WHEN ORDER MATTERS FOR ITERATED STRICT DOMINANCE*

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Abstract: We demonstrate that iterated elimination of strictly dominated strategies is an order dependent procedure. We also prove that order does not matter if strategy spaces are compact and payoff functions continuous. Examples show that this result is tight.

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1. INTRODUCTION

There seems to be widespread belief that the outcome of iterated elimination of strictly dominated strategies (IESDS) does not depend on the order of elimination. Nevertheless, this assertion has not been formally proved. We show that, in fact, order may matter. One of the examples is taken from Stegeman (1990). We also prove that for games with compact strategy spaces and continuous payoff functions order does not matter. This result covers the cases of finite games and their mixed extensions. Examples show that the result is tight.

The work most closely related to ours is that of Gilboa, Kalai & Zemel (1990) (GKZ), and in a separate section we connect to their contribution. GKZ consider a variety of elimination procedures and provide sufficient conditions for order independence. Among the procedures considered by GKZ is a form of IESDS, and they prove that for finite games this procedure is order invariant. GKZ, however, impose a bound on the rate of elimination, that is, they establish invariance for only a subset of possible elimination sequences. It follows from our aforementioned result that this bound is irrelevant for finite games. We generalize this finding. GKZ consider IESDS only for a finite number of eliminations rounds, but in games with infinite strategy spaces it is natural to allow an infinite sequence of elimination rounds, and GKZ's definition is easily generalized to allow this. Given this modification, we prove that GKZ's bound on the rate of elimination is irrelevant for all games with compact strategy spaces and continuous payoff functions.

We show that order may matter for IESDS in Section 2, prove that order does not matter in games with compact strategy spaces and continuous payoff functions in Section 3, discuss the contribution of GKZ in Section 4, and offer concluding remarks in Section 5.

2. WHEN ORDER MATTERS

Our first example is the simplest game we can think of for which order matters for IESDS. The example shows that order can matter if strategy sets are not closed.

Example 1. Consider a one-player game with strategy set $G_1=(0,1)$ and payoff function $u_1:G_1 \to \mathbb{R}$ defined by $u_i(x)=x$ for all $x \in G_1$. In this game every strategy is strictly dominated. For any $x \in G_1$, eliminate in round one all strategies in the set $G_1 \backslash \{x\}$, and only x survives IESDS.

Our next example shows that closing all strategy sets is not enough to ensure order independence. This example shows that using IESDS "to simplify" a two-player game may not be innocuous even if the game possesses a Nash equilibrium. IESDS generates not only ambiguous residual games but also ambiguous sets of Nash equilibria.

Example 2. Consider a two-player game with strategy sets $G_1=G_2=[0,1]$, and payoff functions $u_i:G_i \times G_j \to \mathbb{R}$ with $i,j=1,2$ and $i \neq j$, defined by

$$u_i(x,y) = x \qquad \text{if } x<1$$

$$u_i(1,y) = 0 \qquad \text{if } y<1$$

$$u_i(1,1) = 1$$

The strategy profile (1,1) is the game's unique Nash equilibrium, and every strategy except 1 is strictly dominated. Eliminating $G_i \backslash \{1,x\}$ for some $x<1$, for $i=1,2$, leaves the following 2×2 game, which cannot be further reduced:

	1	***x***
1	1, 1	0, x
x	x, 0	x, x

Suppose one applies an equilibrium selection theory which favours "risk dominance" in the 2×2 game. Then the profile (x,x) is selected iff x is large enough.

The game in Example 2 has discontinuous payoffs. Our final example, taken from Stegeman (1990), shows that even with continuous payoff functions and closed strategy sets order matters. Again, what set of Nash equilibria obtains in the reduced game created by IESDS depends on the order of elimination.

Example 3. Consider a two-player game in which player 1 chooses $x\in\mathbb{R}_+$, 2 chooses $y\in\mathbb{R}_+$, and each player receives the common payoff $u(x,y)=(\max\{x,1-x-y\})/(1+x)$. The payoff function is continuous and has range [0,1]. If y>0 then player 1's optimal action is undefined, and it follows directly that the unique Nash equilibrium of the game is (x,y)=(0,0). One way to perform IESDS is as follows: eliminate every x>0 as it is strictly dominated by some $x'>x$. Given that x=0, every y>0 is then strictly dominated by y=0. IESDS thus eliminates all except Nash play. Another way to perform IESDS is: eliminate every x>0 except x=1, leaving the strategy sets {0,1} for player 1 and $\mathbb{R}_+$ for player 2. No more eliminations are possible and the residual game now has many Nash equilibria: (x,y)=(0,0) and (x,y)=(1,y) for all $y\geq½$

3. WHEN ORDER DOES NOT MATTER

In this section we prove that order does not matter for IESDS in games with compact strategy spaces and continuous payoff functions. Preliminary definitions follow.

Games, subgames, and strict dominance. A game is a triple $G=(I,(G_i)_{i\in I},(u_i)_{i\in I})$, where $I=\{1,2,...n\}$ is the set of players, $G_i\subseteq R^m$ ($G_i\neq\emptyset$) is player i's strategy set for some integer $m\geq1$, and $u_i:\Pi_iG_i\rightarrow\mathbb{R}$ is the payoff to player i. We call the game G compact and continuous if G_i is compact and u_i is continuous $\forall i\in I$. For convenience, assume that the players' strategy sets are disjoint. A subgame of G is a game $H=(I,(H_i)_{i\in I},(u_i')_{i\in I})$, where $H_i\subseteq G_i$ and u_i' is the restriction of u_i to $\Pi_i H_i$, $\forall i\in I$. For any subgame H, let $H_{-i}\equiv\Pi_{j\neq i}H_j$. Let S(G) denote the set of all subgames of G. Given a subgame H of G, and $x,y\in G_i$: $y\succ_H x$ if $u_i(y,s_{-i})>u_i(x,s_{-i})$ $\forall s_{-i}\in H_{-i}$. (The reordering of the arguments of u_i simplifies notation, where no confusion is possible.) The relation $(\succ_H)$ embodies the notion of strict dominance given rivals'options in game H.

Reduction. Consider subgames $H,H'\in S(G)$, such that $H_i'\subseteq H_i$ $\forall i\in I$. $H\to H'$ if, for each $x\in H_i\backslash H_i'$, $\exists y\in H_i$ such that $y\succ_H x$. We use the symbol $\to^*$ as follows: $H\to^*H'$ if there exists a (finite or infinite) sequence of subgames, $A^t\in S(G)$, t=0,1,2..., such that $A^0=H$, $A^t\to A^{t+1}$ for all t, and $H_i'=\cap_t H_i^t$ $\forall i$. H is a maximal $(\to)$-reduction of G if $G\to^*H$ and $H\to H'$ only for $H'=H$.

The following Lemma is the key result behind both of our theorems.

Lemma. If $G\to^*H$ for some compact and continuous game G, and $y\succ_H x$ for some $x,y\in G_i$ and $i\in I$, then $\exists z^*\in H_i$ such that $z\not\succ_H z^*\succ_H x$ $\forall z\in H_i$.

Proof. Given H as described, let $A^t\in S(G)$, t=0,1,2..., be the implied sequence of subgames. Let $Z\equiv\{z\in G_i \mid u_i(z,s_{-i})\geq u_i(y,s_{-i})\ \ \forall s_{-i}\in H_{-i}\}$. Clearly $y\in Z$, and the continuity of u_i and compactness of G_i imply that Z is compact. Define $f:Z\to R$ by $f(z)=u_i(z,s^*)$ for some fixed and arbitrary $s^*\in H_{-i}$. The continuity of u_i, implies that f is continuous, which with Z compact implies that f reaches a maximum f* at some $z^*\in Z$. $z^*\in Z$ and $y\succ_H x$ imply $z^*\succ_H x$. If $z\succ_H z^*$ for some $z\in G_i$, then $u_i(z,s_{-i})>u_i(z^*,s_{-i})$ $\forall s_{-i}\in H_{-i}$, implying $z\in Z$ and $f(z)>f(z^*)=f^*$, a contradiction. Therefore, $z\not\succ_H z^*$ $\forall z\in G_i$ (and hence $\forall z\in H_i$), implying $z\not\succ_{A^t} z^*$ $\forall z\in G_i$, $\forall t$ (because $H_{-i}\subseteq A_{-i}^t$), implying $z^*\in A_i^t$, $\forall t$, implying $z^*\in H_i$. ž

Theorem 1. If G is compact and continuous, then any maximal $(\to)$-reduction of G is unique.

Proof. Let H and H′ be maximal $(\to)$-reductions of G. Given $G\to^*H'$, let $A^t\in S(G)$, t=0,1,2..., be the implied finite or infinite sequence of subgames. Suppose that $H_i\not\subseteq H_i'$ for some i. Then $H_i\not\subseteq A_i^t$ $\forall t>T$, for some T such that A_i^{T+1} is well-defined. Let T take the largest value such that $H_i \subseteq A^T$ $\forall i$. Choose $i \in I$ and $x \in H_i \backslash A_i^{T+1}$. Then $x \in A_i^T \backslash A_i^{T+1}$, implying from $A^T \to A^{T+1}$ that $\exists y \in A_i^T$ such that $y \succ_{A^T} x$, which with $H_i \subseteq A_i^T$ $\forall i$ implies $y \succ_H x$. The Lemma implies $\exists z^* \in H_i$ such that $z^* \succ_H x$, contradicting that H is a maximal reduction. Therefore, $H_i \subseteq H_i'$ $\forall i$. Similarly, $H_i' \subseteq H_i$ $\forall i$, implying $H=H'$. ž

Theorem 1 says that IESDS is an order independent procedure for compact and continuous games. Note that this result covers finite games and their mixed extensions. The three examples of Section 2 show that Theorem 1 is tight with respect to closedness and boundedness of the players' strategy sets, as well as with respect to continuity of the payoff functions.

4. GKZ REDUCTIONS

In this section we connect to the work of GKZ. They define a notion of reduction which bounds the rate of elimination, unlike the textbook ($\to$)-reduction we have considered so far. We shall use the symbol $\Rightarrow$ for GKZ's reduction. Intuitively, the difference between a ($\Rightarrow$)-reduction and a ($\to$)-reduction is that the former, but not the latter, requires that for any strictly dominated strategy x which is eliminated there exists a strategy y which strictly dominates x and which is not eliminated.

GKZ Reduction. Consider subgames $H,H' \in S(G)$, such that $H_i' \subseteq H_i$ $\forall i \in I$. $H \Rightarrow H'$ if, for each $x \in H_i \backslash H_i'$, $\exists y \in H_i'$ such that $y \succ_H x$. We use the symbol $\Rightarrow^*$ as follows: $H \Rightarrow^* H'$ if there exists a (finite or infinite) sequence of subgames, $A^t \in S(G)$, $t=0,1,2...$, such that $A^0=H$, $A^t \Rightarrow A^{t+1}$ for all t, and $H_i'=\cap_t A_i^t$ $\forall i$. H is a maximal ($\Rightarrow$)-reduction of G if $G \Rightarrow^* H$ and $H \Rightarrow H'$ only for $H'=H$.

$H \Rightarrow H'$ and $H \Rightarrow^* H'$ imply, respectively, $H \to H'$ and $H \to^* H'$. The present definition of a maximal ($\Rightarrow$)-reduction is more general than that used by GKZ in that infinite sequences of subgames are allowed. GKZ consider only finite sequences. We now prove that, although

(⇒)-reductions are more restrictive than (→)-reductions, the two produce identical maximal reductions, and hence identical results for IESDS, in compact and continuous games.

Theorem 2. If G is compact and continuous, then $G \to^* H$ if and only if $G \Rightarrow^* H$.

Proof. $G \Rightarrow^* H$ immediately implies $G \to^* H$. Going the other way, suppose $G \to^* H$, and let $A^t \in S(G)$, t=0,1,2..., be the implied sequence of subgames. It is sufficient to show that $A' \Rightarrow A''$ for any two consecutive elements of this sequence. Consider such A' and A''. If $A'=A''$, then $A' \Rightarrow A''$ trivially. If not, then choose $i \in I$ and $x \in A_i' \setminus A_i''$. $A' \to A''$ implies $\exists y \in A_i'$ such that $y \succ_{A'} x$. The Lemma implies that $\exists z^* \in A_i'$ such that $z \nsucc_{A'} z^* \succ_{A'} x$ $\forall z \in A_i'$, and $A' \to A''$ then implies $z^* \in A_i''$. Hence, $x \in A_i' \setminus A_i''$, any $i \in I$, implies $\exists z^* \in A_i''$ such that $z^* \succ_{A'} x$. Therefore, $A' \Rightarrow A''$. ž

Hence, if there is an advantage to GKZ reductions, it must be based on games outside the compact and continuous class. We close this section with a few comments about such games and about GKZ reductions. In Example 1, the problematic (→)-reduction would not be permitted as a (⇒)-reduction, but IESDS based on (⇒)-reductions does not escape the problem of order dependence. To see this, consider the following infinite sequence of (⇒)-reduced strategy sets: (0, 1), [x, 1), $\{x\} \cup [1-(1-x)/2, 1)$, $\{x\} \cup [1-(1-x)/3, 1)$, $\{x\} \cup [1-(1-x)/4, 1)$, ... For any choice of $x \in (0,1)$, the intersection {x} is the strategy set corresponding to a maximal (⇒)-reduction. Order matters. In similar fashion one may readily show that, for IESDS based on (⇒)-reductions, order matters also in the games of Examples 2 and 3.

If one returns to GKZ's original definition, which requires maximal (⇒)-reductions to end in a finite number of steps, then it is not possible to get ambiguous maximal (⇒)-reductions of the games in the Examples 1-3, simply because these games have no maximal (⇒)-reduction in finite steps. The restriction to finite steps seems unnatural, however, because in some games infinite reduction sequences lead to maximal reductions that could not be obtained via a finite number of eliminations. The following well-known example illustrates the point. It requires an infinite sequence of reductions to find the unique maximal (→)-reduction (which by Theorem 2 is also the unique maximal (⇒)-reduction).

Example 4. (Cournot competition) I={1,2}, G_1=G_2=[0,1], u_i:$G_i \times G_j \rightarrow \mathbb{R}$ with *i*,*j*=1,2 and $i \neq j$ defined by u_i(x,y) = x(1-x-y). The following is a infinite sequence of (→)-reduced strategy sets starting with this game: [0, 1], [0, 1/2], [1/4, 1/2], [1/4, 3/8], [5/16, 3/8], ... Taking the intersection we get for each player {1/3} as the strategy set associated with the maximal (→)-reduction. The strategy profile (1/3, 1/3) is the game's unique Nash equilibrium. It is easy to show that while there are many alternative sequences of (→)-reductions, they all require an infinite number of elimination rounds.

Summarizing, complete reduction of some games (including compact and continuous games) requires an infinite sequence of deletions, but the GKZ bound on the rate of deletions does not solve the problem of order dependence if infinite sequences of deletions are permitted. Hence, we cannot find a compelling case for abandoning the standard IESDS definition in favor of one based on GKZ reductions.

5. CONCLUDING REMARKS

Many textbooks do not recommend iterated elimination of *weakly* dominated strategies (IEWDS) as a solution concept, and one important reason is that there are games where order matters for that procedure. Our examples show that the same criticism applies to IESDS. For IEWDS, the finding that order matters has prompted researchers to investigate for what class of games order independence holds, partly on the presumption that it is relatively innocuous to apply IEWDS in those games (see, for example, Marx & Swinkels, 1997). Adopting this view, our result of Section 3 provides consolation: Order does not matter for IESDS in compact and continuous games, so IESDS is a sensible procedure for this large class of games.

It is unclear what is the proper definition and role of iterated strict dominance in games that are not compact and continuous. Our Example 1 shows that there are games for which the concept is intrinsically unsound. The identification of general classes of non-compact/continuous games for which IESDS is an attractive procedure remains an open problem. For compact and continuous games, while we have answered the question of uniqueness, the existence of a maximal reduction remains an open question.

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