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The Washroom Game

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Jan Heufer[1]

The Washroom Game

Abstract

This article analyses a game where players sequentially choose either to become insiders and pick one of finitely many locations or to remain outsiders. They will only become insiders if a minimum distance to the next player can be assured; their secondary objective is to maximise the minimal distance to other players. This is illustrated by considering the strategic behaviour of men choosing from a set of urinals in a public lavatory. However, besides very similar situations (e.g. settling of residents in a newly developed area, the selection of food patches by foraging animals, choosing seats in waiting rooms or lines in a swimming pool), the game might also relevant to the problem of placing billboards attempting to catch the attention of passers-by or similar economic situations. In the non-cooperative equilibrium, all insiders behave as if they cooperated with each other and minimised the total number of insiders. It is shown that strategic behaviour leads to an equilibrium with substantial under-utilization of available locations. Increasing the number of locations tends to decrease utilization. The removal of some locations which leads to gaps can not only increase relative utilization but even absolute maximum capacity.

JEL Classification: C70, H89, R12

Keywords: Efficient design of facilities; location games; privacy concerns; strategic entry prevention; unfriendly seating arrangement; urinal problem

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1 TU Dortmund University. – Views expressed represent exclusively the author's own opinions. *Thanks to Wolfgang Leininger for his support and helpful comments. Thanks to Yoram Bauman for helpful comments. All remaining errors are mine.* *– All correspondence to Jan Heufer TU Dortmund University, Department of Economics and Social Science, 44221 Dortmund, Germany. E-mail: jan.heufer@tu-dortmund.de.*

1 INTRODUCTION

Privacy concerns of individuals are an important aspect of public lavatories or washrooms. Designers of public lavatories may consider lack of privacy as a minor inconvenience for users, and providers of such facilities rarely compete for customers, either because of monopoly power or because the lavatory is provided as a public good. However, if individuals have sufficiently strong preferences for privacy, users may actually prefer to wait or to not use the lavatory at all, as under-utilisation increases privacy. The analysis of the game considered in this article shows that a requirement of minimal distance to the next men using a urinal induces strategic behaviour which leads to substantial under-utilisation due to entry deterrence.

The results obtained here imply that proper washroom design is indeed an important problem, which is also reflected in the guidelines and recommendations of "American Restroom Association".[1] However, the game might also be relevant for (other) economics problems such as placing billboards along a street to catch the attention of passers-by.

Kranakis and Krizanc (2010) consider a similar situation, the "urinal problem". However, in their model, men do not require a minimum distance and instead try to maximise the time their privacy is ensured. More importantly, however, they analyse simple, semi-strategic behavioural rules without regard for any equilibrium notion. A similar problem, the "unfriendly seating arrangement", has already been considered by Freedman and Shepp (1962), Friedman (1964), and Georgiou et al. (2009), but the authors only consider the expected number of persons to sit down when the location choice is random. A recent model considering equilibria in games where players wish to maximise distances on measure spaces is provided by Zhao et al. (2009).

In section 2, we introduce and analyse the game. Section 2.1 provides the basic definitions and assumptions for the game. Section 2.2 derives the equilibrium and Section 2.3 briefly discusses welfare aspects. It is shown that the equilibrium of the game leads to substantial under-utilisation. In most cases, this under-utilisation is below the utilisation achievable by a social planner who can allocate the locations (urinals) to players (men). In fact, in equilibrium, players who choose a location behave as if they cooperated to minimise the utilisation. As the number of locations (urinals) and players (men) increases, the equilibrium utilisation converges. Section 3 discusses applications and possible extensions and generalisations.

[1] See `http://www.americanrestroom.org/design/index.htm`.

2 THE GAME

2.1 *Definitions and Assumptions*

The most simple form of the kind of game considered here consists of a set of equidistant locations from which players can choose one in a sequential manner. The players' objective is to maximise the final distance to the nearest location chosen by another player at the end of the game. Players have the outside option of not choosing any location at all. To provide some interpretation, consider a set of equidistant urinals in a washroom and men who enter the room sequentially. Men dislike to choose a urinal next to another urinal which is already in use. If no urinal providing at least basic privacy is available, each man prefers to leave the room immediately. Each man prefers larger distances to the next man compared to smaller distances. The men enter the bathroom one by one in rapid succession, so men will only consider the privacy they have after no further men decides to use a urinal (e.g., the privacy the first man enjoys before the second man enters is too short to influence the first man's utility).

Formally, the *set of players* is given by $\mathcal{N} = \{1, 2, \ldots, N\}$ with $N \geq 2$. Denote the set of available *locations* (urinals) as $\mathcal{L} = \{1, 2, \ldots, L\}$, where $N \geq L$. The *distance between two locations* k and ℓ is given by $d : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$, with $d(k, \ell) = |k - \ell|$, where $|\cdot|$ denotes the absolute value. Whenever possible generic elements of $\mathcal{N}$ are denoted by i, j and of $\mathcal{L}$ by k, ℓ.

The game has N stages. Let h^t denote the *history* of the game at stage t. At every stage, players $i \in \mathcal{N}$ simultaneously choose an *action* from choice sets $A_i(h^t)$. The history at the start of the game is $h^1 = \varnothing$. The set of available actions for player $i \in \mathcal{N}$ at stage t is

$$A_i(h^t) = \begin{cases} \varnothing & \text{if } i \neq t \\ \mathcal{L} \cup \{-1\} & \text{if } i = t. \end{cases}$$

Hence, players only have a choice when it is their turn, and then they can either choose one of the locations in $\mathcal{L}$ (i.e., $a_i^i \in \mathcal{L}$) or choose the action $a_i^i = -1$, which is interpreted as leaving the room without choosing a location. Also note that this means that a player cannot change his[2] location or leave once he has made a choice. The action profile at stage t is $a^t = (a_1^t, \ldots, a_N^t)$, and the history at stage $t \geq 2$ consists of the action profiles of all previous stages, i.e. $h^t = (a^1, a^2, \ldots, a^{t-1})$. Let $\mathcal{H}^t$ be the set of all possible histories at stage t.

A *strategy* of player i is a plan for how to play at each stage t of the game at which he is to move, for each possible history h^t. Note that player i has no choice at stage $t \neq i$; hence a pure strategy for player i is $s_i : \mathcal{H}^i \to \mathcal{L} \cup \{-1\}$, i.e., it maps the set of all histories at stage

[2] No apologies are necessary for exclusively using the masculine form. Although it has been pointed out to the author that washrooms for women occasionally have similar configurations in some countries, the problem seems to mostly occur in the men's room.

$t = i$ to the set of actions available to player i at that stage. Let S_i denote the strategy set of player i, and $\mathcal{S} = \times_{i \in \mathcal{N}} S_i$. Let s denote a *strategy profile*, i.e. $s = (s_1, \dots, s_N)$.

Let $C^t \subseteq \mathcal{L} \cup \varnothing$ denote the set of all locations at stage t which have been chosen by at least one player $i < t$. Because an action $a_i^i \in \mathcal{L}$ is identified with a location, we will also write $d(\ell, a_i^i)$ if $a_i^i \in \mathcal{L}$. Let $\rho : \mathcal{N} \times \mathcal{L} \to \mathbb{R}$ be defined as

$$\rho_i(\ell) = \begin{cases} \min_{k \in C^i} d(\ell, k) & \text{if } C^i \neq \varnothing \\ v & \text{otherwise,} \end{cases}$$

where $v \geq L$. Thus, $\rho_i(\ell)$ gives the distance to the nearest occupied location when choosing location ℓ after $i-1$ players have chosen. The *utility* of player i is then given by a function $U_i : \mathcal{S} \to \mathbb{R}$, with

$$U_i(s) = U_i(s_i, s_{-i}) = \begin{cases} \rho_N(a_i^i) & \text{if } a_i^i \in \mathcal{L} \\ \omega & \text{otherwise.} \end{cases}$$

For simplicity, we assume that $\omega = 3/2$; generalisations with arbitrary $\omega > 1$ are easily possible but somewhat tedious. Note that ω can be interpreted as the outside option (i.e., the utility obtained when not choosing a location in $\mathcal{L}$). As such, ω can also be interpreted as the required minimal distance to the nearest player and therefore as a parameter determining the degree of privacy needed by a man in a washroom.

This setup results in an extensive form game with complete and perfect information; the *game* is given by a tuple $\Gamma = (\mathcal{N}, \mathcal{S}, \mathcal{U})$, where $\mathcal{U}$ denotes the set of utility functions of all players $i \in \mathcal{N}$. The subgame from stage t on, given history h^t, will be denoted $\Gamma(h^t)$. Let $s_i(h^t)$ be the restriction of s_i to $\Gamma(h^t)$, for $i = 1, \dots, N$. Note that in game $\Gamma(h^t)$ all players are present, but only players $i = t, \dots, N$ are to move.

Definition 1 *A strategy profile s is a subgame-perfect* equilibrium*, denoted $s^* = (s_1^*, \dots, s_N^*)$ if, for every history h^t, $s^*(h^t) = (s_1^*(h^t), \dots, s_N^*(h^t))$ is a Nash equilibrium of $\Gamma(h^t)$, for $t = 1, \dots, N$.*

The *equilibrium capacity* is the cardinality of C^N in equilibrium. Let $c(L)$ denote the set of equilibrium capacities of the equilibria of a game Γ with L locations. The *equilibrium utilisation* is the quotient of an equilibrium capacity and L, i.e., $c(L)/L$. Players who choose an action $a \in \mathcal{L}$ in equilibrium are called *insiders*, while players who choose the action $a = -1$ are called *outsiders*.

Note that the game does not prohibit players from choosing the same location as another player (i.e., two men are allowed to use the same urinal). However, this strategy is strictly dominated by $a = -1$, as $d(\ell, \ell) = 0$ and $U(-1, \cdot) = \omega = 3/2$. As tie-breaking rules, players who are indifferent between $a = -1$ and $a \in \mathcal{L}$ choose $a \in \mathcal{L}$. Note that players have at most two neighbouring players. Suppose a player i chooses $a_i^i \in \mathcal{L}$ but is indifferent between two (or more) optimal locations, say k and ℓ, because player i anticipates that at the end of the game, $\rho_N(k) = \rho_N(\ell)$. In that case, player i will choose the location with the highest $\rho_i(\cdot)$. If players are still indifferent, they will choose one of the optimal locations at random with equal probability.

Let $\mathrm{E}[U_i]$ denote the expected utility of player i. Let $\mathrm{mod}(i, j)$ denote the remainder on division of i by j. Let $\lfloor m \rfloor$ and $\lceil m \rceil$ be the largest integer not greater than m and the smallest integer not less than m, respectively. For $k \in \mathcal{L}$, let $\eta_m(k) = \{\ell \in \mathcal{L} : d(\ell, k) \leq m\}$, and for $K \subseteq \mathcal{L}$, let $\eta_m(K) = \bigcup_{k \in K} \eta_m(k)$. A location k is called a *direct neighbour* of a location ℓ if $d(k, \ell) = 1$; we will also denote $\eta(k) = \eta_1(k)$, the set consisting of k and all direct neighbours of k. Let $\ell_{+m} = \{\ell + 1, \dots, \ell + m\}$ whenever $\ell + m \in \mathcal{L}$, and $\ell_{-m} = \{\ell - 1, \dots, \ell - m\}$ whenever $\ell - m \in \mathcal{L}$ (we only write ℓ_{+m} if $\ell + m \in \mathcal{L}$, and ℓ_{-m} if $\ell - m \in \mathcal{L}$). The locations 1 and L are called *endpoints* of $\mathcal{L}$.

2.2 *Equilibrium Analysis*

We begin with a series of lemmata, which lead to two propostions.

Lemma 1 *For a game Γ with $L \leq 3$ and $\omega = 3/2$ the unique equilibrium capacity is 1. In equilibrium, player 1 is the only insider and obtains a utility of $U_1^* = v$. All other players $i > 1$ are outsiders and obtain a utility of $U_i^* = 3/2$.*

Proof The cases $L = 1$ and $L = 2$ are obvious. For $L = 3$, the action $a_1^1 = 2$ is the unique equilibrium strategy of player 1, because all players $i > 1$ will choose $a_i^i = -1$, thus $U_1^* = v$; note that with $a_1^1 = 1$ or $a_1^1 = 3$ the second player will become an insider, reducing player 1's utility to 2. ■

Let s^* be an equilibrium and let $c_i^* = s_i^*(h^i)$ be the location chosen by player i if he is an insider.

Lemma 2 *In any equilibrium of a game Γ, $\rho_N(c_i^*) \geq 2$ for all players i who are insiders (i.e., the utility of all insiders is at least 2). If player i is an outsider, all players $j > i$ are also outsiders.*

Proof A player i will never choose a location ℓ which is a direct neighbour of a location previously chosen by another player, because $\rho_i(\ell) = 1 \leq \omega = 3/2$. Suppose at player i's turn there is still an available location ℓ with $\rho_i(\ell) \geq 2$. Player i will not have to take into account the choices of players $j > i$ to obtain a utility of 2 (as $\rho_N(\ell)$ will be at least 2), thus he will become an insider and choose the location ℓ unless there are better alternative locations. Then if a player i prefers to remain an outsider it is because there is no location ℓ with $\rho_i(\ell) \geq 2$ at player i's turn. Thus, if player i prefers to remain an outsider, so will all players $j > i$. ■

With Lemma 2 it now makes sense to say that for player i a location ℓ is *free* if neither ℓ nor $\ell - 1$ and $\ell + 1$ (if these locations exist) have been chosen by any players before i, i.e., if $\eta(\ell) \cap C^i = \varnothing$. We need Lemma 2 because it shows that if player i chooses a free location, he is guaranteed to be better off than he would be if he remained an outsider.

Lemma 3 *In any equilibrium of a game Γ with $L > 3$, $\rho_N(c_i^*) \leq 3$ for all players i who are insiders (i.e., the utility of all insiders is at most 3).*

Proof If a player i chooses a location ℓ with $\rho_i(\ell) > 3$, then $\eta_3(\ell) \cap C^i = \varnothing$. More specifically, either (i) $[\ell_{-3} \cup \ell_{+3}] \cap C^i = \varnothing$ or (ii) $[\ell_{-2} \cap C^i = \varnothing, \ell - 3 \notin \mathcal{L}]$ and either $[\ell_{+2} \cap C^i = \varnothing, \ell + 3 \notin \mathcal{L}]$ or $[\ell + 1 \notin C^i, \ell + 2 \notin \mathcal{L}]$ or $\ell + 1 \notin \mathcal{L}$, or (iii) the same as under ii, but with reversed signs. It not not possible that both $\ell - 2$ and $\ell + 2 \notin \mathcal{L}$ because $L > 3$. Then in case (i) $\rho_i(\ell - 2) \geq 2$ and $\rho_i(\ell + 2) \geq 2$, and by Lemma 2 and with $N \geq L$ either some player $j > i$ will choose location $c_j^* = \ell - 2$ or $c_j^* = \ell + 2$, leading to $\rho_N(\ell) = 2$, or two of the following players $j, k > i$ will choose $c_j^* = \ell - 3$ and $c_k^* = \ell + 3$, leading to $\rho_N(\ell) = 3$. In case (ii) $\rho_{i+1}(\ell - 2) = 2$ and some player $j > i$ will choose $c_j^* = \ell - 2$, leading to $\rho_N(\ell) = 2$. In case (iii) $\rho_{i+1}(\ell + 2) = 2$ and some player $j > i$ will choose $c_j^* = \ell + 2$, leading to $\rho_N(\ell) = 2$. ■

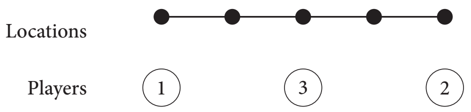
Lemma 3 has another straightforward interpretation: In equilibrium, every player will have another player near him within a distance of at most 3.

Proposition 1 *In all equilibria of a game Γ:*

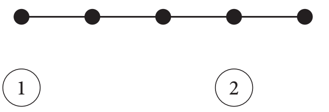
1. *if $L > 3$ the utility of every insider is exactly 3;*
2. *if $mod(L, 3) = 1$ then only and all of the locations $\{\ell \in \mathcal{L} : mod(\ell, 3) = 1\}$ are chosen by insiders;*
3. *if $mod(L, 3) = 2$ then either only and all of the locations $\{\ell \in \mathcal{L} : mod(\ell, 3) = 1\}$ or only and all of the locations $\{\ell \in \mathcal{L} : mod(\ell, 3) = 2\}$ are chosen by insiders;*
4. *if $mod(L, 3) = 0$ then only and all of the locations $\{\ell \in \mathcal{L} : mod(\ell, 3) = 2\}$ are chosen by insiders.*

The proof of Proposition 1 is simple yet tedious and can be found in the appendix.

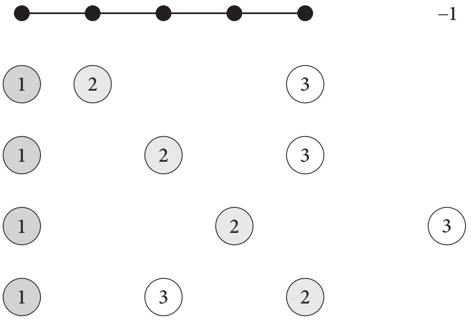
See Figure 1 for a simple example with five locations. In Figure 1(a), the choice of the second player invites a third player to become an insider, leaving all insiders with a utility of 2. Figure 1(b) shows that the second player can avoid this by leaving no free locations, giving all insiders a utility of 3.



(a) Not a subgame-perfect equilibrium.



(b) A subgame-perfect equilibrium.



(c) The strategy of player 3 at h^2.

Figure 1: A game with five locations. In (b), locations of the two players can be switched to obtain a second equilibrium. (c) illustrates the strategy of player 3 at h^2, where player 1 has already chosen location 1. If player 2 chooses locations 2 or 3, player 3 will become an insider and choose location 5, which provides him with utility of 3 or 2, respectively, as player 4 will then remain an outsider. If player 2 chooses location 5, player 3 can obtain utility of 2 by choosing location 3. But if player 2 chooses location 4, player 3 can obtain only a utility of 1 by becoming an insider, which is why he prefers to remain an outsider. This gives player 2 a utility of 3.

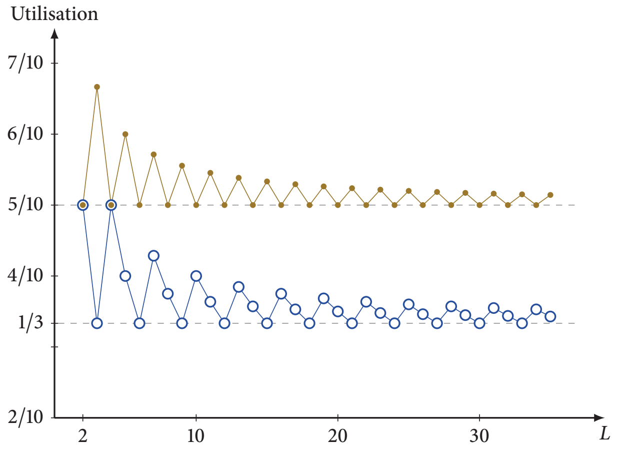


Figure 2: The highest possible utilisation (dots) and the equilibrium utilisation (circles), depending on the number of locations L.

Proposition 2 *The equilibrium capacity of a game* Γ *with* L *locations is unique and given by* $c(L) = 1 + \lfloor (L-1)/3 \rfloor$*; the utilisation converges:* $\lim_{L\to\infty} c(L)/L = 1/3$.

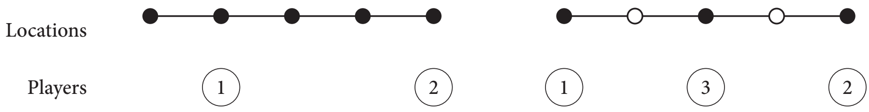
Proof Consider the case $\mathrm{mod}(L,3) = 1$. Let ℓ be the right-most location chosen by an insider. According to Proposition 1, $\mathrm{mod}(\ell,3) = 1$, thus $\ell = L$. It is then easy to see that $c(L) = 1 + \lfloor (L-1)/3 \rfloor$. The same holds for the cases $\mathrm{mod}(L,3) = 0$ and $\mathrm{mod}(L,3) = 2$. Then $c(L)/L = 1/3$ whenever $\mathrm{mod}(L,3) = 0$; $c(L)/L$ gets arbitrarily close to $1/3$ as L increases for $\mathrm{mod}(L,3) \neq 0$. ■

A social planner who delegates players to locations and wishes to maximise the objective function $c(L)$ can place up to $\lceil L/2 \rceil$ players if players are still free to choose the outside option. See Figure 2 for this highest possible utilisation and the equilibrium utilisation. Note that except for $L \in \{1, 2, 4\}$ the strategic behaviour will lead to under-utilisation of the locations.

Note that the strategic non-cooperative behaviour of players keeps down the number of insiders (i.e., the capacity). Players behave as if they cooperated with players who cannot be deterred from becoming insiders against subsequent players, which is a consequence of each player's backwards inductive reasoning.

2.3 *Welfare and Efficiency*

Figure 3 shows an example in which the removal of locations 2 and 4 without changing the distance between the remaining locations (i.e., $d(1,3) = d(3,5) = 2$ and $d(1,5) = 4$) can not only improve relative utilisation, but even absolute maximum capacity. Given $\omega = 3/2$ and



(a) An equilibrium with capacity 2. (b) An equilibrium with capacity 3.

Figure 3: Leaving gaps, represented as circles, can increase absolute maximum capacity in equilibrium.

the tie-breaking rule, if would already be sufficient to leave a gap between urinals such that the distance between two urinals is 3/2 instead of 1.

As was already mentioned, if the objective function of a social planner who determines locations but cannot force players to become insiders is given by the overall capacity (i.e., the total number of insiders), then the number of insiders chosen by the social planner is $\lceil L/2 \rceil$ (see also Figure 2). This can be achieved by assigning location 1 to the first player, location 3 to the second player, and so on.

When "welfare" is to be considered, the question arises in how far the number of insiders should contribute. If a welfare measure only considers the utility of players, then having only one player as insider my even be optimal, depending on the value of v (the utility of an isolated insider). In any case, the sum of players utilities will not generally be maximised by maximising utilisation. Consider the case with $L = 5$. If there are three insiders they will each receive a utility of 2. If there are only two insiders, they will each receive a utility of 3. In either case, all the remaining players will receive a utility of $\omega = 3/2$; thus, the sum of utilities is actually higher with only two insiders, because $3 \cdot 2 + (N-3) \cdot (3/2) < 2 \cdot 3 + (N-2) \cdot (3/2)$.

However, if the utility functions are redefined such that $U(s_i, s_{-i}) = 2 + (\rho_N(i) - 2) \cdot \varepsilon$ whenever player i is an insider, with $\varepsilon \in (0,1)$, then for small enough v and ε the utilitarian welfare ordering is equivalent to an ordering by capacity.

3 DISCUSSION

3.1 *Washroom Design*

Privacy concerns by men using public lavatories is indeed taken seriously by washroom designers. The American Restroom Association stresses that proper design "improves the experience" and hinders "nefarious activities".[3] But the game considered in this article also shows that the low utilisation of a washroom due to the strategic behaviour induced by privacy concerns provides good reasons to assure adequate spacing between urinals and to chose the

[3]See `http://www.americanrestroom.org/design/index.htm`. Also see Aldrich (2004), who gives an historical overview about homosexuality in urban spaces and notes that public urinals throughout history hosted sexual encounters and that, it the 1880s, public urinals in Amsterdam were designed as separated stalls to discourage homosexual activities.

right number of urinals. However, shields between urinals might be a more efficient way to overcome the problem. The "International Plumbing Code" mandates partitions between urinals in public lavatories.[4]

Note, however, that the parameter ω which gives the utility of the outside option of not using a urinal can also be interpreted as a parameter for the urgency of nature's call. If men have different values of ω, and with little room or resources to provide facilities, lack of privacy might actually improve efficiency, as it allocates the scarce resources to those who value it the most.

3.2 *Possible Generalisations and Extensions*

We have considered a case with an outside option utility and thus minimum distance to the nearest player of $\omega = 3/2$. In the symmetric case where each player desires the same minimum distance, the model can be easily generalized to arbitrary ω. One might also consider a game in which ω is different for each player. The values of ω for each player can be either common or private knowledge; in the latter case, players would only know the distribution of ω in the population of potential players.

We have assumed that as long as becoming an insider gives a higher utility than remaining an outsider, players will continue to choose locations. The analysis becomes more complicated when the number of players is not common knowledge and there is a chance that the number of players is far less than the number of locations. For example, with only two players and five locations, players would choose locations $\ell = 1$ and $\ell = 5$ as there is no use in blocking other players.

The more familiar model of spatial competition on a "street" (Hotelling 1929) usually considers a continuous strategy space. It was already noted by Kranakis and Krizanc (2010) that "continuous" urinals with a single central drain exist or existed in the past, so the original framing of the game could to some extent be preserved if one considers a "washroom game" on a line.

A further generalisation would be to consider seating arrangements in a theatre or lecture hall in which, for example, a player wishes to maximise distance to other players to the left or right, and prefers to not sit behind a player in the row before him but does not mind having players sitting behind him.

3.3 *Other Possible Applications*

A straightforward example of how games of the kind considered here can be applied is advertisement. Both geographic and non-geographic interpretations are possible. Consider

[4] See the 2009 International Plumbing Code of the International Code Council, `http://www.iccsafe.org/Store/Pages/Product.aspx?id=3200X09`.

the placement of billboards along a street. Players enter sequentially and choose one position on the street to place one billboard. An isolated billboard will catch a certain "amount" of attention by passers-by. Two billboards very close to each other will result in at most half and possibly even far less than half of the attention of an isolated billboard.[5] The attention each billboard receives increases in the distance to the next closest billboard. This leads to a first mover advantage, as each billboard effectively blocks neighbouring parts of the streets if the effect of a second billboard nearby is too small to be worth the effort. For a non-geographical interpretation consider advertisements targeting a certain demographic group which can represented by a one-dimensional variable with a meaningful definition of distance (e.g. age or education).

A APPENDIX

Proof of Proposition 1 It can be easily seen that the proposition holds for games with $L \leq 3$ and is consistent with Lemma 1.

By Lemma 3, an insider cannot obtain a higher utility than 3 for games with $L > 3$. To show that the proposed strategy profile is indeed an equilibrium it is therefore sufficient to show that they lead to a utility of 3 for every insider. It can be easily checked that in all cases, the locations not chosen by insiders are direct neighbours of locations which are chosen as strategies, thus (by Lemma 2) given the proposed strategy profile, no further players will become insiders. It can also be easily checked that each player obtains a utility of 3.

Showing that the proposed strategy profiles uniquely determine the equilibrium is more tedious. Consider the subgame induced by the choice of a player $i-1$ who is an insider. Let $\mathcal{L}^i = \mathcal{L} \backslash \eta(C^i)$. By Lemma 2, player i will never choose as location a direct neighbour of $\ell \in C^i$, as these locations are strictly dominated by the strategy -1. Iterated elimination of dominated strategies thus leads to a subgame with $S_i = (\mathcal{L}^i) \cup \{-1\}$. Let MaxL^i denote the subset or subsets of $\mathcal{L}^i$ with the highest number of free locations neighbouring each other without interruption by an $\ell \in C^i$, and let $\#\text{MaxL}^i$ denote the cardinality of that set. First consider the following cases:

Case 0: $\#\text{MaxL}^i = 0$ By Lemma 2, all players $j \geq i$ will remain outsiders.

Case 1: $\#\text{MaxL}^i = 1$ Player i will choose an $c_i \in \text{MaxL}^i$ as it offers a utility of $U_i = 2$, which is better than $U(-1, \cdot) = 3/2$.

[5] There might even be the risk that the intended positive effect of a billboard on demand is reversed, as several photographs of juxtapositions circulating on the world wide web suggest (e.g. the billboard advertising an "adult video" store next to a billboard of a church reminding people that "Jesus is watching you"; the billboard advertising "affordable caskets" underneath a billboard promoting a "gun show"; the promotion of a special cheeseburger-deal by a chain of fast food restaurants next to a reminder to call the emergency telephone number in case of heart attacks).

Case 2: $\#\text{MaxL}^i = 2$ (i) If there is no $\ell \in \text{MaxL}^i \cap \{1, L\}$, player i will choose one of the two locations at random and receive a utility of 2 (similar to Case 1). (ii) If there is an $\ell \in \text{MaxL}^i \cap \{1, L\}$, then $4 \in C^i$ or $L-3 \in C^i$, otherwise both 1 and L cannot be in MaxL^i or $\#\text{MaxL}^i < 2$. Player i will choose either $c_i = 1$ or $c_i = L$. This provides a utility of $U_i = 3$ because $2 \in \eta(1)$ and $3 \in \eta(4)$ (or $L-1 \in \eta(L)$ and $L-2 \in \eta(L-3)$, respectively); these locations will not be chosen by any player $j > i$ (by Lemma 2).

Case 3: $\#\text{MaxL}^i = 3$ Player i will choose a location which is in the center of the set (one of the sets in) MaxL^i. This provides a utility of $U_i = 3$: (i) If $c_i \in \{2, L-1\}$, then $5 \in C^i$ (or $L-4 \in C^i$, respectively), $3 \in \eta(c_i)$ and $4 \in \eta(5)$ (or $L-2 \in \eta(s_i)$ and $L-3 \in \eta(L-4)$, respectively). (ii) If $c_i \notin \{2, L-1\}$, then all locations in $\eta_2(c_i)\backslash c_i$ are direct neighbours of a location chosen by some player $j \leq i$. In both cases no location k with $d(c_i, k) < 3$ will be chosen by any player $k > i$ (by Lemma 2).

Suppose player i's choice $c_i = \ell$ leads to a set of free locations to the left or right player of i, i.e., $(\{\ell-2\} \cup \{\ell+2\}) \cap \mathcal{L}^i \neq \varnothing$. If either of these sets consists of only one location (i.e., $\ell - 2$ or $\ell + 2$), some player $j > i$ will choose it (Case 1), leaving player i with a utility of $U_i = 2$. If either of these sets consists of only two locations (i.e., $(\ell-2)_{-1}$ or $(\ell+2)_{+1}$) and does not include 1 or L, then some player $j > i$ will choose either of the two (Case 2.i), leaving player i with an expected utility of $\text{E}[U_i] < 3$. If all of these sets consist of either three locations or two locations including an endpoint of $\mathcal{L}$ (i.e., $(\ell-2)_{-2}$ or $(\ell-2)_{-1}$ with $\ell - 3 = 1$ etc.), then some player $j > i$ will choose $\ell-2$ or $\ell+2$ (Case 2.ii and 3, respectively), leaving player i with a utility $U_i = 3$. Thus, if a player i leaves a set of free locations to his left or right, and one of these sets has either only one element or two elements not including an endpoint, player i will have an expected utility of less than 3. If a player i only leaves sets of free locations to his left and right with two elements including an endpoint or three elements, player i will end up with a utility of 3. Thus, players will not leave "bad" sets with only one element or two elements excluding endpoints if they can avoid it.

Suppose player i's choice $c_i = \ell$ leads to a set of free locations with four elements; say this set is to the right of player i, i.e., $(\ell+2)_{+3}$, and does not contain an endpoint. Some player $j > i$ will choose an element of $(\ell+2)_{+3}$; player j can only leave a set with one or two elements. In either case, some player $k > j$ will choose one of the locations of this set, so player j will end up with a utility of $U_j = 2$. Knowing this, he will (by the tie-breaking rule) choose either $c_j = \ell+3$ or $c_j = \ell+4$ with equal probability. If he chooses $c_j = \ell+4$, some player $k > j$ will choose $c_k = \ell+2$, so either $\rho(\ell) = 2$ or $\rho(\ell) = 3$, which gives $\text{E}[U_i] = 5/2 < 3$. In the same manner, it can be shown that if player i leaves a set of free locations with five elements, then $\text{E}[U_i] = 11/4 < 3$.

Suppose player i's choice $c_i = \ell$ leads to a set with $m > 5$ free locations with $\text{mod}(m, 3) \neq 0$ (which implies $m \geq 7$); say this set is to the right of player i, i.e., $(\ell+2)_{+(m-1)}$, and does not contain an endpoint. Suppose that a (possibly empty) set of players $\{n\}$, with $\#\{n\} =$

$\lfloor (m-5)/3 \rfloor$ and $j > \ldots > j' > j'' > j''' > i$ for all $j, \ldots, j''' \in \{n\}$, choose locations $c_{j'''} = \ell + m$, $c_{j''} = \ell + m - 3$, $c_{j'} = \ell + m - 6$, …, and $s_j = \ell + 7$ (if $\mathrm{mod}(m,3) = 1$) or $s_j = \ell + 8$ (if $\mathrm{mod}(m,3) = 2$). This will give players $j''', \ldots, j'$ a utility of 3. If $\mathrm{mod}(m,3) = 1$, player j is left with the choice of leaving a set with (i) four free locations, (ii) two sets with one and three free locations, respectively, or (iii) two sets with two free locations each. In case (i), as the analysis in the preceding paragraph has shown, $\mathrm{E}[U_j] = 5/2$. In case (ii), $U_j = 2$ because the one free location will be chosen by some player $k > j$. In case (iii), $\mathrm{E}[U_j] = 9/4$ because $U_j = 3$ is only possible if two players after j choose the left- and rightmost free location, respectively. Thus, player j will indeed choose $c_j = \ell + 7$. The same can be shown for the case $\mathrm{mod}(m,3) = 2$. These strategies for players in $\{n\}$ thus form a best response, which, while not the only best response, must be expected to occur with positive probability by player i. Therefore, we end up at a situation with either four or five free location to the right of player i, leaving player i with a utility of less than 3, and $c_i = \ell$ cannot be an equilibrium if player i also has available a strategy which guarantees him a utility of $U_i = 3$. This part of the proofs works analogously for $\mathrm{mod}(m,3) = 2$.

As we have seen, if a player i's choice leads to a set with $m > 0$, $\mathrm{mod}(m,3) \neq 0$, free locations to his left or right, not including an endpoint, then $\mathrm{E}[U_i] < 3$. Now suppose a player $i \geq 2$ can guarantee himself a utility of $U_i = 3$ by leaving a set with $m > 0$, $\mathrm{mod}(m,3) = 0$, free locations to his left or right. Then it is easy to see that a player $i-1$ has the same option. Because a player who only leaves a set with $m = 3$ will end up with a utility of 3, the proof by induction is complete. It is then a straightforward exercise to include subgames with two free locations including an endpoint in the analysis. ■

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