alpha_1\alpha_2}{1-\alpha_2} + \frac{2\alpha_2(\alpha_1+\alpha_2)}{1-\alpha_2^2}\right)\left(\frac{2\alpha_1}{1-\alpha_1} + \frac{2\alpha_1}{1-\alpha_1^2}\right) - \frac{2\alpha_1(\alpha_1+\alpha_2)}{(1-\alpha_1^2)}\frac{4(\alpha_1+\alpha_2)}{(1-\alpha_2)^2(\alpha_2+1)^2}$$

which is always negative. □

### 7.6 Proof of Propositions 3 and 4

Again, we suppress $\alpha_i$'s superscript $k$. From Lemma 1, we have:

$$\frac{dU_3}{dk_1} = \frac{\alpha_2^2}{(\alpha_1+\alpha_2)^2}\frac{d\alpha_1}{dk_1} + \frac{\alpha_1^2}{(\alpha_1+\alpha_2)^2}\frac{d\alpha_2}{dk_1}$$

Since $\alpha_2 = B_2(\alpha_1)$, $(A6)$ and $(A7)$ imply $\frac{dU_3}{dk_1} < 0$ if and only if

$$\frac{\alpha_2}{\alpha_1} - \frac{\alpha_1}{2(\alpha_1+\alpha_2)} \cdot \left[1 - \alpha_2^2 + \frac{(1-\alpha_2^2)^2}{2\alpha_2}\ln\left(\frac{1+\alpha_2}{1-\alpha_2}\right)\right] > 0 \qquad (A8)$$

For $\alpha \in (0,1]$, let $g(\alpha_1) \in (0,1]$ be the $\alpha_2$ that solves

$$\frac{\alpha_2}{\alpha_1} - \frac{\alpha_1}{2(\alpha_1+\alpha_2)} \cdot \left[1 - \alpha_2^2 + \frac{(1-\alpha_2^2)^2}{2\alpha_2}\ln\left(\frac{1+\alpha_2}{1-\alpha_2}\right)\right] = 0$$

First, we show that $g$ is well-defined: for any fixed $\alpha_1$, the left hand side of $(A8)$ is negative for $\alpha_2$ sufficiently close to zero and strictly positive for $\alpha_2 = \alpha_1$. Note that $\frac{\alpha_1}{2(\alpha_1+\alpha_2)}$, $1-\alpha_2^2$, and the last term inside the square bracket are all decreasing in $\alpha_2$. Hence $g$ is well defined. Note also that the left hand side of $(A8)$ is decreasing in $\alpha_1$. Hence $g$ is increasing. Since the terms in the brackets add up to less than 2 it follows that $g(\alpha_1) < \alpha_1$. Let $\hat{\alpha}_2 = g(1)$ and note that $g \leq \hat{\alpha}_2 < 1$. Finally, it is easy to verify that $g$ is continuous.

**Proof of Proposition 3(i):** We first show that for every $\alpha$, there is a unique $k$ such that $\alpha$ is the equilibrium of $W^k$. To see this, let $B_1(\alpha_1, k_1)$ be $i$'s best response to $\alpha_2$ given cost $k_1$. Taking the total derivative of $(A3)$ establishes that $\frac{\partial B_1}{\partial k_1} < 0$ and proves that the mapping that associates an equilibrium $\alpha^k$ with each $k$ is one-to-one and hence invertible. Let $\kappa = (\kappa_1, \kappa_2)$ be the inverse of this mapping. It is straightforward to show that $\kappa$ is continuous and that $\alpha_i > \alpha_i'$ for $i = 1, 2$ implies $\kappa_i(\alpha_1, \alpha_2) < \kappa_i(\alpha_1', \alpha_2')$ for $i = 1, 2$.

Let $z$ be the $k_2$ that solves $B_2(1, k_2) = \hat{\alpha}_2 = g(1)$ and verify using $(A3)$, that $z$ is well defined and let $F(\alpha_1) := \kappa_2(\alpha_1, g(\alpha_1))$. Since $k$ and $g$ are continuous so is $F$. Moreover, $F(1) = z$ and $\lim_{\alpha_1 \to 0} F(\alpha_1) = \infty$ since $\lim_{\alpha_1 \to 0} g(\alpha_1) = 0$. Hence, $F$ is onto. Define $f : I\!R \to I\!R$ such that

$$f(k_2) = \begin{cases} 0 & \text{if } k_2 < z \\ \kappa_1(F^{-1}(k_2), g(F^{-1}(k_2))) & \text{if } k_2 \geq z. \end{cases}$$

Since $g(\alpha_1) < \alpha_1$ it follows that $f(k_2) < k_2$. If $k_2 \to \infty$, then $F^{-1}(k_2) \to 0$ and therefore $\kappa_1(F^{-1}(k_2), g(F^{-1}(k_2))) \to \infty$ as desired.

Let $k = (k_1, k_2)$ and $\alpha^k = (\alpha_1, \alpha_2)$. If $k_1 < f(k_2)$, then $g(\alpha_1) > \alpha_2$ and therefore the voters utility is increasing in $k_1$; if $k_1 > f(k_2)$, then $g(\alpha_1) < \alpha_2$ and therefore the voters utility is decreasing in $k_1$. $\square$

**Proof of Proposition 3(ii):** Let party 1 be the advantaged party. First, we show that the disadvantaged party's cost is increasing in $k_1$ under the conditions stated in Proposition 3(ii). We know from Proposition 3(i) that $\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)}$ is increasing in $k_1$. Moreover, Proposition 1 implies that $\alpha_2$ is increasing in $k_1$. Therefore,

$$c_2(\alpha_1, \alpha_2) = k_2 \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)} \ln \frac{1 + \alpha_2}{1 - \alpha_2}$$

must be increasing in $k_1$.

Next, we show that $c_1(\alpha_1, \alpha_2)$ is increasing in $k_1$. First, note that inequality ($A8$) holds if $\alpha_1 \leq \frac{3}{2}\alpha_2$ and therefore $\alpha_1 > \frac{3}{2}\alpha_2$ under the hypothesis of Proposition 3(ii). Using ($A3$) to solve for $k_1$ we find that, in equilibrium,

$$c_1(\alpha_1, \alpha_2) = \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)^2 \alpha_2 \ln \frac{1+\alpha_1}{1-\alpha_1} \left(\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \ln \frac{1+\alpha_1}{1-\alpha_1} + 2\frac{\alpha_1^2}{1-\alpha_1^2}\right)^{-1}$$

We must show that $c_1$ is increasing in $k_1$. Note that an increase in $k_1$ implies a decrease in $\alpha_1$ and an increase in $\alpha_2$ by Proposition 1. Hence, it is sufficient to show that the above expression is increasing in $\alpha_2$ and decreasing in $\alpha_1$ for $\alpha_1 > \frac{3}{2}\alpha_2$.

The derivative of the above expression with respect to $\alpha_1$ is negative if

$$\frac{4\alpha_1^2(\alpha_1 + \alpha_2)}{1 + \alpha_1 \alpha_2} + \left(\ln \frac{1+\alpha_1}{1-\alpha_1}\right)^2 \frac{\alpha_2^2 (1-\alpha_1)^2 (1+\alpha_1)^2}{(1 + \alpha_1 \alpha_2)(\alpha_1 + \alpha_2)} < 4\alpha_1^2 \ln \frac{1+\alpha_1}{1-\alpha_1}$$

Since the left hand side is increasing in $\alpha_2$, verifying the above inequality for $\alpha_1 = \frac{3}{2}\alpha_2$ and $\alpha_1 \in (0, 1]$ is sufficient. A straightforward calculation reveals this to be the case and similar calculations reveal that the derivative with respect to $\alpha_2$ is positive for $\alpha_1 \geq \frac{3}{2}\alpha_2$. $\square$

**Proof of Proposition 4:** Since $\alpha_1$ is increasing in $k_2$, equation ($A3$) implies

$$\frac{\partial U_3(\alpha^k)}{\partial k_1} + \frac{\partial U_3(\alpha^k)}{\partial k_2} \geq \frac{\alpha_2^2}{(\alpha_1 + \alpha_2)^2} \frac{d\alpha_1}{dk_1} + \frac{\alpha_1^2}{(\alpha_1 + \alpha_2)^2} \left(\frac{d\alpha_2}{dk_1} + \frac{d\alpha_2}{dk_2}\right)$$

Thus, it suffices to show that

$$\frac{\alpha_2^2}{(\alpha_1 + \alpha_2)^2} \frac{d\alpha_1/dk_1}{d\alpha_2/dk_1} + \frac{\alpha_1^2}{(\alpha_1 + \alpha_2)^2} \left(1 + \frac{d\alpha_2/dk_2}{d\alpha_2/dk_1}\right) > 0$$

We have already shown that $\alpha_2 \to 0$ as $k_2 \to \infty$. Substituting for $\frac{d\alpha_1/dk_1}{d\alpha_2/dk_1}$, using ($A6$) and ($A7$), it is straightforward to verify that $\frac{\alpha_2^2}{(\alpha_1+\alpha_2)^2} \frac{d\alpha_1/dk_1}{d\alpha_2/dk_1} \to 0$ as $\alpha_2 \to 0$. Since $\alpha_1$ is bounded away from zero for all $k_2$, the proposition follows if $\left(\frac{d\alpha_2}{dk_2}\right) / \left(\frac{d\alpha_2}{dk_1}\right) \to 0$ as $k_2 \to \infty$. To show this, since $\frac{d\alpha_1}{dk_1}$ bounded away from zero for all $k_2$ and $\frac{d\alpha_2}{dk_1} = \frac{d\alpha_2}{d\alpha_1} \frac{d\alpha_1}{dk_1}$, it suffices to show that $\left(\frac{d\alpha_2}{dk_2}\right) / \left(\frac{d\alpha_2}{d\alpha_1}\right) \to 0$. Equation (A3) yields

$$\frac{\alpha_2}{\alpha_1 + \alpha_2} \left(1 - k_2 \alpha_1 \ln \frac{1+\alpha_2}{1-\alpha_2}\right) = \frac{2\alpha_2^2 k_2}{1 - \alpha_2^2}$$

and therefore:

$$\Big|\left(\frac{d\alpha_2}{dk_2}\right)/\left(\frac{d\alpha_2}{d\alpha_1}\right)\Big| = \Big|(\alpha_2+\alpha_1)\frac{-2\alpha_2(\alpha_1+\alpha_2)-\alpha_1\ln\frac{1+\alpha_2}{1-\alpha_2}+\alpha_2^2\alpha_1\ln\frac{1+\alpha_2}{1-\alpha_2}}{(1-\alpha_2^2)(k_2\alpha_2\ln\frac{1+\alpha_2}{1-\alpha_2}+1)}\Big|$$

Note that $\alpha_2 \to 0$ as $k_2 \to \infty$ and hence the right hand side of the above expression goes to zero as $k_2 \to \infty$. $\square$

## 8. Appendix B: Nonstationary Strategies

In this section, we show that the unique stationary equilibrium of Proposition 1 is also the unique subgame perfect equilibrium of the war of information. Nonstationary Nash equilibria may fail subgame perfection: let $\hat{\alpha}_2 = B_2(1)$ and $\hat{\alpha}_1 = B_1(\hat{\alpha}_2)$, where $B_i$'s are the stationary best response functions of section 2. Hence, $\hat{\alpha}_2$ is party 2's best response to an opponent who never quits and $\hat{\alpha}_1$ is party 1's best response to an opponent who quits at $\hat{\alpha}_2$.

Define the function $a_i : I\!R \to [0,1]$ as follows:

$$a_i(x) = (-1)^{i-1}(1-2p(x))$$

where $p$ is the logistic function. Consider the following strategy profile: $\alpha_2 = \hat{\alpha}_2$ and $\alpha_1 = \hat{\alpha}_1$ if $a_2(X_\tau) < \hat{\alpha}_2$ for all $\tau < t$ and $\alpha_1 = 1$ otherwise. Hence, party 2 plays the stationary strategy $\hat{\alpha}_2$ while party 1 plays $\hat{\alpha}_1$ along any history that does not require party 2 to quit. But, if 2 deviates and does not quit when he is supposed to, party 1 never quits.

First, we verify that the above strategy profile is a Nash equilibrium: Player 1's strategy is optimal by construction. For player 2, quitting before $\alpha$ reaches $\hat{\alpha}_2$ is clearly suboptimal; not quitting at $\hat{\alpha}_2$ is also suboptimal since such a deviation triggers $\alpha_1 = 1$. This strategy profile is not subgame perfect because never quitting after a player 2 deviation is suboptimal: at any $X_t$ such that $a_1(X_t) > \hat{\alpha}_1$, party 1 would be better off quitting.

Below, we define the dynamic war of information $\tilde{W}^k$ and show that the unique equilibrium of Proposition 1 is the only strategy profile in $\tilde{W}^k$ that survives iterative removal of dominated continuation strategies.

Fix any $t > 0$. A (time-$t$) continuation strategy $\gamma_i$ specifies player $i$'s behavior after time $t$ for every possible $X_t$ realization.[18] Let $\Gamma_i$ be the set of all player $i$ continuation strategies. Since our proof relies on a dominance argument, we will not need to specify formally the mapping from continuation strategies to outcomes. It is enough that every continuation strategy profile $\gamma \in \Gamma_1 \times \Gamma_2$ yield a stopping time $T_\gamma \geq t$.

Let $T_\gamma^x$ be the stopping time $T_\gamma$ conditional on $X_t = x$. We assume that $(0, 1] \subset \Gamma_i$; that is, $\Gamma_i$ includes all (stationary) strategies $\alpha_i$ in which player $i$ quits whenever $a_i(X_\tau)$ reaches $\alpha_i$. Given any stopping time $T \geq t$, define player $i$'s payoff as in section 2:

$$v_i(T) = \Pr[(-1)^i \cdot X_T > 0] + \frac{k_i}{2} E \int_{\tau=t}^{T} C_i(p_\tau) d\tau$$

where $C_1 = C$, $C_2 = 1 - C$ and $C$ is as defined in equation (6). For $j \neq i = 1, 2$, $b \in [0, 1]$ and $x$ such that $a_i(x) = b$, let

$$V_i(\gamma, b) = v_i(T_\gamma^x)$$
$$V_i^*(\gamma_j, b) = \sup_{\gamma_i \in \Gamma_i} V_i(\gamma_1, \gamma_2, b)$$

Hence, $V_i$ is player i's continuation utility given the state $x$ and strategy profile $\gamma$ while $V_i^*$ is the highest continuation utility $i$ can attain against strategy $\gamma_j$ given such an $x$. Since a player can always quit, $V_i^* \geq 0$.

We say that continuation strategy $\gamma_i$ is more aggressive than continuation strategy $\hat{\gamma}_i$ ($\gamma_i \succeq_i \tilde{\gamma}_i$) if given any opponent strategy, with probability 1, the game ends later with $\gamma_i$ than with $\hat{\gamma}_i$. In the statements below, it is understood that $j \neq i = 1, 2$.

**Definition:** $\gamma_i \succeq \tilde{\gamma}_i$ *if* $\gamma = (\gamma_i, \gamma_j)$ *and* $\tilde{\gamma} = (\tilde{\gamma}_i, \gamma_j)$ *implies* $\Pr(T_\gamma \geq T_{\tilde{\gamma}}) = 1$.

We do not distinguish between $\gamma_i$ and $\tilde{\gamma}_i$ if $\Pr(T_{(\gamma_i,\gamma_j)} = T_{(\tilde{\gamma}_i,\gamma_j)}) = 1$ for all $\gamma_j \in \Gamma_j$ and view such $\gamma_i$ and $\tilde{\gamma}_i$ as the same strategy. Therefore, $\succeq_i$ is antisymmetric; that is, $\gamma_i \succeq_i \tilde{\gamma}_i$ and $\tilde{\gamma}_i \succeq_i \gamma_i$ implies $\gamma_i = \tilde{\gamma}_i$. Note that $\succeq_i$ ranks all stationary strategies; that is, $\alpha_i \succeq_i \alpha_i'$ if and only if $\alpha_i \geq \alpha_i'$ for all $\alpha_i, \alpha_i' \in (0, 1]$.

**Lemma B:** *If* $\gamma = (\gamma_i, \gamma_j)$, $\tilde{\gamma} = (\tilde{\gamma}_i, \gamma_j)$ *and* $\gamma_i \succeq \tilde{\gamma}_i$, *then* $v_j(T_\gamma) \leq v_j(T_{\tilde{\gamma}})$.

[18] Players may choose different continuation strategies after two $t$-period histories with the same $X_t$.

**Proof: of Lemma B:** Let $A = \{\omega \in \Omega \,|\, T_\gamma \geq T_{\tilde{\gamma}}\}$. Hence, at $\omega \in A$, player $j$'s expenditure with $\tilde{\gamma}$ is less than it is with $\gamma$. If $T_\gamma(\omega) \neq T_{\tilde{\gamma}}(\omega)$, then player $j$ wins at $\omega$ with $\tilde{\gamma}$. Therefore, at every $\omega \in A$, player $j$'s probability of winning is higher and expenditure is lower with $\gamma$ than it is with $\tilde{\gamma}$. Since $\Pr(A) = 1$, the desired conclusion follows. $\square$

For any constant strategy $\alpha_2$, $V_1^*(\alpha_2, b)$ is decreasing in $b$ and is not equal to 0 if and only if $b < B_1(\alpha_2)$. More generally, it is not optimal for player 1 to quit immediately if $V_1^*(\gamma_2, a_1(X_t)) > 0$. Moreover, if there exists $b < a_1(X_t)$ such that $V_1^*(\gamma_2, b') = 0$ for all $b' \geq b$ and for every continuation strategy $\gamma_2$ that player 2 might choose for the remainder of the game, player 1 must quit immediately. To see why the latter statement is true, let $T' \geq t$ be the time at which player 1 quits, $T = \inf\{t' > t \,|\, X_{t'} = x\}$ for $x$ such that $a_1(x) = b$ and set $\tau = \min\{T, T'\}$. If $T' > t$, then $\tau > t$ and since the continuation utility at $\tau$ is 0, player 1's utility at $t$ given $X_t$ is $-k_1(\tau - t)/2 \geq 0$ and hence, $\tau = t$. The two observations above motive the following definition:

**Definition:** *The set $\Gamma_1^* \times \Gamma_2^* \subset \Gamma$ is dynamically rationalizable if for all $\gamma_i \in \Gamma_i^*$ and $b \in [0,1]$, (i) $V_i^*(\gamma_j, b') > 0$ for all $\gamma_j \in \Gamma_j^*$ and $b' < b$ implies $\gamma_i \succeq_i b$ and (ii) $V_i^*(\gamma_j, b') = 0$ for all $\gamma_j \in \Gamma_j^*$ and $b' > b$ implies $b \succeq_i \gamma_i$.*

Hence, if player $i$ knew that player j will only choose continuation strategies from $\Gamma_j^*$ for the rest of the game, then he could conclude that any continuation strategy $\gamma_i$ that does not satisfy (i) and (ii) above is not a best response. That is, as long as the set of remaining continuation strategies is not dynamically rationalizable more strategies can be removed to yield a finer prediction. The proposition below establishes that this procedure must lead to the unique stationary strategy profile.

**Proposition B:** *The unique dynamically rationalizable set of $\tilde{W}^k$ is $\{(\alpha_1^k, \alpha_2^k)\}$.*

**Proof:** Verifying that $\Gamma^* = \{(\alpha_1^k, \alpha_1^k)\}$ is dynamically rationalizable is straightforward. To complete the proof, we will show that there are no other dynamically rationalizable sets. For any dynamically rationalizable $\Gamma^* = \Gamma_1^* \times \Gamma_2^*$, let

$$\bar{a}_i = \inf\{b \in [0,1] \,|\, b \succeq_i \gamma_i \text{ for all } \gamma_i \in \Gamma_i^*\}$$

$$\underline{a}_i = \sup\{b \in [0,1] \,|\, \gamma_i \succeq_i b \text{ for all } \gamma_i \in \Gamma_i^*\}$$

By definition $\bar{a}_i \geq \underline{a}_i$. Let $\bar{b}_i = B_i(\underline{a}_j)$ and $\underline{b}_i = B_i(\bar{a}_j)$. By Lemma B, $V_2^*(\gamma_1, b') \leq V_2^*(\underline{a}_1, b') = 0$ for all $b' > \bar{b}_2$ and all $\gamma_1 \in \Gamma_1^*$. Since $\Gamma^*$ is dynamically rationalizable, we conclude that $\bar{b}_2 \succeq_2 \gamma_2$ for all $\gamma_2 \in \Gamma_2^*$ and hence $\bar{b}_2 \geq \bar{a}_2$. Similarly, since $V_1^*(\gamma_2, b') \geq V_1^*(\bar{a}_2, b') > V_1^*(\bar{a}_2, \underline{b}_1) = 0$ for all $b' < \underline{b}_1$ and all $\gamma_1 \in \Gamma_1^*$, we have $\underline{a}_1 \geq \underline{b}_1$. By symmetry, we have $\bar{b}_i \geq \bar{a}_i$ and $\underline{a}_i \geq \underline{b}_i$ for $i = 1, 2$.

Then, since $B_1$ is nonincreasing, we have $B_1(B_2(\underline{a}_1)) = B_1(\bar{b}_2) \leq B_1(\bar{a}_2) = \underline{b}_1 \leq \underline{a}_1$. Lemma 2 established that $\phi = B_1 \circ B_2$ has a unique fixed point $\alpha_1^k$. Therefore, $\phi(\underline{a}_1) \leq \underline{a}_1$ implies $\underline{a}_1 \geq \alpha_1^k$ and by symmetry, $\underline{a}_2 \geq \alpha_2^k$. Hence, $\alpha_2^k = B_2(\alpha_1^k) \geq B_2(\underline{a}_1) = \bar{b}_2 \geq \bar{a}_2 \geq \underline{a}_2$ and therefore, $\alpha_2^k = \underline{a}_2$ and by symmetry $\alpha_1^k = \underline{a}_1$. Then, $\alpha_1^k = \underline{a}_1 \leq \bar{a}_1 \leq \bar{b}_1 = B_1(\underline{a}_2) = B_1(\alpha_2^k) = \alpha_1^k$. This proves that $\alpha_i^k = \bar{a}_i = \underline{a}_i$ for $i = 1, 2$. Since $\succeq_i$ is antisymmetric, we have $\Gamma^* = \{(\alpha_1^k, \alpha_2^k)\}$ as desired. $\square$

# 9. Appendix C: Extensions

## 9.1 Proof of Lemma 3

The proof is similar to that of Lemma 1: let $c_1(y \,|\, \mu)$ be player 1's expenditure given the strategy profile $y = (y_1, y_2)$ and the drift $\mu$. Hence,

$$c_1(y) = \frac{c_1(y \,|\, {}^1\!/\!_2) + c_1(y \,|\, {}^{-1}\!/\!_2)}{2} \tag{C1}$$

First, we will show that

$$\begin{aligned} c_1(y \,|\, {}^1\!/\!_2) &= k_1 \frac{1 - e^{-y_2}}{1 - e^{y_1 - y_2}} (e^{y_1}(2 - y_1^2) - 2(y_1 + 1)) \\ c_1(y \,|\, {}^{-1}\!/\!_2) &= k_1 \frac{e^{y_2} - 1}{e^{y_2} - e^{y_1}} (2e^{y_1}(1 - y_1) + y_1^2 - 2) \end{aligned} \tag{C2}$$

For $z_1 < 0 < z_2$, let $P(z_1, z_2)$ be the probability that a Brownian motion $X_t$ with drift $\mu$ and variance 1 hits $z_2$ before $z_1$ given that $X_0 = 0$. Harrison (1985, p. 43) shows that

$$P(z_1, z_2) = \frac{1 - e^{2\mu z_1}}{1 - e^{-2\mu(z_2 - z_1)}} \tag{C3}$$

For $z_1 < 0 < z_2$, let

$$C(z_1, z_2 \,|\, \mu) = E \int_0^T X_t dt$$

where $X_t$ is a Brownian motion with drift $\mu$ and $T$ is the random time at which $X_t = z_1$ or $X_t = z_2$. Harrison (1985), Proposition 3 provides an expression for $E\int_0^T e^{-\lambda t} X_t dt$. Taking the limit of that expression as $\lambda \to 0$ yields

$$C(z_1, z_2 \,|\, {}^1\!/\!_2) = \frac{z_2(z_2 - 2 - 2z_1) + e^{z_1}(z_2 - z_1)(z_1 - z_2 + 2) + e^{z_1 - z_2} z_1(z_1 + 2)}{1 - e^{z_1 - z_2}}$$
$$C(z_1, z_2 \,|\, {}^{-1}\!/\!_2) = \frac{z_1(z_1 - 2) + e^{z_1 - z_2} z_2(-2z_1 + z_2 + 2) + e^{-z_2}(-z_1 + z_2 + 2)(z_1 - z_2)}{1 - e^{z_1 - z_2}}.$$

To compute $c_1(y \,|\, \mu)$, let $\epsilon \in (0, y_2]$ and assume that player 1 bears the cost until $X_t \in \{y_1, \epsilon\}$. If $X_t = \epsilon$, then player 2 bears the cost until $X_{t+\tau} \in \{0, y_2\}$. If $X_{t+\tau} = 0$, then the process repeats with player 1 bearing the cost until $X_{t+\tau+\tau'} \in \{-y_1, \epsilon\}$ and so on. Clearly, this yields an upper bound to $c_1(y \,|\, \mu)$. Let $D^\epsilon(\mu)$ denote that upper bound and note that

$$D^\epsilon(\mu) = k_1 C(y_1, \epsilon \,|\, \mu) + P(y_1, \epsilon)(1 - P(-\epsilon, y_2 - \epsilon))D^\epsilon(\mu)$$

Substituting for $C(y_1, \epsilon \,|\, \mu)$ and taking the limit as $\epsilon \to 0$ establishes that the right-hand side of $(C2)$ is an upper bound for the left-hand side. We can compute analogous lower bound which converges to the right hand side of equation $(C2)$ as $\epsilon < 0$ converges to 0. This establishes equation $(C2)$.

Recall that $p(y_i) = \frac{1}{1+e^{-y_i}}$ and $\alpha_1 = 1 - 2p(y_1), \alpha_2 = 2p(y_2) - 1$. Substituting these expressions into $(C1), (C2)$ yields

$$c_1(y) = k_1 \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \left( \left( \ln \frac{1 + \alpha_1}{1 - \alpha_1} \right)^2 + \frac{2}{\alpha_1} \ln \frac{1 + \alpha_1}{1 - \alpha_1} - 4 \right)$$

The win probability is the same as in Lemma 1. □

## 9.2 Discounting

We define

$$a = (1/\sigma^2)[(\mu^2 + 2\sigma^2 r)^{1/2} - \mu] = ((1/2)^2 + 2r)^{1/2} - (1/2)$$
$$b = (1/\sigma^2)[(\mu^2 + 2\sigma^2 r)^{1/2} + \mu] = ((1/2)^2 + 2r)^{1/2} + (1/2)$$

Let $x_1 = e^{y_1}$ and $y_2 = e^{-y_2}$. Since, $y_1 < 0 < y_2$, we have $x_i \in [0, 1]$ with a lower $x_i$ indicating a larger (in absolute value) threshold.

**Lemma C:** *Player $i$'s utility is*

$$U_i = \frac{1-x_i^{a+b}}{1-(x_ix_j)^{a+b}}\frac{x_j^a+x_j^b}{2} - \frac{k_i}{4r}\frac{(1-x_i^a)(1-x_i^b)(1-x_j^{a+b})}{1-(x_ix_j)^{a+b}}$$

*for $i=1,2$, $j \neq i, j=1,2$.*

**Proof:** To compute the expenditure, we follow the same approach as in the proof of Lemma 1: fix $\mu$ and let $E[C(y)\,|\,\mu]$ be player 1's expenditure given $\mu$. To compute $E[C(y\,|\,\mu)]$, let $\epsilon \in (0, y_2]$ and assume that player 1 bears the cost until $X_t \in \{y_1, \epsilon\}$. If $X_t = \epsilon$, then player 2 bears the cost until $X_{t+\tau} \in \{0, y_2\}$. If $X_{t+\tau} = 0$, then the process repeats with player 1 bearing the cost until $X_{t+\tau+\tau'} \in \{y_1, \epsilon\}$ and so on. Clearly, this calculation yields an upper bound $C^\epsilon$ for $E[C(y)\,|\,\mu]$. Let $\tau_1$ be such that $X_{\tau_1} \in \{y_1, \epsilon\}$ given the initial state 0. Let $\tau_2$ be the random time when $X_t \in \{0, y_2\}$ given the initial state $\epsilon$. Then, by the strong Markov property of Brownian motion, we have

$$C^\epsilon = \frac{k_1}{2}E\int_0^{\tau_1} e^{-rt}dt + E[e^{-r\tau_1}\,|\,X_{\tau_1}=\epsilon]E[e^{-r\tau_2}\,|\,X_{\tau_2}=0]C^\epsilon$$

By Proposition 3-2-18 in Harrison (1985, p. 40-41) we have

$$E[e^{-r\tau_1}] = \frac{e^{-a\epsilon} - e^{by_1}e^{a(y_1-\epsilon)}}{1-e^{b(y_1-\epsilon)}e^{a(y_1-\epsilon)}}$$

and

$$E[e^{-r\tau_2}] = \frac{e^{-b\epsilon} - e^{-a(y_2-\epsilon)}e^{-by_2}}{1-e^{-by_2}e^{-ay_2}}$$

and by Proposition 3-5-3 in Harrison (1985, p. 49) we have

$$E\left[\int_0^{\tau_1} e^{-rt}dt\right] = \frac{1}{r}\left(1 - \frac{e^{-a\epsilon} - e^{by_1}e^{a(y_1-\epsilon)}}{1-e^{b(y_1-\epsilon)}e^{a(y_1-\epsilon)}} - \frac{e^{by_1} - e^{-a\epsilon}e^{b(y_1-\epsilon)}}{1-e^{b(y_1-\epsilon)}e^{a(y_1-\epsilon)}}\right)$$

Let

$$\bar{C}^\epsilon = \frac{C^\epsilon(y\,|\,{}^1\!/\!_2) + C^\epsilon(y\,|\,{}^{-1}\!/\!_2)}{2}$$

Then, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\bar{C}^\epsilon =& \frac{k_1(1-e^{-(a+b)y_2})}{4r}\frac{1-e^{ay_1}-e^{by_1}+e^{(a+b)y_1}}{1-e^{(a+b)(y_1-y_2)}}\\ =& \frac{k_1}{4r}\frac{(1-x_1^a)(1-x_1^b)(1-x_2^{a+b})}{1-(x_1x_2)^{a+b}}\end{aligned}$$

We can compute an analogous lower bound that converges to the same limit as $\epsilon < 0$ converges to 0. Hence, the expression above is player 1's expenditure.

Next, we compute the utility of winning. Let $T$ be time when the game ends; that is the time first $t$ such that $X_t \in \{y_1, y_2\}$. Then,

$$\begin{aligned}2E[e^{-rT} \mid X_T = y_2] =& E[e^{-rT} \mid 1/2, X_T = y_2] + E[e^{-rT} \mid -1/2, X_T = y_2] \\ =& \frac{e^{-ay_2} - e^{by_1} e^{a(y_1-y_2)}}{1 - e^{b(y_1-y_2)} e^{a(y_1-y_2)}} + \frac{e^{-by_2} - e^{ay_1} e^{b(y_1-y_2)}}{1 - e^{a(y_1-y_2)} e^{b(y_1-y_2)}} \\ =& \frac{1 - x_1^{a+b}}{1 - (x_1 x_2)^{a+b}} \left(x_2^a + x_2^b\right)\end{aligned}$$

This completes the proof of Lemma C. ☐

Note that the coefficients $a, b$ are functions of $r$. Letting $r \to 0$, we obtain the payoffs calculated in Lemma 1.

$$\begin{aligned}&\lim_{r\to 0} \left( \frac{1 - x_i^{a+b}}{1 - (x_i x_j)^{a+b}} \frac{x_j^a + x_j^b}{2} - \frac{k_i}{4r} \frac{(1 - x_i^a)(1 - x_i^b)(1 - x_j^{a+b})}{1 - (x_i x_j)^{a+b}} \right) \\ &= \frac{1 - x_i}{2(1 - x_i x_j)} \left(1 + x_j + k_i(1 - x_j) \ln x_i\right) \\ &= \frac{\alpha_i}{\alpha_1 + \alpha_2} \left(1 - k_i \alpha_j \ln\left(\frac{1 + \alpha_i}{1 - \alpha_i}\right)\right)\end{aligned}$$

where $\alpha_i = \frac{1-x_i}{1+x_i}$.

**Proof of Proposition 5:** If we rescale the original signal $\hat{X}$ and let $X = \delta \hat{X}$, then $a = (1/\delta)\hat{a}$. Hence, we can choose $\delta > 0$ so that the rescaled signal satisfies $a + b = 1$ and consider the game with the rescaled signal. Since $X$ and $\hat{X}$ provide the same information, the game with the rescaled signal is equivalent to the original game.

Player $i$'s payoff is

$$U_i(x_i, x_j) = \frac{1 - x_i}{1 - x_i x_j} \frac{x_j^a + x_j^{1-a}}{2} - \frac{k_i}{4r} \frac{1 - x_i^a - x_i^{1-a} + x_i}{1 - x_i x_j} (1 - x_j)$$

Let $K = \frac{k_i}{2r}$. Then, the first order condition can be written as follows:

$$x_j^a + x_j^{1-a} = K h(x_i, x_j)$$

where

$$h(x_i, x_j) = \frac{1}{x_i^a}\left(1 - a + ax_ix_j + ax_i^{2a-1} - (1 + x_j)x_i^a + (1 - a)x_jx_i^{2a}\right)$$

Let $h_i$ denote the partial derivative of $h$ with respect to its $i$'th argument. We have

$$\begin{aligned} h_1 =& -\frac{1}{x_1^{a+2}}a(1 - a)(x_1 + x_1^{2a})(1 - x_1x_2) \\ h_2 =& \frac{1}{x_1^a}\left(ax_1 - x_1^a + (1 - a)x_1^{2a}\right) \end{aligned}$$

Note that $h_1 < 0$ which implies that the second order condition is satisfied and that $dx_i/dK > 0$ at any solution to the first order condition. We conclude that the first order condition has a unique solution. Moreover, it is straightforward to verify that $x_i > 0$ for all $x_j \in [0, 1]$ and $K > 0$ and that $x_i < 1$ for all $x_j > 0$.

Next, we show that $dx_i/dx_j < 0$ and find a convenient bound for $|dx_i/dx_j|$.

$$\frac{dx_i}{dx_j} = -\frac{Kh_2 - ax_j^{a-1} - (1 - a)x_j^{-a}}{Kh_1} < 0$$

since $h_2 < 0$ (which in turn follows from the fact that $0 < a < 1$) and $h_1 < 0$. Also, $ax_j^{a-1} + (1 - a)x_j^{-a} \leq \frac{x_j^a + x_j^{1-a}}{2x_j}$. Therefore, the first order condition yields

$$\left|\frac{dx_i}{dx_j}\right| \leq \frac{h_2 - h/(2x_j)}{h_1}$$

and

$$\begin{aligned} \left|\frac{dx_i}{dx_j}\right|\left|\frac{dx_j}{dx_i}\right| \leq& \frac{h_2(x_i, x_j) - h(x_i, x_j)/(2x_j)}{h_1(x_i, x_j)} \cdot \frac{h_2(x_j, x_i) - h(x_j, x_i)/(2x_i)}{h_1(x_j, x_i)} \\ =& \frac{f(x_i, x_j)f(x_j, x_i)}{4a^2(1 - a)^2(x_i + x_i^{2a})(x_j + x_j^{2a})(1 - x_ix_j)^2} \end{aligned}$$

where

$$f(x_i, x_j) = -x_i(1 - a) + x_i^{1+a}(1 - x_j) - ax_i^{2a} + x_i^{1+2a}x_j(1 - a) + ax_jx_i^2$$

To prove uniqueness, we show that $\left|\frac{dx_i}{dx_j}\right|\left|\frac{dx_j}{dx_i}\right| \leq 1$. For that,

$$f(x_i, x_j) \leq 2a(1 - a)(x_i + x_i^{2a})(1 - x_ix_j)$$

is sufficient. Establishing the inequality for $x_j = 0, 1$ is straightforward and since $f$ is linear in $x_j$, the inequality holds for all $x_j \in [0, 1]$. This completes the proof of uniqueness. To see part (ii), note that $x_i$ is increasing in $K$ and hence $|y_i|$ is decreasing in $k_i$. Moreover, $x_i$ is decreasing in $x_j$ and hence $|y_i|$ is decreasing in $|y_j|$. $\square$

**Proof of Proposition 6:** By the first order condition, $x_j$ stays bounded away from zero along any sequence in which $K_j$ stays bounded away from zero. Therefore, the first order condition for $x_i$ implies that $x_i$ converges to zero as $K_i$ converges to zero. This and the first order condition for $x_j$ ensure that $x_j$ converges to 1 as $K_i$ converges to zero. $\square$

## 10. Appendix D: Asymmetric Information

For $z < x < 0$, let $P_x^i(z)$ be the probability that $X_t^i$ hits 0 before it hits $z$ and $T_x^i(z)$ the expected time $X_t^i$ spends until it hits either 0 or $z$. As noted in the proof of Lemma 1, Harrison (1985 p. 43 and 52) shows that

$$\begin{aligned} P_x^i(z) &= \frac{1 - e^{(2i-1)(z-x)}}{1 - e^{(2i-1)z}} \\ T_z^i(x) &= 2(2i-1)[z - x - zP_z(x)] \end{aligned} \tag{D1}$$

Define $\Pi_x^i(z) = P_x^i(z) - \frac{k}{2} T_x^i(z)$. Then, the above equations yield

$$\begin{aligned} \Pi_x^0(z) &= \frac{1 - e^{x-z}}{1 - e^{-z}}(1 - kz) + k(z - x) \\ \Pi_x^1(z) &= \frac{1 - e^{z-x}}{1 - e^{z}}(1 + kz) - k(z - x). \end{aligned} \tag{D2}$$

Recall that $z_*$ is the unique negative solution to

$$e^{z_*} + z_* = \frac{k+1}{k} \tag{**}$$

### 10.1 Proof of Proposition 7

Given any real number $z$ and two stochastic processes $\hat{Y}, \hat{Z}$ such that $\hat{Y}_0 < \hat{Z}_0$, consider the following optimization problem: the party incurs flow cost $k/2$ as long as the $z < \hat{Y}_t < \hat{Z}_t$. The game ends if $\hat{Y}$ hits $z$ or if $\hat{Z} - \hat{Y}$ hits 0. In the latter case, the party gets an additional payoff of 1. Let $\hat{Y}_0 = x$. Let $W_x(z, \hat{Y}, \hat{Z})$ be the payoff that the type-$i$

party would get in this single person game and $T_x(x, \hat{Y}, \hat{Z})$ be the expected time until the ends. Also, let $V_x(\hat{Y}, \hat{Z}) = \sup_z W_x(z, \hat{Y}, \hat{Z})$. If $\hat{Z}$ is the constant 0, we omit it and write $W_x(z, \hat{Y})$, $T_x(z, \hat{Y})$ and $V_x(\hat{Y})$.

Note that $T_x^i(z) = T_x(z, X^0)$ and $\Pi_x^0(z) = W_x(z, X^0)$. Hence, taking a derivative with respect to $z$ in $(D2)$ and using $(**)$ reveals that the unique maximizer of $W_x(\cdot, X^0)$ is $z_*$ and $V_x(X^0) = W_x(z_*, X^0) > 0$ for all $x < z_*$ while $W_z(y, X^0) < 0$ for all $y < z \leq z_*$.[19]

**Fact:** $\hat{Z} \geq \hat{Z}'$ for all $\omega, t$ implies $W_x(z, \hat{Y}, \hat{Z}) \leq W_x(z, \hat{Y}, \hat{Z}')$.

To see why the fact is true, note that given any $\omega$, the game ends with $\hat{Z}'$ no latter than with $\hat{Z}$ and the party wins with $\hat{Z}'$ if it wins with $\hat{Z}$.

Since $X_t^0 = Y_t^0 = z$ implies $L_t^{0z_*} = p(z_*)$ for any $z < z_*$, the strategy $F_{z_*}$ is optimal for the type 0 party if and only if quitting when $L^{0z_*}$ reaches $p(z_*)$ and never quitting are both optimal. Since $U_x^0(G_z, L^{0z_*}) = W_{p(x)}(p(z), L^{0z_*}, 1/2) = W_x(z, X^0)$ whenever $z \geq z_*$, by the definition of $z_*$, $U_x^0(G_z, L^{0z_*}) < U_x^0(G_{z_*}, L^{0z_*})$ for all $z > z_*$. Hence, to conclude the proof that $F_{z_*}$ is optimal for the type 0 party, it is enough to verify that $U_x^0(0, L^{0z_*}) = U_x^0(G_{z_*}, L^{0z_*})$ or equivalently that $T_{p(z_*)}(0, L_t^{0z_*}, 1/2) \cdot k/2 = 1$. Let $a(\epsilon) = T_{z_*+\epsilon}(0, L_t^{0z_*}, 1/2)$. It follows $(D1)$ above that

$$T^0_{z_*}(z_* - \epsilon) + (1 - P^0_{z_*}(z_* - \epsilon)) \cdot a(\epsilon) \geq a(\epsilon) \geq T^0_{z_*+\epsilon}(z_*) + (1 - P^0_{z_*+\epsilon}(z_*)) \cdot a(\epsilon).$$

Hence, $a(\epsilon)$ is bounded between $\frac{T^0_{z_*}(z_*-\epsilon)}{P^0_{z_*}(z_*-\epsilon)}$ and $\frac{T^0_{z_*+\epsilon}(z_*)}{P^0_{z_*+\epsilon}(z_*)}$. Taking limits establishes that $a(0) = 2(e^{-z_*} + z_* - 1)$. Hence, the expected delay cost until winning, given the strategy profile $\alpha$ and current voter belief $p(z_*)$, is $a(0) \cdot k/2 = 1$. Therefore, the type-0 party's continuation utility at belief state $p(z_*)$ is 0. Since never quitting is optimal for the type-0 party, it is also optimal for the type 1 party.

Next, we prove that $(F_{z_*}, 0)$ is the unique equilibrium. For any cdf $G$, $x$ is *a point of increase of* $G$ if for every $\epsilon > 0$, there exists $y, y' \in (x - \epsilon, x + \epsilon)$ such that $G(y) < G(y')$. Let $\alpha = (G^0, G^1)$ be any equilibrium and define $x^i = \infty$ if $G^i(x) < 1$ for all $x$ and $x^i = \inf\{x \,|\, G(z) = 1\}$ otherwise. Note that $\alpha$ is an equilibrium if and only if for $i = 0, 1$ $U_x^i(G_z^i, L^{i\alpha}) \geq U_x^i(G_y^i, L^{i\alpha})$ for every point of increase $-z$ of $G^i$ and every $y$. Clearly, if $x^i < \infty$, then it is a point of increase of $G^i$.

[19] If $x \leq z_*$, then any $z \geq x$, including $z_*$, amounts to same action: quitting immediately. Hence, we call $z_*$ the unique optimal strategy.

If $x^1 < x^0$, then the first time $X^i$ reaches $-x_1$, the voter's current belief becomes 0 and stays at 0 until the probability that the type 0 party quits reaches 1. Then, the type 0 party would have been better off with the strategy $z = -x^1$. If $x^0 < \infty$, then, the party wins as soon as $X_t^i < -x^0$ which means quitting at $-x^0$ is not optimal for party 0. It follows that $x^0 = x^1 = \infty$. Which means that $\hat{G} = 0$ (i.e., never quitting) is an optimal strategy for the type 0 party and therefore it is the unique optimal strategy for the type 1 party. Hence, $G^1 = 0$.

By definition, $U_x^0(G_z, L^{0\alpha}) = W_x(z, X^0, \log(1 - G^0(-Y^0))$. Since $\log(1 - G^0(-Y^0)) < 0$, the fact above ensures that $U_x^0(G_z, L^{0\alpha}) \geq W_x(z_*, X^0) = V_x(X^0) > 0$ for all $x > z_*$. Therefore, it is not optimal for the type-0 party 0 to quit before $z_*$. Hence, $G^0(-z) = 0$ for all $z > z_*$. Next, suppose $G^0(-z) > 1 - e^{z-z_*}$ for some $z < z_*$. We can assume, without loss of generality that $-z$ is a point of increase of $G^0$. Then, choose $\epsilon > 0$ such that $G^0(-z) > 1 - e^{z-z_*-\epsilon}$.

Consider any $\omega, t$ such that $X_t^0 = Y_t^0 = z$. Note that the type-0 party's continuation utility at $(\omega, \tau)$ is no less than $W_z(z - \epsilon, X^0 - \log(1 - G^0(-Y^0))$ since quitting as soon as $X^0$ reaches $z - \epsilon$ is a feasible strategy. Since $\log(1 - G^0(-Y^0)) \leq \log(1 - G^0(-z))$, the fact above implies that the type-0 party's continuation utility at $z$ is no less than $W_z(z - \epsilon, X^0, \log(1 - G^0(-z))$ which by the same fact is no less than $W_z(z - \epsilon, X^0, -z + z_* + \epsilon) = W_{z_*+\epsilon}(z_*, X^0) > 0$. It follows that quitting at $z$ is not optimal for the type-0 party contradicting the fact that $-z$ is a point of increase of $G^0$. Hence, $G^0(-z) \leq 1 - e^{z-z_*}$ for all $z < z_*$.

Finally, suppose $G^0(-z) < 1 - e^{z-z_*}$ for some $z < z_*$. If $G(-x) = G(-z)$ whenever $-x > -z$, let $y = -\infty$, otherwise let $y = -\min\{-x \mid G^0(-x) > G^0(-z)\}$. Then, if $y = z$ let $y_* < y$ be any point of increase of $G^0$ such that $G^0(y_*) < 1 - e^{z-z_*}$. (The right-continuity of $G^0$ and the fact that $y = z$ ensures such a $z$ exists.) Otherwise, let $y_* = y$ and note that $y_* < z$. The optimality of $G^0$ implies that $G_{y_*}$ is also optimal for party 0. Hence, by the fact above, we have $U_z^0(G^0, L^{0\alpha}) = U_z^0(G_{y_*}, L^{0\alpha}) = W_z(y_*, X^i, \log(1 - G(-z)) \leq W_z(y_*, X^0, -z + z_*) = W_{z_*}(y_* - z + z_*, X^0) < 0$ contradicting the optimality of $G_{y_*}$. Hence, $G^0(-z) = 1 - e^{z-z_*}$ for all $z < z*$ as desired. $\square$

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