Numerical Representation of Incomplete and Nontransitive Preferences and Indifferences on a Countable Set

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Abstract: This note considers preference structures over countable sets which allow incomparable outcomes and nontransitive preferences and indifferences. Necessary and sufficient conditions are provided under which such a preference structure can be represented by means of a utility function and a threshold function.

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1 Introduction

To handle nontransitive indifference relations, Luce [5] introduced his well-known threshold model in which a decision maker prefers one outcome over another if and only if the increase in utility exceeds a certain nonnegative threshold. Formally, denoting the set of outcomes by X, strict preference by $\succ$ and indifference by $\sim$, each $x \in X$ is assigned a utility $u(x)$ and a threshold $t(x) \geq 0$ such that for all $x, y \in X$:

$$\begin{aligned} x \succ y \quad &\Leftrightarrow \quad u(x) > u(y) + t(y) \\ x \sim y \quad &\Leftrightarrow \quad \begin{cases} u(x) \leq u(y) + t(y), \\ u(y) \leq u(x) + t(x). \end{cases} \end{aligned}$$

In this note, also incomparability between outcomes and nontransitivity of strict preferences is allowed. Incomparabilities arise if the decision maker is not capable to compare outcomes, finds it unethical to do so, or thinks that outcomes are comparable, but lacks the information to do so. Fishburn [4] motivates nontransitive preferences. In this case the double implications above are replaced by single implications, so that we want for all $x, y \in X$:

$$\begin{aligned} x \succ y \quad &\Rightarrow \quad u(x) > u(y) + t(y) \\ x \sim y \quad &\Rightarrow \quad \begin{cases} u(x) \leq u(y) + t(y), \\ u(y) \leq u(x) + t(x) \end{cases} \end{aligned}$$

Our main theorem gives necessary and sufficient conditions for the existence of functions u and t as above on a broad class of preference structures over a countable set of alternatives. As a corollary, a representation theorem of interval orders (See [1] and [2]) is obtained.

2 Definitions

A *preference structure* on a set X is a pair $(\succ, \sim)$ of binary relations on X such that

- For each $x, y \in X$, at most one of the following is true: $x \succ y, y \succ x, x \sim y$;
- The relation $\sim$ is reflexive and symmetric.

The first condition implies that $\succ$ is anti-symmetric (if $x \succ y$, then not $y \succ x$). With $\succ$ interpreted as strict preference and $\sim$ as indifference, this leads to a very general type of preferences in which neither strict preference, nor indifference is assumed to be transitive and in which a decision maker may have pairs $x, y \in X$ which he cannot compare.

Consider a set X with preference structure $(\succ, \sim)$. A *path* in X is a finite sequence $(x_1, \ldots, x_m)$ of elements of X such that for each $k = 1, \ldots, m-1$, either $x_k \succ x_{k+1}$ or $x_k \sim x_{k+1}$. In the first case, we speak of a $\succ$-*connection* between x_k and x_{k+1}, in the second case of a $\sim$-*connection* between x_k and x_{k+1}. A *cycle* in X is a path $(x_1, \ldots, x_m)$ in X with at least two different elements of X and $x_1 = x_m$.

A path $(x_1, \ldots, x_m)$ in X has two consecutive $\sim$-connections if for some $k = 1, \ldots, m-2$: $x_k \sim x_{k+1}$ and $x_{k+1} \sim x_{k+2}$ or — in case the path is a cycle — if $x_1 \sim x_2$ and $x_{m-1} \sim x_m = x_1$.

Denote by $\rhd$ the composition of $\succ$ and $\sim$, i.e., for each $x, y \in X$:

$$x \rhd y \quad \Leftrightarrow \quad (\exists z \in X \ : \ x \succ z, \text{ and } z \sim y).$$

Since $\sim$ is reflexive, $x \succ y$ implies $x \rhd y$. The relation $\rhd$ is *acyclic* if its transitive closure is irreflexive, i.e., if there is no finite sequence $(x_1, \ldots, x_m)$ of elements of X such that $x_1 = x_m$ and for each $k = 1, \ldots, m-1$: $x_k \rhd x_{k+1}$.

A special case of a preference structure is an interval order (Fishburn, [2]). The preference structure $(\succ, \sim)$ is an *interval order* if for each $x, y \in X$

$$x \sim y \quad \Leftrightarrow \quad (\text{ not } x \succ y \text{ and not } y \succ x), \tag{1}$$

and for each $x, x', y, y' \in X$

$$(x \succ y \text{ and } x' \succ y') \quad \Rightarrow \quad (x \succ y' \text{ or } x' \succ y).$$

In interval orders, exactly one of the claims $x \succ y, y \succ x, x \sim y$ is true. Define the binary relation $\succeq$ on X by taking for each $x, y \in X$:

$$x \succeq y \quad \Leftrightarrow \quad \text{not } y \succ x.$$

Then it is easily seen that a preference structure satisfying (1) is an interval order if and only if for each $x, x', y, y' \in X$:

$$x \succ x' \succeq y' \succ y \quad \Rightarrow \quad x \succ y. \tag{2}$$

Hence, interval orders have transitive strict preference $\succ$. The preference structure of an interval order can be identified with the relation $\succ$, since the relations $\sim$ and $\succeq$ follow from $\succ$.

Lemma 2.1 *Let $\succ$ be an interval order on a set X. Then the relation $\rhd$ is acyclic.*

Proof. Suppose, to the contrary, that there exists a cycle $(x_1, y_1, x_2, y_2, \ldots, x_{m-1}, y_{m-1}, x_m)$ such that for each $k = 1, \ldots, m-1$: $x_k \succ y_k$ and $y_k \sim x_{k+1}$. Then $x_1 \succ y_1$ by definition. Moreover, $x_1 \succ y_1 \sim x_2 \succ y_2$, so (2) implies $x_1 \succ y_2$. Similarly, one shows that $x_1 \succ y_k$ for each $k = 1, \ldots, m-1$. In particular, $x_1 \succ y_{m-1}$. However, by definition of the cycle, $y_{m-1} \sim x_m = x_1$, so $x_1 \sim y_{m-1}$ by symmetry of $\sim$. But at most one of the two possibilities $x_1 \succ y_{m-1}$ and $x_1 \sim y_{m-1}$ is true, a contradiction. $\square$

Some additional conventions and matters of notation: $\subseteq$ denotes weak set inclusion, $\subset$ denotes proper set inclusion. Summation over an empty set yields zero. The infimum of the empty set equals infinity. $\mathbb{N}$ denotes the set of positive integers, $\mathbb{Q}$ the set of rationals, $\mathbb{R}$ the set of reals, $\mathbb{R}_+$ the set of nonnegative reals.

3 The representation theorem

This section contains the main theorem and an application of this theorem to obtain a well-known characterization of interval orders.

Theorem 3.1 *Let X be a countable set and $(\succ, \sim)$ a preference structure on X. The following claims are equivalent.*

(a) *There exist functions $u : X \to \mathbb{R}$ and $t : X \to \mathbb{R}_+$ such that for all $x, y \in X$:*

$$x \succ y \;\Rightarrow\; u(x) > u(y) + t(y)$$
$$x \sim y \;\Rightarrow\; \begin{cases} u(x) \leq u(y) + t(y), \\ u(y) \leq u(x) + t(x) \end{cases}$$

(b) *The relation $\rhd$ is acyclic;*

(c) *Every cycle in X contains at least two consecutive $\sim$-connections.*

Proof.
(a) $\Rightarrow$ (b): Assume (a) holds and suppose that $\rhd$ is cyclic. Take a sequence $(x_1, \ldots, x_m)$ of points in X such that $x_1 = x_m$ and for each $k = 1, \ldots, m-1 : x_k \rhd x_{k+1}$. Then for each such k there exists a $y_k \in X$ such that $x_k \succ y_k$ and $y_k \sim x_{k+1}$, which implies $u(x_k) > u(y_k) + t(y_k) \geq u(x_{k+1})$. Hence $u(x_1) > u(x_2) > \ldots > u(x_m) = u(x_1)$, a contradiction.

(b) $\Rightarrow$ (c): Suppose $(x_1, \ldots, x_m)$ is a cycle in X without two consecutive $\sim$-connections. W.l.o.g. $x_1 \succ x_2$. Let $(y_1, \ldots, y_n)$ with $n \leq m$ be the sequence of points in X obtained by removing from $(x_1, \ldots, x_m)$ all those points x_k $(k = 1, \ldots, m-1)$ satisfying $x_k \sim x_{k+1}$, i.e., all those points that are indifferent to the next point in the cycle. Notice that by construction $y_1 = x_1$, $y_n = x_m = x_1$, and for each $k = 1, \ldots, n-1$ there exists an $l \in \{1, \ldots, m-1\}$ such that

- either $y_k = x_l$ and $y_{k+1} = x_{l+1}$, in which case $y_k \succ y_{k+1}$, which implies $y_k \rhd y_{k+1}$,
- or $y_k = x_l$ and $y_{k+1} = x_{l+2}$, in which case $y_k \succ x_{l+1}$ and $x_{l+1} \sim y_{k+1}$, which also implies $y_k \rhd y_{k+1}$.

But then the sequence $(y_1, \ldots, y_n)$ indicates that $\rhd$ is cyclic.

(c) $\Rightarrow$ (a): Assume (c) holds. Since X is countable, write $X = \{x_k \mid k \in \mathbb{N}\}$. Call a path from x to y a good path if it does not contain two consecutive $\sim$-connections. Define for each $x \equiv x_k \in X$:

$$\begin{aligned}
S(x) &:= \{n \in \mathbb{N} \mid \text{there exists a good path from } x \text{ to } x_n \text{ starting with a } \succ\text{-connection}\}, \\
T(x) &:= \{n \in \mathbb{N} \mid \text{there exists a good path from } x \text{ to } x_n\}, \\
u(x) &:= \textstyle\sum_{n \in S(x)} 2^{-n}, \\
v(x) &:= \textstyle\sum_{n \in T(x)} 2^{-n}, \\
t(x) &:= 2^{-k-1} + v(x) - u(x).
\end{aligned}$$

We proceed to prove that u and t defined above give the desired representation.

- Clearly $S(x) \subseteq T(x)$, so $v \geq u$ and $t > 0$.
- Let $x, x_k \in X, x \succ x_k$. Then $T(x_k) \subseteq S(x)$. Moreover, $k \in S(x)$, but $k \notin T(x_k)$, since by assumption every cycle in X has two consecutive $\sim$-connections. Hence $T(x_k) \subset S(x)$ and $k \in S(x) \setminus T(x_k)$. So $u(x) = v(x_k) + \sum_{n \in S(x) \setminus T(x_k)} 2^{-n} \geq v(x_k) + 2^{-k} > v(x_k) + 2^{-k-1} = u(x_k) + t(x_k)$.
- Let $x, y \in X, x \sim y$. Then $S(y) \subseteq T(x)$. Hence $u(x) + t(x) > v(x) \geq u(y)$ and similarly $u(y) + t(y) \geq u(x)$.

This completes the proof. □

Remark 3.2 Luce [5] considers nonnegative threshold functions, Fishburn [2] and Bridges [1] consider positive threshold functions. Our statement of (c) involves nonnegative threshold functions $t : X \to \mathbb{R}_+$. However, in the proof that (c) implies (a) we actually construct a positive function. Clearly, the proof that (a) implies (b) — and hence the theorem — also holds if t were required to be positive rather than nonnnegative. The theorem was formulated with nonnegative threshold functions for intuitive reasons: there seems to be no reason to require that sufficiently perceptive decision makers need to have a positive threshold above which they can perceive changes in utility.

An immediate corollary of this theorem is a well-known representation theorem of interval orders. See Fishburn [2, Theorem 4] and Bridges [1, Theorem 2].

Theorem 3.3 *Let X be a countable set and $\succ$ a binary relation on X. The following claims are equivalent.*

(a) *The relation $\succ$ is an interval order;*

(b) *There exist functions $u, v : X \to \mathbb{R}, v \geq u$, such that for each $x, y \in$, $x \succ y$ if and only if $u(x) > v(y)$;*

(c) *There exist functions $u, t : X \to \mathbb{R}, t > 0$, such that for each $x, y \in$, $x \succ y$ if and only if $u(x) > u(y) + t(y)$.*

Proof. Obviously (c) $\Rightarrow$ (b) $\Rightarrow$ (a). That (a) $\Rightarrow$ (c) follows from Lemma 2.1, Remark 3.2, and Theorem 3.1. That $u(x) > u(y) + t(y)$ implies $x \succ y$ is clear: $y \succ x$ implies $u(y) + t(y) > u(y) > u(x) + t(x) > u(x)$ and $x \sim y$ implies $u(y) + t(y) \geq u(x)$. In interval orders exactly one of the claims $x \succ y, y \succ x, x \sim y$ holds, so one must have that $x \succ y$. □

4 Uncountable Sets

In Theorem 3.1, the proof that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) holds for arbitrary, not necessarily countable, sets X. Moreover, it is easy to see that also (c) implies (b) for arbitrary sets. However, acyclicity of $\rhd$ does not imply the existence of the desired functions u, t if the set X is uncountable. This is not surprising: it is usually necessary to require additional assumptions to guarantee the existence of preference representing functions on uncountable sets. The purpose of this section is to indicate that such assumptions are not straightforward. Fishburn [3] discusses representations of interval orders on uncountable sets.

The existence of functions u, t as in part (a) of Theorem 3.1 implies that

$$\forall x, y \in X : \quad x \rhd y \quad \Rightarrow \quad u(x) > u(y). \tag{3}$$

Hence, the existence of a function $u : X \to \mathbb{R}$ satisfying (3) is a necessary condition. However, it is not sufficient. Suppose such a function u exists. Without loss of generality,

u is bounded (take $x \mapsto \arctan(u(x))$ if necessary). The function $t : X \to \mathbb{R}_+$ then has to satisfy for each $x, y \in X$, if $y \succ x$, then $u(y) - u(x) > t(x)$ and if $y \sim x$, then $u(y) - u(x) \leq t(x)$.

Define $\mathcal{S}(x) := \sup\{u(y) - u(x) \mid y \sim x\}$ and $\mathcal{I}(x) := \inf\{u(y) - u(x) \mid y \succ x\}$. Let $y \succ x, z \sim x$. Then $u(y) > u(z)$, so $\mathcal{S}(x) \leq \mathcal{I}(x)$. Notice also that $\mathcal{S}(x) \geq u(x) - u(x) = 0$. So if $\mathcal{S}(x) < \mathcal{I}(x)$, one can take $t(x) \in [\mathcal{S}(x), \mathcal{I}(x))$. However, if $\mathcal{S}(x) = \mathcal{I}(x)$, then the only candidate for $t(x)$ equals $\mathcal{S}(x)$. But to make sure that $u(y) - u(x) > t(x)$ for all y with $y \succ x$, we need the additional property that the infimum $\mathcal{I}(x)$ is not achieved.

The next example shows that in some cases there exists a function $u : X \to \mathbb{R}$ satisfying (3), but in which the last property is not satisfied.

Example 4.1 Take $X = \mathbb{R}$ and define for each $x, y \in \mathbb{R}$:

$$x \succ y \quad \Leftrightarrow \quad x \geq y + 1,$$

$$x \sim y \quad \Leftrightarrow \quad |x - y| < 1.$$

Then

$$x \rhd y \Leftrightarrow \exists z \in \mathbb{R} : (x \geq z + 1, |z - y| < 1) \Leftrightarrow \exists z \in \mathbb{R} : x \geq z + 1 > y > z - 1 \Leftrightarrow x > y.$$

So $\rhd$ is acyclic and the set of functions preserving the order $\rhd$ is the set of strictly increasing functions $u : \mathbb{R} \to \mathbb{R}$. For every strictly increasing function u and every $x \in X$ we have that $\mathcal{I}(x) = \inf_{y \geq x+1} u(y) - u(x) = u(x+1) - u(x)$. Hence the infimum is achieved. This means that a function t exists if and only if there is an increasing function u such that

$$\forall x \in \mathbb{R} : \quad \mathcal{S}(x) < u(x+1) - u(x).$$

Suppose such a u does exist. We will derive a contradiction by constructing an injective function f from the uncountable set $\mathbb{R} \setminus \mathbb{Q}$ to the countable set $\mathbb{Q}$. For each $x \in \mathbb{R} \setminus \mathbb{Q}$, take $f(x) \in (\mathcal{S}(x), u(x+1) - u(x)) \cap \mathbb{Q}$. In order to show that f is injective, let $x, y \in \mathbb{R} \setminus \mathbb{Q}, x < y$. Then $f(x) < u(x+1) - u(x) < \sup\{u(z) - u(y) \mid z < y + 1\} = \mathcal{S}(y) < f(y)$.

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