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Strategy-proofness and population-monotonicity for house allocation problems ☆

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Abstract

We study a simple model of assigning indivisible objects to agents, such as dorm rooms to students, or offices to professors, where each agent receives at most one object and monetary compensations are not possible. For these problems population-monotonicity, which requires that agents are affected by population changes in the same way, is a compelling property because tentative assignments are made in many typical situations, which may have to be revised later to take into account the changing population. We completely describe the allocation rules satisfying *population-monotonicity*, *strategy-proofness*, and *efficiency*. The characterized rules assign the objects by an iterative procedure in which at each step no more than two agents "trade" objects from their hierarchically specified "endowments."

JEL classification: D63; D70

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1. Introduction

We study the problem of allocating heterogeneous indivisible objects among a group of agents when each agent receives at most one object and monetary compensations are

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not possible. The classical literature on such problems is concerned with the case of initial property rights. The problem in which each agent initially owns an object, known as the *housing market*, is due to Shapley and Scarf (1974).[1] More recently, the focus of interest has shifted to the case where there are no initial property rights and the set of objects is regarded as the social endowment, called the *house allocation problem*. The house allocation problem is the subject of recent papers by Abdulkadiroğlu and Sönmez (1999), Ehlers (2002), Pápai (2000), and Svensson (1999). All these papers study the requirement that no agent can manipulate the allocation to his advantage by lying about his preference relation. This property, called *strategy-proofness*, has also been analyzed in a probabilistic setting by Zhou (1990) and Bogomolnaia and Moulin (2001).[2]

In addition to *efficiency* and *strategy-proofness*, we study the property of *population-monotonicity* (Thomson, 1983). When a change in the population is exogenous, it would be unfair if the agents who were not responsible for this change were treated unequally. *Population-monotonicity* represents this idea of solidarity, and requires that if some agents leave, then as a result either all remaining agents (weakly) gain or they all (weakly) lose. When the allocation rule is *efficient*, *population-monotonicity* implies that if the population expands then everyone is weakly worse off, and if the population shrinks then everyone is weakly better off. *Population-monotonicity* is a compelling requirement for house allocation problems, because tentative or actual assignments are made in many typical applications, which may have to be revised later to take into account the changing population. For example, if tentative preliminary dorm room assignments are revised because additional students apply for the dorms, it would be rather unreasonable to give some students better dorm rooms than initially as a result of the extra dorm applications. Also, imagine that there is a list of term paper topics that can be assigned to students in a class, each of which is different from the others. Suppose that the term paper topics have been assigned to students at the beginning of the semester and it turns out that a few students dropped the course by the 6-week mark. Would it be reasonable to assign any remaining student a harder topic, in their opinion, than they had initially, just because some students have dropped the course? Finally, consider the example of assigning offices to professors. Suppose that a new wing of the building is built with new offices, and professors are offered a choice between staying in their old office or moving to the new wing. Some professors who signed up for an office in the new wing change their mind the last second (say, because of the inconvenience of actually moving). In this case, it would clearly be quite unfair to the remaining professors who are moving to the new wing if any one of them were assigned a worse office than they were assigned before. Similar examples of house allocation problems, in which the population may change after a tentative assignment is made, abound.

Although *population-monotonicity* is a rather important property in this context, it is also a very demanding property when coupled with *efficiency* and *strategy-proofness*. Our characterization result yields a class of allocation rules, called *restricted endowment inheritance*

[1] Further papers on the housing market include Roth and Postlewaite (1977), Wako (1984), Ma (1994), Abdulkadiroğlu and Sönmez (1998), and Konishi et al. (2001) (for a survey, see Moulin, 1995).

[2] Further studies that are concerned with the house allocation problem are Ergin (2000), Hylland and Zeckhauser (1979), and Svensson (1994); a related assignment problem with "deadlines" is studied by Bogomolnaia and Moulin (2002) and Crès and Moulin (2001).

rules, which are essentially hierarchical rules in the following sense. They allow “trading” of the objects by at most two agents at a time, where these two agents share the objects available at the given stage of the procedure as “endowments.” More precisely, such a rule partitions the set of agents into singletons and pairs, and chooses a “priority” ordering over these singletons and pairs. If the rule is based on a partition of all singletons, then it is a *serial dictatorship* (see, for example, Ergin, 2000; Svensson, 1999), in which each agent gets his favorite object among the objects that were not assigned to agents who precede this agent in the priority ordering. At the other extreme of this set of allocation rules are the ones that are based on a partition of the agents exclusively into pairs. These rules allow that, according to the priority ordering of the selected pairs, each pair “trades” objects, where each of the remaining objects is the “endowment” of one or the other agent in the pair.

Restricted endowment inheritance rules constitute a subclass of the *fixed endowment hierarchical exchange rules* introduced by Pápai (2000). In Section 2, we define these rules, which we call simply *endowment inheritance rules*. Section 3 contains the characterization of all rules satisfying *population-monotonicity*, *strategy-proofness*, and *efficiency*. In Section 4, we give a short conclusion.

2. Endowment inheritance rules

2.1. The model

Let $N \equiv \{1, 2, \ldots, n\}$ denote a finite set of agents, $n \geq 2$. Let K denote a set of objects and $k \equiv |K|$. Let 0 represent the *null object*.[3] Each agent $i \in N$ is equipped with a strict preference relation R_i over $K \cup \{0\}$. In other words, R_i is a linear order over $K \cup \{0\}$. Given $x, y \in K \cup \{0\}$, $x\, P_i\, y$ means that agent i strictly prefers x to y. Let $\mathcal{R}$ denote the class of all linear orders over $K \cup \{0\}$, and $\mathcal{R}^N$ the set of (*preference*) *profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Let $\mathcal{R}_0 \subsetneq \mathcal{R}$ denote the class of preference relations where the null object is the worst object. That is, if $R_i \in \mathcal{R}_0$, then all the objects are “goods”: for all $x \in K,\;\; x\, P_i\, 0$. Since, for the time being, the set of agents and the set of objects are fixed, $\mathcal{R}^N$ completely describes the *set of* (*house allocation*) *problems*.

An *allocation* is a list $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in K \cup \{0\}$, and none of the objects in K is assigned to more than one agent. Note that 0, the null object, can be assigned to any number of agents and that not all objects in K have to be assigned. An (*allocation*) *rule* φ is a function choosing for every $R \in \mathcal{R}^N$ an allocation, denoted by $\varphi(R)$. Given $i \in N$, we call $\varphi_i(R)$ the *allotment* of agent i at $\varphi(R)$.

2.2. Basic properties

The first property requires that the rule chooses only (Pareto) efficient allocations.

Efficiency: For all $R \in \mathcal{R}^N$, there is no allocation a such that for all $i \in N$, $a_i\, R_i\, \varphi_i(R)$ with strict preference holding for some $j \in N$.

[3] “Receiving the null object” means “not receiving any object”.

Given $R \in \mathcal{R}^N$ and $M \subseteq N$, let R_M denote the profile $(R_i)_{i \in M}$. It is the restriction of R to the set M. We also use the notations $R_{-i} = R_{N\setminus\{i\}}$ and $R_{-i,j} = R_{N\setminus\{i,j\}}$. For example, $(\bar{R}_i, R_{-i})$ denotes the profile obtained from R by replacing R_i by $\bar{R}_i$.

The second property requires that no agent ever benefits from misrepresenting his preference relation.

Strategy-proofness: For all $R \in \mathcal{R}^N$, all $i \in N$, and all $\bar{R}_i \in \mathcal{R}$, $\varphi_i(R) R_i \varphi_i(\bar{R}_i, R_{-i})$.

2.3. Endowment inheritance rules

Endowment inheritance rules (Pápai, 2000) are based on *Gale's top trading cycle algorithm*, which identifies the unique core allocation in a housing market (Roth and Postlewaite, 1977). We describe this algorithm first. Given a housing market, let every agent point to the agent who owns his first-ranked house. This way we can identify the *top trading cycles*, cycles of agents who wish to trade with each other in a feasible manner. Let every agent in a top trading cycle receive his favorite house, and remove these agents from the market with their allotted houses. Next, identify the top trading cycles in the reduced market, carry out the corresponding trades, etc. Repeat this procedure until all agents are allotted a house.

Endowment inheritance rules allot objects to agents using an iterative procedure that is similar to the top trading cycle algorithm, except that it also specifies the property rights of the objects in an iterative hierarchical manner. Each object is the initial individual "endowment" of an agent and we apply a round of top trading cycle exchange to these endowments. Given that multiple endowments are allowed, after the agents in top trading cycles are removed from the market with only their allotted objects, their unallocated endowments are re-assigned as endowments to agents who are still in the market. In other words, these objects that are left behind are "inherited" as new endowments by agents who have not received their allotments yet. Notice that then each remaining object is the endowment of some remaining agent and the top trading cycle algorithm is well-defined at the second stage. We determine the allotments of agents who are in top trading cycles in this round, remove them with their allotted objects, and determine the endowments of the remaining agents for the next stage. And so on, until for each agent we have specified an allotment this way.

The initial endowments and the hierarchical endowments at later rounds are determined using a so-called *endowment inheritance table*, which consists of a permutation of the agents for each object, indicating the order of inheritance for the particular object. Thus, each endowment inheritance rule is defined by an endowment inheritance table. We formally define the class of endowment inheritance rules and illustrate such a rule in an example.[4]

Let Π^N denote the set of all one-to-one functions from N to N. Given $x \in K$, $\pi_x \in \Pi^N$, and $i, j \in N$, $\pi_x(i) < \pi_x(j)$ means that agent i is ranked higher than agent j with respect to object x. The function π_x indicates the inheritance of object x. Furthermore,

[4] We refer the reader to Pápai (2000) for a detailed discussion of these rules.

$\pi \equiv (\pi_x | \pi_x \in \Pi^N)_{x \in K}$ is an *endowment inheritance table* that shows the inheritance of each object.

Endowment inheritance rule φ^π: Let $R \in \mathcal{R}^N$. Then $\varphi^\pi(R)$ is defined in at most $m \equiv \min\{n, k\}$ stages. Given $i \in N$ and $R \in \mathcal{R}^N$, we give recursive definitions of the associated *hierarchical endowments* $E_t(i, R)$, the *top choices* $T_t(i, R)$, *trading cycles* $S_t(i, R)$, *assigned individuals* $W_t(R)$, and *assigned non-null objects* $F_t(R)$, all of which are indexed by $t \in \{1, \ldots, m\}$, the corresponding *stage*. For every profile $R \in \mathcal{R}^N$ and stage t, let $W^t(R) \equiv \cup_{z=1}^t W_z(R)$ and $F^t(R) \equiv \cup_{z=1}^t F_z(R)$. Let $W^0(R) \equiv \emptyset$ and $F^0(R) \equiv \emptyset$.

Stage t: If agent $i \in N \setminus W^{t-1}(R)$ is ranked highest with respect to object $x \in K \setminus F^{t-1}(R)$ among all agents in $N \setminus W^{t-1}(R)$, then x belongs to his hierarchical endowment at stage t. The null object is part of each agent's endowment.

*t*th *Hierarchical endowments*:

$$E_t(i, R) \equiv \left\{ x \in K \setminus F^{t-1}(R) | i = \arg \min_{j \in N \setminus W^{t-1}(R)} \{\pi_x(j)\} \right\} \cup \{0\}.$$

Next, each agent $i \in N \setminus W^{t-1}(R)$ identifies his top choice in $(K \cup \{0\}) \setminus F^{t-1}(R)$.

Top choices:

$$T_t(i, R) = x \Leftrightarrow x \in (K \cup \{0\}) \setminus F^{t-1}(R) \text{ and for all } y \in (K \cup \{0\}) \setminus F^{t-1}(R),\ xR_i y.$$

A trading cycle consists of a set of agents in $N \setminus W^{t-1}(R)$ who would like to exchange objects from their hierarchical endowments in a "cyclical way" such that each of them receives his top choice.

Trading cycles:

$$S_t(i, R) \equiv \begin{cases} \{j_1, \ldots, j_g\}, & \text{if } \{j_1, \ldots, j_g\} \subseteq N \setminus W^{t-1}(R) \text{ such that} \\ & |\{j_1, \ldots, j_g\}| = g \text{ and for all } v \in \{1, \ldots, g\}, \\ & T_t(j_v, R) \in E_t(j_{v+1}, R) \text{ where } i = j_1 = j_{g+1}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Agents in a trading cycle are assigned their top choices.[5]

Assigned individuals: $W_t(R) \equiv \{i \in N | S_t(i, R) \neq \emptyset\}$.

Assigned non-null objects: $F_t(R) \equiv \{T_t(i, R) \in K | i \in W_t(R)\}$.

Note that for all $R \in \mathcal{R}^N$ there exists a last stage $t^* \leq m$ such that either $W^{t^*}(R) = N$ or $F^{t^*}(R) = K$ and for all $t < t^*$, $W^t(R) \neq N$ and $F^t(R) \neq K$.

Given endowment inheritance table π, for all $R \in \mathcal{R}^N$ the allocation chosen by the *endowment inheritance rule* φ^π is defined as follows. For all $i \in N$,

$$\varphi_i^\pi(R) \equiv \begin{cases} T_t(i, R), & \text{if for some } t \in \{1, \ldots, m\},\ i \in W_t(R), \\ 0, & \text{otherwise.} \end{cases}$$

[5] Note that if an agent's top choice is the null object, he forms a trading cycle with himself, i.e. he is assigned the null object in his own hierarchical endowment.

Example 1. Let $N \equiv \{1, 2, 3, 4, 5\}$ and $K \equiv \{a, b, c, d, e, f\}$. Consider the endowment inheritance rule defined by the following endowment inheritance table $\pi \equiv (\pi_x | \pi_x \in \Pi^N)$.

π_a	π_b	π_c	π_d	π_e	π_f
2	1	2	1	2	2
1	2	1	2	1	1
3	3	3	3	3	3
4	4	4	5	5	4
5	5	5	4	4	5

R_1	R_2	R_3	R_4	R_5
a	b	b	d	a
f	f	e	a	b
d	a	c	b	f
e	c	a	c	c
c	d	d	e	d
b	e	f	f	e

Associated with each object is a permutation of the agents (given by the column corresponding to the object). For example, the first column shows that object a is agent 2's initial endowment, which is (possibly) inherited by 1, 3, 4, and 5, in this order. We illustrate the use of this table for the preference profile $R \in \mathcal{R}_0^N$ given above, which shows the rankings of objects from the top down for each agent.

Stage 1: The initial endowments are given by the first row of the endowment inheritance table. The endowments are $E_1(1, R) = \{b, d\}$ for 1 and $E_1(2, R) = \{a, c, e, f\}$ for 2, and $\emptyset$ for 3, 4, and 5. Then $T_1(1, R) = a$, $T_1(2, R) = b$, $T_1(3, R) = b$, $T_1(4, R) = d$, and $T_1(5, R) = a$ are the top choices of the agents in $K \cup \{0\}$. Hence, $\{1, 2\}$ is the only cycle at Stage 1 under which 1 receives a from 2 and 2 receives b from 1, i.e. $S_1(1, R) = S_1(2, R) = \{1, 2\}$, $W_1(R) = \{1, 2\}$, and $F_1(R) = \{a, b\}$.

Stage 2: Since agents 1 and 2 already received their allotments, objects c, d, e, and f are left behind from 1's and 2's initial endowments. These objects are inherited by 3, i.e. $E_2(3, R) = \{c, d, e, f\}$ and $E_2(4, R) = E_2(5, R) = \emptyset$. Then, 3 picks his top choice, object e, among the remaining objects. So, $S_2(3, R) = \{3\}$, $W_2(R) = \{3\}$, and $F_2(R) = \{e\}$.

Stage 3: Now only 4 and 5 remain in the market. Agent 4 inherits $\{c, f\}$ and 5 inherits $\{d\}$, i.e. $E_3(4, R) = \{c, f\}$ and $E_3(5, R) = \{d\}$. Because $T_3(4, R) = d$ and $T_3(5, R) = f$, 4 and 5 form a trading cycle and receive their top choices in $\{c, d, f\}$, i.e. $S_3(4, R) = S_3(5, R) = \{4, 5\}$, $W_3(R) = \{4, 5\}$, and $F_3(R) = \{d, f\}$. Then $\varphi^\pi(R) = (a, b, e, d, f)$ are the allotments to $(1, 2, 3, 4, 5)$.

3. The result

We extend the model to allow population changes. Let $P \equiv \{1, \ldots, p\}$, $p \geq 3$, be the finite set of potential agents. Let $\mathcal{P}$ denote the set of non-empty subsets of P. In this context, a *rule* is a function φ that associates with each set of agents $N \in \mathcal{P}$ and each preference profile $R \in \mathcal{R}^N$ an allocation $\varphi(R) = (\varphi_i(R))_{i \in N}$.

Population-monotonicity:[6] For all $N \in \mathcal{P}$, all $R \in \mathcal{R}^N$, and all $M \subseteq N$, either [for all $i \in M$, $\varphi_i(R_M) R_i \varphi_i(R)$] or [for all $i \in M$, $\varphi_i(R) R_i \varphi_i(R_M)$].

[6] For a survey on *population-monotonicity* see Thomson (1995).

Our first lemma states that as a result of *population-monotonicity* and *efficiency*, when some agents leave the economy, none of the remaining agents loses. The proof is omitted.

Lemma 1. *Let φ be a rule satisfying population-monotonicity and efficiency. If $N \in \mathcal{P}$, $R \in \mathcal{R}^N$, and $\emptyset \neq M \subseteq N$, then for all $i \in M$, $\varphi_i(R_M) R_i \varphi_i(R)$.*

We characterize the class of rules that are *population-monotonic*, *strategy-proof*, and *efficient*. It turns out that this class is a subclass of endowment inheritance rules: a rule satisfying these properties must be a so-called *restricted endowment inheritance rule*.

A restricted endowment inheritance rule is an endowment inheritance rule in which at most two agents are allowed to trade at a time, or more precisely, no more than two agents can be endowed with objects at any stage of the procedure. This is ensured if the endowment inheritance table can be "partitioned" such that the elements of the partition are either single rows or two adjacent rows: any single row in the partition contains one agent only and any two adjacent rows in the partition contain two agents only. In Example 1, we have rows 1 and 2 as an element of the partition with agents 1 and 2, then row 3 with agent 3, and finally, rows 4 and 5with agents 4 and 5. It is clear that initially agents 1 and 2 will be allotted some objects, whether or not they trade, then agent 3 gets his favorite object among the remaining objects, and finally, agents 4 and 5 get their allotments.

Restricted endowment inheritance rules are similar to serial dictatorships, in comparison with endowment inheritance rules that are not restricted this way. In a serial dictatorship, there exists a hierarchy of the agents specified a priori, such that agents receive their favorite object from the set of objects that remain after we remove all the objects from the market that are allotted to agents who are ranked higher in the hierarchy. For restricted endowment inheritance rules, we choose one or two agents among the remaining agents at each step of the procedure (as opposed to always choosing one agent in serial dictatorships), and allocate the favorite remaining object to the agent if he is chosen alone, and to one or both agents if chosen in a pair. In the latter case, if one agent does not receive his first-ranked object among the remaining objects (which means that it is the other chosen agent's favorite object as well, and it has been allocated to him), then he receives his second-ranked object. In other words, while we have a single dictator at each stage of the procedure for serial dictatorships, restricted endowment inheritance rules allow the choice of "twin-dictators" as well as ordinary (single) dictators at any given stage of the procedure.

Restricted endowment inheritance rules: Let $\pi = (\pi_x | \pi_x \in \Pi^P)_{x \in K}$ be such that

(a) for all $j \in P$, $|\{\pi_x^{-1}(j) | x \in K\}| \leq 2$ and
(b) for all $j \in P$, if $|\{\pi_x^{-1}(j) | x \in K\}| = 2$, then either
$\{\pi_x^{-1}(j) | x \in K\} = \{\pi_x^{-1}(j-1) | x \in K\}$ or $\{\pi_x^{-1}(j) | x \in K\} = \{\pi_x^{-1}(j+1) | x \in K\}$.

Let $N \in \mathcal{P}$, $|N| = n$, and $R \in \mathcal{R}^N$. Define the collection of injective functions $\pi^N = (\pi_x^N : N \to \{1, \ldots, n\} | \pi_x^N \in \Pi^N)_{x \in K}$ that is induced by π as follows: let $x \in K$ and $N = \{i_1, \ldots, i_n\}$ be such that $\pi_x(i_1) < \pi_x(i_2) < \cdots < \pi_x(i_n)$. Then, $\pi_x^N(i_1) \equiv 1$, $\pi_x^N(i_2) \equiv 2, \ldots, \pi_x^N(i_n) \equiv n$. The *restricted endowment inheritance rule* φ^π is defined for all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$ by $\varphi^\pi(R) \equiv \varphi^{\pi^N}(R)$, where $\varphi^{\pi^N}(R)$ is defined as in Section 2.3.

Theorem 1. *On the domain $\mathcal{R}^N$ ($\mathcal{R}_0^N$), restricted endowment inheritance rules are the only rules satisfying population-monotonicity, strategy-proofness, and efficiency*.

The proof of Theorem 1 is in Appendix A. The following examples show logical independence of the axioms in Theorem 1.

Population-monotonicity: An endowment inheritance rule which is not a restricted endowment inheritance rule (defined appropriately for variable population) satisfies *strategy-proofness* and *efficiency*, but not *population-monotonicity*.

Strategy-proofness: Fix $y \in K$ and an ordering σ of P, say $\sigma = (1, 2, \dots, p)$. For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}_0^N$, the rule is a serial dictatorship relative to the following ordering: first, the rule orders according to σ the agents that rank y *last*, and second, the remaining agents are ordered according to σ. This rule violates *strategy-proofness* because announcing a preference relation under which y is ranked last may be profitable. However, the rule satisfies *population-monotonicity* and *efficiency*.

Efficiency: The rule that assigns for all profiles to each agent the null object satisfies *population-monotonicity* and *strategy-proofness*, but not *efficiency*.

4. Conclusion

In this study, we demonstrate that guaranteeing *population-monotonicity* for the allocation of indivisible objects comes with a serious price. Whereas without this solidarity property agents can "trade" objects arbitrarily, once individual property rights are assigned, the imposition of this property restricts the assignment of individual property rights and therefore "trading" to at most two agents at a time, thereby rendering the selected allocation rules essentially hierarchical. One can intuitively see that *population-monotonicity* is violated if there is individual ownership: if I trade with an agent originally who then leaves the market, then in the new setup the new owner of the object that I obtained previously may want to trade with someone else, and I end up worse off, even though there are fewer agents now for the same resources. Given the structure of this simple indivisible goods allocation problem, our result suggests a general trade-off between solidarity and individual property rights.

Appendix A. Proof of Theorem 1

By Pápai (2000), restricted endowment inheritance rules satisfy *strategy-proofness* and *efficiency*. It is easy to check that they also satisfy *population-monotonicity* by considering how the allotments of the agents are constructed. In proving the converse, let φ be a rule satisfying *population-monotonicity*, *strategy-proofness*, and *efficiency*. We give all proofs for the domain $\mathcal{R}_0$, since the proof for the larger domain $\mathcal{R}$ is completely analogous.

First, for all $x \in K$, we inductively define $\pi_x \in \Pi^P$. Second, we show that $\pi = (\pi_x)_{x \in K}$ satisfies conditions (a) and (b) in the definition of a restricted endowment inheritance rule. Third, we prove that $\varphi = \varphi^\pi$.

Step 1 (Construction of the endowment inheritance table). Given $x \in K$, let $R^x \in \mathcal{R}_0^P$ be such that for all $i \in P$ and all $y \in K$, $xR_i y$. By *efficiency*, for some $j \in P$, $\varphi_j(R^x) = x$. Define $\pi_x(j) \equiv 1$ and $\pi_x^{-1}(1) \equiv j$. Given $t \in \{1, \dots, p\}$, let $N_t^x \equiv P \setminus \{\pi_x^{-1}(l) \mid l \in \{1, \dots, t\}\}$. By *efficiency*, for some $j \in N_t^x$, $\varphi_j(R^x_{N_t^x}) = x$. Define $\pi_x(j) \equiv t+1$ and $\pi_x^{-1}(t+1) \equiv j$. This inductive definition yields a one-to-one function $\pi_x : P \to P$. Note that given $N \in \mathcal{P}$ there exists $j \in N$ such that for all $i \in N$, $\pi_x(j) \le \pi_x(i)$. Thus, by $\varphi_j(R^x_{N^x_{\pi_x(j)-1}}) = x$ and *population-monotonicity*,

$$\varphi_j(R_N^x) = x. \tag{1}$$

Let $x \in K$, $N \in \mathcal{P}$, and $i \in N$. We say that φ *respects the minimal right of agent* i *for object* x *in* N if for all $R \in \mathcal{R}_0^N$, $\varphi_i(R) R_i x$. A rule φ *respects minimal rights for agent* i *in* N if there exists $x \in K$ such that φ respects the minimal right of i for x in N.

Lemma 2. *For all $x \in K$ and all $N \in \mathcal{P}$, there exists some $j \in N$ such that φ respects the minimal right of agent j for object x in N.*

Proof. Let $x \in K$ and $j \in N$ be such that for all $i \in N$, $\pi_x(j) \le \pi_x(i)$. We prove that φ respects the minimal right of j for x in N. Let $R \in \mathcal{R}_0^N$.

If $x\, P_j \varphi_j(R)$, then by *strategy-proofness*, $\varphi_j(R_j^x, R_{-j}) \ne x$. By *efficiency*, for some $i \in N \setminus \{j\}$, $\varphi_i(R_j^x, R_{-j}) = x$. By *strategy-proofness*, $\varphi_i(R_j^x, R_i^x, R_{-j,i}) = x$. By *population-monotonicity*, $\varphi_i(R_j^x, R_i^x) = x$. This is a contradiction to $\pi_x(j) < \pi_x(i)$ and (1). □

Step 2 ($\pi \equiv (\pi_x)_{x \in K}$ satisfies (a) and (b)). Given $N \in \mathcal{P}$ and $x \in K$, let $\pi_x^N : N \to \{1, \dots, |N|\}$ denote the one-to-one function which is induced by π, i.e. for all $i, j \in N$, $\pi_x^N(i) \le \pi_x^N(j) \Leftrightarrow \pi_x(i) \le \pi_x(j)$.

Lemma 3. *Let $N \in \mathcal{P}$. We have* (i) *φ respects minimal rights of at most two agents in N, i.e., $|\{(\pi_x^N)^{-1}(1) | x \in K\}| \le 2$; and* (ii) *if $|\{(\pi_x^N)^{-1}(1) | x \in K\}| = 2$, then $\{(\pi_x^N)^{-1}(2) | x \in K\} = \{(\pi_x^N)^{-1}(1) | x \in K\}$.*

Proof.

(i) Suppose that φ respects minimal rights of more than two agents in N. Hence, $k \ge 3$ and $|N| \ge 3$. Without loss of generality, let $1, 2, 3 \in N$ and $x_1, x_2, x_3 \in K$ be such that for all $i \in \{1, 2, 3\}$, $\pi_{x_i}^N(i) = 1$ (agent i has a minimal right for x_i in N). Let $R \in \mathcal{R}_0^N$ be such that for all $y \in K \setminus \{x_1, x_2, x_3\}$,

$$x_3\, P_1\, x_1\, P_1\, x_2\, P_1\, y,$$
$$x_1\, P_2\, x_2\, P_2\, x_3\, P_2\, y,$$
$$x_1\, P_3\, x_3\, P_3\, x_2\, P_3\, y.$$

Since agents 1 and 3 have minimal rights for x_1 and x_3 in N, *efficiency* implies $\varphi_1(R) = x_3$ and $\varphi_3(R) = x_1$. Thus, because 2 has a minimal right for x_2 in N, $\varphi_2(R) = x_2$.

Hence, by *population-monotonicity*,

$$\varphi_2(R_{N\setminus\{1\}}) = x_2 \text{ and } \varphi_3(R_{N\setminus\{1\}}) = x_1. \tag{2}$$

Let $R' = (R'_1, R_{-1}) \in \mathcal{R}_0^N$ be such that for all $y \in K \setminus \{x_1, x_2, x_3\}$, $x_2 P'_1 x_1 P'_1 x_3 P'_1 y$. Since agents 1 and 2 have minimal rights for x_1 and x_2, *efficiency* implies $\varphi_1(R') = x_2$ and $\varphi_2(R') = x_1$. Thus, because 3 has a minimal right for x_3 in N, $\varphi_3(R') = x_3$. Hence, by *population-monotonicity*, $\varphi_2(R'_{N\setminus\{1\}}) = x_1$ and $\varphi_3(R'_{N\setminus\{1\}}) = x_3$. Since $R'_{N\setminus\{1\}} = R_{N\setminus\{1\}}$, the previous fact contradicts (2).

(ii) Let $\{i, j\} = \{(\pi_x^N)^{-1}(1) | x \in K\}$ and $y \in K$. Then $\pi_y^N(i) = 1$ or $\pi_y^N(j) = 1$. Let $\pi_y^N(i) = 1$ and $x \in K$ be such that $\pi_x^N(j) = 1$. Let $R_j \in \mathcal{R}_0$ be such that for all $x' \in K \setminus \{x, y\}$, $y R_j x R_j x'$. Because φ respects the minimal right of i for y in N and of j for x in N, *efficiency* implies that $\varphi_i(R_i^x, R_j, R^y_{N\setminus\{i,j\}}) = x$ and $\varphi_j(R_i^x, R_j, R^y_{N\setminus\{i,j\}}) = y$. Thus, by *strategy-proofness*, $\varphi_j(R_i^x, R^y_{N\setminus\{i\}}) = y$. By *population-monotonicity*, $\varphi_j(R^y_{N\setminus\{i\}}) = y$ and $\pi_y^{N\setminus\{i\}}(j) = 1$. Hence, $\pi_y^N(j) = 2$, the desired conclusion. □

By Lemma 3, $|\{\pi_x^{-1}(1)|x \in K\}| \leq 2$ and if $|\{\pi_x^{-1}(1)|x \in K\}| = 2$, then $\{\pi_x^{-1}(2)|x \in K\} = \{\pi_x^{-1}(1)|x \in K\}$. Let $P^1 \equiv P \setminus \{\pi_x^{-1}(1)|x \in K\}$ and $l_1 \equiv |\{\pi_x^{-1}(1)|x \in K\}|$. By Lemma 3, $|\{(\pi_x^{P^1})^{-1}(1)|x \in K\}| \leq 2$ and if $|\{(\pi_x^{P^1})^{-1}(1)|x \in K\}| = 2$, then $\{(\pi_x^{P^1})^{-1}(2)|x \in K\} = \{(\pi_x^{P^1})^{-1}(1)|x \in K\}$. Thus, by definition, $|\{\pi_x^{-1}(l_1+1)|x \in K\}| \leq 2$ and if $|\{\pi_x^{-1}(l_1 + 1)|x \in K\}| = 2$, then $\{\pi_x^{-1}(l_1 + 2)|x \in K\} = \{\pi_x^{-1}(l_1 + 1)|x \in K\}$. Now, by induction, Lemma 3 implies that π satisfies (a) and (b).

Step 3 ($\varphi = \varphi^\pi$). Suppose that $\varphi \neq \varphi^\pi$. Then there exist $N \in \mathcal{P}$ and $R \in \mathcal{R}_0^N$ such that $\varphi(R) \neq \varphi^\pi(R)$. Hence, by *efficiency*, there exists $i_1 \in N$ such that $\varphi^\pi_{i_1}(R) P_{i_1} \varphi_{i_1}(R)$. Let $\varphi^\pi_{i_1}(R) \equiv x_1 (\neq 0)$. By *strategy-proofness*, $\varphi_{i_1}(R^{x_1}_{i_1}, R_{-i_1}) \neq x_1$. By *efficiency*, for some $i_2 \in N$ we have $\varphi_{i_2}(R^{x_1}_{i_1}, R_{-i_1}) = x_1$. By *strategy-proofness*, $\varphi_{i_2}(R^{x_1}_{i_1}, R^{x_1}_{i_2}, R_{-i_1,i_2}) = x_1$. Hence, by *population-monotonicity*, $\varphi_{i_2}(R^{x_1}_{i_1}, R^{x_1}_{i_2}) = x_1$ and by definition of π, $\pi_{x_1}(i_2) < \pi_{x_1}(i_1)$.

By *strategy-proofness* and $\varphi^\pi_{i_1}(R) = x_1$, $\varphi^\pi_{i_1}(R^{x_1}_{i_1}, R_{-i_1}) = x_1$. Thus, by $\pi_{x_1}(i_2) < \pi_{x_1}(i_1)$, $\varphi^\pi_{i_2}(R^{x_1}_{i_1}, R_{-i_1}) P_{i_2} x_1$. Let $\varphi^\pi_{i_2}(R^{x_1}_{i_1}, R_{-i_1}) \equiv x_2 (\neq 0)$ and $R^{x_2}_{i_2} \in \mathcal{R}_0$ be such that for all $y \in K \setminus \{x_1\}$, $x_2 P^{x_2}_{i_2} x_1 R^{x_2}_{i_2} y$. Then using the same arguments as above there is some $i_3 \in N$ such that $\pi_{x_2}(i_3) < \pi_{x_2}(i_2)$, $\varphi_{i_3}(R^{x_1}_{i_1}, R^{x_2}_{i_2}, R_{-i_1,i_2}) = x_2$, and $\varphi^\pi_{i_3}(R^{x_1}_{i_1}, R^{x_2}_{i_2}, R_{-i_1,i_2}) P_{i_3} x_2$. By *strategy-proofness*, $\varphi_{i_2}(R^{x_1}_{i_1}, R^{x_2}_{i_2}, R_{-i_1,i_2}) = x_1$. Hence, by *efficiency*, $i_3 \neq i_1$ (otherwise i_1 and i_2 would be strictly better off by switching their objects).

Let $\varphi^\pi_{i_3}(R^{x_1}_{i_1}, R^{x_2}_{i_2}, R_{-i_1,i_2}) \equiv x_3 (\neq 0)$.[7] So, for any $l \geq 1$, we inductively obtain i_l, i_{l+1}, $i_l \neq i_{l+1}$, and $0 \neq x_l \in K$ such that $\pi_{x_l}(i_{l+1}) < \pi_{x_l}(i_l)$ and $i_{l+1} \neq i_{l-1}$. Because N is finite, at some point there will be a "cycle", i.e. there exist $i_k, i_t \in N$ such that $k < t$ and $\pi_{x_t}(i_k) < \pi_{x_t}(i_t)$. Without loss of generality, let $i_1 = i_k$ and consider the sets $\{i_1, \ldots, i_t\} \subseteq N$ and $\{x_1, \ldots, x_t\} \subseteq K$ such that for all $l \in \{1, \ldots, t-1\}$, $\pi_{x_l}(i_{l+1}) < \pi_{x_l}(i_l)$ and

[7] Now we would choose $R^{x_3}_{i_3} \in \mathcal{R}_0$ such that for all $y \in K \setminus \{x_3\}$, $x_3 P^{x_3}_{i_3} x_2 R^{x_3}_{i_3} y$.

$\pi_{x_t}(i_1) < \pi_{x_t}(i_t)$. Let $u \equiv \pi_{x_1}(i_1)$. By Step 2, π satisfies conditions (a) and (b). Thus, by $\pi_{x_1}(i_2) \leq u-1$ and (b), $\pi_{x_2}(i_2) \leq u$. Similarly, for all $l \in \{1, \dots, t\}$, $\pi_{x_l}(i_l) \leq u$. Next, (b) also implies $\pi_{x_t}(i_1) \geq u-1$. Thus, since $u \geq \pi_{x_t}(i_t) > \pi_{x_t}(i_1) \geq u-1$, $\pi_{x_t}(i_t) = u$. This implies that for all $l \in \{1, \dots, t\}$, $\pi_{x_l}(i_l) = u$. Because $i_1 \neq i_3$, we have $|\{i_1, \dots, i_t\}| \geq 3$. Now the two previous facts contradict (a).

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