ISSN 1936-5098

CAE Working Paper #11-01

Failure of a Credit System: Implications of the Large Deviation Theory

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February 2011

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April 2010

1 Introduction

϶ Suppose each of $N_A = N/A$ customers ($n \geq 1$) receives a loan of A dollars from a bank, and the probability is p that the customer will return RA dollars to the bank at the end of the year ($R > 1$), and the probability is $1 - p$ that he or she will default, returning no money to the bank ($0 < p < 1$). Assume also that apart from the N dollars the bank lends each year, it has a backup asset of M dollars per year ($M \geq 0$). We will say that the bank fails at the end of T years if

$$RA \sum_{j=1}^{T} S_j \leq NT - MT, \tag{1}$$

where S_j is the number of customers in the jth period returning RA dollars to the bank. We assume S_j, $1 \leq j \leq T$, are independent, as are the N_A customers in each period. Then S_j has the binomial distribution $Binom(N_A, p)$, $1 \leq j \leq T$, and $\sum_{j=1}^{T} S_j$ is also $Binom(N_A T, p)$. The *probability of bank failure at the end of period* T is then

$$Q(T) \equiv P\left(\sum_{j=1}^{T} S_j \leq \frac{(N-M)T}{RA}\right), \tag{2}$$

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which may also be represented as

$$P\left(\sum_{i=1}^{N_A T} Y_i \leq \frac{(N-M)\,T}{RA}\right), \tag{3}$$

where Y_i, $1 \leq i \leq N_A T$, are i.i.d., with $P(Y_i = 1) = p$, and $P(Y_i = 0) = 1 - p$. Note that $EY_i = p\ \forall i$.

Consider two cases.

Case I: $p < \frac{1}{R}\left(1 - \frac{M}{N}\right)$, **Case II** : $p > \frac{1}{R}\left(1 - \frac{M}{N}\right)$.

Case I. In this case, writing $\delta = \frac{1}{R}\left(1 - \frac{M}{N}\right) - p$, $\delta > 0$, we may express (3) as

$$Q_I(T) = P\left(\frac{\sum_{i=1}^{N_A T} Y_i}{N_A T} \leq p + \delta\right) \tag{4}$$

By the theory of large deviations (see, e.g., Bhattacharya and Waymire (2007), Theorem 4.8, pp. 54-55),

$$1 - Q_I(T) = e^{-\lambda N_A T}\left(1 + o(1)\right) \text{ as } N_A T \equiv \frac{NT}{A} \to \infty, \tag{5}$$

where, writing $m(h) = Ee^{hY_i} = pe^h + (1-p)$, one has

$$\lambda = c^*(p+\delta),\ c^*(x) := \sup_{h \in R}\{xh - \ln m(h)\}. \tag{6}$$

Clearly, $c^*(x) \geq 0$. Also, for $0 < |x| < 1$, $xh - \ln m(h) \to -\infty$ as $h \to \pm\infty$. Hence $c^*(x)$ may be obtained by solving (for h) the equation

$$0 = \frac{d}{dh}\{xh - \ln m(h)\} = x - \frac{pe^h}{m(h)}, \text{ or, } pe^h = \frac{1-p}{1-x} - (1-p) = \frac{(1-p)x}{1-x}. \tag{7}$$

or $h = \ln\left\{\frac{(1-p)x}{p(1-x)}\right\}$.

Then

$$c^*(x) = x\ln\left\{\frac{(1-p)x}{p(1-x)}\right\} - \ln\left(\frac{1-p}{1-x}\right), \tag{8}$$

and

$$\begin{aligned}
\lambda &= c^*(p+\delta) = \frac{1}{R}\left(1 - \frac{M}{N}\right)\ln\left\{\frac{1-p}{p} \cdot \frac{\left(1 - \frac{M}{N}\right)/R}{1 - \left(1 - \frac{M}{N}\right)/R}\right\} - \ln\left\{\frac{1-p}{1 - \left(1 - \frac{M}{N}\right)/R}\right\} \\
&= B\ln\left(\frac{1-p}{p} \cdot \frac{B}{1-B}\right) - \ln\left(\frac{1-p}{1-B}\right),
\end{aligned}$$

$$B := \left(1 - \frac{M}{N}\right)\frac{1}{R}. \tag{9}$$

Thus the probability that the bank does not fail at the end of period T is $e^{-\lambda NT/A}(1+o(1))$, which goes to zero exponentially fast as $NT/A \to \infty$. One may, in this case prove the stronger result that

$$1 - Q_I^*(T) = \{e^{-\lambda}NT/A\}(1+o(1)), \tag{10}$$

$$Q_I^*(T) := P(\text{the bank fails at some period } t,\ 1 \le t \le T). \tag{11}$$

Remark 1. Note that λ (in (9)) does not depend on A. Thus the exponent $\lambda NT/A$ decreases as A increases, so that $e^{-\lambda NT/A}$ increases as A increases. This shows that, in Case I, the probability of bank failure increases as A increases. That is, with the same capital outlay of N dollars per year, the same probability $1-p$ of default by a customer, and the same expected revenue NRp per year (i.e., the same interest rate $R-1$ charged to a customer), the probability of bank failure rises as the amount of loan per customer rises. One may think of this as the effect of higher *volatility*, since $var\left(RAS_j\right) = R^2A^2N_Ap(1-p) = \left(R^2Np(1-p)A\right)$ (although $ERAS_j = RNp$ is not affected by A).

Remark 2. Note that

$$\frac{d}{dx}c^*(x) = \ln\left\{\frac{(1-p)x}{p(1-x)}\right\},\ (0 < x < 1) \tag{12}$$

Hence $(d/dx)c^*(x) > 0$ if $(1-p)x > p(1-x)$, and $(d/dx)c^*(x) < 0$ if $(1-p)x < p(1-x)$. Since $(1-p)(p+\delta) > p(1-p-\delta)$,

$$\lambda = c^*(p+\delta) \equiv c^*\left(\frac{1}{R}\left(1-\frac{M}{N}\right)\right) \text{ decreases as } R \text{ increases.} \tag{13}$$

In other words, the chance of bank failure decreases as R (or the interest rate) increases - a rather obvious conclusion, but with a precise calculation of the rates.

Case II. Assume now that $p > \frac{1}{R}\left(1-\frac{M}{N}\right)$. Then one may rewrite (3) as

$$Q_{II}(T) \equiv P\left(\sum_{i=1}^{N_AT} Z_i \ge -\frac{(N-M)T}{RA}\right), \tag{14}$$

where $Z_i = -Y_i,\ 1 \le i \le N_AT$, are i.i.d., $P(Z_i = -1) = p$, $P(Z_i = 0) = 1-p$, $EZ_i = -p$, $m(h) = Ee^{hZ_i} = pe^{-h} + 1 - p$. One now has

$$\frac{1}{R}\left(1-\frac{M}{N}\right) = p - \delta, \delta := p - \frac{1}{R}\left(1-\frac{M}{N}\right). \tag{15}$$

Then, by the large deviation principle,

$$\begin{aligned} Q_{II}(T) &\equiv P\left(\frac{\sum_{i=1}^{N_A T} Z_i}{N_A T} \geq -p+\delta\right) = e^{-\lambda N_A T}(1+o(1)) \qquad (16) \\ &= e^{-\lambda NT/A}(1+o(1)), \text{ as } N_A T \to \infty, \end{aligned}$$

where

$$\lambda = c^*(-p+\delta), \, c^*(x) := \sup_h \left[xh - \ln\left(pe^{-h}+1-p\right)\right]. \qquad (17)$$

By symmetry, or by direct calculation as in (7), (8), one may show that, in this case,

$$\begin{aligned} \lambda &\equiv c^*(-p+\delta) = (p-\delta)\ln\left\{\frac{1-p}{p}\cdot\frac{p-\delta}{1-p+\delta}\right\} - \ln\frac{1-p}{1-p+\delta} \\ &= B\ln\left\{\frac{1-p}{p}\cdot\frac{B}{1-B}\right\} - \ln\left(\frac{1-p}{1-B}\right) = B\ln\left(\frac{B}{p}\right) - (1-B)\ln\left(\frac{1-p}{1-B}\right) \qquad (18) \\ B &:= \frac{1}{R}\left(1-\frac{M}{N}\right). \end{aligned}$$

Remark 3. Since λ in (18) does not involve A, it follow that the (exponentially small) probability of bank failure, as given by (16), increases as A increases (showing the effect of volatility). Also, as in Remark 2, if R increases then the probability of bank failure decreases, since the revenue grows (given that p remains the same). The relation (18), however, refines this obvious fact.

A numerical illustration.

Case II. $N = 1000$, $T = 5$, $p = 0.9$, $R = 1.2$

(a) $A = 10$ [$N_A = 100$]. Then

$$\begin{aligned} \lambda &= \frac{1}{1.2}\ln\left(\frac{1}{(1.2)(0.9)}\right) - \frac{2}{1.2}\ln\left(\frac{(0.1)(1.2)}{0.2}\right) \\ &= 0.02101 \\ Q_{II}(T) &\approx e^{-(0.02101)500} = e^{-10.505} = 0.00027 \end{aligned}$$

(b) $A = 100$ [$N_A = 10$]. Then

$$Q_{II}(T) \approx e^{-1.0505} = 0.35$$

.

The calculation in (b) for the approximate probability of ruin is better done using the central limit theorem, rather than large deviations. For in this case the Normal approximation to the probability is

$$P(Z > 1.58) = 0.057,$$

where Z is a standard Normal random variable.

In Case II, the probability of bank failure before or in period T is (for $M = 0$)

$$\begin{aligned} Q_I^*(T) &= P(\text{Bank failure occurs at the end of period 1}) \\ &\quad + P(\text{First failure occurs at the end of period 2}) \\ &\quad + \cdots + P(\text{First failure occurs at the end of period T}) \\ &\leqslant e^{-\lambda N_A}(1+o(1)) + e^{-2\lambda N_A}(1+o(1)) + \cdots + e^{-T\lambda N_A}(1+o(1)) \\ &= e^{-\lambda N_A} + e^{-2\lambda N_A} + \cdots + e^{-T\lambda N_A} + o(e^{-\lambda N_A}) \\ &\approx e^{-\lambda N_A} \end{aligned} \tag{19}$$

On the other hand, obviously,

$$Q_I^*(T) \geq P(\text{Bank fails at the end of period 1}) = e^{-\lambda N_A}(1+o(1)). \tag{20}$$

It follows from (19) and (20) that

$$Q_I^*(T) = e^{-\lambda N_A}(1+o(1)) \tag{21}$$

We consider next the more realistic model in which the probability p depends on the state θ of nature. Given the state θ that obtains, the customers behave independently with regard to loan repayment, with a common probability p_θ of repayment. The distribution of customers is thus exchangeable. It is also assumed that the sequence $\theta_n : n \geqslant 1$ of values of θ. For simplicity, let θ have two possible values $\theta = a_1$ (e.g., 'normal rainfall') and $\theta = a_2$ ('drought'). Let $\pi(a_i) = Prob(\theta = a_i)$, $i = 1,2$. Assume $p_{a_1} > \frac{1}{R}\left(1 - \frac{M}{N}\right)$, $p_{a_2} < \frac{1}{R}\left(1 - \frac{M}{N}\right)$. In one period (i.e., $T = 1$), the probability of bank failure is

$$Q(1) = \sum_{i=1}^{2} \pi(a_1) \cdot P\left(S_1 \leqslant \frac{N-M}{RA} | \theta = a_i\right). \tag{22}$$

By the preceding (see (6),(10), and (16), (18)),

$$Q(1) = \pi(a_1) \cdot e^{-\lambda_{a_1} N/A}(1+o(1)) + \pi(a_2)(1 - e^{-\lambda_{a_2} NT/A}(1+o(1)), \tag{23}$$

where, with $B = \frac{1}{R}\left(1-\frac{M}{N}\right)$ as in (10), one has (see (10) and (18)).

$$\lambda_{a_1} = B\ln\left(\frac{B}{p_{a_1}}\right) - (1-B)\ln\left(\frac{1-p_{a_1}}{1-B}\right),$$

$$\lambda_{\theta_2} = B\ln\left(\frac{1-p_{a_2}}{p_{a_2}}\cdot\frac{B}{1-B}\right) - \ln\left(\frac{1-p_{a2}}{1-B}\right). \tag{24}$$

For the case $T = 2$, the corresponding failure probability is

$$\begin{aligned} Q(2) &= \sum_{i,j=1}^{2} \pi(a_i)\cdot\pi(a_j)P\left(S_1+S_2 \leqslant \frac{2(N-M)}{RA}|\theta_1 = a_i, \theta_2 = a_j\right) \\ &= \pi(a_1)^2 e^{\frac{-2\lambda_{a_1}N}{A}}(1+o(1)) + \pi(a_2)^2\cdot[1-e^{-\frac{2\lambda_{a_2}N}{A}}(1+o(1))] \\ &+ 2\pi(a_1)\pi(a_2)P\left(S_1+S_2 \leqslant \frac{2(N-M)}{RA}|\theta_1 = a_i, \theta_2 = a_j\right). \end{aligned} \tag{25}$$

The last summand on the right side in (25) may be expressed as

$$2\pi(a_1)\pi(a_2)\sum_{y=0}^{N_A}\binom{N_A}{y}p_{a_1}^y\cdot(1-p_{a_1})^{N_A-y}\cdot P\left(S_2 \leqslant \frac{2(N-m)}{RA}|\theta_2 = a_2\right). \tag{26}$$

For asymptotics, one may use a number of approximations to (26) (or the last term in (25)).

Consider two cases. First, suppose $\left(\frac{p_{a_1}+p_{a_2}}{2}\right) > \frac{1}{R}\left(1-\frac{M}{N}\right)$. Then, by Bernstein's inequality( ), we can show that $P(S_1+S_2 \leq \frac{2(N-M)}{RA}|\theta_1 = a_1, \theta_2 = a_2)$ is exponentially small, namely, $O(exp{-}cN_A)$ for some positive constant c. In this case,

$$Q(2) = \pi(a_2)^2 + 0(e^{-c'N_A}) \tag{27}$$

for some constant $c' > 0$ Thus $Q(2)$ is essentially $\pi(a_2)^2$. Secondly, suppose $\left(\frac{p_{a_1}+p_{a_2}}{2}\right) < \frac{1}{R}\left(1-\frac{M}{N}\right)$. Then, again by Bernstein's inequality, one can show that, $P\left(S_1+S_2 \leq \frac{2(N-M)}{RA}|\theta_1 = a_1, \theta_2 = a_2\right) = 1-\delta_{N_A}$, where $\delta_{N_A} \to 0$ exponentially fast with N_A. In this case,

$$Q(2) = \pi(a_2)^2 + 2\pi(a_1)\pi(a_2) + o(e^{-c''N_A}) \tag{28}$$

for same positive constant c''.

In the general case of T periods ($T > 1$), one may express the failure probability as

$$
\begin{aligned}
Q(T) = {} & \pi(a_2)^T + \binom{T}{1}\pi^{T-1}(a_2)\pi(a_1)\cdot \\
& P\left(S_1 + \cdots + S_T \leqslant \frac{T(N-m)}{RA} | \theta_1 = a_1, \theta_i = a_2 \text{ for } 2 \leqslant i \leqslant T\right) \\
& + \binom{T}{2}\pi^{T-2}(a_2)\pi^2(a_1)\cdot \\
& P\left(S_1 + \cdots + S_T \leqslant \frac{T(N-m)}{RA} | \theta_1 = a_1, \theta_2 = a_1, \theta_i = a_2 \text{ for } 3 \leqslant i \leqslant T\right) \\
& + \cdots + \binom{T}{r}\pi^{T-r}(a_2)\pi^r(a_1)\cdot \\
& P\left(S_1 + \cdots + S_T \leqslant \frac{T(N-m)}{RA} | \theta_i = a_1, \text{ for } 1 \leqslant i \leqslant r,\ \theta_i = a_2 \text{ for } r+1 \leqslant i \leqslant T\right) \\
& + \cdots + \pi^T(a_1)\cdot \\
& P\left(S_1 + \cdots + S_T \leqslant \frac{T(N-m)}{RA} | \theta_i = a_1, \text{ for } 1 \leqslant i \leqslant T\right).
\end{aligned}
\tag{29}
$$

Assume, for simplicity, that $rp_{a_1} + (T-r)p_{a_2}$ does not equal $\frac{T(N-M)}{RA}$ for any r. Again we consider several cases. Suppose r, $0 \le r \le T-1$, is the largest integer such that,

$$\text{case } r : rp_{a_1} + (T-r)p_{a_2} < \frac{T(N-M)}{RA} (r = 0, 1, \cdots, T-1). \tag{30}$$

Then

$$Q(T) = \sum_{j=0}^{r} \binom{T}{j} \pi^j(a_1)\pi^{T-j}(a_2) + o(1) \text{ for } r = 0, 1, \cdots, T-1. \tag{31}$$

The error $o(1)$ is of the order $\exp -c_r \cdot N_A$, where $c_r > 0$ can be estimated using Bernstein's inequality. Note c_r is increasing in r.