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A Further Exploration of Some Computational Issues in Equilibrium Business Cycle Theory

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Abstract

This paper revisits some of the issues involved in the comparison of alternative computational procedures within the context of a dynamic stochastic general equilibrium model. The framework in question is a more general one, in which a "standard" or relatively simple model is nested as a special case. Results of numerical experiments suggest that different computational methods may be used interchangeably in the case of the standard model, but not in the case of the more general model. Varying a preference parameter allows us to compare what happens to solutions using alternative procedures as one moves away from the special case to the more general framework. On the basis of the numerical experiments conducted, we find that not only do differences in solutions become larger, but answers to several economic issues of interest can yield qualitatively different answers depending on the solution method used. Examples of such issues include how second moment features change as one varies the parameters of a model, and the relative contribution of different types of stochastic shocks to fluctuations in variables.

1. Introduction

There are several methods of approximating the numerical solution to dynamic stochastic general equilibrium models. Depending on the type of approximation used these methods yield different types of approximation error. Some methods, in principle, allow for reduction in this error by means of refining the solution. However, the process of refinement is typically associated with a decline in computational convenience. The literature on computational methods has therefore often focused on the extent to which the trade off between computational convenience and accuracy is significant.

The focus of this paper is on one of the extant approaches to assessing this significance. This approach involves comparing the solutions to a model using two different computational methods; one which allows for improvement in accuracy via refinement and another which is associated with computational simplicity but not amenable to refinement. The works of Danathine, Donaldson and Mehra (1989) and Christiano (1990) suggest that the level of accuracy is not compromised by using computationally inexpensive methods. For example, Danathine, Donaldson and Mehra (1989) compare the solution obtained by using a quadratic approximation of the return function around the steady state, to the solution obtained by using standard value iteration in which continuous variables are discretised by using a grid. They find the solutions to be very similar, and that the process of refining the solution by increasing the fineness of the grid does not yield significant gains in accuracy. However, as the authors suggest, their results are subject to the caveat that the model in question is a fairly standard one and a similar comparison using a more complicated model is likely to yield a different answer. Other literature involving comparisons of this kind, such as Taylor and Uhlig (1990) also concludes that it is advisable for researchers to perform a "check" on their solutions using alternative computational procedures.

One of the outcomes of the exercise conducted in this paper is essentially an "accuracy check" of the type conducted in Danathine, Donaldson and Mehra(1989), applied to a more complex dynamic stochastic general equilibrium framework. However, the framework we consider allows us to address another interesting issue, which is the focus of this paper. Here, we are able to examine what happens to the *differences* in solutions using alternative procedures as one moves away from a simple to a more general and complex framework. To be specific, the framework we consider nests a relatively "standard" framework as a special case, and varying a preference parameter allows us to look at shifts in the difference alternative computational procedures as we move away from the special case to the more general model.

The procedures considered in this paper are the linear approximation method of King, Plosser, and Rebelo (1988), and the method of parameterized expectations due to Den Haan and Marcet (1990). The latter method involves a polynomial approximation, and allows for refinement of the solution by increasing the degree of the polynomial. The model these methods are applied to is the dynamic stochastic general equilibrium model studied in Lahiri (2002), which includes monetary and technological shocks and a *variable* or *endogenous* rate of time preference. Time preference is endogenous in the sense that the representative agent's discount factor depends on current utility. As mentioned above, a standard fixed time preference model is nested as a special case. To elaborate, when the value of a parameter τ , to be described later, is set equal to zero, the endogenous time preference model coincides with the fixed time preference case.

As expected, the results here confirm that the interchangeability of the two methods depends on the model. Specifically, the two solution methods yield almost identical results when applied to the nested constant time preference model but significant differences emerge as one increases the variability in the rate of time preference. Also, an interesting feature of these experiments is the way in which results using the alternative procedures change as one moves away from the fixed time preference model to the endogenous time preference case, which involves increasing the value of the parameter τ in the interval (0,1). For example, if we are interested in how the second moment features of economic variables changes as we increase τ , the two procedures can yield very different answers. The reason underlying the differences is related to how the two procedures estimate the relative significance of the monetary and technological shocks, and their income and substitution effects.

In particular, the feature described above is of interest because it provides an example of a case in which computational issues become important in a *qualitative* sense; answers to how economic variables respond to the model's parameters or stochastic shocks would depend on the solution method used, especially in situations where the model had significant non-linearities in preferences or technology. In this case the presence of an endogenous discount factor changes the magnitude of income effects of technology and monetary shocks. For a range of values of τ , particularly in the case of technology shocks, the income effects cancel out the substitution effects, and this tends to weaken the relative significance of technology shocks[1]. However, the range of values for which this happens is different for the two procedures considered here. This is because increasing τ seems to increase the variability in the rate of time preference at a faster rate in the case of the parameterized expectations approach.

In what follows, Section 2 outlines the model, which is the same as that of Lahiri (2002). Section 3 outlines the solution methods, which are described in greater detail in King, Plosser, and Rebelo (1988) and Den Haan and Marcet (1990). Section 4 compares the results of both methods in the fixed and variable time preference cases. Section 5 concludes.

2. The Model

The economy described below is the monetary business cycle model used in Lahiri(2002), which is a variant of the Cooley and Hansen (1989) cash-in-advance framework, with endogenous time preference introduced *via* a discount factor that depends on utility. Preferences of the continuum of identical infinitely lived households are formulated in accordance with Epstein's (1983) concept of *stationary cardinal utility*. The representative household in this economy therefore desires to maximize expected lifetime utility given by

$$E\left\{\sum_{t=0}^{\infty}\left[\prod_{\tau=0}^{t-1}\beta\big(u(c_\tau,1-h_\tau)\big)\right]u(c_t,1-h_t)\right\}, \tag{1}$$

where $\beta\big(u(c_t,1-h_t)\big)$ must be of the form $e^{-\phi(u(c_t,1-h_t))}$ and $u(c_t,1-h_t)$ represents the households period-t momentary utility, defined over consumption, c_t, and leisure $1-h_t$. The function u must be negative, strictly increasing with $\ln(-u)$ convex in the composite consumption-leisure good. It is also required that ϕ is positive, increasing, strictly concave and that $u'e^{\phi(u)}$ is nonincreasing[2]. The endogenous discount factor attached to $u(c_t,1-h_t)$, (i.e. the term $\prod_{\tau=0}^{t-1}e^{-\phi(u(c_\tau,1-h\tau))}$), reflects the endogenous impatience that characterizes this framework: an increase in current period utility causes the household to discount future periods more heavily.

In particular, the functional forms for the period utility and discount functions used here are described as follows:

$$u(c_t,1-h_t)=\frac{c_t^{1-\sigma}-1}{1-\sigma}-Bh_t \tag{2}$$

$$\beta\big(u(c_t,1-h_t)\big)=e^{-(\eta+\tau u)},\ 0<\tau<1,\ \eta>-u, \tag{3}$$

1 This was one of the key motivations underlying the model used in Lahiri (2002); an endogenous rate of time preference allowed for a larger contribution of monetary shocks in business cycles, in addition to preserving the models ability to match features of the data.

2 These conditions ensure the existence, stability, and uniqueness of a steady state distribution for the state variables. Other implications of these conditions, as shown in Epstein (1983), is that the composite consumption-leisure good, and that deviations from the fixed time preference set up are not too great. Other equilibrium models that use such preferences are the studies of Gomme and Greenwood (1995), Dolmas and Wynne (1994), and Mendoza (1991).

where work effort enters the period utility function in a manner consistent with the "indivisible labor" assumption of Hansen (1985), and nests the Cooley and Hansen (1989) log utility specification as the case in which $\sigma = 1$.

In period *t* households possess nominal money balances m_{t-1}, carried over from the previous period, which are, in addition, augmented by a lump-sum transfer from the government. This transfer is equal to the increase in money supply, where the aggregate money supply, M_t, is determined according the following rule[3]:

$$M_t = g_t M_{t-1} \quad . \tag{4}$$

Here, the growth rate of money, g_t, evolves according to:

$$\log(g_{t+1}) = \alpha \log(g_t) + \xi_{t+1}. \tag{5}$$

ξ_{t+1} is *i.i.d* normal with mean $(1-\alpha)\log(\bar{g})$ and variance σ_ξ^2, and $\log(\bar{g})$ represents the unconditional mean of $\log(g_t)$.

Thus the total amount of money balances held by a household, at the beginning of period *t,* including the monetary transfer from the government, is the amount

$$m_{t-1} + (g_t - 1)M_{t-1}. \tag{6}$$

There is a cash in advance constraint on the purchase of the non-storable consumption good, which ensures that money will be valued in equilibrium. Expenditure on the consumption good, therefore cannot exceed the total money balances available to the household, i.e.,

$$p_t c_t \le m_{t-1} + (g_t - 1)M_{t-1}. \tag{7}$$

The representative firm in the economy hires labor and capital from the households to produce a composite consumption-investment good. There is a standard neoclassical aggregate production function of the Cobb-Douglas form, which combines capital (K_t) and labor input (H_t) to yield output (Y_t):

$$Y_t = e^{z_t} K_t^{\theta} H_t^{1-\theta}, \tag{8}$$

where θ is the factor income share of capital and e^{z_t} represents a shock to technology in period t. The random variable z_t follows the process:

$$z_{t+1} = \gamma z_t + \varepsilon_{t+1}, \tag{9}$$

where ε_{t+1} is an *i.i.d.* random variable with mean zero and variance σ_ε^2.

The competitive firm maximizes profit, which is given by $Y_t - w_t H_t - r_t K_t$. The variables w_t and r_t represent the wage and rental rates paid for the use of labor and capital services of the

[3] Capital letters denote aggregate economy wide per capita variables which an individual household regards as being outside its sphere of influence, while lower case letters denote variables specific to the household.

households. The first order conditions for the firm's profit maximization problem imply that w_t and r_t are given by:

$$w_t = (1-\theta)e^{z_t} K_t^{\theta} H_t^{-\theta}; \qquad (10)$$
$$r_t = \theta e^{z_t} K_t^{\theta-1} H_t^{1-\theta}. \qquad (11)$$

In every period *t*, household expenditures consist of consumption (c_t), investment (i_t), and the amount of money balances $\left(\frac{m_t}{p_t}\right)$ that are to be carried over to the next period. These expenditures must not exceed total household income, which is the sum of income earned from labor and capital services, money balances carried over from the previous period, and the lump-sum monetary transfer from the government. Households therefore maximize expected lifetime utility subject to (7) and a sequence of budget constraints of the following form:

$$c_t + x_t + \frac{m_t}{p_t} \le w(z_t, K_t, H_t)h_t + r(z_t, K_t, H_t)k_t + \frac{m_{t-1} + (g_t - 1)M_{t-1}}{p_t}, \qquad (12)$$

where household investment expenditure in period-*t* is given by

$$x_t = k_{t+1} - (1-\delta)k_t. \qquad (13)$$

In equation (13) k_t is the household's capital stock in period-*t* and δ is the rate at which the capital stock depreciates.

For a value of *g* greater than one, both M_t and p_t will grow without bound. In order to make the household's problem stationary, some of the variables need to be transformed. To that end, we define $\hat{m}_t = \dfrac{m_t}{M_t}$ and $\hat{p}_t = \dfrac{p_t}{M_t}$. Furthermore, stationary cardinal utility permits a recursive formulation of the household's problem described above. The problem, then, can be restated as:

$$V(z_t, g_t, \hat{m}_{t-1}, k_t, K_t) = \max_{c_t, h_t, \hat{m}_t, k_{t+1}} \left\{ u(c_t, 1-h_t) + \beta(u(c_t, 1-h_t)) \underset{t}{E}[V(z_{t+1}, g_{t+1}, \hat{m}_t, k_{t+1}, K_{t+1}) / z_t, g_t, \hat{m}_{t-1} k_t, K_t] \right\}$$

(14)

subject to

$$c_t + k_{t+1} - (1-\delta)k_t + \frac{\hat{m}_t}{\hat{p}_t} = w(z_t, K_t, H_t)h_t + r(z_t, K_t, H_t)k_t + \frac{g_t - 1 + \hat{m}_{t-1}}{g_t \hat{p}_t}, \qquad (15)$$

and

$$c_t = \frac{g_t - 1 + \hat{m}_{t-1}}{g_t \hat{p}_t}. \qquad (16)$$

In addition, the representative household's decisions must be consistent with the laws of motion for the aggregate state variables, given by

$$K_{t+1} = (1-\delta)K_t + X_t, \tag{17}$$
$$z_{t+1} = \gamma z_t + \varepsilon_{t+1}, \tag{18}$$
$$\log(g_{t+1}) = \alpha \log(g_t) + \xi_{t+1}, \tag{19}$$

as well as the economy-wide aggregate decision rules perceived by the households:

$$H_t = H(z_t, g_t, K_t),$$
$$X_t = X(z_t, g_t, K_t),$$

and

$$\hat{P}_t = \hat{P}_t(z_t, g_t, K_t). \tag{20}$$

In equilibrium, aggregate per capita quantities turn out to be equal to the choices of the representative household. In particular, it must be the case that $h_t = H_t$, $k_t = K_t$, $x_t = X_t$, and $\hat{m}_{t-1} = \hat{m}_t = 1$. Since the cash in advance constraint is assumed to be binding in equilibrium, we also have $c_t = \dfrac{1}{\hat{P}_t}$.

3. Computational Procedures

A. The Linear Approximation Method

Due to the presence of money, equilibrium allocations in this economy are not necessarily Pareto optimal. Consequently, the competitive equilibrium for this economy cannot be computed indirectly by solving a social planner's problem. The approach followed here is essentially that of King, Plosser and Rebelo (1988b), which involves a linear approximation around steady state of the optimization conditions for the household's problem. Henceforth, we will refer to this approximation method as the KPR approach. Before forming this approximation, however, certain market clearing conditions for the economy need to be imposed. Another modification to the standard procedure involved here is that, to begin with, the household's problem is set up as a dynamic programming problem, as described in Section 2. The resulting optimization conditions are then transformed into a system that is suitable for linearization, in a manner suggested by Dolmas and Wynne (1994).

The first order conditions for the dynamic programming problem are then given by the following set of equations:

$$u_1(c_t, 1-h_t)\{1 + \beta'(u(c_t, 1-h_t)) \underset{t}{E} V(t+1)\} - \lambda_{1t} - \lambda_{2t} = 0, \tag{21}$$
$$-u_2(c_t, 1-h_t)\{1 + \beta'(u(c_t, 1-h_t)) \underset{t}{E} V(t+1)\} + \lambda_{1t} w(t) = 0, \tag{22}$$
$$\beta(u(c_t, 1-h_t)) \underset{t}{E} \frac{(\lambda_{1t+1} + \lambda_{2t+1})}{\hat{p}_{t+1} g_{t+1}} - \frac{\lambda_{1t}}{\hat{p}_t} = 0, \tag{23}$$

$$\beta\left(u(c_t,1-h_t)\right)\underset{t}{E}\left[r(t+1)+1-\delta\right]\lambda_{1t+1}-\lambda_{1t}=0, \tag{24}$$

$$c_t+k_{t+1}-(1-\delta)k_t+\frac{\hat{m}_t}{\hat{p}_t}=w(z_t,K_t,H_t)h_t+r(z_t,K_t,H_t)k_t+\frac{g_t-1+\hat{m}_{t-1}}{g_t\hat{p}_t}, \tag{25}$$

$$c_t=\frac{g_t-1+\hat{m}_{t-1}}{g_t\hat{p}_t}, \tag{26}$$

where λ_{1t} and λ_{2t} are the Lagrange multipliers associated with the period-*t* budget constraint and cash-in-advance constraint respectively.

Again, noting that in equilibrium aggregate per capita quantities coincide with the choices of the representative household, we make the substitutions $h_t=H_t$, $k_t=K_t$, and $\hat{m}_{t-1}=\hat{m}_t=1$. The first order conditions above can then be transformed into the economy's equilibrium characterization. In addition, define $\mu_t=\underset{t}{E}V(t+1)$. Then the system of equations (21)-(26) is transformed into the following:

$$u_1(C_t,1-H_t)\{1+\beta'\left(u(C_t,1-H_t)\right)\mu_t\}-\lambda_{1t}-\lambda_{2t}=0, \tag{27}$$

$$-u_2(C_t,1-H_t)\{1+\beta'\left(u(C_t,1-H_t)\right)\mu_t\}+\lambda_{1t}w(t)=0, \tag{28}$$

$$\beta\left(u(C_t,1-H_t)\right)E\frac{\left(\lambda_{1t+1}+\lambda_{2t+1}\right)C_{t+1}}{g_{t+1}}-\lambda_{1t}C_t=0, \tag{29}$$

$$\beta\left(u(C_t,1-H_t)\right)E\left[r(t+1)+1-\delta\right]\lambda_{1t+1}-\lambda_{1t}=0, \tag{30}$$

$$C_t+K_{t+1}-(1-\delta)K_t=w(z_t,K_t,H_t)H_t+r(z_t,K_t,H_t)K_t. \tag{31}$$

Furthermore, the Bellman equation implies the following law of motion for the variable μ_t:

$$\mu_t=E\left[u(C_{t+1},1-H_{t+1})+\beta\left(u(C_{t+1},1-H_{t+1})\right)\mu_{t+1}\right] \tag{32}$$

Transversality conditions are given by:

$$\lim_{t\to\infty}\prod_{s=0}^{t-1}\beta\left(u(c_s,1-h_s)\right)k_{t+1}\lambda_{1t}=0,$$

$$\lim_{t\to\infty}\prod_{s=0}^{t-1}\beta\left(u(c_s,1-h_s)\right)\hat{m}_t\lambda_{2t}=\lim_{t\to\infty}\prod_{s=0}^{t-1}\beta\left(u(c_s,1-h_s)\right)\lambda_{2t}=0,$$

and, $$\lim_{t\to\infty}\prod_{s=0}^{t-1}\beta\left(u(c_s,1-h_s)\right)\mu_t=0.$$

Equations (27)-(32) represent a system that can be solved using the technique described in King, Plosser, and Rebelo (1988a,b) and their Technical Appendix. The first step is to obtain a linear approximation of the above system around the steady state. Each of the above conditions is expressed in terms of percentage deviations of variables from their steady state values. Let $\hat{x}$ denote the percentage deviation of variable *x* from its steady state value. Then (27) and (28) can be written as:

$$\left[\xi_{cc}+\Phi\upsilon_{uu}\xi_{c}\right]\hat{C}_t-\left[\left\{\xi_{cl}+\Phi\upsilon_{uu}\xi_{l}\right\}\frac{H}{1-H}\right]\hat{H}_t=\frac{\lambda_1}{\lambda_1+\lambda_2}\hat{\lambda}_{1t}+\frac{\lambda_2}{\lambda_1+\lambda_2}\hat{\lambda}_{2t}-\Phi\hat{\mu}_t, \tag{27'}$$

$$\left[\xi_{lc}+\Phi\upsilon_{uu}\xi_{c}\right]\hat{C}_t-\left[\left\{\xi_{ll}+\Phi\upsilon_{uu}\xi_{l}\right\}\frac{H}{1-H}-\theta\right]\hat{H}_t=\hat{\lambda}_{1t}+\theta\hat{K}_t+\hat{z}_t, \tag{28'}$$

where steady state values of variables C_t, H_t, etc. are denoted by C, H, respectively, and,

$$\xi_{cc}=\frac{u_{11}(C,1-H)}{u_1(C,1-H)}*C\ ; \qquad \xi_{cl}=\frac{u_{12}(C,1-H)}{u_1(C,1-H)}*(1-H)\,;$$

$$\xi_{lc}=\frac{u_{21}(C,1-H)}{u_2(C,1-H)}*C\ ; \qquad \xi_{ll}=\frac{u_{22}(C,1-H)}{u_2(C,1-H)}*(1-H),$$

where ξ_{ij} is interpreted as the elasticity of marginal utility of *i* with respect to *j* ; $i,j=C,1-H$. We also have,

$$\upsilon_{uu}=\frac{\beta''\left(u(C,1-H)\right)}{\beta'\left(u(C,1-H)\right)}*u(C,1-H),$$

and, $$\Phi=\frac{\beta'(u(C,1-H))\mu}{1+\beta'(u(C,1-H))\mu}$$

We can similarly define

$$\upsilon_{u}=\frac{\beta'(u(C,1-H))}{\beta\left(u(C,1-H)\right)}*u(C,1-H),$$

and $$\xi_c=\frac{u_1(C,1-H)}{u\ (C,1-H)}*C\ , \qquad \xi_l=\frac{u_2(C,1-H)}{u\ (C,1-H)}*(1-H),$$

so that linearising (29)-(32) yields:

$$\frac{\lambda_1}{\lambda_1+\lambda_2}\hat{\lambda}_{1t+1}+\frac{\lambda_1}{\lambda_1+\lambda_2}\hat{\lambda}_{2t+1}-\hat{g}_{t+1}+\hat{C}_{t+1}=\hat{C}_t\left[1-\upsilon_u\xi_c\right]+\upsilon_u\xi_l\frac{H}{1-H}\hat{H}_t+\hat{\lambda}_{1t}, \tag{29'}$$

$$\hat{\lambda}_{1t+1}+\frac{r}{r+1-\delta}\left[\hat{z}_{t+1}+(\theta-1)\hat{K}_{t+1}+(1-\theta)\hat{H}_{t+1}\right]=\upsilon_u\xi_l\frac{H}{1-H}\hat{H}_t+\hat{\lambda}_{1t}-\upsilon_u\xi_c\hat{C}_t, \tag{30'}$$

$$\frac{K}{Y}\hat{K}_{t+1}=\hat{z}_t+\theta\hat{K}_t+(1-\delta)\frac{K}{Y}\hat{K}_t+(1-\theta)\hat{H}_t-\frac{C}{Y}\hat{C}_t, \tag{31'}$$

and,

$$\hat{\mu}_t-\left[1-\beta(u(C,1-H))+\beta(u(C,1-H))\upsilon_u\right]\xi_c\hat{C}_{t+1}\hat{H}_{t+1}$$
$$+\left[1-\beta(u(C,1-H))+\beta(u(C,1-H))\upsilon_u\right]\xi_l\frac{H}{1-H}\hat{H}_{t+1}=\beta(u(C,1-H))\hat{\mu}_{t+1}. \tag{32'}$$

The linearised first order conditions can then be combined to yield the *fundamental difference equation*:

$$\begin{pmatrix} \hat{K}_{t+1} \\ \hat{\lambda}_{1t+1} \\ \hat{\lambda}_{2t+1} \\ \hat{\mu}_{t+1} \end{pmatrix} = W \begin{pmatrix} \hat{K}_{t} \\ \hat{\lambda}_{1t} \\ \hat{\lambda}_{2t} \\ \hat{\mu}_{t} \end{pmatrix} + R \begin{pmatrix} \hat{z}_{t+1} \\ \hat{g}_{t+1} \end{pmatrix} + Q \begin{pmatrix} \hat{z}_{t} \\ \hat{g}_{t} \end{pmatrix}, \tag{33}$$

where W is a 4×4 matrix, and R and Q are 4×2 matrices. The procedure to compute the solution to this difference equation is the same as that outlined in King, Plosser and Rebelo (1988a).

B. The Method of Parameterized Expectations

The approach followed here is essentially a variation of Den Haan and Marcet (1990), which involves a polynomial approximation of the expectations part of the stochastic Euler equations of the household's problem. We will refer to this method as the PE approach. To see how the method is applied, consider the following version of the equations (27)-(31):

$$C_t^{-\sigma}\left\{1 - \underset{t}{E}\,\tau\,\beta\big(u(C_t, 1-H_t)\big)V(t+1)\right\} - \lambda_{1t} - \lambda_{2t} = 0 \tag{34}$$

$$-B\left\{1 - \underset{t}{E}\,\tau\,\beta\big(u(C_t, 1-H_t)\big)V(t+1)\right\} + (1-\theta)\lambda_{1t} e^{Z_t} K_t^{\theta} H_t^{-\theta} \tag{35}$$

$$-\lambda_{1t} C_t + \underset{t}{E}\,\beta\big(u(C_t, 1-H_t)\big)\frac{(\lambda_{1t+1} + \lambda_{2t+1})C_{t+1}}{g_{t+1}} = 0 \tag{36}$$

$$-\lambda_{1t} + \underset{t}{E}\,\beta\big(u(C_t, 1-H_t)\big)\left\{\theta e^{Z_{t+1}} K_t^{\theta-1} H_t^{1-\theta} + 1 - \delta\right\}\lambda_{1t+1} = 0 \tag{37}$$

$$C_t + K_{t+1} - (1-\delta)K_t = e^{Z_t} K_t^{\theta} H_t^{1-\theta} \tag{38}$$

Here λ_{1t} and λ_{2t} are the Lagrangian multipliers associated with the household budget and cash-in-advance constraints respectively. To solve the model, we need to form an approximation for the terms involving expectations in equations (34), (35), (36) and (37). To that end, we let the term $\underset{t}{E}\,\tau\,\beta\big(u(C_t, 1-H_t)\big)V(t+1)$ be approximated by $\psi(z, g, K; \mu)$, and the terms $\underset{t}{E}\,\beta\big(u(C_t, 1-H_t)\big)\frac{(\lambda_{1t+1} + \lambda_{2t+1})C_{t+1}}{g_{t+1}}$ and $\underset{t}{E}\,\beta\big(u(C_t, 1-H_t)\big)\left\{\theta e^{Z_{t+1}} K_t^{\theta-1} H_t^{1-\theta} + 1 - \delta\right\}\lambda_{1t+1}$ by $\xi(z, g, K; \omega)$ and $\nu(z, g, K; \pi)$ respectively. The functions $\psi, \xi,$ and ν are polynomials in *z*, *g*, and *K*, while $\mu, \omega,$ and π represent the respective vectors of polynomial coefficients. As is conventional, we choose the degree of the polynomial by examining how the results change by increasing the degree of the polynomial. That is, if the solution does not change much between an $n\,th$ and $(n+1)\,th$ degree polynomial, then the $n\,th$ degree polynomial is considered a good approximation. It turns out that for the model in question a second degree polynomial is a good one. The procedure for forming the approximation is as follows. Starting with an initial guess for the vectors $\mu, \omega,$ and π, say $\mu_0, \omega_0,$ and π_0, we can solve the first order conditions above to construct a time series for consumption, hours and the capital stock, for a given series of monetary and technology shocks. Specifically, we can solve for C_t by dividing equation (36) by equation (37). Given K_0 and z_0, we can use (35) and (37) to solve for H_t, and then use (38) to solve for K_{t+1}. Once a series has been constructed, it can be used is to run three nonlinear least squares regressions to estimate a new set of coefficients $\mu_1, \omega_1,$ and π_1. For example we get an estimate for μ_1 by running a nonlinear regression of the series $\tau\beta(u)V(t+1)$ on the function ψ. (For details on the procedure used for the nonlinear regressions, see Den Haan and Marcet (1990), and Pindyck and Rubinfeld (1987)). The next step is to construct a new series using a linear combination of the coefficients μ_0 and μ_1. The new series is used to run another regression to compute μ_2, and so on, until estimates from successive iterations have converged. However, in this case, we need convergence in all of the three polynomial coefficients $\mu, \omega,$ and π.

4. Results

In this section we compare the second moment features of the model using the two procedures described in Section 3. In order to shed light on the differences in the second moment features using the two methods, we also look at the relative contribution of monetary shocks in the two cases. Values of the preference and technology parameters are assigned following the convention of choosing parameters based on observed features of the data. As in Lahiri (2002), and Cooley and Hansen (1989), we set $\theta = 0.36$, $B = 2.86$, $\alpha = 0.48$, $\gamma = 0.95$, $\sigma = 1$, $\sigma_\varepsilon = 0.00721$, and $\sigma_\xi = .009$. The parameter η is set to ensure that the steady state value of the discount function coincides with the fixed discount factor β where $\beta = 0.99$.

Next, we need to choose the range of values for τ for which we want to compare the two solution methods. Stability conditions require a choice in the range $0 \le \tau \le 1$, where $\tau = 0$ coincides with the special case of fixed time preference. However, it is important to restrict the range of comparison a bit further. The results for Lahiri (2002) suggest that the interval [0,.026] is appropriate for comparison, as in this range the endogenous time preference models provide a reasonable match to the data[4]. Here, from the point of view of providing more information, we look at a larger range of values, namely $\tau \in [0, 0.3]$.

Figure 1 presents the percentage standard deviations of variables in the range $0 \le \tau \le 0.3$, using the two methods described above. Figure 2 presents the correlations of variable with output. The unbroken line represents the second moments computed using the PE approach, and the dashed line represents the corresponding moments for the KPR approach[5]. In the case of percentage standard deviations the differences seem to get larger as τ increases over this range, while in the case of correlations with output the biggest differences, with the exception of consumption, emerge in an intermediate range of values. In particular, even if we consider the "empirically plausible" range of $0 \le \tau \le 0.026$, there are some significant differences in the second moment features of some of the variables. Specifically, the volatility of consumption increases with more variability in time preference if computed using the PE approach, while it decreases if computed using the KPR approach. There are also differences in volatilities of other variables,

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[4] The choice of this range was based on results using the PE approach, the computational method used for that paper.

[5] As mentioned in the previous section, a second degree polynomial was found to be a good approximation in the parameterized expectations case. However, as far as the second moment features of the model are concerned, there is no difference, in a *graphical* sense, using simulations based on a first degree or second degree polynomial approximation. The figures are therefore based on a first degree polynomial approximation, as this was computationally more convenient.

but these are not as striking as in the case of consumption. However, it is intuitively surprising that there are any differences at all given that the difference in the volatility of the discount factor is not very significant. Of course, the differences get magnified as τ increases over a larger range but the comparison for higher values of τ is not very meaningful. As mentioned before, the model does not match the data very well in those ranges.

Likewise, in the range of $\tau \in [0,0.3]$ there are differences in the computation of correlations with output of other variables, which are presented in Figure 2. In the "empirically plausible" range of [0,0.026], the striking differences are with respect to consumption, and to some extent, work effort and capital. Again, the differences become more dramatic for a larger range of values of τ.

The reason for such striking differences may be related to the fact that the model in question consists of two types of stochastic shocks, and the two methods estimate their relative contribution differently. As discussed in Lahiri (2002), the percentage contribution of monetary shocks changes as τ increases. This is relevant here in the sense that the two methods yield a different level of contribution of the monetary shock. Consequently the cyclical features change in a different manner as τ increases. Briefly, the presence of endogenous time preference leads to the magnification of income effects of technology shocks. For a range of values of τ, the income effects of the technology cancel out its substitution effects leading to a weak response of variables to these shocks. This causes the relative contribution of monetary shocks to increase[6]. Of course, this feature is observed regardless of which computational technique is used. However, the magnitudes of income and substitution effects change at a different rate in each case; consequently, the relative contribution changes in a different way.

The relative contribution of monetary shocks for different values of τ is presented in Table 1. Here, our focus will again be on the subset of values in [0,0.026], for reasons described above. Comparing the way in which the relative contribution of shocks changes, with the way in which second moment features change, provides a partial explanation of the differences using the two approaches. Consider, for example, the standard deviation of consumption and its correlation with output. For the range we are looking at, the second moments move in the PE case move in the opposite direction to the KPR case. Specifically, in the PE case, the standard deviation and the correlation with output *increases* as τ increases in [0, 0.026], while the corresponding moments in the KPR case decrease as τ increases in [0, 0.026]. Looking at the corresponding range of values in Table 1, for the case of consumption, we see that the percentage contribution of monetary shocks (and consequently technology shocks) moves in opposite directions when computed using the alternative approaches. In the case of other variable, second moment features and the underlying patterns for the relative contributions of shocks is quite similar for the range in question.

Why differences arise in the case of consumption and not particularly in the case of other variables is also related with the way in which money enters the model and how monetary shocks impact on the discount factor. The largest impact of monetary shocks, in terms of percentage contribution to fluctuations, is on consumption; the presence of a cash-in-advance constraint on

[6] Here, we wish to focus primarily on computational issues; for further details on the *economic* aspects of the model see Lahiri (2002).

consumption is the obvious reason. Loosely speaking, the impact on consumption in turn affects the endogenous discount factor via the impact of consumption on utility. Now the presence of endogenous time preference is the very feature that makes the model more complex, by means of introducing an additional non-linearity in the preferences of the representative individual. The estimation of the impact on the discount factor is, of course, likely to be very different not only for the computational procedures being examined here but for all types of procedures. The effects of different types of stochastic shocks would then translate into different outcomes for the variables of the model. Here, there are only two shocks, and the differences are likely to be the most significant with respect to consumption because of the "first-order" impact of the monetary shock via the cash-in-advance constraint.

While the inferences drawn above are based on a specific example, there are some obvious implications of a generic kind. As models are made more complex by means of incorporating either more types of stochastic shocks or additional non-linearities in preferences or technology, answers to economic issues of interest are likely to hinge critically on the type of approximation method used. Furthermore the diversity in conclusions is not likely to be merely of a quantitative type, but of a qualitative type. For example, it is conceivable that varying parameters of a model could lead to variation of cyclical features of a completely different nature, depending on the type of approximation used. As noted earlier, researchers have sometimes recommended the use of methods in which the solution is amenable to refinement. However, there is usually a limit to the degree of refinement, and there is also the question of possible differences in results when the comparison is made vis-à-vis procedures that allow for refinement. It would therefore be advisable for researchers to check their results using several rather than a few alternative procedures.

5. Concluding Remarks

This paper re-examined some of the issues involved in the comparison of alternative computational procedures within the context of a dynamic stochastic general equilibrium model. The framework studied was a more general one, in which a "standard" or relatively simple model was nested as a special case. Results of numerical experiments suggested that different computational methods could be used interchangeably in the case of the standard model, but not in the case of the more general model. Varying a preference parameter allowed us to compare what happened to solutions using alternative procedures as one moved away from the special case to the more general framework. On the basis of the numerical experiments conducted, we found that not only did differences in solutions become larger, but answers to several economic issues of interest could yield qualitatively different answers depending on the solution method used. Examples of such issues include how second moment features change as one varies the parameters of a model, and the relative contribution of different types of stochastic shocks to fluctuations in variables.

While these conclusions were based on a specific example, they apply in a more general sense. The results seem to suggest that addition of more parameters, a larger number of stochastic shocks, incorporation of features that add to the non-linearities in technology or preferences contribute to the emergence of larger differences in solutions using alternative methods. Not only could the effect of these features estimated differently, but their interaction could also be estimated subject to different types of approximation error. I such situations, it would be advisable for researchers to check their solutions using several alternative approaches.

References

Cooley, T. F. and Gary D. Hansen, 1989, "The Inflation Tax and the Business Cycle," *American Economic Review*, 79, 733-748.

Christiano, L.J., 1990, "Linear Quadratic Approximation and Value Function Iteration: A Comparison", *Journal of Business Economics and Statistics*, 8, 99-113.

Danthine, J., Donaldson, J.B., and R. Mehra, 1989, "On Some Computational Aspects of Equilibrium Business Cycle Theory", *Journal of Economic Dynamics and Control*, 13, 449-470

Den Haan, W.J. and A. Marcet, 1990, "Solving the Stochastic Growth Model by Parameterizing Expectations," *Journal of Business and Economic Statistics*, 8, 31-34.

Dolmas, J. and M. Wynne, 1994, "Fiscal Policy in More General Equilibrium," Federal Reserve Bank of Dallas.

Epstein, L.G., 1983, "Stationary Cardinal Utility and Optimal Growth Under Uncertainty," *Journal of Economic Theory*, 31, 133-152.

Gomme, P., and J. Greenwood, 1995, "On the Cyclical Allocation Of Risk," *Journal of Economic Dynamics and Control*.

King, R.G., Plosser, C.I., and S.T. Rebelo, 1988a, "Production, Growth and Business Cycles, I: The Basic Neoclassical Model," *Journal of Monetary Economics*, 21, 195-232.

King, R.G., Plosser, C.I., and S.T. Rebelo, 1988b, "Production, Growth and Business Cycles, II: New Directions," *Journal of Monetary Economics*, 21, 309-341.

Lahiri, R., 2002, "The Inflation Tax, Variable Time Preference, and the Business Cycle" *Macroeconomic Dynamics,* 6, 496-522.

Mendoza, E., 1991, "Real Business Cycles in a Small Open Economy," *American Economic Review*, 81, 797-818.

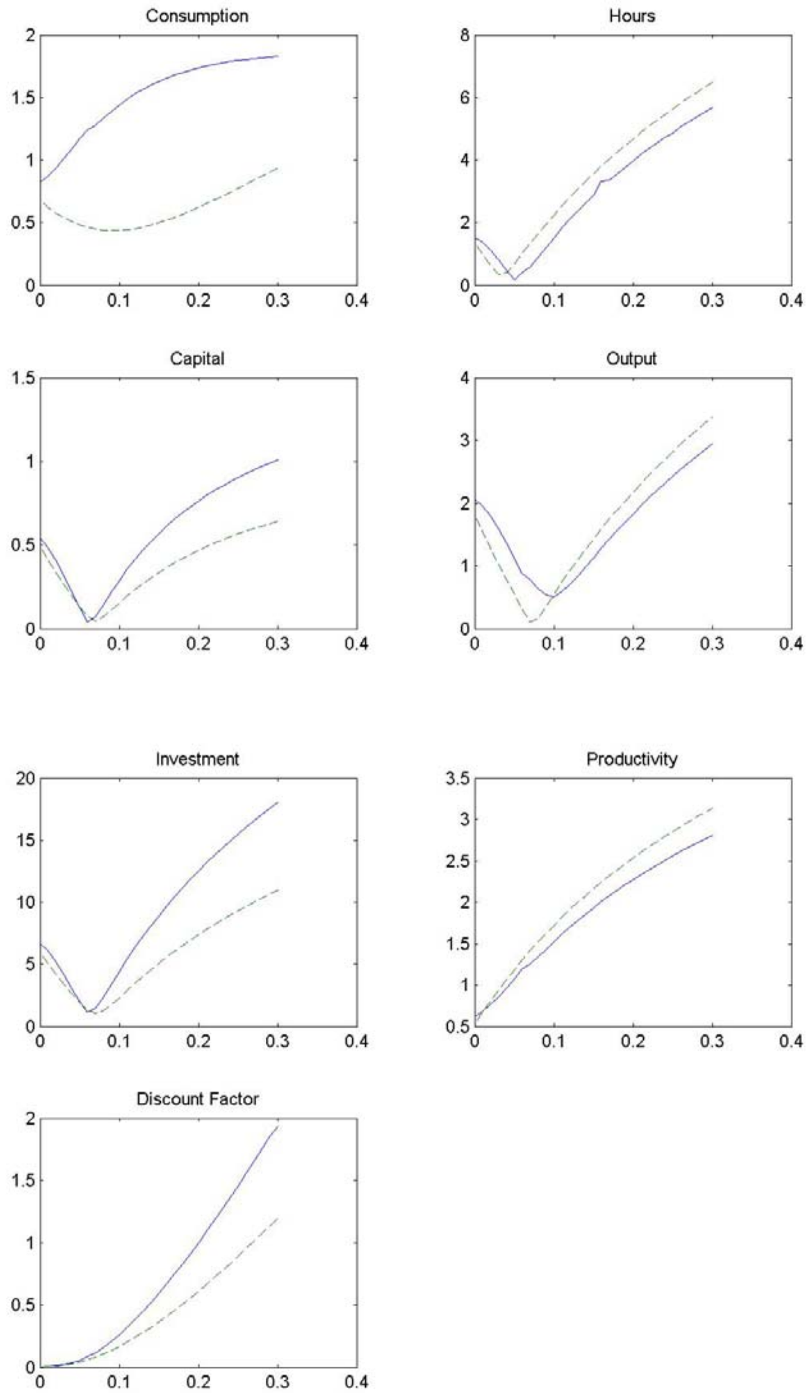

Figure 1. Percentage Standard Deviations Of Variables for $0 \le \tau \le 0.3$,

— PE Approach

--- KPR Approach

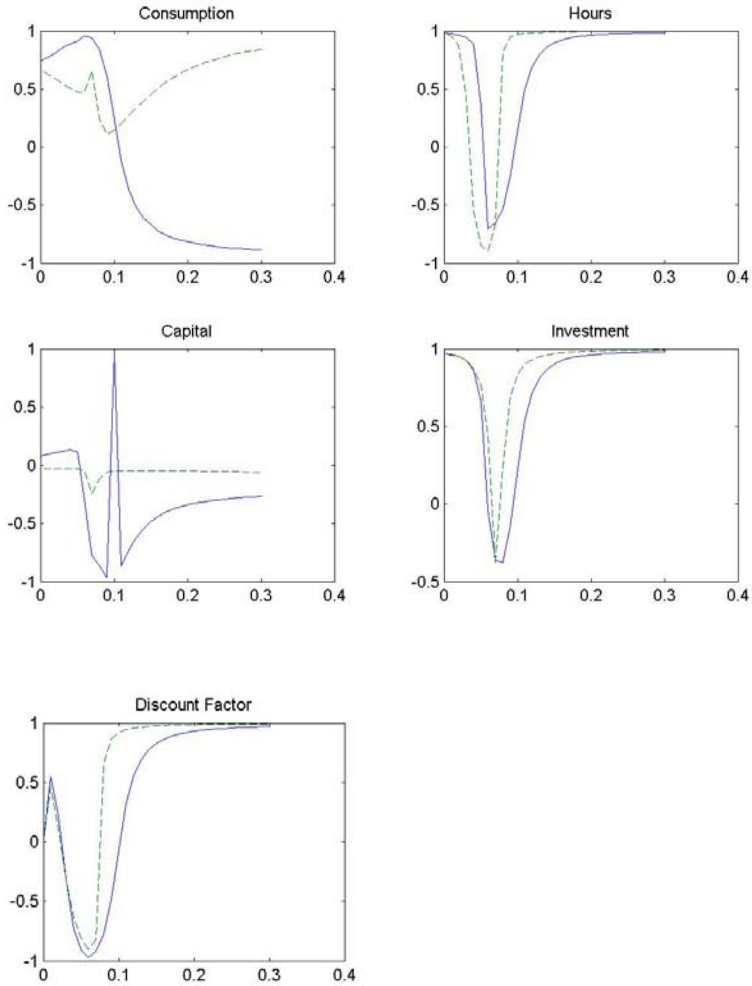

Figure 2. Correlations of Variables with Output for $0 \leq \tau \leq 0.3$,

— PE Approach

--- KPR Approach

Table 1: Percentage Contribution of the Monetary Shock to Fluctuations in Variables

τ						
	0	0.01	0.02	0.026	0.03	0.04
Output						
PE Method	2.7	2.6	3.3	3.6	4.8	6.6
KPR Method	1.6	1.9	2.8	3.6	4.3	6.8
Consumption						
PE Method	62.1	58.6	54.0	51.3	49.6	45.5
KPR Method	63.7	64.4	74.8	78.5	80.1	85.1
Investment						
PE Method	10.1	21.5	24.9	28.1	31.0	42.1
KPR Method	19.2	22.3	26.5	30.2	32.6	41.9
Capital Stock						
PE Method	12.2	13.9	16.1	18.2	20.1	28.0
KPR Method	11.3	13.3	15.9	17.9	20.0	26.5
Hours						
PE Method	5.1	5.4	5.6	9.9	12.3	24.3
KPR Method	3.5	6.0	11.8	18.5	24.6	24.9
Productivity						
PE Method	5.7	5.6	5.8	5.9	6.0	5.9
KPR Method	6.0	5.5	5.2	5.1	5.0	4.9
Discount Factor						
PE Method	0.0	54.2	63.6	66.9	64.8	51.3
KPR Method	0.0	59.6	68.8	71.3	68.3	58.5