EUROPEAN CENTRAL BANK
EUROSYSTEM

WORKING PAPER SERIES
NO 1294 / FEBRUARY 2011

THE NORMALIZED CES PRODUCTION FUNCTION

THEORY AND EMPIRICS

by Rainer Klump, Peter McAdam and Alpo Willman

EUROPEAN CENTRAL BANK

EUROSYSTEM

WORKING PAPER SERIES

NO 1294 / FEBRUARY 2011

THE NORMALIZED CES PRODUCTION FUNCTION

THEORY AND EMPIRICS[1]

by Rainer Klump[2], Peter McAdam[3] and Alpo Willman[4]

In 2011 all ECB publications feature a motif taken from the €100 banknote.

1 We thank Cristiano Cantore, Jakub Growiec, Olivier de La Grandville, Miguel León-Ledesma and Ryuzo Sato for comments, past collaborations and support.

2 Goethe University, Frankfurt am Main & Center for Financial Studies.

3 European Central Bank, Kaiserstrasse 29, D-60311 Frankfurt am Main, Germany: email: peter.mcadam@ecb.europa.eu. McAdam is also visiting professor at the University of Surrey.

4 European Central Bank, Kaiserstrasse 29, D-60311 Frankfurt am Main, Germany: email: Alpo.Willman@ecb. europa.eu

Address
Kaiserstrasse 29
60311 Frankfurt am Main, Germany

Postal address
Postfach 16 03 19
60066 Frankfurt am Main, Germany

Telephone
+49 69 1344 0

Internet
http://www.ecb.europa.eu

Fax
+49 69 1344 6000

Information on all of the papers published in the ECB Working Paper Series can be found on the ECB's website, http://www.ecb.europa.eu/pub/scientific/wps/date/html/index.en.html

ISSN 1725-2806 (online)

CONTENTS

Abstract. The elasticity of substitution between capital and labor and, in turn, the direction of technical change are critical parameters in many fields of economics. Until recently, though, the application of production functions with non-unitary substitution elasticities (i.e., non Cobb Douglas) was hampered by empirical and theoretical uncertainties. As has recently been revealed, "normalization" of production functions and production-technology systems holds out the promise of resolving many of those uncertainties. We survey and critically assess the intrinsic links between production (as conceptualized in a macroeconomic production function), factor substitution (as made most explicit in Constant Elasticity of Substitution functions) and normalization (defined by the fixing of baseline values for relevant variables). First, we recall how the normalized CES function came into existence and what normalization implies for its formal properties. Then we deal with the key role of normalization in recent advances in the theory of business cycles and of economic growth. Next, we discuss the benefits normalization brings for empirical estimation and empirical growth research. Finally, we identify promising areas of future research on normalization and factor substitution.
Keywords. Normalization, Constant Elasticity of Substitution Production Function, Factor-Augmenting Technical Change, Growth Theory, Identification, Estimation.

Non-technical Summary

Substituting scarce factors of production by relatively more abundant ones is a key element of economic efficiency and a driving force of economic growth. A measure of that force is the elasticity of substitution between capital and labor which is the central parameter in production functions, and in particular CES (Constant Elasticity of Substitution) ones. Until recently, the application of production functions with non-unitary substitution elasticities (i.e., non Cobb Douglas) was hampered by empirical and theoretical uncertainties.

As has recently been revealed, "normalization" of production functions and production-technology systems holds out the promise of resolving many of those uncertainties and allowing elements as the role of the substitution elasticity and biased technical change to play a deeper role in growth and business-cycle analysis. Normalization essentially implies representing the production function in consistent indexed number form. Without normalization, it can be shown that the production function parameters have no economic interpretation since they are dependent on the normalization point and the elasticity of substitution itself.

This feature significantly undermines estimation and comparative-static exercises, among other things. Due to the central role of the substitution elasticity in many areas of dynamic macroeconomics, the concept of CES production functions has recently experienced a major revival. The link between economic growth and the size of the substitution elasticity has long been known. As already demonstrated by Solow (1956) in the neoclassical growth model, assuming an aggregate CES production function with an elasticity of substitution above unity is the easiest way to generate perpetual growth. Since scarce labor can be completely substituted by capital, the marginal product of capital remains bounded away from zero in the long run.

Nonetheless, the case for an above-unity elasticity appears empirically weak and theoretically anomalous. However, when analytically investigating the significance of non-unitary factor substitution and non-neutral technical change in dynamic macroeconomic models, one faces the issue of "normalization", even though the issue is still not widely known. The (re)discovery of the CES production function in normalized form in fact paved the way for the new and fruitful, theoretical and empirical research on the aggregate elasticity of substitution which has been witnessed over the last years. In La Grandville (1989b) and Klump and de La Grandville (2000) the concept of normalization was introduced in order to prove that the aggregate elasticity of substitution between labor and capital can be regarded as

an important and meaningful determinant of growth in the neoclassical growth model.

In the meantime this approach has been successfully applied in a series of theoretical papers to a wide variety of topics. Further, as Klump et al. (2007a, 2008) demonstrated, normalization also has been a breakthrough for empirical research on the parameters of aggregate CES production functions, in particular when coupled with the system estimation approach. Empirical research has long been hampered by the difficulties in identifying at the same time an aggregate elasticity of substitution as well as growth rates of factor augmenting technical change from the data. The received wisdom, in both theoretical and empirical literatures, suggests that their joint identification is infeasible. Accordingly, for more than a quarter of a century following Berndt (1976), common opinion held that the US economy was characterized by aggregate Cobb-Douglas technology, leading, in turn, to its default incorporation in economic models (and, accordingly, the neglect of possible biases in technical progress). Translating normalization into empirical production-technology estimations allows the presetting of the capital income share (or, if estimated, facilitates the setting of reasonable initial parameter conditions); it provides a clear correspondence between theoretical and empirical production parameters and allows us ex post validation of estimated parameters.

Here we analyze and survey the intrinsic links between production (as conceptualized in a macroeconomic production function), factor substitution (as made most explicit in CES production functions) and normalization.

Until the laws of thermodynamics are repealed, I shall continue to relate outputs to inputs - i.e. to believe in production functions.
Samuelson (1972) (p. 174)

All these results, negative and depressing as they are, should not surprise us. Bias in technical progress is notoriously difficult to identify.
Kennedy and Thirwall (1973) (p. 784)

The degree of factor substitution can thus be regarded as a determinant of the steady state just as important as the savings rate or the growth rate of the labor force.
Klump et al. (2008) (p. 655)

1 Introduction

Substituting scarce factors of production by relatively more abundant ones is a key element of economic efficiency and a driving force of economic growth. A measure of that force is the elasticity of substitution between capital and labor which is the central parameter in production functions, and in particular CES (Constant Elasticity of Substitution) ones. Until recently, the application of production functions with non-unitary substitution elasticities (i.e., non Cobb Douglas) was hampered by empirical and theoretical uncertainties. As has recently been revealed, "normalization" of production functions and production-technology systems holds out the promise of resolving many of those uncertainties and allowing considerations as the role of the substitution elasticity and biased technical change to play a deeper role in growth and business-cycle analysis. Normalization essentially implies representing the production function in consistent indexed number form. Without normalization, it can be shown that the production function parameters have no economic interpretation since they are dependent on the normalization point and the elasticity of substitution itself. This feature significantly undermines estimation and comparative-static exercises, among other things.

Let us first though place the importance of the topic in perspective. Due to the central role of the substitution elasticity in many areas of dynamic macroeconomics, the concept of CES production functions has recently experienced a major revival. The link between economic growth and the size of the substitution elasticity has long been known. As already demonstrated by Solow (1956) in the neoclassical growth model, assuming an aggregate CES production function with an elasticity of substitution above unity is the easiest way to generate perpetual growth. Since scarce labor can be completely substituted by capital, the marginal product of capital remains bounded away from zero in the long run. Nonetheless, the case for an above-unity elasticity appears empirically weak and theoretically anomalous.[1]

It has been shown that integration into world markets is also a feasible way for a country to increase the effective substitution between factors of production and thus pave the way for sustained growth (Ventura (1997), Klump (2001), Saam (2008)). On the other hand, it can be shown in several variants of the standard neoclassical (exogenous) growth model that introducing an aggregate CES production functions that with an elasticity of substitution below unity can generate multiple growth equilibria, development traps and indeterminacy (Azariadis (1996), Klump (2002), Kaas and von Thadden (2003)), Guo and Lansing (2009)).

Public finance and labor economics are other fields where the elasticity of substitution has been rediscovered as a crucial parameter for understanding the impact of policy changes. This relates to the importance of factor substitution possibilities for the demand for each input factor. As pointed out by Chirinko (2002), the lower the elasticity of substitution, the smaller the response of business investment to variations in interest rates caused by monetary or fiscal policy.[2] In addition, the welfare effects of tax policy changes specifically, appear highly sensitive to the assumed values of the substitution elasticity. Rowthorn (1999) also stresses its importance in macroeconomic analysis of the labor market and, in particular, how incentives for higher investment formation exercise a significant effect on unemployment when the elasticity of substitution departs from unity.

Indeed, there is now mounting empirical evidence that aggregate production is better characterized by a non-unitary elasticity of substitution (rather than unitary or above unitary), e.g., Chirinko et al. (1999), Klump et al. (2007a), León-Ledesma et al. (2010a). Chirinko (2008)'s recent survey suggests that most evidence favors elasticities ranges of 0.4-0.6 for the US. Moreover, Jones (2003, 2005)[3] argued that capital shares exhibit such protracted swings and trends in many countries as to

---

[1]The critical threshold level for the substitution elasticity (to generate such perpetual growth) can be shown to be increasing in the growth of labor force and decreasing in the saving rate, see La Grandville (1989b).

[2]This may be one reason why estimated investment equations struggle to identify interest-rate channels.

[3]Jones' work essentially builds on Houthakker (1955)'s idea that production combinations reflect the (Pareto) distribution of innovation activities, Jones proposes a "nested" production function. Given such parametric innovation activities, this will exhibit a (far) less than unitary substitution elasticity over business-cycle frequencies but asymptote to Cobb-Douglas.

be inconsistent with Cobb-Douglas or CES with Harrod-neutral technical progress (see also Blanchard (1997), McAdam and Willman (2011a)).

This coexistence of capital and labor-augmenting technical change, has different implications for the possibility of balanced or unbalanced growth. A balanced growth path - the dominant assumption in the theoretical growth literature - suggests that variables such as output, consumption, etc tend to a common growth rate, whilst key underlying ratios (e.g., factor income shares, capital-output ratio) are constant, Kaldor (1961). Neoclassical growth theory suggests that, for an economy to posses a steady state with positive growth and constant factor income shares, the elasticity of substitution must be unitary (i.e., Cobb Douglas) or technical change be Harrod neutral.

As Acemoglu (2009) (Ch. 15) comments, however, there is little reason to assume technical change is necessarily labor augmenting.[4] In models of "biased" technical change (e.g., Kennedy (1964), Samuelson (1965), Acemoglu (2003), Sato (2006)), scarcity, reflected by relative factor prices, generates incentives to invest in factor-saving innovations. In other words, firms reduce the need for scarce factors and increase the use of abundant ones. Acemoglu (2003) further suggested that while technical progress is necessarily labor-augmenting along the balanced growth path, it may become capital-biased in transition. Interestingly, given a below-unitary substitution elasticity this pattern promotes the stability of income shares while allowing them to fluctuate in the medium run.

However, when analytically investigating the significance of non-unitary factor substitution and non-neutral technical change in dynamic macroeconomic models, one faces the issue of "normalization", even though the issue is still not widely known. The (re)discovery of the CES production function in normalized form in fact paved the way for the new and fruitful, theoretical and empirical research on the aggregate elasticity of substitution which has been witnessed over the last years.

In La Grandville (1989b) and Klump and de La Grandville (2000) the concept of normalization was introduced in order to prove that the aggregate elasticity of substitution between labor and capital can be regarded as an important and meaningful determinant of growth in the neoclassical growth model. In the meantime this approach has been successfully applied in a series of theoretical papers (Klump (2001), Papageorgiou and Saam (2008), Klump and Irmen (2009), Xue

[4]Moreover, that a BGP cannot coexist with capital augmentation is becoming increasingly questioned in the literature, see Growiec (2008), La Grandville (2010), Leon-Ledesma and Satchi (2010).

and Yip (2009), Guo and Lansing (2009), Wong and Yip (2010)) to a wide variety of topics.

A particular striking example of how neglecting normalization can significantly bias results and how explicit normalization can help to overcome those biases is presented in Klump and Saam (2008). The effect of a higher elasticity of substitution on the speed of convergence in a standard Ramsey type growth model is shown to double if a non-normalized (or implicitly normalized) CES function is replaced by a reasonably normalized one.

Further, as Klump et al. (2007a, 2008) demonstrated, normalization also has been a breakthrough for empirical research on the parameters of aggregate CES production functions,[5] in particular when coupled with the system estimation approach. Empirical research has long been hampered by the difficulties in identifying at the same time an aggregate elasticity of substitution as well as growth rates of factor augmenting technical change from the data. The received wisdom, in both theoretical and empirical literatures, suggests that their joint identification is infeasible. Accordingly, for more than a quarter of a century following Berndt (1976), common opinion held that the US economy was characterized by aggregate Cobb-Douglas technology, leading, in turn, to its default incorporation in economic models (and, accordingly, the neglect of possible biases in technical progress).[6]

Translating normalization into empirical production-technology estimations allows the presetting of the capital income share (or, if estimated, facilitates the setting of reasonable initial parameter conditions); it provides a clear correspondence between theoretical and empirical production parameters and allows us ex post validation of estimated parameters. In a series of papers, León-Ledesma et al. (2010a,b) showed the empirical advantages in estimating and identifying production-technology systems when normalized. Further, McAdam and Willman (2011b) showed that normalized factor-augmenting CES estimation, in the context of estimating "New Keynesian" Phillips curves, helped better identify the volatility in the driving variable (real marginal costs) that most previous researchers had not detected.

Here we analyze the intrinsic links between production (as conceptualized in a

[5]It should be noted that the advantages of re-scaling input data to ease the computational burden of highly nonlinear regressions has been the subject of some study, e.g., ten Cate (1992). And some of this work was in fact framed in terms of production-function analysis, De Jong (1967), De Jong and Kumar (1972). See also Cantore and Levine (2011) for a novel discussion of alternative but equivalent ways to normalize.

[6]It should be borne in mind, however, that Berndt's result concerned only the US manufacturing sector.

macroeconomic production function), factor substitution (as made most explicit in CES production functions) and normalization. The paper is organized as follows. In section 2 we recall how the CES function came into existence and what this implies for its formal properties. Sections 3 and 4 will deal with the role of normalization in recent advances in the theory of business cycles and economic growth. Section 5 will discuss the merits normalization brings for empirical growth research. The last section concludes and identifies promising area of future research.

2 The general normalized CES production function and variants

It is common knowledge that the first rigid derivation of the CES production function appeared in the famous Arrow et al. (1961) paper (hereafter *ACMS*).[7] However, there were important forerunners, in particular the explicit mentioning of a CES type production technology (with an elasticity of substitution equal to 2) in the Solow (1956) article (done, Solow wrote, to add a "bit of variety") on the neoclassical growth model. There was also the hint to a possible CES function in its Swan (1956) counterpart (on the Swan story see Dimond and Spencer (2008)).[8] Shortly before, though, Dickinson (1954) (p. 169, fn 1) had already made use of a CES production technology in order to model "a more general kind of national-income function, in which the factor shares are variable" compared to the Cobb-Douglas form. It has even been conjectured that the famous and mysterious tombstone formula of von Thünen dealing with "just wages" can be given a meaningful economic interpretation if it is regarded as derived from an implicit CES production function with an elasticity of substitution equal to 2 (see Jensen (2010)).

In this section we want to demonstrate, that (and how) the formal construction of a CES production function is intrinsically linked to normalization. The function

[7]It is still not widely known that the famous Arrow et al. (1961) paper was in fact the merging of two separate submissions to the *Review of Economics and Statistics* following a paper from Arrow and Solow, and another from Chenery and Minhas.

[8]In the inaugural ANU Trevor Swan Distinguished Lecture, Peter L. Swan (Swan (2006)) writes, "While Trevor was at MIT he pointed out that a production function Solow was utilizing had the constant elasticity of substitution, CES, property In this way, the CES function was officially born. Solow and his coauthors publicly thanked Trevor for this insight (see Arrow et al, 1961)."

may be defined as follows:

$$Y_t = F(K_t, N_t) = C\left[\pi K_t^{\frac{\sigma-1}{\sigma}} + (1-\pi) N_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \quad (1)$$

where distribution parameter $\pi \in (0,1)$ reflects capital intensity in production; C is an efficiency parameter and, σ, is the elasticity of substitution between capital, K, and labor, N. Like all standard CES functions, equation (1) nests a Cobb-Douglas function when $\sigma \to 1$; a Leontief function with fixed factor proportions when $\sigma = 0$; and a linear production function with perfect factor substitution when $\sigma = \infty$.

The construction of such an aggregate production technology with a CES property starts from the formal definition of the elasticity of substitution which had been introduced independently by Hicks (1932) and Robinson (1933) (on the differences between both approaches to the concept see Hicks (1970)). It is there defined (in the case of two factors of production, capital and labor) as the elasticity of K/N with respect to the marginal rate of substitution between K and N (the percentage change in factor proportions due to a change in the marginal rate of technical substitution) along an isoquant:[9]

$$\sigma \in [0, \infty] = \frac{d(K/N)/(K/N)}{d(F_N/F_K)/(F_N/F_K)} = \frac{d\log(K/N)}{d\log(F_N/F_K)} \quad (2)$$

As Hicks notes this concept of elasticity can be equally expressed in terms of the second derivative of the production function, but only under the assumption of constant returns to scale (due to Euler's theorem).

Since under this assumption the marginal factor productivities would also equal factor prices and the marginal rate of substitution would be identical with the wage/capital rental ratio, the elasticity of substitution can also expressed as the elasticity of income per person y with respect to the marginal product of labor in efficiency terms (or the real wage rate, w), i.e., Allen's theorem (Allen (1938)).

Given that income per person is a linear homogeneous function $y = f(k)$ of the capital intensity $k = K/N$, the elasticity of substitution can *also* be defined as:

$$\sigma = \frac{dy}{dw} \cdot \frac{w}{y} = -\frac{f'(\kappa)\left[f(\kappa) - \kappa f'(\kappa)\right]}{\kappa f(\kappa) f''(\kappa)} \quad (3)$$

[9]Alternatively, the substitution elasticity is sometimes expressed in terms of the parameter of factor substitution, $\rho \in [-1, \infty]$, where $\rho = \frac{1-\sigma}{\sigma}$.

Although it is rarely stated explicitly, the elasticity of substitution is implicitly always defined as a point elasticity. This means that it is related to one particular baseline point on one particular isoquant (see our Figures 1 and 2 below). From there a whole system of non-intersecting isoquants is defined which all together create the CES production function. Even if it is true that a given and constant elasticity of substitution would not change along a given isoquant or within a given system of isoquants, it is also evident that changes in the elasticity of substitution would of course alter the system of isoquants. Following such a change in the elasticity of substitution the old and the new isoquant are not intersecting at the baseline point but are tangents, if the production function is normalized. And they should not intersect because given the definition of the elasticity of substitution (i.e. the percentage change in factor proportions due to a change in the marginal rate of technical substitution) at this particular point (as in all other points which are characterized by the same factor proportion) the old and the new CES function should still be characterized by the same factor proportion and the same marginal rate of technical substitution.

Just as there are two possible definitions of σ following (3) - from $\frac{dy}{dw} \cdot \frac{w}{y}$ and from $-\frac{f(k)[f(k)-kf'(k)]}{kf''(k)f(k)}$ - thus there are two ways of uncovering the normalized production function. These, we cover in the following two sub-sections.

2.1 Derivation via the Power Function

Let us start from the definition $\sigma = \frac{d\log(y)}{d\log(w)} = \frac{dy}{dw} \cdot \frac{w}{y}$, integration of which gives the power function,

$$y = cw^{\sigma} \tag{4}$$

where c is some integration constant.[10] Under the assumption of constant returns to scale (or perfectly competitive factor and product markets), and applying the profit-maximizing condition that the real wage equals the marginal product of labor, and with the application of Allen's theorem, we can transform this equation into the form $y = c\left(y - k\frac{dy}{dk}\right)^{\sigma}$.

Accordingly, after integration and simplification, this leads us to a production

[10] ACMS started from the empirical observation that the relationship between per-capital income and the wage rate might best be described with the help of such a power function. Note, $\sigma = 1$ implies a linear relationship between y and w which would, in turn, imply that labor's share of income was constant. However, instead of a linear $y - w$ scatter plot, they found a concave relationship in the US data. The authors then tested a logarithmic and power relationship and concluded that $\sigma < 1$. Integration of power function (4) then leads to a production function with constant elasticity of substitution, consistent with definitions (2), (3).

function with the constant elasticity of substitution function (see La Grandville (2009), p. 83ff for further details):

$$y = \left[\beta k^{\frac{\sigma-1}{\sigma}} + \alpha\right]^{\frac{\sigma}{\sigma-1}} \tag{5}$$

and,

$$Y = \left[\beta K^{\frac{\sigma-1}{\sigma}} + \alpha L^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{6}$$

in the extensive form.

It should be noted that (5) and (6) contain the two constants of integration β and $\alpha = c^{-\frac{1}{\sigma}}$, where the latter directly depends on σ. Identification of these two constants make use of baseline values for the power function (4) and for the functional form (5) at the given baseline point in the system of isoquant. In a dynamic setting this baseline point must (as we will see later) also be regarded as a particular point in time, $t = t_0$:

$$y_0 = c w_0^{\sigma} \tag{7}$$

$$y_0 = \left[\beta k^{\frac{\sigma-1}{\sigma}} + \alpha\right]^{\frac{\sigma}{\sigma-1}} \tag{8}$$

Together with (5) this leads to the normalized CES production function,

$$y = y_0 \left[\pi_0 \left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi_0)\right]^{\frac{\sigma}{\sigma-1}} \tag{9}$$

and,

$$Y = Y_0 \left[\pi_0 \left(\frac{K}{K_0}\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi_0)\left(\frac{N}{N_0}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{10}$$

in the extensive form. Parameter $\pi_0 = \frac{y_0 - w_0}{y_0} = \frac{r_0 K_0}{Y_0}$ denotes the capital share in total income at the point of normalization.[11] As a test of consistent normalization, we see from (10) that for $t = t_0$ we retrieve $Y = Y_0$.

[11] Under perfect competition, this distribution parameter is equal to the capital income share but, under *imperfect* competition with non-zero aggregate mark-up, it equals the share of capital income over total factor income.

2.2 Derivation via the Homogenous Production Function

It was shown by Paroush (1964), Yasui (1965) and McElroy (1967) that the rather narrow assumption of Allen's theorem is not essential for the derivation of the CES production function which can start directly from the original Hicks definition (2). This definition can be transformed into a second-order differential equation whose solution also implies two constants of integration.

Following Klump and Preissler (2000) we start with the definition of the elasticity of substitution in the case of linear homogenous production function $Y_t = F(K_t, N_t) = N_t f(k_t)$ where $k_t = K_t/N_t$ is the capital-labor ratio in efficiency units. Likewise $y_t = Y_t/N_t$ represents per-capita production.

The definition of the substitution elasticity, $\sigma = -\frac{f'(k)[f(k)-kf'(k)]}{kf''(k)f(k)}$, can then be viewed as a second-order differential equation in k having the following general CES production function as its solution (intensive and extensive forms):

$$y_t = a\left[k_t^{\frac{\sigma-1}{\sigma}} + b\right]^{\frac{\sigma}{\sigma-1}} \tag{11}$$

$$Y_t = a\left[K_t^{\frac{\sigma-1}{\sigma}} + bN_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{12}$$

where parameters a and b are two arbitrary constants of integration with the following correspondence with the parameters in equation (1): $C = a(1+b)^{\frac{\sigma}{\sigma-1}}$ and $\pi = 1/(1+b)$.

A meaningful identification of these two constants is given by the fact that the substitution elasticity is a point elasticity relying on three baseline values: a given capital intensity $k_0 = K_0/N_0$, a given marginal rate of substitution $[F_K/F_N]_0 = w_0/r_0$ and a given level of per-capita production $y_0 = Y_0/N_0$. Accordingly, (1) becomes,

$$Y_t = Y_0\left[\pi_0\left(\frac{K_t}{K_0}\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi_0)\left(\frac{N_t}{N_0}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{13}$$

where $\pi_0 = r_0K_0/(r_0K_0 + w_0N_0)$ is the capital income share evaluated at the point of normalization. Rutherford (2003) calls (13) (or (10)) the "calibrated form".

2.3 A Graphical Representation

Normalization as understood by La Grandville (1989b), Klump and de La Grandville (2000) and Klump and Preissler (2000) is again nothing else but identifying these two arbitrary constants in an economically meaningful way. Normalizing means the fixing (in the $K - N$ plane as in **Figure 1**) of a baseline point (which can be thought of as a point in time at, $t = t_0$), characterized by specific values of N, K, Y and the marginal rate of technical substitution μ_0 - in which isoquants of CES functions with different elasticities of substitution but with all other parameters equal - are tangents.

Normalization is helpful to clarify the conceptual relationship between the elasticity of substitution and the curvature of the isoquants of a CES production function (see La Grandville (1989a) for a discussion of various misunderstandings on this point). Klump and Irmen (2009) point out that in the point of normalization (and only there), there exists an inverse relationship between the elasticity of substitution and the curvature of isoquant of the normalized CES production function. This relationship has also an interpretation in terms of the degree of complementarity of both input factors. At the normalization point, a higher elasticity of substitution implies a lower degree of complementarity between the input factors. The link between complementarity between input factors and the elasticity of substitution is also discussed in Acemoglu (2002) and in Nakamura and Nakamura (2008).

Equivalently (in the $k - y$ plane as in **Figure 2**) the baseline point can be characterized by specific values of k, y and the marginal productivity of capital (or the real wage rate). If base values for these three variables are selected this means of course that also a baseline value for the elasticity of production with respect to capital input is fixed which (under perfect competition) equals capital share in total income.

2.4 Normalization As A Means To Uncover Valid CES Representations

Normalization thus creates specific 'families' of CES functions whose members all share the same baseline point but are distinguished by the elasticity of substitution (and *only* the elasticity of substitution).

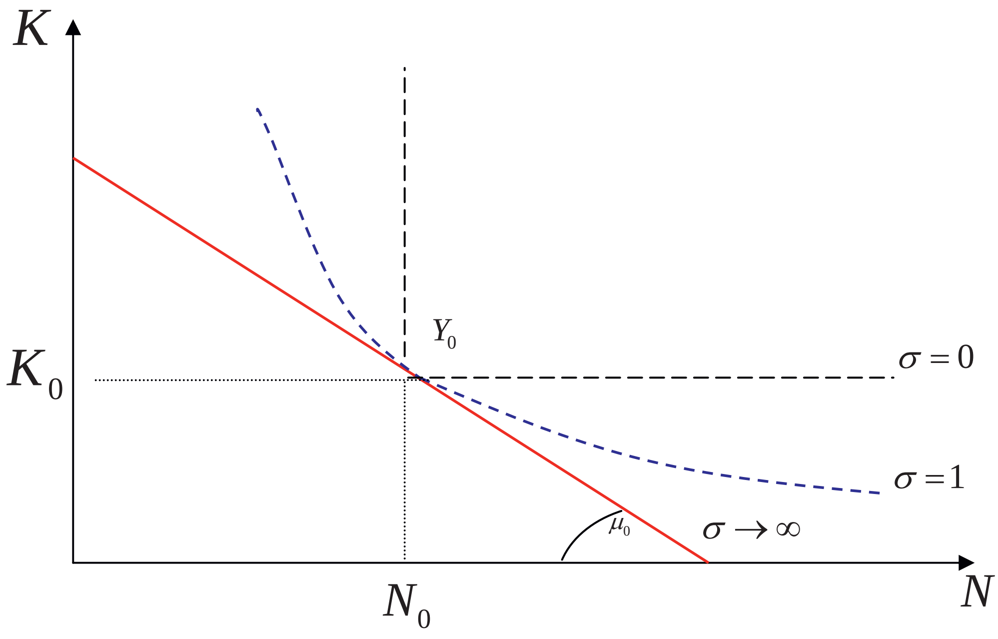

Figure 1. Isoquants of Normalized CES Production Functions

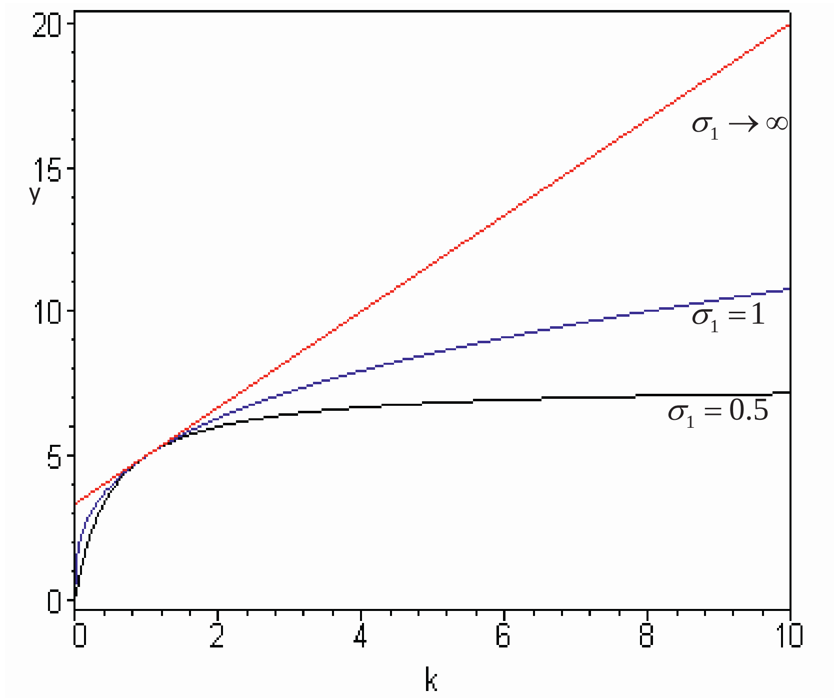

Figure 2. Normalized per-capita CES production functions

As shown in Klump and Preissler (2000), normalization also helps to distinguish those variants of CES production functions which are functionally identical with the general form (1) from those which are inconsistent with (5) in one way or another. Consider, first, the "standard form" of the CES production function, as it was introduced by ACMS, restated below:

$$Y = C\left[\pi K^{\frac{\sigma-1}{\sigma}} + (1-\pi)\, N^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{14}$$

This variant is clearly identical with (10), albeit (and this is a crucial aspect) with the "substitution parameter" C and the "distribution parameter" π being defined in the following way (solving for completeness in terms of both ρ and σ):

$$C(\sigma,\cdot) = Y_0\left[\pi_0 K_0^{\rho} + (1-\pi_0)\, N_0^{\rho}\right]^{\rho} = Y_0\left[\frac{r_0 K_0^{1/\sigma} + w_0 N_0^{1/\sigma}}{r_0 K_0 + w_0 N_0}\right]^{\frac{\sigma}{\sigma-1}} \tag{15}$$

$$\pi(\sigma,\cdot) = \frac{\pi_0 K_0^{\rho}}{\pi_0 K_0^{\rho} + (1-\pi_0)\, N_0^{\rho}} = \frac{r_0 K_0^{1/\sigma}}{r_0 K_0^{1/\sigma} + w_0 N_0^{1/\sigma}} \tag{16}$$

Expressions (15) and (16) reveal that, in the non-normalized case, both "parameters" (apart from being dependent on the scale of the normalized variables) change with variations in the elasticity of substitution, unless K_0 and N_0 are exactly equal, implying $k_0 = 1$.

This makes the non-normalized form in general inappropriate for comparative static exercises in the substitution elasticity. It is the interaction between the normalized efficiency and distribution terms and the elasticity of substitution which guarantees that within one family of CES functions the members are only distinguished by the elasticity of substitution. Given the accounting identity (and abstracting from the presence of an aggregate mark-up),

$$Y_0 = r_0 K_0 + w_0 N_0 \tag{17}$$

it also follows from this analysis that treating C and π in (14) as deep parameters is equivalent to assuming $k_0 = 1$. In the case $\sigma \to 0$, we have a perfectly symmetrical Leontief function.

As explained in Klump and Saam (2008) the Leontief case can serve as a benchmark for the choice of the normalization values for k_0 in calibrated growth models. The baseline capital intensity corresponds to the capital intensity that would be efficient if the economy's elasticity of substitution were zero. For $k <$

k_0 the economy's relative bottleneck resides in this case in its capacity to make productive use of additional labor, as capital is the relatively scarce factor. For $k > k_0$ the same is true for capital and labor is relatively scarce. Since the latter case is most characteristic for growth model of capitalist economies, calibrations of these model can be based on the assumption $k > k_0$.

In the following sub-sections, we will illustrate how normalization can reveal whether certain production functions used in the literature are legitimate.

2.4.1 David and van de Klundert (1965) Version

Consider the CES variant proposed by David and van de Klundert (1965):

$$Y = \left[(BK)^{\frac{\sigma-1}{\sigma}} + (AN)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{18}$$

This variant is identical with (10) as long as the two "efficiency levels" are defined in the following way:

$$B = \frac{Y_0}{K_0} \pi_0^{\frac{\sigma}{\sigma-1}} \tag{19}$$

$$A = \frac{Y_0}{N_0} (1 - \pi_0)^{\frac{\sigma}{\sigma-1}} \tag{20}$$

Again, it is obvious, that the efficiency levels change directly with the elasticity of substitution.

2.4.2 Ventura (1997) Version

Consider now a CES variant used by Ventura (1997):

$$Y = \left[K^{\frac{\sigma-1}{\sigma}} + (AN)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{21}$$

At first glance (21) could be regarded as a special case of (14) with B being equal to one. With a view on the normalized efficiency level it becomes clear, however, that $B = 1$ is not possible for given baseline values and a changing elasticity of substitution. Given that Ventura (1997) makes use of (21) in order to study the impact of changes in the elasticity of substitution on the speed of convergence, in the light of this inconsistency his results should be regarded with particular caution. As shown in Klump (2001), Ventura's result are unnecessarily restrictive; working with a correctly normalized CES technology leads to much more general results.

2.4.3 Barro and Sala-i-Martin (2004) Version

Next consider the CES production function proposed by Barro and Sala-i-Martin (2004):

$$Y = C\left[\pi\left(BK\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi)\left((1-B)\,N\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{22}$$

Normalization is helpful in this case in order to show that (22) can be transformed without any problems into (10) and/or (14) so that the terms B and $1-B$ simply disappear. If for any reason these two terms are considered necessary elements of a standard CES production function, they cannot be chosen independently from the normalized values for C and π, but they remain independent from changes in σ.

2.5 The Normalized CES Function with Technical Progress

So far we have treated efficiency levels as constant over time. If we now consider factor-augmenting technical progress one has to keep in mind the intrinsic links between rising factor efficiency in the distribution of income. This brings us to one further justification for normalizing CES production function which is closely related to the concept of neutral technical progress and was first articulated by Kamien and Schwartz (1968). Normalization implies that there may be considered a reference (or representative) value for the capital income share (and thus for income distribution) at some given point. Technical progress that does not changes income distribution over time is called Harrod-Neutral technical progress. There are many other types of classifiable neutral technical change, however, that would not have this effect.[12] So the whole concept of whether technical progress is neutral with respect to the income distribution, relies on the idea that one has to check whether or not a given income distribution at one point in time remains constant. This given income distribution, which is used to evaluate possible distribution effect of technical progress, is exactly the income distribution in the baseline point of normalization at a fixed point in time, $t = t_0$.

[12] See the seminal contribution of Sato and Beckmann (1970) for such a classification.

2.5.1 Constant growth rates of normalized factor efficiency levels

As shown in Klump et al. (2007a), a normalized CES production function with factor-augmenting technical progress can be written as,

$$Y = \left[\left(E_t^K K \right)^{\frac{\sigma-1}{\sigma}} + \left(E_t^N N \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (23)$$

where E_t^K and E_t^N represent the levels of efficiency of both input factors.[13]

Thus, whereas the ACMS specification seems to imply that technological change is always Hicks-neutral, the above specification allows for different growth rates of factor efficiency. To circumvent problems related to Diamond-McFadden's Impossibility theorem (Diamond et al. (1978); Diamond and McFadden (1965)), we assume a certain functional form for the growth rates of both efficiency levels and define:

$$E_t^i = E_0^i e^{\gamma_i (t-t_0)} \quad (24)$$

where γ_i denotes growth in technical progress associated with factor i and t represents a (typically linear) time trend. The combination $\gamma_K = \gamma_N > 0$ denotes Hicks-Neutral technical progress; $\gamma_K > 0, \gamma_N = 0$ yields Solow-Neutrality; $\gamma_K = 0, \gamma_N > 0$ represents Harrod-Neutrality; and $\gamma_K > 0 \neq \gamma_N > 0$ indicates general factor-augmenting technical progress.[14]

E_0^i are the fixed points of the two efficiency levels, taken at the common baseline time, $t = t_0$. Again, normalization of the CES function implies that members of the same CES family should all share the same fixed point and should in this point and at that time of reference only be characterized by different elasticities of substitution. In order to ensure that this property also holds in the presence of

[13] In the case where there is such technical progress, the question of whether σ is greater than or below unity takes on added importance. Recall, when $\sigma < 1$, factors are "gross complements" in production and "gross substitutes" otherwise. Thus, it can be shown that with gross substitutes, substitutability between factors allows both the augmentation and bias of technological change to favor the same factor. For gross complements, however, a capital-augmenting technological change, for instance, increases demand for labor (the complementary input) more than it does capital, and vice versa. By contrast, when $\sigma = 1$ an increase in technology does not produce a bias towards either factor (factor shares will always be constant since any change in factor proportions will be offset by a change in factor prices).

[14] Neutrality concepts associate innovations to related movements in marginal products and factor ratios. An innovation is Harrod-Neutral if relative input shares remain unchanged for a given capital-output ratio. This is also called labor-augmenting since technical progress raises production equivalent to an increase in the labor supply. More generally, for $F(X_i, X_j, ..., A)$, technical progress is X_i-augmenting if $F_A A = F_{X_i} X_i$.

growing factor efficiencies, it follows that:

$$E_0^N = \frac{Y_0}{N_0}\left(\frac{1}{1-\pi_0}\right)^{\frac{\sigma}{1-\sigma}} \;;\; E_0^K = \frac{Y_0}{K_0}\left(\frac{1}{\pi_0}\right)^{\frac{\sigma}{1-\sigma}} \tag{25}$$

Note that at $t = t_0$, $e^{\gamma_i(t-t_0)} = 1$. This ensures that at the common fixed point the factor shares are not biased by the growth of factor efficiencies but are just equal to the distribution parameters π_0 and $1 - \pi_0$

Inserting equations (24) and the normalized values (25) into (23), leads to a normalized CES function that can be rewritten in the following form that again resembles the ACMS variant:

$$Y = \left[\pi_0\left(\frac{Y_0}{K_0}\cdot e^{\gamma_K(t-t_0)}\cdot K_t\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi_0)\left(\frac{Y_0}{N_0}\cdot e^{\gamma_N(t-t_0)}\cdot N_t\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{26}$$

or equivalently,

$$Y = Y_0\left[\pi_0 K_0^{\frac{1-\sigma}{\sigma}}\left(K_t\cdot e^{\gamma_K(t-t_0)}\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi_0)\, N_0^{\frac{1-\sigma}{\sigma}}\left(N_t\cdot e^{\gamma_N(t-t_0)}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{27}$$

In this specification of the normalized CES function, with factor augmenting technical progress, the growth of efficiency levels for capital and labor is now measured by the expressions $K_0e^{\gamma_K(t-t_0)}$ and $N_0e^{\gamma_N(t-t_0)}$, respectively, and t_0 is the baseline year. Again, we see from (27) that for $t = t_0$ we retrieve $Y = Y_0$.

Special cases of (27) are the specifications used by Rowthorn (1999), Bentolila and Saint-Paul (2003) or Acemoglu (2002), where $N_0 = K_0 = Y_0 = 1$ is implicitly assumed, or by Antràs (2004) who sets $N_0 = K_0 = 1$. Caballero and Hammour (1998), Blanchard (1997) and Berthold et al. (2002) work with a version of (27) where in addition to $N_0 = K_0 = 1$, $\gamma_K = 0$ is also assumed so that technological change is only of the labor-augmenting variety.

It is also worth noting that for constant efficiency levels $\gamma_N = \gamma_K = 0$ our normalized function (27) is formally identical with the CES function that Jones (2003) (p. 12) has proposed for the characterization of the "short term". In his terminology, the normalization values k_0, y_0, and π_0 are "appropriate" values of the fundamental production technology that determines long-run dynamics. This long-run production function is then considered to be of a Cobb-Douglas form with constant factor shares equal to π_0 and $1-\pi_0$ and with a constant exogenous growth rate. Actual behavior of output and factor input is thus modeled as permanent fluctuations around "appropriate" long-term values. For a similar approach in which steady state Cobb-Douglas parameter values are used to normalized a CES production function see Guo and Lansing (2009).

2.5.2 Growth Rates in Normalized Technical Progress Functions: Time-Varying Frameworks

Following recent theoretical discussion about possible biases in technical progress (e.g., Acemoglu (2002)), it is not clear that growth rates of technical progress components should always be constant. An innovation of Klump et al. (2007a) was to allow deterministic but time-varying technological progress terms where curvature or decay terms could be uncovered from the data in economically meaningful ways. For this they used a Box and Cox (1964) transformation in a normalized context where (with $E_t^i = e^{g_i}$):

$$g_i(\gamma_i, \lambda_i, t, t_0) = \frac{\gamma_i}{\lambda_i} t_0 \left(\left[\frac{t}{t_0} \right]^{\lambda_i} - 1 \right), \ i = K, N \tag{28}$$

Curvature parameter λ_i determines the shape of the technical progress function. For $\lambda_i = 1$, technical progress functions, g_i, are the (textbook) linear specification; if $0 < \lambda_i < 1$ they are exponential; if $\lambda_i = 0$ they are log-linear and $\lambda_i < 0$ they are hyperbolic functions in time. Note, the re-scaling of γ_i and t by the fixed point value t_0 in (28) allows us to interpret γ_N and γ_K directly as the rates of labor- and capital-augmenting technical change at the fixed-point period.

Asymptotically, function (28) would behave as follows:

$$g_i(\gamma_i, \lambda_i, t, t_0) = \begin{cases} \lim_{t \to \infty} g_i = \infty & \text{for} \quad \lambda_i \geq 0 \\ \lim_{t \to \infty} g_i = -\frac{\gamma_i}{\lambda_i} t_0 & \text{for} \quad \lambda_i < 0 \end{cases}$$

$$\frac{\partial g_i}{\partial t} = \gamma_i \left(\frac{t}{t_0} \right)^{\lambda_i - 1} \Rightarrow \begin{cases} \frac{\partial g_i}{\partial t} = \gamma_i & \text{for} \quad \lambda_i = 1 \\ \frac{\partial g_i}{\partial t} = 0 & \text{for} \quad \lambda_i < 1 \end{cases}$$

This framework allows the data to decide on the presence and dynamics of factor-augmenting technical change rather than being imposed a priori by the researcher. If, for example, the data supported an asymptotic steady state, this would arise from the estimated dynamics of these curvature functions (i.e., labor-augmenting technical progress becomes dominant (linear), that of capital absent or decaying).

In addition, as McAdam and Willman (2011a) pointed out, the framework also allows one to nest various strands of economic convergence paths towards the steady state. For instance, the combination,

$$\gamma_N > 0, \lambda_N = 1 \; ; \; \gamma_K = 0, \lambda_K = 0 \tag{29}$$

coupled with the assumption, $\sigma >> 1$, corresponds to that drawn upon by Caballero and Hammour (1998) and Blanchard (1997), in explaining the decline in the labor income share in continental Europe.

Another combination speculatively termed "Acemoglu-Augmented" Technical Progress by McAdam and Willman (2011a), can be nested as,

$$\gamma_N, \gamma_K > 0; \lambda_N = 1, \lambda_K < 1 \tag{30}$$

where $\sigma < 1$ is more natural.

Consider two cases within (30). A "weak" variant, $\lambda_K < 0$, implies that the contribution of capital augmentation to TFP is bounded with its growth component returning rapidly to zero; a "strong" case, where $0 < \lambda_K < 1$, capital imparts a highly persistent contribution with (asymptotic convergence to) a zero growth rate. Both cases are asymptotically consistent with a balanced growth path (BGP), where TFP growth converges to that of labor-augmenting technical progress, γ_N. The interplay between $|\gamma_N - \gamma_K|$ and λ_K, λ_N and can thus be considered sufficient statistics of BGP divergence. Normalization, moreover, makes this kind of classification quite natural since we are looking at biases in technical progress relative to some average or representative point.

3 The elasticity of substitution as an engine of growth

Although one of the first references to a CES structure of aggregate production appears in the Solow (1956) paper it had been for a long time impossible to answer the question of what effect changes in the substitution elasticity had on the steady-state values in the standard neoclassical growth model. Common sense would certainly suggest that easier factor substitution - via helping to overcome decreasing returns - should lead to a higher level of development. But a formal proof of this conjecture seemed out of reach. In fact, when Harbrecht (1975) tried to answer this question with the help of a (non-normalized) David and van de

Klundert (1965) CES variant, he found the contrary result! His analysis was, of course, biased by the interaction between a changing elasticity of substitution and the efficiency parameters of the CES function which is not compensated for as in the normalized version.

Already some years earlier, as mentioned in section 2.4, Kamien and Schwartz (1968) had presented a proof of the central relationship between the substitution elasticity and output but only for the special case in which the baseline values for K and N were equal. Their proof is based on the General Mean property of the CES function, which had already been recognized by ACMS.

A General Mean of order p is defined as,

$$M(p) = \left[\sum_{i=1}^{n} f_i x_i^p \right]^{\frac{1}{p}} \tag{31}$$

where $x_i, ...x_n$ are positive numbers (of the same dimension) and where the weights $f_i, ...f_n$ sum to unity. Special cases of the General Mean are the arithmetic, the geometric and the harmonic means where the order p would be 1, 0, and -1 respectively. If p tends to $-\infty$, the mean becomes the minimum of the numbers $(x_i, ...x_n)$.

One of the most important theorems about a General Mean is that it is an increasing function of its order (Hardy et al. (1934), p. 26 f.; Beckenbach and Bellman (1961), p. 16-18; see also the proof in La Grandville (2009), p. 111-113). More exactly it says that the mean of order p of the positive values x_i with weights f_i is a strictly increasing function in p unless all the x_i are equal. With the two factors K and N (and implicit normalization $K_0 = N_0$) this leads to the following statement:

> *Enlargement of the elasticity of substitution results in an increase in output from every combination of factors except that for which the capital labor ratio is equal to one.* (Kamien and Schwartz (1968), p. 12)

Of course, this result can be generalized provided that all numbers have the same dimension which is precisely achieved by normalizing numbers of different dimensions.

La Grandville (1989b) developed a graphical representation of normalized CES structures. He demonstrated that the general relationship between the elasticity of substitution and the level of development is usually positive. Moreover, when there are two factors of production, numerical results suggests that the function has a *single* inflection point, La Grandville and Solow (2006): in other words, between its limiting values, $\lim_{p\in(-\infty,\infty)} M(p)$, the function $M(p)$ is first convex then concave. For typical production-function weights, (i.e., $f_1 = 0.4; f_2 = 1 - f_1$) that inflection point occurs around $p \approx 0$ (i.e., the Cobb-Douglas neighborhood). This means that within some relevant interval around that even small perturbations of the substitution elasticity (however such a change may be implemented) might have extremely large implications for an economy. In short raising, your elasticity of substitution can raise your growth rate and its effect may be potentially even larger than that traditionally studied in the case of improvements in the savings rate and/or technical progress (such reasoning is reflected in the third quote that started our paper).

The formal proof for the conjecture was then presented by Klump and de La Grandville (2000), based on a very general normalized CES production function. An alternative proof is presented in Klump and Irmen (2009) who also deal with normalized CES functions in a Diamond-type version of the neoclassical growth model. It distinguishes efficiency and distribution effects of changes in the elasticity of substitution which can work in different directions if not all individuals have the same savings pattern so that redistribution matters. The interaction of both effects creates an acceleration effect for capital accumulation which can have a positive or a negative effect on the steady state. It can be shown, however, that even in this setting a higher elasticity of substitution leads to a higher steady state level as long as the efficiency effect dominates the distribution effect which is the most likely case.

Klump and Preissler (2000) extend the analysis of the standard neoclassical growth model with a normalized CES production function by calculating the effect of a changing elasticity of substitution on the speed of convergence towards the steady state. Earlier studies of this problem, e.g., Ramanathan (1975), which were not considering normalization had not derived convincing results. With an explicitly normalized CES production function, it is possible to show that an increase in the elasticity of substitution reduces the speed of convergence if the steady state value of the capital intensity is higher than its baseline value (which seems the most likely case).

Klump (2001) presents the analysis of a Ramsey type (intertemporal optimizing) growth model with a normalized CES production function. He is able to prove that as long as the steady state value of the capital intensity is higher than its baseline value the comparative static effect of a change in the elasticity of substitution on the steady state is strictly positive. The result were only recently reproduced by Xue and Yip (2009) using a different approach. For the effect of the elasticity of substitution on the speed of adjustment the same results as in the Solow model can be derived in the Ramsey model (Klump and Saam (2008)). This result holds irrespective of the value of the elasticity of substitution, whereas Ventura (1997) making use of a non-normalized CES production function could only generate meaningful results for $\sigma < 1$ values.

Temple (2008) has criticized the use of normalized CES functions for calculating convergence effects of a higher factor substitution because of an unclear economic meaning of the chosen baseline value for the capital intensity. However, as has been clarified by Klump and Saam (2008) the essence of normalization does not consist in the arbitrary choice of baseline values but in forcing the researcher to give an explicit statement about the relationship between baseline and steady state (ss) values. As growth models are generally motivated by the idea that labor is relatively scarce in the steady state it seems reasonable to normalize such that $k^{ss} > k_0$. The setting may be different in the business cycle literature, where fluctuations around the (typically zero growth) steady state are studied. In this case it makes sense to use steady state values as normalization parameters (Guo and Lansing (2009), Cantore et al. (2010), hereafter CLMW).

Finally, Irmen (2010) is able to show in an endogenous growth framework with a normalized CES production function that the steady state growth rate of output per worker increases with the elasticity of substitution between efficient capital and efficient labor. All analysis confirm that the elasticity of substitution is among the most powerful determinants of capital accumulation and growth as long as normalized CES production functions are used. La Grandville (2009, 2010) suggests that changes in the elasticity of substitution have a much higher effect on social welfare than changes in the rate of technical progress.

4 Estimated Normalized Production Function

Previous sections of this paper introduced the concept of normalization and its importance in theoretical analysis. Here we discuss, how the idea of normalization should be applied in *empirical* analysis and, more importantly, whether it alleviates the estimation of the parameters of the CES production function? We show that its merits are strong especially if system approach (containing cross-equation restrictions) is used. In this context the scepticism on the proper identification of the elasticity of substitution and technical progress from each other aroused by the famous Diamond-McFadden impossibility theory largely looses its practical importance. In fact we argue that a general factor-augmenting specification results in markedly less biased estimates of substitution elasticity parameter compared to the case where an a priori neutrality constraint is imposed. In the context of single equation approach, in turn, normalization is of lesser use.

An added problem[15], however, is that often the predictions of different elasticity and technical change combinations can have similar implications for variables of interest, such as factor income shares and factor ratios. Notwithstanding, whether factor income movements are driven by high or low substitution elasticities and with different combinations of technical change is profoundly important in terms of their different implications for, e.g., growth accounting, inequality, calibration in business-cycle models, public policy issues etc.

By way of illustration, **Tables 1** and **2** present an overview of empirical results obtained for the elasticity of substitution. We concentrate on the results from time-series or panel studies on aggregate data. In the case of the US, which has been widely studied, it is possible to find values of the elasticity of substitution above unity (with Harrod-neutral technical progress), at unity (with Hicks-neutral progress) and below unity (with Hicks-neutral progress and with technical progress augmenting both factors). The situation for other countries is no better; for Germany, values of above, below and at unity have been estimated. Using information about the degree of factor substitution from other sources does not re-solve this puzzle, either. It has been recognized, for example, by Lucas (1969) that older time-series studies for the US have generally provided lower estimates than cross-section studies that were supportive of the Cobb-Douglas function. More recent cross section analysis based on micro data that were used to estimate the relationship between business capital formation and user costs (e.g., Chirinko et al. (1999)) estimate very low elasticities of substitution ranging from 0.25-0.40. A drawback of these kinds of studies, however, is their inability to quantify any growth rate(s) of technical progress.

That there should be diversity in production function estimates - even for countries whose data properties are relatively stable and well-understood - is not surprising. It doubtless reflects the familiar trapdoor of empirical pitfalls: data quality; a priori modeling choices (such as whether to test for certain types of factor neutrality or impose them); the performance of various estimators (e.g., single equation, systems) and algorithms; as well as more prosaic data problems (e.g., outliers, uncertain auto-correlation, structural breaks, quality improvements, measurement errors etc).

[15]See the discussion in León-Ledesma et al. (2010b) and possible observational equivalence in examining income share developments and inferring the associated bias in technical progress.

Table 1.
Empirical Studies of Aggregate Elasticity of Substitution and Technological Change in the US

Study	SAMPLE[a]	Assumption on Technological Change	Estimated Elasticity of Substitution: σ	Estimated Annual Rate Of Efficiency Change		
				Neutral: $\gamma_N = \gamma_K$	Labor-augmenting: γ_N	Capital-Augmenting: γ_K
Arrow et al. (1961)	1909-1949	Hicks-Neutral	0.57	1.8	-	-
Kendrick and Sato (1963)	1919-1960	Hicks-Neutral	0.58	2.1	-	-
Brown and De Cani (1963)	1890-1918	Factor Augmenting	0.35	Labor saving ($\gamma_N - \gamma_K$ = 0.48)		
	1919-1937		0.08	Labor saving ($\gamma_N - \gamma_K$ = 0.62)		
	1938-1958		0.11	Labor saving ($\gamma_N - \gamma_K$ = 0.36)		
	1890-1958		0.44	?		
David and van de Klundert (1965)	1899-1960	Factor Augmenting	0.32	-	2.2	1.5
Bodkin and Klein (1967)	1909-1949	Hicks-neutral	0.50-0.70	1.4-1.5		
Wilkinson (1968)	1899-1953	Factor Augmenting	0.50	Labor saving ($\gamma_N - \gamma_K$ = 0.51)		
Sato (1970)	1909-1960	Factor Augmenting	0.50 – 0.70	-	2.0	1.0
Panik (1976)	1929-1966	Factor Augmenting	0.76	Labor saving ($\gamma_N - \gamma_K$ = 0.27)		
Berndt (1976)	1929-1968	Hicks-neutral	0.96-1.25	?	-	-
Kalt (1978)	1929-1967	Factor Augmenting	0.76	-	2.2	0.01
Antràs (2003)	1948-1998	Hicks-neutral	0.94-1.02	1.14	-	-
		Factor-augmenting	0.80	Labor saving ($\gamma_N - \gamma_K$ = 3.15)		
Klump et al (2007b)[b]	1953-2002	Factor-augmenting	0.7	-	-	-
León-Ledesma et al. (2010)	1960- 2004	Factor-augmenting	0.60-0.70	-	1.60	0.70

Notes:

[a] All studies are estimated on annual frequency data except León-Ledesman et al. (2010b) which is quarterly. Although, to aid comparability, we annualized their estimates for technical change in the table.

[b] We do not report technical change estimates for Klump et al (2007b) since they estimate with structural breaks in their technical progress terms. For reasons of space, the reader is referred to León-Ledesman et al. (2010a) for the original citations.

Table 2.
Empirical Studies of Aggregate Elasticity of Substitution in Selected Other Countries

Study	Countries	Sample (Frequency)	Assumption For Technological Change	Estimated Elasticity Of Substitution: σ
Lewis and Kirby (1988)	Australia	1967-1987 (Weekly)	Hicks-Neutral	0.78
Easterly and Fischer (1995)	Soviet Union	1950-1987 (Annual)	Hicks-Neutral	0.37
Andersen et al. (1999)	Panel of 17 OECD countries	1966-1996 (Annual)	Hicks-Neutral	1.12
Bolt and van Els (2000)	Austria Belgium Germany Denmark Spain Finland France Italy Netherlands Sweden UK US Japan	1971-1996 (Quarterly)	Hicks-Neutral	0.24 0.78 0.53 0.61 1 0.34 0.73 0.52 0.27 0.68 0.6 0.82 0.3
Duffy and Papageorgiou (2000)	Panel of 82 developed and developing countries	1960-1987 (Annual)	Hicks-Neutral	1.4
Ripatti and Vilmunen (2001)	Finland	1975-1999 (Quarterly)	Factor Augmenting	0.6
Willman (2002)	Euro area	1970-1997 (Quarterly)	Solow Neutral	0.95-1.05
McAdam and Willman (2004)	Germany	1983-1999 (Quarterly)	Hicks Neutral	0.7, 1, 1.2
Berthold et al. (2002)	US Germany France	1970-1995 (Semi-Annual)	Harrod-Neutral	1.15 1.45 2.01
Bertolila and Saint-Paul (2003)	13 industries in 12 OECD countries	1972-1993	Harrod-Neutral	1.06
McAdam and Willman (2004a)	Germany	1983-1999 (Quarterly)	Hicks Neutral	0.7, 1, 1.2
Klump et al (2007b)	Euro Area	1970-2003 (quarterly)	Factor-Augmenting	0.7
Luoma and Luoto (2010)	Finland	1902-2004 (annual)	Factor-Augmenting	0.5

Notes: For reasons of space, the reader is referred to Klump et al. (2007b) for most of these original citations.

At a simple level, normalization removes the problem that arises from the fact that labor and capital are measured in different units - although as we have seen its importance goes well beyond that. Under Cobb-Douglas, normalization plays no role since, due to its multiplicative form, differences in units are absorbed by the scaling constant. The CES function, by contrast, is highly non-linear, and so, unless correctly normalized, excluding technical progress, out of its three key parameters - the efficiency parameter, the distribution parameter, the substitution elasticity - only the latter is "deep". The other two parameters turn out to be affected by the size of the substitution elasticity and factor income shares.

If one compares the normalized with the non-normalized function, as before, i.e.,

$$Y_t = C\left[\pi\left(\Gamma_t^K K_t\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi)\left(\Gamma_t^N N_t\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

$$Y_t = Y_0\left[\pi_0\left(\frac{\Gamma_t^K K_t}{\Gamma_0^K K_0}\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi_0)\left(\frac{\Gamma_t^N N_t}{\Gamma_0^N N_0}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

we may be unsure as to where the estimation benefits of normalization derive. After all, both equations contain the same number of parameters. In fact the latter equation seemingly *adds* complexity by incorporating normalized reference points into the estimation (the empirical choice of the normalization point is a particular aspect discussed in section (4.2)).

The answer as to why normalization should improve matters empirically reflects the following. The distribution and efficiency parameters (respectively, $Y_0; \pi_0$ and $C; \pi$) can now either be pre-set prior to estimation or at least have a deep interpretation in terms of the data (i.e., the representative capital income share). Effectively normalization allows us to reduce the number of freely estimated parameters by two.

This follows straightforwardly from our earlier analysis. In the non-normalized formulation the parameters C and π above have no clear theoretic or empirical meaning. Instead, they are composite parameters conditional on, besides the selected fixed points, the elasticity of substitution (re-stating equations (15) and (16)):

$$C(\sigma, \cdot) = Y_0\left[\frac{r_0 K_0^{1/\sigma} + w_0 N_0^{1/\sigma}}{r_0 K_0 + w_0 N_0}\right]^{\frac{\sigma}{\sigma-1}}$$

$$\pi(\sigma,\cdot)=\frac{r_0K_0^{1/\sigma}}{r_0K_0^{1/\sigma}+w_0N_0^{1/\sigma}}$$

The additional merit in using the normalized instead of non-normalized form is that all parameters have a clear empirical correspondence. In particular, the distribution parameter is identified as the capital income share of total factor income at the fixed point. Hence, a suitable choice for the fixed point may alleviate the estimation of the deep parameters and, to repeat, makes the estimated production function suitable, for example, for comparative static analysis.

Table 3 presents some consistent sets of (deterministic) initial values for generating data and the implied ranges of the true values of C and π and for $\sigma \in [0.2, 1.3]$. In the first row we assumed $K_0 = N_0 = 1$. This allows us to solve Y_0 from the first row - with initial values of $\Gamma_0^K = \Gamma_0^N = 1$. In fact this represents a special case because indexing by the point of normalization equaling one is neutral implying that the true value of $C = 1$ and $\pi = \pi_0 = r_0 = 0.3\ \forall\sigma$ (this, in turn, implies solving the normalized real wage rate as $\frac{(1-\pi_0)Y_0}{N_0}$). In this special case it does not matter, if the same initial values of parameters are used, whether the system is estimated in normalized or non-normalized form. Although this particular normalization point generates the odd case whereby the user cost (essentially, the sum of the real interest rate plus depreciation rates) equals capital income share.

In all other cases, however, this is not so. To illustrate, in these other cases we have adjusted the initial conditions for output to make them consistent with an initial (and arguably more reasonable) value for r (the real user cost of capital) equal to 5%. The sample average normalization insulates the normalized system from the effects of changes in initial values in generating the data but the true values of composite parameters C and π vary widely: $C \in [0.16, 0.49]$, $\pi \in [0.29, 0.99]$. Thus, we confirm that the actual income distribution of the data is completely unrelated to the true value of π.

This illustrates the difficulty that a practitioner faces when trying to estimate non-normalized forms since the actual data scarcely gives any guidelines for appropriate choices for the initial parameter values of C and π. As León-Ledesma et al. (2010a) have documented, that results in serious estimation problems. They estimated normalized and non-normalized forms where in the latter case the initial parameter values for C and π are selected randomly from their given range in Table 3. When $\widehat{C}$ and $\widehat{\pi}$ substantially depart from their true, theoretical values, there are significant and quantitatively important biases in the estimated substitution elasticity and technical change.

Table 3. Consistent Normalization Values

N_0	π_0	r_0	K_0	$Y_0^* = Y_0 = \frac{r_0}{\pi_0} K_0$	$w_0 = \frac{(1-\pi_0)Y_0}{N_0}$	C		π	
						when $\sigma = 1.3$	when $\sigma = 0.2$	when $\sigma = 1.3$	when $\sigma = 0.2$
1	0.3	0.3	1	1	0.7	1		0.4	
1	0.3	0.05	5	0.833	0.583	0.546	0.225	0.228	0.996
1	0.3	0.05	8	1.333	0.933	0.788	0.225	0.210	0.999

Notes: C and π in the final two columns are calculated according to equations (15) and (16) for $\sigma \in [0.2,1.3]$.

Outside of the "special case" note the following partial derivatives showing how ceteris paribus changes in initial values change these last two parameters, C and π:

$$C_\sigma, C_{Y_0}, C_{K_0}, C_{w_0} > 0, C_{N_0} < 0$$

$$\pi_\sigma, \pi_{N_0} < 0, \pi_{K_0}, \pi_{w_0} > 0$$

4.1 Estimation Forms

The recognition of normalization says nothing specifically about the way production and production-technology should be estimated and how normalization impacts those estimation choices. Typical estimation forms found in the literature include: the non-linear CES production function; the linear first-order conditions of profit maximization; linear approximation of the CES function; and "system" estimation incorporating the production function and the first-order conditions.[16]

To proceed let us express the CES function, equation (27), in log form:

$$\log\left(\frac{Y}{Y_0}\right) = \frac{\sigma}{\sigma-1}\log\left[\pi_0\left(e^{\gamma_K(t-t_0)}\frac{K_t}{K_0}\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi_0)\left(e^{\gamma_N(t-t_0)}\frac{N_t}{N_0}\right)^{\frac{\sigma-1}{\sigma}}\right] \quad (32)$$

From this we can derive the marginal profit-maximization conditions,[17]

$$\log(r) = \underbrace{\log\left(\pi_0\frac{Y_0}{K_0}\right) + \frac{1}{\sigma}\log\left(\frac{K_0}{Y_0}\right)}_{\alpha_r} + \frac{1}{\sigma}\log\left(\frac{Y}{K}\right) + \frac{\sigma-1}{\sigma}\left(\gamma_K(t-t_0)\right) \quad (33)$$

$$\log(w) = \underbrace{\log\left((1-\pi_0)\frac{Y_0}{N_0}\right) + \frac{1}{\sigma}\log\left(\frac{N_0}{Y_0}\right)}_{\alpha_w} + \frac{1}{\sigma}\log\left(\frac{Y}{N}\right) + \frac{\sigma-1}{\sigma}\left(\gamma_N(t-t_0)\right) \quad (34)$$

Where, as before, γ_N and γ_K are the respective growth rates of labor and capital augmenting technical progress. Equations (33) and (34) represent the first-order conditions with respect to capital and labor respectively.

Estimation of production and technology parameters based on the first-order conditions and other single-equation approach are hampered by the fact that they only admit estimates of technical progress terms contained by their presumed

[16] We confine ourselves to constant-returns production functions. This is largely done to be consistent with much of the aggregate evidence (e.g., Basu and Fernald (1997)).

[17] Given that the real user cost and real interest rate can be sometimes negative in historical samples (particularly in the 1970s), the user cost conditions is usually expressed in levels rather than logarithms. Note, the last two conditions in some estimation cases are merged in many papers:

$$\log\left(\frac{K_t}{N_t}\right) = \alpha_i + \sigma\log\left(\frac{w_t}{r_t}\right) + (\gamma_N - \gamma_K)(1-\sigma)\,t$$

$$\log\left(\frac{K_t}{N_t}\right) = \alpha_j - \frac{\sigma}{1-\sigma}\log\frac{r_t K_t}{w_t N_t} + (\gamma_N - \gamma_K)\,t$$

FOC choice (in that sense any *bias* in technical progress is, by definition, not separately identifiable). This apparent drawback is presumably compensated by their tractable form and estimation linearity. Accordingly, these forms are common (more common, for instance, than direct non-linear CES estimation): e.g., equation (33) has been widely used in the investment literature (e.g., Caballero (1994)) and (34) was the form used by ACMS amongst many others.

A notable feature of the above three equations is that if estimated in single-equation mode, the normalization points (denoted by the curly lower brackets) are absorbed by the respective constants, α_r and α_w. Thus, from an estimation stand point, it is only when the non-linear CES function is estimated directly or where the system approach is used, does formal normalization play an empirical role.

Another possible vehicle of estimation is the Kmenta (1967) approximation (which became an important, if apparently unacknowledged, pre-cursor to the translog form). This is a Taylor-series expansion of the log CES production function around $\sigma = 1$.[18] However, we can also express it in normalized form:[19]

$$y_t = \pi_0 k_t + \lambda k_t^2 \\ + \underbrace{\pi_0 \left[1 + \frac{2\lambda}{\pi_0} k_t\right] \gamma_K \widetilde{t} + (1 - \pi_0) \left[1 - \frac{2\lambda}{1 - \pi_0} k_t\right] \gamma_N \widetilde{t} + \lambda \left[\gamma_K - \gamma_N\right]^2 \cdot \widetilde{t}^2}_{tfp} \tag{35}$$

where $\widetilde{t} = t - t_0$, $y_t = \log[(Y_t/Y_0) / (N_t/N_0)]$, $k_t = \log[(K_t/K_0) / (N_t/N_0)]$, $tfp = \log(TFP)$ and $\lambda = \frac{(\sigma-1)\pi_0(1-\pi_0)}{2\sigma}$. Equation (35) shows that the output-labor ratio can be decomposed into capital deepening and technical progress, weighted by factor shares and the substitution elasticity (where $sign\,(\lambda) = sign\,(\sigma - 1)$ and $\lim_{\sigma \in [0,\ \infty]} \lambda \in \left[-\infty, \frac{1}{2}\pi_0 (1 - \pi_0)\right]$). In addition, (35) shows that, when $\sigma \neq 1$ and

[18] Linearization around a unitary substitution is algebraically the most convenient form, as can be easily verified.

[19] The expressions for Log(TFP) for the restricted neutrality cases would be:

$$\text{Harrod}: (1 - \pi) \left[1 - \frac{2\lambda}{(1 - \pi)} k_t\right] \gamma_N \cdot \widetilde{t} + \lambda \gamma_N^2 \cdot \widetilde{t}^2$$

$$\text{Solow}: \pi \left[1 + \frac{2\lambda}{\pi} k_t\right] \gamma_K \cdot \widetilde{t} + a \gamma_K^2 \cdot \widetilde{t}^2$$

$$\text{Hicks}: \gamma \cdot \widetilde{t},\ \text{where } \gamma = \gamma_K = \gamma_N$$

$$\text{Cobb Douglas}: \pi \gamma_K \widetilde{t} + (1 - \pi)\, \gamma_N \cdot \widetilde{t}$$

Though, strictly speaking in the latter case, individual technical change cannot be identified in the Cobb-Douglas case, inter alia, (Barro and Sala-i-Martin (2004), p78-80).

$\gamma_K \neq \gamma_N > 0$, additional (quadratic[20]) curvature is introduced into the estimated production function.

With the predetermined normalization point, the advantage of (35) over the Kmenta approximation of the non-normalized CES is - as usual - that, since all variables appear in indexed form, the estimates are invariant to a change in units of measurement. Another advantage is that in the neighborhood of the normalization point (i.e., $K_t = K_0, N_t = N_0, \pi = \pi_0$) and without σ deviating "too much" from unity, as the approximation also assumes. The terms including the normalized capital intensity and multiplying linear trend have only second-order importance and, without any significant loss of precision, can be dropped, yielding,

$$\begin{aligned} y &= \pi_0 k_t + \lambda k \\ &\quad + \underbrace{\underbrace{\left[\pi_0 \gamma_K + (1-\pi_0)\gamma_N\right]}_{\theta} \widetilde{t} + \underbrace{\lambda\left[\gamma_K - \gamma_N\right]^2}_{\epsilon} \widetilde{t}^2}_{tfp} \end{aligned} \tag{36}$$

Estimation of equation (36) yields 4 parameters, $\pi_0, \widehat{\lambda}, \widehat{\theta}, \widehat{\epsilon}$, for 4 primitives, $\pi_0, \sigma, \gamma_K, \gamma_N$. Using π_0 allows us to identify σ from composite parameter λ, i.e., $\sigma = \left(1 - \frac{2\widehat{\lambda}}{\pi_0(1-\pi_0)}\right)^{-1}$. However, without a priori information on which one of two technical progress components dominates and, in addition, that the signs of estimates λ and ϵ are (or are constrained to be) the same, one cannot identify γ_K and γ_N. This leads to the following weak identification result: for $\gamma_N > \gamma_K$ we obtain $\gamma_N = \widehat{\theta} + \pi_0\sqrt{\frac{\widehat{\epsilon}}{\widehat{\lambda}}}$ and $\gamma_N = \widehat{\theta} - (1-\pi_0)\sqrt{\frac{\widehat{\epsilon}}{\widehat{\lambda}}}$ and for $\gamma_N < \gamma_K$ we obtain, $\gamma_N = \widehat{\theta} - \pi_0\sqrt{\frac{\widehat{\epsilon}}{\widehat{\lambda}}}$ and $\gamma_N = \widehat{\theta} + (1-\pi_0)\sqrt{\frac{\widehat{\epsilon}}{\widehat{\lambda}}}$. Given this, even under the helpful environment of normalization, we can say that although the Kmenta approximation can be used to estimate σ, it cannot effectively identify the direction of the biased technical change.[21] However, the approximation is a useful vehicle to, *ex post*, calculate TFP.

4.2 The Point of Normalization - Literally!

To be empirically operational, *the point of normalization* must be defined (i.e., what these Y_0, K_0's are in practice). If the estimation data were deterministic,

[20] Quadratic or higher depending on the order of the approximation.

[21] The Kmenta approximation, both empirically and in terms of general identification, has enjoyed limited success (see, Kumar and Gapinski (1974), Thursby (1980), León-Ledesma et al. (2010a)

this would be unproblematic: every sample point would be equally suitable for the point of normalization. For instance, in theoretic settings, the normalization point is often fashioned around the non-stochastic steady state (e.g., Klump and Saam (2008), CLMW).

However, since actual data is inevitably stochastic (and the intensity with which factors are utilized is unobserved) this convenience does not carry over because the production function does not hold exactly in any sample point. Therefore, to diminish the size of cyclical and stochastic components in the point of normalization, an appealing procedure is to define the normalization point in terms of sample averages for the underlying variables - typically geometric averages for growing variables (such as factors of production) and arithmetic ones for approximately stationary variables (e.g., factor income shares, the real interest rate and user cost).[22]

The nonlinearity of the CES function, however, in turn, implies that the sample average of production need not exactly coincide with the level of production implied by the production function with sample averages of the right hand variables *even with a deterministic DGP*. To circumvent this problem, Klump et al. (2007a) introduced an additional parameter ξ whose expected value is around unity (which we might call the normalization constant[23]). Hence, we can define $Y_0 = \xi\overline{Y}$, $K_0 = \overline{K}$, $N_0 = \overline{N}$, $\pi_0 = \overline{\pi}$ and $t_0 = \overline{t}$ where the bar refers to the respective sample average (geometric or, as in the last two, arithmetic).

An advantage of the normalized system over the non-normalized system, in turn, is that the distribution parameter $\overline{\pi}$ has a clear data-based interpretation. Therefore, it can either be fixed prior to estimation or, at least, the sample average can be used as a very precise initial value of the distribution parameter. Likewise a natural choice for the initial value of normalization constant, ξ, is one. Estimated values of these two parameters should not deviate much from their initial values without casting serious doubts on the reasonableness of estimation results. In the non-normalized case, by contrast, no clear guidelines exist in choosing the initial values of distribution parameter π and efficiency parameter C. In the context of non-liner estimation this may imply an enormous advantage of the normalized over the non-normalized system.

[22]Klump et al. (2007a,b) define it in terms of long-run averages, and Jones (2003) in terms of appropriate values.

[23]Only in the log-linear case of Cobb-Douglas would one expect ξ to exactly equal unity. Hence, in choosing the sample average as the point of normalization we lose precision because of the CES's non-linearity. If, alternatively, we choose the sample mid-point as the normalization point, we should also lose because of stochastic (and in actual data, cyclical) components that would also imply non-unitary ξ.

5 Normalization in Growth and Business Cycle Models

Production functions are a key part of business-cycle and growth macro-models. The two main competing models in the macro profession are the RBC (real business cycle) model and the NK (New Keynesian) model. Both imply relatively tight, theory-led dynamics and are furnished with a rich number of shocks which displace the agent from his optimal work, leisure, investment and employment choices. The standard RBC model is a variant of the representative agent neoclassical model, where business cycles are due to non-monetary sources (primarily, changes in technology). The NK one supplements that with various real and nominal rigidities to better capture the data.

However, what both models tend to share in practice is a focus on Cobb-Douglas aggregate production. This is especially puzzling given that such models tend to be motivated as *business-cycle* frameworks. Yet over business-cycle frequencies one might precisely expect relatively little (and presumably below unitary) factor substitutability as well as the presence of *non-neutral* technical change to capture factor income share developments. By introducing and assessing non-unitary production forms, the potential for a better understanding of technology and policy transmission and for a richer decomposition of historical time series is likely to be considerable.

The introduction of normalized technologies in simple business-cycle models is relatively straightforward and can be illustrated using the canonical RBC model. The model is relatively well known and can therefore be introduced compactly. The standard model with CES production technology in the supply side would be given by (omitting the expectations operator for simplicity):

$$\Lambda_t = \beta \Lambda_{t+1}[1 + r_{t+1} - \delta] \tag{37}$$

$$w_t = \upsilon \frac{N_t^{\varsigma}}{\Lambda_t} \tag{38}$$

$$Y_t = Y_0 \left[\pi_0 (\frac{K_t}{K_0} e^{\gamma_t^K})^{\frac{\sigma-1}{\sigma}} + (1-\pi_0)(\frac{N_t}{N_0} e^{\gamma_t^N})^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{39}$$

$$w_t = (1-\pi_0)\left(\frac{Y_0}{N_0}e^{\gamma_t^N}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y_t}{N_t}\right)^{\frac{1}{\sigma}} \quad (40)$$

$$r_t = \pi_0\left(\frac{Y_0}{K_0}e^{z_t^K}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y_t}{K_t}\right)^{\frac{1}{\sigma}} \quad (41)$$

$$C_t + K_t - (1-\delta)K_{t-1} \leq Y_t \quad (42)$$

$$\gamma_t^j = \rho_j\gamma_{t-1}^j + \eta_t \quad (43)$$

where Λ_t, w_t and r_t are, respectively, the marginal utility of consumption (C_t), wages and the interest rate (all expressed in real terms). Parameters β, δ and υ represent, respectively, the discount factor, the capital depreciation rate and a scaling constant. Processes γ_t^j are technology shocks - as equation (43) shows usually modeled as a stationary AR(1) process - for $j = K, N$ (i.e., capital-augmenting and labor-augmenting shocks respectively). Equations (37) and (38) represent the household's optimal consumption and labor supply choices given, for example, the separable utility function,

$$U_t = \frac{C_t^{1-\sigma_c}}{1-\sigma_c} - \upsilon\frac{N_t^{1+\varsigma}}{1+\varsigma} \quad (44)$$

where σ_c is the coefficient of relative risk aversion and ς is the inverse of the Frisch elasticity. This particular utility function implies $\Lambda_t = C_t^{-\sigma_c}$. If the researcher wanted to simulate this model conditional on different values of the substitution elasticity, he/she would do the following:

(i) Pre-set key normalization parameters: $r_0 = \frac{1}{\beta} - 1 + \delta$, and, for some given N_0, K_0, π_0, solve out $Y_0 = \frac{r_0}{\pi_0}K_0$ and $w_0 = (1-\pi_0)\frac{Y_0}{N_0}$ following Table 3 (for temporary shocks, these normalization points will be chosen to be the same as the presumed steady state);

(ii) Reset leisure scaling parameter υ to equate the real wage expressions in (38) and (40), implying $\upsilon = \frac{(1-\alpha_0)r_0^{\sigma_c}}{(r_0-\delta\alpha_0)^{\sigma_c}}$;

In this simple way, conventional dynamic exercises can be performed on the model (e.g., examining the effect of technology shocks) which are robust to changes in the substitution elasticity can be legitimately made.

In this vein, CLMW, looked at the relationship between technology shocks and hours worked - a key controversy between NK and RBC explanation of the business cycle - by expressing both models in consistent normalized form. They showed that, depending on the value of the substitution elasticity and the source of the shock (capital or labor-augmenting), both models could generate positive or negative hours responses (thus, largely overturning conventional wisdom on the mechanisms in the models).

6 Conclusions and Future Directions

The elasticity of substitution between capital and labor represents a key parameter in many fields of economics: e.g., business-cycle and growth outcomes, income distribution, stabilization policy, labor market dynamics etc. The empirical evidence - for the US and other developed economies - is clear in rejecting the unitary-elasticity Cobb-Douglas specification in favor of (generally below unity) CES aggregate production functions.[24] When investigating the ramifications of a non-unitary substitution elasticity, one inevitably necessarily faces the issue of normalization.

The importance of explicitly normalizing CES functions was discovered in the seminal paper by La Grandville (1989b), then further explored by Klump and de La Grandville (2000), Klump and Preissler (2000), La Grandville and Solow (2009), and first empirically implemented and investigated by Klump et al. (2007a). Normalization starts from the observation that a family of CES functions whose members are distinguished only by different elasticities of substitution needs a common benchmark point. Since the elasticity of substitution is defined as a point elasticity, one needs to fix benchmark values for the level of production, the inputs of capital and labor and for the marginal rate of substitution, or equivalently for per-capita production, capital intensity and factor income shares.

Overall, we can say that,

(a) Normalization is necessary for identifying in an economically meaningful way the constants of integration which appear in the solution to the differential equation from which the CES function is derived (and thus makes it suitable for comparative static analysis);

(b) Normalization helps to distinguish among the various functional forms, which have been developed in the CES literature and thus which CES production functions are legitimate;

[24] Duffy and Papageorgiou (2000) suggest developing countries may be empirically represented by an above-unity aggregate substitution elasticity.

(c) Normalization is necessary for securing the basic property of CES production in the context of growth theory, namely the strictly positive relationship between the substitution elasticity and the output level given the CES function's representation as a "General Mean";

(d) In situations where the researchers wishes to gauge the sensitivity of results (steady-state or dynamic) to variations in the substitution elasticity, normalization is imperative.

(e) Normalization alleviates the estimation of the deep parameters of the aggregate production function, in particular the elasticity of substitution and the growth rates of factor augmenting technical progress.

and finally,

(e) Normalization is convenient when biases in the direction of technical progress are to be empirically determined, since it fixes a benchmark value for factor income shares. This is important when it comes to an empirical evaluation of changes in income distribution arising from technical progress. If technical progress is biased in the sense that factor income shares change over time the nature of this bias can only be classified with regard to a given baseline value.

In our view there are at least four promising and related areas for future research on normalization:

(1) Following Jones (2005) one can regard a macroeconomic production function as a reduced form which should be derived from micro-foundations. This conjecture has been taken up by Growiec (2008) who shows that a CES production function can be linked to Weibull distributions of unit factor productivities, whereas a Cobb-Douglas function responds to Pareto distributions. It is still an open question, however, how the normalization values of the CES function can be linked in a meaningful way to the parameters of the underlying unit factor productivity distributions.

(2) A better understanding of the micro-foundations of the normalized CES function would also help to better understand reasons for possible international differences and intertemporal changes in the elasticity of substitution. Both seem to be linked to some deeper economic, social and cultural parameters as well as to

the level of development measured by capital per units of labor or income per capita (Klump and de La Grandville (2000); Duffy and Papageorgiou (2000); Masanjala and Papageorgiou (2004); Mallick (2010), Wong and Yip (2010)).

(3) Given the suggestion by La Grandville (2009, 2010) that an increase in the elasticity of substitution has a much stronger effect on aggregate wealth than an increase in the rate of technical progress one finally wonders why the empirical evidence seems to suggest that at least in the medium run the elasticity of substitution for a given country appears rather stable. There seems to be some kind of "sigma-augmenting" technical progress at work whose exact mechanisms are not yet understood. Kamien and Schwartz (1968) had already pointed out that changes in the elasticity of substitution have similar effects on the distribution of income as the augmentation in factor efficiency. They might therefore induce similar correction effects which are worth being analyzed in more detail.

(4) In business cycle and growth models, the Cobb Douglas aggregate production function is the default choice. However, the convenience/centrality of Cobb-Douglas production functions in macro is likely to be obscuring important issues. We would therefore expect that normalization - which leads to a simplification and deepening of CES properties - to be more widely used in such models. In so doing, its use should shed light on the propagation and decomposition of business cycle shocks and policy transmissions.

To conclude our survey, we re-stress that production functions are ubiquitous in theoretical and empirical models, and ubiquitously Cobb Douglas! This appears to us not only an unjustifiable simplification but an impediment to understanding various economic phenomena. The paper by CLMW is a welcome addition in fashioning otherwise standard Real Business Cycle and New Keynesian models around a normalized CES supply side and showing the extraordinary impact in terms of over-turning the prediction that these separate models made for the technology-hours correlation. But this is merely one example. We would hope therefore that an understanding of normalized CES functions would be an integral part in building macro-economic models, in the same way as nominal and real rigidities have become.

References

Acemoglu, D. (2002). Directed technical change. *Review of Economic Studies*, 69:781–809.

Acemoglu, D. (2003). Labor- and capital-augmenting technical change. *Journal of the European Economic Association*, 1:1–37.

Acemoglu, D. (2009). *Introduction to Modern Economic Growth*. MIT Press.

Allen, R. G. D. (1938). *Mathematical Analysis for Economists*. MacMillian and Co Ltd, London.

Antràs, P. (2004). Is the US Aggregate Production Function Cobb-Douglas? New Estimates of the Elasticity of Substitution. *Contributions to Macroeconomics*, 4:Article 4.

Arrow, K. J., Chenery, H. B., Minhas, B. S., and Solow, R. M. (1961). Capital-labor substitution and economic efficiency. *Review of Economics and Statistics*, 43(3):225–250.

Azariadis, C. (1996). The economics of poverty traps - Part One: Complete markets. *Journal of Economic Growth*, 1:449–486.

Barro, R. J. and Sala-i-Martin, X. (2004). *Economic Growth*. 2nd Edition, MIT Press, Cambridge, MA.

Basu, S. and Fernald, J. (1997). Returns to Scale in U.S. Manufacturing: Estimates and Implications. *Journal of Political Economy*, 105(2):249–283.

Beckenbach, E. F. and Bellman, R. (1961). *Inequalities*. Berlin: Springer.

Bentolila, S. and Saint-Paul, G. (2003). Explaining movements in the labor share. *Contributions to Macroeconomics*, 3:Article 9.

Berndt, E. R. (1976). Reconciling alternative estimates of the elasticity of substitution. *Review of Economics and Statistics*, 58(1):59–68.

Berthold, N., Fehn, R., and Thode, E. (2002). Falling labour share and rising unemployment: Long-run consequences of institutional shocks? *German Economic Review*, 3:431–459.

Blanchard, O. J. (1997). The Medium Run. *Brookings Papers on Economic Activity*, 2:89–158.

Box, G. and Cox, D. (1964). An analysis of transformations. *Journal of the Royal Statistical Society*, 26, Series B:211–243.

Caballero, R. J. (1994). Small sample bias and adjustment costs. *Review of Economics and Statistics*, 85:153–65.

Caballero, R. J. and Hammour, M. L. (1998). Jobless growth: Appropriablity, factor substitution and unemployment. *Carnegie-Rochester Conference Proceedings*, 48:51–94.

Cantore, C., León-Ledesma, M. A., McAdam, P., and Willman, A. (2010). Shocking Stuff: Technology, Hours and Factor Substitution. ECB Working Paper 1278.

Cantore, C. and Levine, P. (2011). Getting Normalization Right. Mimeo, University of Surrey.

Chirinko, R. S. (2002). Corporate taxation, capital formation, and the substitution elasticity between labor and capital. *National Tax Journal*, 60:339–355.

Chirinko, R. S. (2008). σ: The Long and Short of It. *Journal of Macroeconomics*, 30(2):671–686.

Chirinko, R. S., Fazzari, S. M., and Meyer, A. P. (1999). How responsive is business capital formation to its user cost? *Journal of Public Economics*, 74(1):53–80.

David, P. A. and van de Klundert, T. (1965). Biased efficiency growth and capital-labor substitution in the US, 1899-1960. *American Economic Review*, 55(3):357–394.

De Jong, F. J. (1967). *Dimensional Analysis for Economists*. North Holland.

De Jong, F. J. and Kumar, T. K. (1972). Some considerations on a class of macroeconomic production functions. *De Economist*, 120(2):134–152.

Diamond, P. A. and McFadden, D. (1965). Identification of the Elasticity of Substitution and the bias of Technical Change: An Impossibility Theorem. Working Paper No. 62, University of California Berkeley.

Diamond, P. A., McFadden, D., and Rodriguez, M. (1978). Measurement of the elasticity of substitution and bias of technical change. In Fuss, M. and McFadden, D., editors, *Production Economics, Vol. 2*, pages 125–147. Amsterdam and North Holland.

Dickinson, H. D. (1954). A Note on Dynamic Economics. *Review of Economic Studies*, 22(3):169–179.

Dimond, R. W. and Spencer, B. J. (2008). Trevor Swan and the neoclassical growth model. *NBER Working Paper 13950*, 99(2):256–259.

Duffy, J. and Papageorgiou, C. (2000). A cross-country empirical investigation of the aggregate production function specification. *Journal of Economic Growth*, 5(1):86–120.

Growiec, J. (2008). Production Functions and Distributions of Unit Factor Productivities: Uncovering the Link. *Economics Letters*, 101(1):87–90.

Guo, J. T. and Lansing, K. J. (2009). Capital-labour substitution and equilibrium indeterminacy. *Journal of Economic Dynamics and Control*, 33(12):1991–2000.

Harbrecht, W. (1975). Substitutionselastizitat und Gleichgewichtswachstum. *Jahrbucher fur Nationalokonomie und Statistik*, 189:190–201.

Hardy, G. H., Littlewood, J. E., and Pólya, G. (1934). *Inequalities*. Cambridge University Press, 2nd ed. 1952.

Hicks, J. R. (1932). *The Theory of Wages*. London: Macmillan.

Hicks, J. R. (1970). Elasticity of substitution again: Substitutes and Complements. *Oxford Economic Papers*, 22:289–296.

Houthakker, H. S. (1955). The Pareto Distribution and the Cobb-Douglas Production Function in Activity Analysis. *Review of Economic Studies*, 23:27–31.

Irmen, A. (2010). Steady-state growth and the elasticity of substitution. Working Paper 0496, University of Heidelberg.

Jensen, B. S. (2010). Production functions and growth theory - von Thünen and Solow. Mimeo, University of Southern Denmark.

Jones, C. I. (2003). Growth, Capital Shares, and a New Perspective on Production Functions. Mimeo, University of California Berkeley.

Jones, C. I. (2005). The shape of production functions and the direction of technical change. *Quarterly Journal of Economics*, 120(2):517–549.

Kaas, L. and von Thadden, L. (2003). Unemployment, factor substitution, and capital formation. *German Economic Review*, 4:475–495.

Kaldor, N. (1961). Capital accumulation and economic growth. In Lutz, F. A. and Hague, D. C., editors, *The Theory of Capital*. St. Martin's Press.

Kamien, M. I. and Schwartz, N. L. (1968). Optimal Induced technical change. *Econometrica*, 36(1):1–17.

Kennedy, C. (1964). Induced bias in innovation and the theory of distribution. *Economic Journal*, 74:541–547.

Kennedy, C. and Thirwall, A. P. (1973). Technological Change and the Distribution of Income: A Belated Comment. *International Economic Review*, 14(3):780–784.

Klump, R. (2001). Trade, money and employment in intertemporal optimizing models of growth. *Journal of International Trade and Economic Development*, 10(4):411–428.

Klump, R. (2002). Verteilung und Wirtschaftswachstum: Eine neoklassische Verallgemeinerung des Wachstumsmodell von Kaldor. In Menkhoff, L. and Sell, F., editors, *Zur Theorie, Empirie und Politik der Einkommensverteilung*, pages 11–26. Berlin-Heidelberg-New York.

Klump, R. and de La Grandville, O. (2000). Economic growth and the elasticity of substitution: Two theorems and some suggestions. *American Economic Review*, 90(1):282–291.

Klump, R. and Irmen, A. (2009). Factor substitution, income distribution and growth in a generalized neoclassical model. *German Economic Review*, 10(4):464–479.

Klump, R., McAdam, P., and Willman, A. (2007a). Factor Substitution and Factor Augmenting Technical Progress in the US. *Review of Economics and Statistics*, 89(1):183–92.

Klump, R., McAdam, P., and Willman, A. (2007b). The Long-Term SucCESs of the Neo-Classical Growth Model. *Oxford Review of Economic Policy*, 23(1):94–114.

Klump, R., McAdam, P., and Willman, A. (2008). Unwrapping Some Euro Area Growth Puzzles: Factor Substitution, Productivity and Unemployment. *Journal of Macroeconomics*, 30(2):645–666.

Klump, R. and Preissler, H. (2000). CES production functions and economic growth. *Scandinavian Journal of Economics*, 102(1):41–56.

Klump, R. and Saam, M. (2008). Calibration of normalized CES production functions in dynamic models. *Economics Letters*, 99(2):256–259.

Kmenta, J. (1967). On Estimation of the CES Production Function. *International Economic Review*, 8(2):180–189.

Kumar, T. K. and Gapinski, J. H. (1974). Nonlinear estimation of the CES Production Function Parameters: A Monte Carlo. *Review of Economics and Statistics*, 56(4):563–567.

La Grandville, Olivier de. (1989a). Curvature and the elasticity of substitution: Straightening it out. *Journal of Economics*, 66(1):23–34.

La Grandville, Olivier de. (1989b). In Quest of the Slutzky Diamond. *American Economic Review*, 79:468–481.

La Grandville, Olivier de. (2009). *Economic Growth: A Unified Approach*. Cambridge University Press.

La Grandville, Olivier de. (2010). How Much Should a Nation Save? A New Answer. Mimeo.

La Grandville, Olivier de. and Solow, R. M. (2006). A conjecture on general means. *Journal of Inequalities in Pure and Applied Mathematics*, 7(3).

La Grandville, Olivier de. and Solow, R. M. (2009). Capital-labour substitution and economic growth. In *Economic Growth: A Unified Approach*. Cambridge University Press. Chapter 5.

León-Ledesma, M. A., McAdam, P., and Willman, A. (2010a). Identifying the Elasticity of Substitution with Biased Technical Change. *American Economic Review*, 100(4):1330–1357.

León-Ledesma, M. A., McAdam, P., and Willman, A. (2010b). In Dubio pro CES: Supply Estimation with Mis-Specified Technical Change. European Central Bank, Working Paper No. 1175.

Leon-Ledesma, M. A. and Satchi, M. (2010). A note on balanced growth with a less than unitary elasticity of substitution. Studies in Economics 1007, Department of Economics, University of Kent.

Lucas, R. E. (1969). Labor-capital substitution in US manufacturing. In Harberger, A. C. and Bailey, M. J., editors, *The taxation of income from capital*, pages 223–274. Washington, D. C.: Brookings Institution.

Mallick, D. (2010). Capital-labor substitution and balanced growth. *Journal of Macroeconomics*, 32(4):1131–1142.

Masanjala, W. H. and Papageorgiou, C. (2004). The Solow model with CES technology: Nonlinearities and parameter heterogeneity. *Journal of Applied Econometrics*, 19:171–201.

McAdam, P. and Willman, A. (2011a). Medium Run Redux. *Macroeconomic Dynamics*, (forthcoming).

McAdam, P. and Willman, A. (2011b). Technology, Utilization and Inflation: Re-assessing the New Keynesian Fundament. *ECB Working Paper forthcoming*.

McElroy, F. W. (1967). Notes on the CES production function. *Econometrica*, 35(154-156).

Nakamura, H. and Nakamura, M. (2008). Constant-elasticity-of-substitution production function. *Macroeconomic Dynamics*, 12(5):694–701.

Papageorgiou, C. and Saam, M. (2008). Two-level CES production technology in the Solow and Diamond growth models. *Scandinavian Journal of Economics*, 110(1):119–143.

Paroush, J. (1964). A note on the CES production function. *Econometrica*, 32(213-156).

Ramanathan, R. (1975). The elasticity of substitution and the speed of convergence in growth models. *Economic Journal*, 85:612–613.

Robinson, J. (1933). *The Economics of Imperfect Competition*. MacMillian and Co Ltd, London.

Rowthorn, R. (1999). Unemployment, wage bargaining and capital-labour substitution. *Cambridge Journal of Economics*, 23(4):413–425.

Rutherford, T. (2003). Lecture notes on constant elasticity functions. Mimeo, University of Colorado.

Saam, M. (2008). Openness to trade as a determinant of the macroeconomic elasticity of substitution. *Journal of Macroeconomics*, 30:691–702.

Samuelson, P. A. (1965). A Theory of Induced Innovations along Kennedy-Weisacker Lines. *Review of Economics and Statistics*, 47(4):344–356.

Samuelson, P. A. (1972). *Collected Scientific Papers Vol.2*. MIT Press, Cambridge, MA.

Sato, R. (2006). *Biased Technical Change and Economic Conservation Laws*. Springer.

Sato, R. and Beckmann, M. J. (1970). The estimation of biased technical change and the production function. *Review of Economic Studies*, 35(1):57–266.

Solow, R. M. (1956). A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70(1):65–94.

Swan, P. L. (2006). ANU Inaugural Trevor Swan Distinguished Lecture. Available at `http://www.crawford.anu.edu.au/pdf/seminars`.

Swan, T. (1956). Economic growth and capital accumulation. *Economic Record*, 32(334-361).

Temple, J. R. W. (2008). The calibration of CES production functions. Discussion Paper 606, Department of Economics, University of Bristol.

ten Cate, A. (1992). Data Scaling With Highly Non-Linear Regression. *Computational Statistics*, 7:59–65.

Thursby, J. (1980). Alternative CES estimation techniques. *Review of Economics and Statistics*, 62(2):259–299.

Ventura, J. (1997). Growth and interdependence. *Quarterly Journal of Economics*, 62:57–84.

Wong, T. N. and Yip, C. K. (2010). Indeterminacy and the elasticity of substitution in one-sector models. *Journal of Economic Dynamics and Control*, 34(4):623–635.

Xue, J. and Yip, C. K. (2009). Economic growth and factor substitution: A clarification. Working Paper, Department of Economics, Chinese University of Hong Kong.

Yasui, T. (1965). The CES production function: A Note. *Econometrica*, 33:646–648.