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“Fair marriages”: An impossibility

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ABSTRACT

For marriage markets [Gale, D. and Shapley, L.S., 1962, College admissions and the stability of marriage, American Mathematical Monthly 69, 9–15.] so-called fair matchings do not always exist. We show that restoring fairness by using monetary transfers is not always possible: there are marriage markets where no amount of money can guarantee the existence of a fair allocation.

1. Introduction

For the classical marriage model (introduced in Gale and Shapley, 1962) efficiency and envy-freeness are not always compatible, i.e., fair matchings do not always exist. However, for many allocation of indivisible goods models (see Velez, 2008, and references therein), fairness can be restored if a sufficiently large amount of money is available for distribution/compensation as well. Interpreting the agents as the objects to be allocated, one might try to restore fairness for marriage markets in a similar fashion. We prove that there are marriage markets where no amount of money can guarantee the existence of a fair allocation.

2. Marriage markets: the classical model

There is a finite set of agents N, which can be partitioned into a set of women W and a set of men M. We denote a generic agent by i, a generic woman by w, and a generic man by m. Each agent i has a complete and transitive preference relation $\succeq_i$ over the agents on the other side of the market and being alone, i.e., a woman w has preferences over $M\cup\{w\}$ and a man m has preferences over $W\cup\{m\}$. A *marriage market* (Gale and Shapley, 1962) is a pair $(N,(\succeq_i)_{i\in N})$ such that $N=W\cup M$ and $W\cap M=\emptyset$.[1] A *matching* for marriage market $(N,(\succeq_i)_{i\in N})$ is a function $\mu{:}N\to N$ such that (i) $\mu(w)\notin M\Rightarrow\mu(w)=w$, (ii) $\mu(m)\notin W\Rightarrow\mu(m)=m$, and (iii) $\mu(w)=m\Leftrightarrow\mu(m)=w$. If $\mu(i)\neq i$ then we call $\mu(i)$ agent i's *mate*. If $\mu(i)=i$ then we call i a *single*.

3. Marriage markets with money

3.1. Value functions, money, and quasi-linear utilities

We assume that each agent i's preference relation is represented by a *value function* v_i. Hence, $v_w{:}M\cup\{w\}\to\mathbb{R}$ and $v_m{:}W\cup\{m\}\to\mathbb{R}$ are such that $v_w(m)$ represents the value for woman w of being matched to man m, $v_m(w)$ represents the value for man m of being matched to woman w, and $v_i(i)$ represents the value for agent i of being single. Furthermore, for all agents i, j, $k\in N$, $j\succeq_i k$ if and only if $v_i(j)\geq v_i(k)$. Let $v=(v_i)_{i\in N}$ denote the list of agents' value functions.

Next, we assume that apart from matching the agents, we can also distribute an amount of *money* $\Omega\in\mathbb{R}$ among the agents in N.[2] Since the set of agents is fixed throughout this paper, an *economy* is a pair $e\equiv(v;\Omega)$. We denote the set of all economies by $\mathcal{E}$. A *feasible allocation* for economy $e\equiv(v;\Omega)$ is a pair $z=(\mu;\nu)$ consisting of a matching μ and a vector $\nu\in\mathbb{R}^N$ such that $\sum\nu_i=\Omega$.[3] The *bundle* received by agent i at z is $z_i=(\mu(i);\nu_i)$.

We assume that agents only care about their own consumptions and that preferences over bundles are quasi-linear: agent i's preference relation over feasible allocations and over bundles is

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[1] Note that we do not require preferences to be strict.

[2] Results would not change if we restrict the model to only allow for $\Omega\in\mathbb{R}_+$.

[3] Again, results would not change if we restrict the model to only allow for $\Omega\in\mathbb{R}_+$ and $\nu\in\mathbb{R}^N_+$.

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represented by a *utility function* u_i such that for all feasible allocations $z \equiv (\mu;\nu)$, $u_i(z) = u_i(\mu(i);\nu_i) = v_i(\mu(i)) + \nu_i$.

An allocation z is (*Pareto*)-*efficient* if it is feasible and there is no other feasible allocation that Pareto-dominates it, i.e., there exists no z' such that for all $i \in N$, $u_i(z') \geq u_i(z)$ and for some $j \in N$, $u_j(z') > u_j(z)$. It is easy to see that because of quasi-linearity of preferences, allocation $z \equiv (\mu;\nu)$ is efficient if and only if matching μ is efficient, i.e., there exists no μ' such that for all $i \in N$, $v_i(\mu'(i)) \geq v_i(\mu(i))$ and for some $j \in N$, $v_j(\mu'(j)) > v_j(\mu(j))$.

An allocation z is *envy-free* if it is feasible and each agent finds his bundle at least as desirable as that of each other agent (Foley, 1967), i.e., a feasible allocation $z \equiv (\mu;\nu)$ satisfies *no-envy* if

(a.1) no woman w prefers to be matched to another woman $\overline{w}$'s mate and consume $\nu_{\overline{w}}$:
if $\mu(\overline{w}) \in M$, then $u_w(z) = u_w(\mu(w);\nu_w) \geq u_w(\mu(\overline{w});\nu_{\overline{w}})$;
(a.2) no woman w envies a single woman $\overline{w}$:
if $\mu(\overline{w}) = \overline{w}$, then $u_w(z) = u_w(\mu(w);\nu_w) \geq u_w(w;\nu_{\overline{w}})$;
(b.1) no man m prefers to be matched to another man $\overline{m}$'s mate and consume $\nu_{\overline{m}}$:
if $\mu(\overline{m}) \in W$, then $u_m(z) = u_m(\mu(m);\nu_m) \geq u_m(\mu(\overline{m});\nu_{\overline{m}})$; and
(b.2) no man m envies a single man $\overline{m}$:
if $\mu(\overline{m}) = \overline{m}$, then $u_m(z) = u_m(\mu(m);\nu_m) \geq u_m(m;\nu_{\overline{m}})$.

Allocations that are efficient and envy-free are sometimes called *fair* (Varian, 1974).

A *solution* associates with each economy a non-empty set of feasible allocations.

The *Pareto solution P* associates with each economy its set of efficient allocations. To show that for all $e \in \mathcal{E}$, $P(e) \neq \emptyset$ one can define a serial dictatorship allocation as follows: based on a fixed order, men can sequentially choose their partner, maybe respecting individual rationality,[4] and if $\Omega > 0$, then some fixed woman $\tilde{w} \in W$ receives the full amount Ω, but if $\Omega < 0$, then some fixed man $\tilde{m} \in M$ receives it.

The *no-envy solution F* associates with each economy its set of envy-free allocations. To show that for all $e \in \mathcal{E}$, $F(e) \neq \emptyset$ one can always assign the feasible allocation where all agents are single and Ω is equally divided among all agents.

4. Fairness: an impossibility

The matching model we have introduced is closely related to indivisible goods economies where a set of indivisible objects and an amount of money has to be allocated among a set of agents (see for instance Svensson, 1983; Maskin, 1985; Alkan et al., 1991). In these models, typically fair allocations exist if either consumptions of money are unbounded below or the amount of money available is large enough. In a recent paper, Velez (2008) proves this existence result for a general model that in addition to the previously studied models includes situations with externalities (e.g., inequality aversion or altruism).

Our matching model shares similar features to the indivisible object models mentioned above; the only difference is that in our model the agents are also the objects that have to be assigned. Given the discrete character of the original marriage market model as introduced in Section 2, it is easy to see that efficiency and envy-freeness might not be compatible: consider a three agent example with two women who would like to be matched to the same man. Hence, our question is if similarly as for the indivisible objects allocation model a sufficiently large amount of money Ω (or unbounded consumptions of money from below) would restore the possibility for fair allocations. The next theorem answers this question.

Table 1
A profile of value functions for which no fair allocation exists.

i	$v_{w_1}(i)$	$v_{w_2}(i)$	$v_{m_1}(i)$	$v_{m_2}(i)$
w_1	0	–	3	3
w_2	–	0	1	2
m_1	3	3	0	–
m_2	2	1	–	0

Theorem 1. *There exist profiles of value functions v such that for all* $e \equiv (v;\Omega)$,

$$P(e) \cap F(e) = \emptyset.$$

Proof. Let $N = \{w_1, w_2, m_1, m_2\}$ and v as described in Table 1.

The only efficient matchings in this market are μ such that $\mu(w_1) = m_1$ and $\mu(w_2) = m_2$ and μ' such that $\mu'(w_1) = m_2$ and $\mu'(w_2) = m_1$. Hence,

$$P(e) = \left\{(\mu;\nu) \,|\, \nu \in \mathbb{R}^N \text{ such that } \sum \nu_i = \Omega\right\} \cup \left\{(\mu';\nu) \,|\, \nu \in \mathbb{R}^N \text{ such that } \sum \nu_i = \Omega\right\}.$$

Case 1. Let $z \equiv (\mu;\nu)$. To avoid that w_2 envies w_1, we need that $u_{w_2}(\mu(w_2);\nu_{w_2}) = 1 + \nu_{w_2} \geq 3 + \nu_{w_1} = u_{w_2}(\mu(w_1);\nu_{w_1})$. Hence, for $z \in F(e)$ we need $\nu_{w_2} \geq 2 + \nu_{w_1}$ to hold. But then, $u_{w_1}(\mu(w_1);\nu_{w_1}) = 3 + \nu_{w_1} \leq 1 + \nu_{w_2} < 2 + \nu_{w_2} = u_{w_1}(\mu(w_2);\nu_{w_2})$ and w_1 envies w_2. Thus, $z \notin F(e)$.

Case 2. Let $z \equiv (\mu';\nu)$. To avoid that m_1 envies m_2, we need that $u_{m_1}(\mu'(m_1);\nu_{m_1}) = 1 + \nu_{m_1} \geq 3 + \nu_{m_2} = u_{m_1}(\mu'(m_2);\nu_{m_2})$. Hence, for $z \in F(e)$ we need $\nu_{m_1} \geq 2 + \nu_{m_2}$ to hold. But then, $u_{m_2}(\mu(m_2);\nu_{m_2}) = 3 + \nu_{m_2} \leq 1 + \nu_{m_1} < 2 + \nu_{m_1} = u_{m_2}(\mu'(m_1);\nu_{m_1})$ and m_2 envies m_1. Thus, $z \notin F(e)$.

Since Cases 1 and 2 cover all efficient allocations, it follows that $P(e) \cap F(e) = \emptyset$. □

Note that in the proof of Theorem 1 we have not used any information on Ω (its size or sign) or feasible transfers (transfers described by ν could be negative as well). Furthermore, in a two-sided model with an equal number of men and women where feasibility excludes single agents, even efficiency can be omitted from Theorem 1.

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References

Alkan, A., Demange, G., Gale, D., 1991. Fair allocation of indivisible goods and criteria of justice. Econometrica 59, 1023–1039.
Foley, D., 1967. Resource allocation and the public sector. Yale Economic Essays 7, 45–98.
Gale, D., Shapley, L.S., 1962. College admissions and the stability of marriage. American Mathematical Monthly 69, 9–15.
Maskin, E., 1985. On the fair allocation of indivisible goods. In: Feiwel, G.R. (Ed.), Arrow and the Foundations of the Theory of Economic Policy. MacMillan, London, pp. 341–349.
Svensson, L.-G., 1983. Large indivisibilities: an analysis with respect to price equilibrium and fairness. Econometrica 51, 939–954.
Varian, H., 1974. Equity, envy and efficiency. Journal of Economic Theory 9, 63–91.
Velez, R., 2008, Fairness and externalities, Mimeo.

[4] A matching μ is *individually rational* if no agent would prefer to be single, i.e., for all $i \in N$, $\mu(i) \succeq_i i$.