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Abstract

This paper presents a model of partnership formation. A set of agents wants to conduct some business or other activities. Agents may act alone or seek a partner for cooperation and need in the latter case to consider with whom to cooperate and how to share the profit in a collaborative and competitive environment. We provide necessary and sufficient conditions under which an equilibrium can be attained. In equilibrium, the partner formation and the payoff distribution are endogenously determined. Every agent realizes his full potential and has no incentive to deviate from either staying independent or from the endogenously determined partner and payoff. The partnership formation problem contains the classical assignment market problem as a special case.

Keywords: Partnership formation, equilibrium, indivisibility, assignment market.

JEL classification: C62, C72, D02.

1 Introduction

Partnership is one of the most common and fundamental relation patterns in society. There is a set of agents who wish to conduct business or some other economic activities. Each agent may act alone or seek a partner for cooperation. When an agent acts as a sole proprietor, he gains a certain payoff by himself, which could be his outside option.

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When an agent cooperates with another agent, they obtain a joint payoff and this value has to be divided between the two partners. Different partners generate different joint payoffs and thus may lead to a different payoff share for an agent. In this collaborative and competitive environment, each agent has to evaluate what is more profitable, acting alone or seeking a partner. In the latter case, he has to decide with whom to cooperate and how to share the joint payoff in a satisfactory way. We may imagine that after an initial period of negotiation and bargaining, a number of partners and independents will be formed. Under proper circumstances, this process will reach an equilibrium state in which every agent is satisfied in the sense that no further more favorable deals could be obtained. In equilibrium, all agents realize their full potential and have no incentive to deviate from either staying independent or cooperating with the endogenously determined partner.

Our analysis is closely related to the models on assignment markets studied by Koopmans and Beckmann (1957), Shapley and Shubik (1972), Shapley and Scarf (1974), Crawford and Knoer (1981), Kelso and Crawford (1982), Svensson (1983), Quinzii (1984), Kaneko and Yamamoto (1986), and Yamamoto (1987). In their classic papers, Koopmans and Beckmann (1957) and Shapley and Shubik (1972) investigate assignment markets from the viewpoint of equilibrium theory and cooperative game theory, respectively. In such markets, there are many buyers and sellers, and transactions are bilateral, namely, bring together a buyer and a seller of a single commodity. By means of duality theory in linear programming it is shown that the set of Walrasian equilibrium price vectors is a nonempty lattice and coincides with the core. Shapley and Scarf (1974) consider a swap market model without monetary transfers and show the existence of a core allocation. Svensson (1983), Quinzii (1984), Kaneko and Yamamoto (1986), and Yamamoto (1987) extend the models of Koopmans and Beckmann (1957), Shapley and Shubik (1972) by allowing nonlinear utilities in both money and items.

Crawford and Knoer (1981) propose a price adjustment process for the assignment markets studied by Koopmans and Beckmann (1957), Shapley and Shubik (1972) and prove that their process converges to an equilibrium. Kelso and Crawford (1982) examine a job assignment model in which each firm can hire many workers but each worker is allowed to work only at one firm. They prove through a salary adjustment process that there exists an equilibrium, if every firm views all workers as substitutes. This latter condition is called gross substitutes and has been widely used in auction, matching and housing models, see for instance, Roth and Sotomayor (1990), Gul and Stacchetti (1999, 2000), Milgrom (2000), Ausubel (2006), Sun and Yang (2006), and Ostrovsky (2008).

This paper presents a new model, the partnership formation problem that subsumes and extends the assignment models studied by Koopmans and Beckmann (1957), Shapley and Shubik (1972), and Crawford and Knoer (1981). In these assignment models the role

of an agent is exogenously given and each agent is either a buyer (firm) or a seller (worker). So, all agents are exogenously split into two disjoint groups. Agents in the same group do not have any involvement with each other and cannot work together as partners. In the current model, the role of an agent need not be exogenously given and each agent may stay alone or work together with someone else. Precisely, because it is possible for an agent to form a partnership with anyone else, this creates not only more opportunities for agents to form partners but also more obstacles to cooperate. We provide necessary and sufficient conditions for the existence of an equilibrium in the partnership formation problem. Second, more importantly, we give a complete characterization of the set of solutions to the problem and offer a general sufficient condition. The condition is always satisfied by the assignment market models, and so it explains why the assignment models always possess an equilibrium.

In the partnership formation problem, permitted coalitions only consist of at most two individuals. Simple as they are, such coalitions are compelling, relatively easy to form, and widely observed in real life. For example, most transactions, trade, and merger occur bilaterally. Furthermore, conditions for equilibrium existence are rather mild and thus can be easily satisfied. In addition, an analysis on partnership formation may yield useful insights into many practical situations how stable partnerships can be built and such an analysis is also a necessary step to the study of more general coalition formation problems. From a different perspective, the following three papers deal with the latter problem. Hart and Kurz (1983) examine a general coalition formation problem. Assuming that players' prospects in various coalition structures are evaluated by a coalition structure value, they study stable coalition structures using a strategic form game. Aumann and Myerson (1988) investigate endogenous formation of cooperation structure under which players' payoffs follow the Myerson value, i.e., the Shapley value in graph games. Qin (1996) considers a cooperation-formation game in which players choose independently with whom they wish to cooperate in a given coalitional game, and players' payoffs are specified by a solution imposed on the coalitional game. He shows how cooperation evolves under best-response and fictitious-play learning processes.

The rest of the paper proceeds as follows. In Section 2 we present the model. In Sections 3 we establish all existence results. In Section 4 we conclude.

2 The model

Suppose there are n agents and let $N = \{1, 2, \dots, n\}$ denote the set of agents. Each agent wishes to engage in a business activity and seeks a partner to cooperate or acts alone. If two agents i and j in N , $j \neq i$, cooperate, they make a total payoff of $v(\{i, j\})$. The value

$v(\{i, j\})$ may differ for different pairs i and j . If agent $i \in N$ acts alone, he will have a payoff of $v(\{i\})$. We call $v(\cdot)$ the *value function*. For any positive integer k , let I_k denote the set $\{1, \dots, k\}$.

Definition 2.1 An assignment on N is a partition $P = \{U_1, \dots, U_k\}$ of N satisfying that $|U_h| \leq 2$ for every $h \in I_k$.

When, for an assignment $P = \{U_1, \dots, U_k\}$ on N , $|U_h| = 2$ for some $h \in I_k$, we say that the two agents in U_h are partner of each other or are being matched in P , and when $|U_h| = 1$, we say that the single agent in U_h is a sole proprietor or an independent in P . An assignment is a complete matching if every agent has a partner. A payoff vector is a vector $r \in \mathbb{R}^n$ with r_i the payoff or revenue of player $i \in N$.

In this economic environment, while bearing in mind that any other agent j could be both his partner and competitor, each agent $i \in N$ has to contemplate whether to act alone or seek a partner and how to share the joint payoff in case of cooperation. If agent i acts alone, he will get a payoff of $v(\{i\})$. An agreement or contract between agents i and j would specify how the joint payoff $v(\{i, j\})$ should be divided when the agreement is signed by both agents who consent to be partner of each other. Of course, a rational agent i will not rush to form a partnership with another agent j merely because agent j could offer him immediately a payoff share higher than his own value $v(\{i\})$. He will instead try to squeeze payoff gain as much as possible from any other agent and ultimately enter a partnership with someone until he has contented himself that no better contracts could be obtained with someone else. We look for an equilibrium state in which every agent will be satisfied with staying independent or cooperating with a specific partner and with his payoff share and therefore has no incentive to deviate. This problem is called the *partner formation problem* and is denoted by (N, v) .

Definition 2.2 An allocation (P, r) for the partner formation problem (N, v) consists of an assignment P on N and a payoff vector $r \in \mathbb{R}^n$ satisfying that $r_i = v(\{i\})$ if agent $i \in N$ is an independent in P and $r_j + r_h = v(\{j, h\})$ if agents $j \in N$ and $h \in N$ are partners of each other in P .

An allocation is therefore an assignment and a payoff vector such that the payoff of a sole proprietor is equal to his own value and if two agents are matched they get a total payoff equal to their joint value. The following equilibrium concept yields for each partnership formation problem a stable set of allocations.

Definition 2.3 An equilibrium for the partner formation problem (N, v) is an allocation (P^*, r^*) satisfying that $r_i^* \geq v(\{i\})$ for every $i \in N$ and $r_j^* + r_h^* \geq v(\{j, h\})$ for every $j, h \in N, j \neq h$.

An allocation is an equilibrium if every agent receives a payoff at least equal to the value when he is an independent and every pair of agents receives a total payoff at least equal to the value they get when they are matched. So, no agent has an incentive to deviate from being an independent or from the endogenously determined partner and payoff.

Definition 2.4 An assignment $\bar{P} = \{\bar{U}_1, \dots, \bar{U}_k\}$ on N is socially optimal for the partnership formation problem (N, v) if

$$\sum_{i=1}^k v(\bar{U}_i) \geq \sum_{j=1}^l v(U_j)$$

for any assignment $P = \{U_1, \dots, U_l\}$ on N .

The next lemma shows that an equilibrium assignment is socially optimal.

Lemma 2.5 If an allocation (P^*, r^*) is an equilibrium for the partner formation problem (N, v) , then the assignment P^* is socially optimal.

Proof. Let $P^* = \{U_1^*, \dots, U_k^*\}$ and take any assignment $P = \{U_1, \dots, U_l\}$. Let $K = \{U_i \mid |U_i| = 1, i \in I_l\}$ and $L = \{U_i \mid |U_i| = 2, i \in I_l\}$. Since (P^*, r^*) is an allocation, we have $\sum_{i=1}^k v(U_i^*) = \sum_{i \in N} r_i^*$. From the definition of an equilibrium, we get

$$\begin{aligned} \sum_{i=1}^k v(U_i^*) &= \sum_{i \in N} r_i^* \\ &= \sum_{\{i\} \in K} r_i^* + \sum_{\{j, h\} \in L} (r_j^* + r_h^*) \\ &\geq \sum_{\{i\} \in K} v(\{i\}) + \sum_{\{j, h\} \in L} v(\{j, h\}) \\ &= \sum_{j=1}^l v(U_j). \end{aligned}$$

This shows that the equilibrium assignment P^* is socially optimal. \square

The following example shows that an equilibrium allocation may not always exist. There are three agents with payoffs given by $v(\{i\}) = 0$ for $i = 1, 2, 3$, $v(\{1, 2\}) = 3$, $v(\{1, 3\}) = 5$, $v(\{2, 3\}) = 3$. It is easy to see that the assignment $P = \{\{1, 3\}, \{2\}\}$ is the unique socially optimal assignment. It is sufficient to consider the allocation (P, r) where $r = (r_1, r_2, r_3) = (r_1, 0, 5 - r_1)$ with $r_1 \geq 0$. Then $r_1 + r_2 < 3 = v(\{1, 2\})$ if $r_1 < 3$ and $r_2 + r_3 < 3 = v(\{2, 3\})$ if $r_1 > 2$. This demonstrates that (P, r) cannot be an equilibrium.

In the classical assignment problem as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1972), Crawford and Knoer (1981), there are two types of agents: a group N_1 of buyers (firms) and a group N_2 of sellers (workers). When a buyer $i \in N_1$ and a seller $j \in N_2$ cooperate, they can achieve a value of $v(\{i, j\})$. When each agent $i \in N_1 \cup N_2$ acts independently, he gets a payoff of $v(\{i\})$. As will be shown in Section 3, this problem can be formulated as a special case of the partnership formation problem.

3 Existence results

In this section we state conditions under which the partnership formation problem has an equilibrium. First, we give necessary and sufficient conditions. Consider the following linear programming problem:

$$(P) \quad \min \quad \sum_{i=1}^n r_i \\ \text{s.t.} \quad \sum_{i \in U} r_i \geq v(U), \forall U \subseteq N, |U| \leq 2.$$

Its dual linear programming problem is given by

$$(D) \quad \max \quad \sum_{U \subseteq N, |U| \leq 2} \lambda_U v(U) \\ \text{s.t.} \quad \lambda_{\{h\}} + \sum_{l \neq h} \lambda_{\{h,l\}} = 1, \forall h \in N \\ \lambda_U \geq 0, \forall U \subseteq N, |U| \leq 2.$$

The following theorem establishes an intimate relation between the partnership formation problem and an integer linear programming problem.

Theorem 3.1 *The partnership formation problem has an equilibrium if and only if the dual linear programming problem (D) has an integer optimal solution.*

Proof. Let $\lambda_{U_1^*} = 1, \dots, \lambda_{U_k^*} = 1, \lambda_U = 0$ for all other $U \subseteq N, |U| \leq 2$, constitute an integer optimal solution of the dual linear programming problem (D), for some U_1^*, \dots, U_k^* with $|U_i^*| \leq 2$ for all $i \in I_k$. By the n equality constraints of (D), every $h \in N$ appears in U_1^*, \dots, U_k^* exactly once. This implies that $P^* = \{U_1^*, \dots, U_k^*\}$ is a partition of N and thus an assignment on N . By the well-known duality theorem of linear programming theory (see Dantzig (1963)) and since both (P) and (D) have feasible solutions, the minimal value of the primal linear programming problem (P) is equal to the maximal value $\sum_{i=1}^k v(U_i^*)$ of the dual problem (D). So, there is an optimal solution $r^* = (r_1^*, \dots, r_n^*)$ of the problem (P) such that $\sum_{i=1}^n r_i^* = \sum_{i=1}^k v(U_i^*)$. By the complementarity theorem of linear programming theory, we know that for every $i \in I_k, \sum_{j \in U_i^*} r_j^* = v(U_i^*)$. Moreover, from (P) it follows that $\sum_{j \in U} r_j^* \geq v(U)$ for any $U \subset N, |U| \leq 2$. This shows that (P^*, r^*) is an equilibrium.

Suppose that (P^*, r^*) is an equilibrium with assignment $P^* = \{U_1^*, \dots, U_k^*\}$. By definition of an equilibrium r^* is a feasible solution of the primal problem (P) and $\sum_{i=1}^n r_i^* = \sum_{i=1}^k v(U_i^*)$. We will show that $\sum_{i=1}^k v(U_i^*)$ is the minimal value of the primal problem (P). Let $K = \{U_i^* \mid |U_i^*| = 1, i \in I_k\}$ and $L = \{U_i^* \mid |U_i^*| = 2, i \in I_k\}$. Take any feasible solution $r = (r_1, \dots, r_n)$ of (P). Because $r_i \geq v(\{i\})$ for every $i \in N$ and $r_j + r_h \geq v(\{j, h\})$ for all $j, h \in N, j \neq h$, we have

$$\begin{aligned} \sum_{i=1}^n r_i &= \sum_{\{i\} \in K} r_i + \sum_{\{j,h\} \in L} (r_j + r_h) \\ &\geq \sum_{\{i\} \in K} v(\{i\}) + \sum_{\{j,h\} \in L} v(\{j, h\}) \\ &= \sum_{i=1}^k v(U_i^*). \end{aligned}$$

Hence, by the duality theorem and since both (P) and (D) have feasible solutions, the maximal value of the dual problem (D) is equal to $\sum_{i=1}^k v(U_i^*)$. Define $\lambda_{U_1^*} = 1, \dots, \lambda_{U_k^*} = 1$, and $\lambda_U = 0$ for all other $U \subseteq N$, $|U| \leq 2$. Then the complementarity theorem implies that these parameters yield an integer optimal solution of the dual problem (D). \square

The theorem gives a necessary and sufficient condition under which an equilibrium exists. We now give a more natural sufficient condition under which an equilibrium exists. A sequence $S = (i_1, \dots, i_l)$ of l different elements of N is a *proper ordering* if $i_1 < i_j$, for all $j > 1$, and $i_2 < i_l$ when $l \geq 3$. For $i, j \in N$, $i \neq j$, let $M(i, j) = v(\{i, j\}) - v(\{i\}) - v(\{j\})$ be the marginal value of i and j of becoming partners.

Assumption 3.2 *For any proper ordering $S = (i_1, \dots, i_l)$ with l odd and $l \geq 3$, it holds that (i) there exists some j , $1 \leq j \leq l$, such that*

$$M(i_{j-1}, i_j) + M(i_j, i_{j+1}) < M(i_{j-1}, i_{j+1}),$$

or (ii) there exist j and h between 1 and l with $j \geq h + 2$ such that

$$M(i_h, i_{h+1}) + M(i_j, i_{j+1}) < M(i_h, i_{j+1}) + M(i_{h+1}, i_j),$$

where $k - 1 = l$ when $k = 1$ and $k + 1 = 1$ when $k = l$.

This assumption says intuitively that, for any (proper) ordering of the agents with odd length, there is some agent with respect to whom the marginal value of his two neighbors in the ordering of becoming partners exceeds the total marginal values of them of becoming partners with him, and if there is not such an agent there are two nonconsecutive agents in the ordering with respect to whom the total marginal values of becoming partners with the successor in the ordering of the other exceeds the total marginal value of becoming partners with their own successor in the ordering. Under this assumption we have the following theorem.

Theorem 3.3 *Under Assumption 3.2 the partnership formation problem (N, v) has an equilibrium.*

To prove the theorem, we need to introduce an auxiliary result concerning the structure of the set W of feasible solutions to the dual linear programming problem (D). Clearly, W is a nonempty bounded polyhedron and is therefore a polytope in \mathbb{R}^m with $m = \frac{1}{2}n(n-1)$. An element x of \mathbb{R}^m will be denoted by

$$x = (x_{\{1\}}, x_{\{2\}}, \dots, x_{\{n\}}, x_{\{1,2\}}, \dots, x_{\{1,n\}}, x_{\{2,3\}}, \dots, x_{\{2,n\}}, \dots, x_{\{n-2,n\}}, x_{\{n-1,n\}}).$$

Let \mathcal{C} be the family of collections $C = \{S_1, \dots, S_k\}$ of proper orderings on N with $S_h = (i_1^h, \dots, i_{l_h}^h)$, $h \in I_k$, satisfying that $\{T_1, \dots, T_k\}$ is a partition of N , where $T_h =$

$\{i_1^h, \dots, i_{l_h}^h\}$, $h \in I_k$. For such $C \in \mathcal{C}$, the vector $q(C) \in \mathbb{R}^m$ is defined by, for all $h \in I_k$, $q_{T_h}(C) = 1$ if $l_h = 1$ or 2 , $q_{\{i_j^h, i_{j+1}^h\}}(C) = \frac{1}{2}$ for $j = 1, \dots, l_h$ if $l_h \geq 3$, where $j + 1 = 1$ when $j = l_h$, and $q_U(C) = 0$ for all other $U \subseteq N$, $|U| \leq 2$. Clearly, $q(C)$ is an element of W for any $C \in \mathcal{C}$. Finally, let \mathcal{C}^0 be the subfamily of collections $C = \{S_1, \dots, S_k\}$ in \mathcal{C} of proper orderings on N with odd length or length equal to 2.

The next result gives a complete and useful characterization of the polytope W and also presents a new class of polytopes whose vertices have components with values of only 0, $\frac{1}{2}$, and 1. It tells us the precise structure of the polytope W and thus enables us to know how to achieve an integer optimal solution to the problem (D) and thus an equilibrium to the partnership formation problem.

Lemma 3.4 *The points $q(C)$, $C \in \mathcal{C}^0$, are all the vertices of the polytope W .*

Proof. The polytope W is equal to the set

$$\{x \in \mathbb{R}^m \mid Ax = e(N), x \geq 0\},$$

where A is an $n \times m$ -matrix with U th column equal to $A_U = \sum_{i \in U} e(i)$ for $U \subset N$, $|U| \leq 2$, with $e(i)$, $i \in N$, the i th unit vector in \mathbb{R}^n , and $e(N) = \sum_{i=1}^n e(i)$. We first show that for every $C \in \mathcal{C}^0$ the point $q(C)$ is a vertex of W . So, let $C = \{S_1, \dots, S_k\}$ be a collection of proper orderings on N with $S_h = (i_1^h, \dots, i_{l_h}^h)$, $h \in I_k$, satisfying for all $h \in I_k$ that l_h is odd or equal to 2. Let the matrix B be defined as follows. For $h \in I_k$, the matrix B contains column $A_{\{i_1^h\}}$ if $S_h = (i_1^h)$, the two columns $A_{\{i_1^h, i_2^h\}}$ and $A_{\{i_2^h\}}$ if $S_h = (i_1^h, i_2^h)$, and the l_h columns $A_{\{i_j^h, i_{j+1}^h\}}$ for $j = 1, \dots, l_h$, where $j + 1 = 1$ when $j = l_h$. Clearly, the matrix B is a square matrix. Since l_h is odd whenever $l_h > 2$, the matrix B is of full rank. Moreover it holds that $q(C) = B^{-1}e(N)$. Therefore, the matrix B is a feasible basis matrix and $q(C)$ is a vertex of W .

To prove the converse, take any feasible basis matrix $B = (A_{U_1}, \dots, A_{U_n})$ of the matrix A , so $x = B^{-1}e(N) \geq 0$. Suppose there is a row in B with only one element equal to one and let this element correspond to the U th column of the matrix A . Then at the basic solution x corresponding to basis matrix B it must hold that $x_U = 1$. Let H be equal to N . If $U = \{i_1\}$ for some $i_1 \in H$, perform a singleton-elimination: define sequence $S = (i_1)$, delete i_1 from H , and delete from the matrix B the row indexed by i_1 and the column indexed by U . If $U = \{i_1, i_2\}$ for some $i_1, i_2 \in H$, where without loss of generality $i_1 < i_2$, perform a pair-elimination: define sequence $S = (i_1, i_2)$, delete from the set H both indices i_1 and i_2 , and delete from the matrix B the two rows indexed by i_1 and i_2 and any column indexed by some U' containing i_1 or i_2 or both. Since $x_{\{i_1, i_2\}} = 1$ it holds that $x_{\{i_1, j\}} = 0$ for all $j \in H$, $j \neq i_2$, and $x_{\{h, i_2\}} = 0$ for all $h \in H$, $h \neq i_1$.

This singleton- and pair-elimination of indices from H and rows and columns of B is continued until no row of the remaining B is left that contains only one 1. In this way

a family of sequences of one or two elements of N being a partition of the set $N \setminus H$ is generated. We now prove that in each step of the elimination process the reduced matrix B has full column-rank. By definition the initial matrix B is regular and therefore has full column-rank. Clearly, by a singleton-elimination the new matrix B has full column-rank again, because the column and the row of B being deleted are both unit vectors with a 1 as common element. At a pair-elimination step, two rows and one or more columns of B are deleted. If more than one column is deleted from B , then, for the two rows that are deleted, we have that one row is a unit vector and its 1 belongs to a column that is deleted, and from the other row all ones belong to columns that are also deleted. Therefore the remaining matrix B must have full column-rank. If only one column is deleted, the two rows that are deleted are identical unit vectors. Since B has full column-rank and two rows are identical, B must have at least one row more than columns and the remaining columns still have full rank and therefore also the new matrix B .

After the singleton- and pair-elimination process the remaining matrix B has full column-rank and therefore its number of columns is at most equal to its number of rows. Moreover, each row of B has at least two ones, while by definition every column has at most two ones. This can only be the case if the matrix B is a square matrix, has full rank, and every row and column of B has precisely two ones. Let i_1 be the smallest index in the remaining index set H , then there exist j and h in H such that $\{i_1, j\}$ and $\{i_1, h\}$ are indices of columns of B . Notice that $j \neq h$ because B has full rank. Take $i_2 = j$ if $j < h$ and $i_2 = h$ if $j > h$. Now there must exist a chain $\{i_1, i_2\}, \dots, \{i_{l-1}, i_l\}, \{i_l, i_1\}$ of indices of columns of the matrix B for some $l \geq 3$. Next a chain-elimination is performed: define sequence $S = (i_1, \dots, i_l)$, delete from the set H the indices i_1, \dots, i_l , and delete from B the l rows indexed by i_1, \dots, i_l and the l columns indexed by the indices of the chain. We repeat this chain-elimination on the remaining H and B until the index set H has become empty. Clearly, any sequence $S = (i_1, \dots, i_l)$ obtained at a chain-elimination step is a proper ordering with $l \geq 3$. We still have to prove that S has odd length. Suppose that S has even length, then the columns of the initial matrix B indexed by $\{i_1, i_2\}, \dots, \{i_{l-1}, i_l\}, \{i_l, i_1\}$ are linearly dependent. This contradicts the fact that the matrix B has full rank. For each proper ordering $S = (i_1, \dots, i_l)$ obtained in a chain-elimination step, it holds at the solution x that $x_{\{i_h, i_{h+1}\}} = \frac{1}{2}$ for all $h \in I_l$, where $h + 1 = 1$ when $h = l$.

Let C be the complete collection of proper orderings obtained at any singleton-, pair- and chain-elimination step. Then C is an element of \mathcal{C}^0 and $q(C) = B^{-1}e(N)$. Since the matrix B is any feasible basis, this implies that every vertex of W is equal to $q(C)$ for some $C \in \mathcal{C}^0$, which completes the proof. \square

Now we are ready to prove Theorem 3.3. In the proof, two basic ideas will be used. That is, case (i) of Assumption 3.2 implies that for the players i_1, \dots, i_l in the chain induced by

the sequence $S = (i_1, \dots, i_l)$, they can do better by breaking the chain into a single player i_j and linking players i_{j-1} and i_{j+1} to form a new sub-chain. Case (ii) of the assumption implies that for the players i_1, \dots, i_l in the chain induced by the sequence $S = (i_1, \dots, i_l)$, they can do better by breaking the chain into two sub-chains. The chain is broken between players i_j and i_{j+1} and between i_h and i_{h+1} and then player i_j is linked to player i_{h+1} to form one sub-chain and player i_{j+1} is linked to player i_h to form another sub-chain. In both cases the sub-chains have smaller length than the original chain.

Proof of Theorem 3.3: Suppose the problem has no equilibrium, then the dual linear programming problem has only non-integer optimal solutions. By Lemma 3.4 there exists a vertex $q(C)$, $C \in \mathcal{C}^0$, solving the dual problem (D), where $C = \{S_1, \dots, S_k\}$ is a collection of proper orderings of odd length or length equal to 2. Because $q(C)$ is not an integer vector, we can assume without loss of generality that $S_k = (i_1, \dots, i_l)$ for some odd number l , $l \geq 3$. From Assumption 3.2 it follows that in case (i) there exists some j , $1 \leq j \leq l$, such that

$$M(i_{j-1}, i_j) + M(i_j, i_{j+1}) < M(i_{j-1}, i_{j+1}),$$

and in case (ii) there exist j and h between 1 and l with $j \geq h + 2$ such that

$$M(i_h, i_{h+1}) + M(i_j, i_{j+1}) < M(i_h, i_{j+1}) + M(i_{h+1}, i_j),$$

where $p - 1 = l$ when $p = 1$ and $p + 1 = 1$ when $p = l$.

In case (i), we obtain

$$v(\{i_{j-1}, i_j\}) + v(\{i_j, i_{j+1}\}) < v(\{i_{j-1}, i_{j+1}\}) + 2v(\{i_j\}).$$

If in case (i) it holds that $l = 3$, take $\bar{C} = \{S_1, \dots, S_{k-1}, \bar{S}_k, \bar{S}_{k+1}\}$, where \bar{S}_k is a proper reordering of the sequence (i_{j-1}, i_{j+1}) , i.e., $\bar{S}_k = (i_{j-1}, i_{j+1})$ when $i_{j-1} < i_{j+1}$ and $\bar{S}_k = (i_{j+1}, i_{j-1})$ when $i_{j-1} > i_{j+1}$, and $\bar{S}_{k+1} = (i_j)$. Since \bar{C} is a collection of proper orderings, the point $q(\bar{C})$ is an element of W . The difference of the value of the dual linear programming problem at $q(\bar{C})$ and the vertex $q(C)$ equals

$$v(\{i_j\}) + v(\{i_{j-1}, i_{j+1}\}) - \left(\frac{1}{2}v(\{i_{j-1}, i_j\}) + \frac{1}{2}v(\{i_j, i_{j+1}\}) + \frac{1}{2}v(\{i_{j-1}, i_{j+1}\}) \right) > 0.$$

Therefore $q(C)$ is not an optimal solution of problem (D).

If in case (i) it holds that $l \geq 5$, take $\bar{C} = \{S_1, \dots, S_{k-1}, \bar{S}_k, \bar{S}_{k+1}\}$, where \bar{S}_k is a proper reordering of the sequence $(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_l)$ and $\bar{S}_{k+1} = (i_j)$. Since \bar{C} is a collection of proper orderings, the point $q(\bar{C})$ is an element of W . The difference of the value of the dual linear programming problem at $q(\bar{C})$ and the vertex $q(C)$ equals

$$v(\{i_j\}) + \frac{1}{2}v(\{i_{j-1}, i_{j+1}\}) - \left(\frac{1}{2}v(\{i_{j-1}, i_j\}) + \frac{1}{2}v(\{i_j, i_{j+1}\}) \right) > 0.$$

Therefore $q(C)$ is not an optimal solution of problem (D).

In case (ii), we obtain

$$v(\{i_h, i_{h+1}\}) + v(\{i_j, i_{j+1}\}) < v(\{i_h, i_{j+1}\}) + v(\{i_{h+1}, i_j\}).$$

If in case (ii) it holds that $j = h+2$, take $\bar{C} = \{S_1, \dots, S_{k-1}, \bar{S}_k, \bar{S}_{k+1}\}$, where \bar{S}_k is a proper reordering of the sequence (i_{h+1}, i_{h+2}) , i.e., $\bar{S}_k = (i_{h+1}, i_{h+2})$ when $i_{h+1} < i_{h+2}$ and $\bar{S}_k = (i_{h+2}, i_{h+1})$ when $i_{h+1} > i_{h+2}$, and \bar{S}_{k+1} is a proper reordering of $(i_1, \dots, i_h, i_{h+3}, \dots, i_l)$. Since \bar{C} is a collection of proper orderings, the point $q(\bar{C})$ is a vertex of W . The difference of the value of the dual linear programming problem at $q(\bar{C})$ and the vertex $q(C)$ equals

$$v(\{i_{h+1}, i_{h+2}\}) + \frac{1}{2}v(\{i_h, i_{h+3}\}) - (\frac{1}{2}v(\{i_h, i_{h+1}\}) + \frac{1}{2}v(\{i_{h+1}, i_{h+2}\}) + \frac{1}{2}v(\{i_{h+2}, i_{h+3}\})) > 0.$$

Therefore $q(C)$ is not an optimal solution of problem (D).

Finally, if in case (ii) it holds that $j > h+2$, take $\bar{C} = \{S_1, \dots, S_{k-1}, \bar{S}_k, \bar{S}_{k+1}\}$, where \bar{S}_k is a proper reordering of the sequence $(i_1, \dots, i_h, i_{j+1}, \dots, i_l)$ and \bar{S}_{k+1} is a proper reordering of the sequence (i_{h+1}, \dots, i_j) . Since \bar{C} is a collection of proper orderings, the point $q(\bar{C})$ is a vertex of W . The difference of the value of the dual linear programming problem at $q(\bar{C})$ and the vertex $q(C)$ equals

$$\frac{1}{2}v(\{i_h, i_{j+1}\}) + \frac{1}{2}v(\{i_{h+1}, i_j\}) - (\frac{1}{2}v(\{i_h, i_{h+1}\}) + \frac{1}{2}v(\{i_j, i_{j+1}\})) > 0.$$

Therefore $q(C)$ is not an optimal solution of problem (D).

This shows that the dual linear programming problem (D) doesn't have any non-integer optimal solution, and therefore it must have an integer optimal solution. Consequently, the partnership formation problem has an equilibrium. \square

Next, we investigate what condition can ensure the existence of a complete matching in equilibrium. A value-function v is said to be super-additive if $v(\{i, j\}) > v(\{i\}) + v(\{j\})$ for all $i, j \in N$, $i \neq j$. Super-additivity implies that all agents are complementary to each other, cooperation gives a higher joint value than the sum of the individual values. The following result asserts that if the value function satisfies super-additivity, all agents, except one in case the number of agents is odd, are being matched in equilibrium.

Theorem 3.5 *Suppose the value function of the partnership formation problem is super-additivity. Then, at any equilibrium, there is a complete matching if the number of agents is even, whereas precisely one agent is not being matched if the number of agents is odd.*

Proof. Suppose n is even, and let (P^*, r^*) be an equilibrium. If agent $i \in N$ is an independent in P^* , then there must be another independent in P^* , say, $j \in N$, $j \neq i$. Since both i and j are independents in P^* , it holds that $r_i^* = v(\{i\})$ and $r_j^* = v(\{j\})$. But then

$r_i^* + r_j^* = v(\{i\}) + v(\{j\}) < v(\{i, j\})$, which contradicts that (P^*, r^*) is an equilibrium. Hence, there can be no independents in P^* .

Suppose n is odd, and let (P^*, r^*) be an equilibrium. It is clear that at least one agent must be an independent in P^* . If more than one agent is an independent in P^* , let i and j be two independents in P^* . This means that $r_i^* = v(\{i\})$ and $r_j^* = v(\{j\})$. But then $r_i^* + r_j^* = v(\{i\}) + v(\{j\}) < v(\{i, j\})$, which also contradicts that (P^*, r^*) is an equilibrium. Hence, precisely one agent is an independent in P^* . \square

Finally, we show that the classical assignment problem mentioned in Section 2 can be formulated as a partnership formation problem satisfying Assumption 3.2. The classical assignment problem consists of two disjoint groups of agents, N_1 and N_2 . Agents that belong to the same group cannot be matched. If agent $i \in N_1$ and agent $j \in N_2$ are matched, their value is equal to $v(\{i, j\})$. Agents that are not being matched can get some value by their own, $v(\{i\})$ for agent $i \in N$, where $N = N_1 \cup N_2$ is the set of all agents. An equilibrium is then as described before, except that no two agents of the same group can be matched. To show that this problem fits the framework above, we assign artificially a value to any pair of agents in a same group. Take any positive number a satisfying $a > 2 \max\{|v(\{i, j\}) - v(\{i\}) - v(\{j\})| | i \in N_1, j \in N_2\}$, then we define for any $i, j \in N_1$ and any $i, j \in N_2, j \neq i$,

$$v(\{i, j\}) = -a + v(\{i\}) + v(\{j\}).$$

Observe that $M(i, j) = -a$ whenever i and j belong to the same group. The next theorem shows that an equilibrium of this induced partnership formation problem always exists and is an equilibrium of the assignment problem.

Theorem 3.6 *Consider the induced partnership formation problem as described above. Then this problem satisfies Assumption 3.2. Therefore it has an equilibrium, and any equilibrium of it is an equilibrium of the assignment problem.*

Proof. To show that the induced partnership formation problem satisfies Assumption 3.2, let $S = (i_1, \dots, i_l)$ be a proper ordering on the set N of all agents with l odd and $l \geq 3$, and let $T = \{i_1, \dots, i_l\}$. Suppose that $T \subset N_1$ or $T \subset N_2$. Then $M(i_1, i_2) = M(i_2, i_3) = M(i_1, i_3) = -a$, and therefore case (i) of Assumption 3.2 is satisfied because $a > 0$. Next, suppose $T \cap N_1 \neq \emptyset$ and $T \cap N_2 \neq \emptyset$. Since l is odd and $l \geq 3$, there exists $j \in I_l$ such that i_{j-1} and i_j in N_1 and i_{j+1} in N_2 or such that i_{j-1} and i_j in N_2 and i_{j+1} in N_1 , where $j - 1 = l$ when $j = 1$ and $j + 1 = 1$ when $j = l$. In both cases $M(i_{j-1}, i_j) = -a$ and therefore

$$M(i_{j-1}, i_j) < -2 \max\{|M(i, j)| | i \in N_1, j \in N_2\} \leq M(i_{j-1}, i_{j+1}) - M(i_j, i_{j+1}),$$

and so case (i) of Assumption 3.2 is satisfied. Hence, from Theorem 3.3 it follows that the induced partnership formation problem has an equilibrium.

Let (P^*, r^*) be any equilibrium of the induced partnership formation problem and let $P^* = \{U_1^*, \dots, U_k^*\}$. We still have to prove that for all $h \in I_k$ it holds that if $|U_h^*| = 2$ then $U_h^* \cap N_1 \neq \emptyset$ and $U_h^* \cap N_2 \neq \emptyset$. Suppose $U_h^* \subset N_1$ or $U_h^* \subset N_2$ and let $U_h^* = \{i_1, i_2\}$. From the definition of equilibrium it follows that

$$r_{i_1}^* + r_{i_2}^* = v(\{i_1, i_2\}) = -a + v(\{i_1\}) + v(\{i_2\}) < v(\{i_1\}) + v(\{i_2\}),$$

because $a > 0$. On the other hand, in equilibrium it must hold that $r_{i_1}^* \geq v(\{i_1\})$ and $r_{i_2}^* \geq v(\{i_2\})$, which implies $r_{i_1}^* + r_{i_2}^* \geq v(\{i_1\}) + v(\{i_2\})$, yielding a contradiction. Therefore, an equilibrium of the induced partnership formation problem is an equilibrium for the assignment problem. \square

4 Conclusion

In this paper we have introduced the partnership formation problem and proposed an equilibrium solution for the problem. The equilibrium solution endogenously provides answers to the question of partnership formation and that of payoff division. We have identified necessary and sufficient conditions and established equilibrium existence theorems for the model. In particular, Assumption 3.2 is very intuitive and general and is satisfied by the classical assignment models.

The current study leaves us with some natural questions. As mentioned earlier, for the assignment market models studied by Koopmans and Beckmann (1957), Shapley and Shubik (1972), Crawford and Knoer (1981) have proposed a competitive adjustment process which mimics the behavior of real competitive markets and finds an equilibrium. It is of considerable interest and difficulty to develop a similar competitive process that imitates the behavior of real partnership formation and endogenously yields partnership structure and payoff distribution in finitely many rounds of negotiations. It is also equally challenging to investigate what happens when uncertainty or imperfect information is introduced into the payoff that each agent or each pair of agents can achieve.

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