THE UNIVERSITY of York

Discussion Papers in Economics

No. 11/27

All-pay aspects of decision making under public scrutiny

By

Thomas Giebe and Paul Schweinzer

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

All-pay aspects of decision making under public scrutiny*

Thomas Giebe† Paul Schweinzer‡

Abstract

We study decision making processes with non-standard all-pay structures. In our first group of applications, individual members of some institution—due to public pressure or expectation—propose reductions of their own income, e.g., corporate board members reducing their bonus payments in an economic downturn, or politicians who reduce their expense allowances after some scandal. In such situations, the most aggressive proposal might carry the day and win public credibility or moral kudos for the proposer. Everyone, however, suffers the cost of that winning proposal. In the second group of applications, all participants bear the total cost of all bids as, e.g., under the filibuster strategy of delaying legislative action. The common features of these situations are a winner-take-all structure, non-standard payment rules, and nonvoluntary participation. We find that, in the equilibria of these games, everybody suffers a loss (with the possible exception of the winner).
(JEL *C7, D7.* Keywords: *Auctions, Contests, Truth-telling.*)

1 Introduction

We analyse decision making processes with all-pay structures which are different from the classic all-pay auction or war-of-attrition settings. Our main idea is to explore situations where decision makers take actions against their own narrow self interest because background 'participation' considerations force them to maintain public goodwill. In our first class of applications we consider situations where decision makers of some institution find themselves under public pressure to reevaluate the status quo regarding, e.g., excessive bonus payments, expense allowances, subsidies, or vacation regulations. Situations might be conceivable in which public scrutiny forces decision makers to accept the most stringent proposal, conveying on the proposer a gain in public credibility.[1] This public esteem may arise from the proposer's connection with the righteous policy while competing, less daring proposals may be forgotten. Another example involving only a subset of the public is the US debate on a deal to raise the federal debt ceiling of July 2011. There, threatened withdrawal of support by parts of

*Thanks to Peter Simmons and Bipasa Datta for helpful comments. †Microeconomics, Technische Universität Berlin, Str. des 17. Juni 135, 10623 Berlin, Germany, thomas.giebe@tu-berlin.de. ‡Department of Economics, University of York, Heslington, York YO10 5DD, United Kingdom, paul.schweinzer@york.ac.uk. (12-Dec-2011)

[1] "The criminal justice system has become a tool of the Kremlin and its commercial allies. Russians of all sorts loathe such cronyism. Both Mr Putin and Mr Medvedev have talked about tackling graft, but done nothing. If they took action, they would lose some power, but win kudos." (The Economist, *The Cracks Appear*, 10-Dec-2011)

the Republican Tea party basis can be argued to have led the Republican side into a similar position: "the G.O.P. has just demonstrated its willingness to risk financial collapse unless it gets everything its most extreme members want."[2] At any rate, in our examples, all participants sustain the cost of the winning proposal. Importantly, in these situations, nonparticipation (in a public debate or the vote of a corporate board) might either be inconsequential if players are bound to the decision anyway or may lead to direct and prohibitively large payoff consequences. We model this type of situation as an all-pay auction where the highest bid wins and all participants pay that winning bid. We find that, in the symmetric equilibrium, there is systematic overbidding, i.e., either all players make a loss or only the winner can avoid a loss.[3]

In the second class of applications, we study all-pay structures where all participants pay (a function of) the sum of all efforts. An example is filibustering, the parliamentary tactic of extended speechmaking, typically employed for the purpose of delaying legislative action or 'talking out a bill'.[4] This payment rule allows the study of decision making processes where through perseverance, extended speaking, etc, players try to convince each other to follow their respective lead, or, simply, to frustrate the others' proposals. In the end one player (or group of players) wins while all participants bear the total cost of all participants' effort, e.g., in terms of time or loss of reputation in the public opinion. Again, voluntary participation is not an issue, since the single decision that we study is part of a larger game in which the players have overwhelming incentives to participate.

We analyse these applications in the framework of uniform-price versions of the classic war of attrition. Formally, a two-player war of attrition is equivalent to a second-price all-pay auction in which the highest bidder wins the prize, and both players pay the loser's bid. In the first 'expenses' application discussed above, we use an all-pay auction where all players pay the highest bid. In the second 'filibuster' application we use an all-pay-all auction where each player's payment depends on the sum of all bids.

In both cases we find that, for certain distributions of valuations, the symmetric equilibrium we derive exhibits truthful bidding, i.e., players bid their true valuations. This implies that the winner's equilibrium profit is zero while everyone else suffers a loss. In order to shed more light on these truth-telling results, we study payment rules which are more general than those described above. In the symmetric equilibria of the games we study we generally find that all players suffer a loss, possibly with the exception of the winner. The details depend on the distribution of valuations and the number of players. It is well known in the literature that rent-seeking behaviour may lead to welfare losses. An example are young graduates who are attracted by the large bonuses paid in the financial sector although they might be socially more gainfully employed elsewhere.[5] Viewed from this angle and focusing on situations where a reduction in the decision makers' payoffs is welfare-

[2] Paul Krugman, *The President Surrenders*, op-ed, New York Times, 31-Jul-2011.

[3] An economic application of this idea is advertising on the basis of (differentiated) Bertrand price competition. For instance, Google's strategy of providing free, web-based email and office-application services has the same flavour of binding competitors to their lowest price offer.

[4] One of the first known practitioners of the filibuster was the Roman senator Cato the Younger. In debates over legislation proposed by Julius Caesar, Cato obstructed the measure by speaking continuously until nightfall. As the Roman Senate had a rule requiring all business to conclude by dusk, Cato's purposefully long-winded speeches were an effective device to forestall a vote.

[5] For details and a wealth of further applications of the basic ideas of our model, see Frank and Cook (1995).

increasing, a result of our simple auction-theoretic model is that forcing the players to debate in public and rewarding the most aggressive proposal may improve welfare although decision makers independently maximise their own payoffs.

It appears that situations with the general flavour outlined in our examples have not previously been studied formally. Moreover, it may come as a surprise that a small modification of the well-studied all-pay auction framework can be used to characterise the main strategic incentives players face in this type of interaction.

Literature

Our paper relates to the study of all-pay auction games. In a first-price all-pay auction, each player pays their own bid, with the prize being awarded to the highest bidder. There are not too many references to this concept in the recent literature. The ones that we are aware of were made by Krishna and Morgan (1997), Hopkins and Kornienko (2007), Kaplan, Luski, Sela, and Wettstein (2002), and Klose and Kovenock (2011a) as well as by Konrad (2008) in the survey literature. Krishna and Morgan (1997) study the standard, second-prize war of attrition, both Hopkins and Kornienko (2007) and Kaplan, Luski, Sela, and Wettstein (2002) explore pricing rules which are different from ours, and Klose and Kovenock (2011a) is a study under complete information. Although Klose and Kovenock (2011b) investigate a similar theme as we do—that extremism can sometimes drive out moderation—both their analysis and research questions are wholly different. Güth and van Damme (1986) and Amann and Leininger (1996) discuss hybrid versions between the first- and second-price sealed-bid all-pay auctions. In this hybrid format, the prize is won by the highest bidder but the loser pays her bid while the winner pays a convex combination of her own and the losers bid. The type of auction we consider relates to those with externalities; for recent contributions to this field see Maasland and Onderstal (2007) or, again, Klose and Kovenock (2011a).

The two papers which are closest to our analysis are Baye, Kovenock, and de Vries (2005) and Goeree and Turner (2000). The latter introduce the all-pay-all auction format—in which all bidders pay a weighted sum of all bids—as optimal format when all bidders receive some share of the amount of money raised. Thus, each bidder has an incentive to drive up the price because every participant receives a fixed share of the auction's revenue. The paper by Baye, Kovenock, and de Vries (2005) studies litigation systems. As in our paper, their payoffs are functions of players' expenditures. They study so called fee-shifting rules which are relevant in the legal context. Our payment rules, however, violate their assumption (A3) 'internalised legal cost,' which says that each individual effort is borne by exactly one party.

To the best of our knowledge, however, the uniform pricing rules in all-pay auctions presented in this paper have not been previously studied in the literature. In particular, our finding that truth-telling can be an equilibrium feature of the war of attrition in certain environments is a novelty.

2 The model

There is a set $\mathcal{N}$ of $n > 1$ risk-neutral, ex-ante identical bidders $i \in \mathcal{N}$ who each value an object at θ_i.[6] These valuations are privately known; all the opponents and the auctioneer know is the symmetric distribution $F(\theta)_{[0,1]}$, with strictly positive density $f(\theta) = F'(\theta)$, from which valuations are drawn. We denote order statistics as follows: The k-th highest of s independent draws from cdf F is $\Theta_{(k:s)}$ and its distribution is denoted by $F^{(k:s)}$. Admissible bids are positive; a zero bid is interpreted as nonparticipation. In that case, a bidder realises the symmetric and commonly known outside option $\bar{u} \in (-\infty, \infty)$.

We model our first 'expenses' application as an all-pay auction where every participant pays (a fraction of) the highest bid. Thus, all participants are bound by the decision of the player who is willing to bear the largest cost. This player wins and realises her valuation. For our second 'filibuster' application, we analyse an all-pay auction where every participant pays (a function of) the sum of all bids. Thus, each participant suffers from the combined effort of all participants. For both games, we derive the symmetric equilibrium. A feature of these arising equilibria is that, under certain conditions, equilibrium bids exhibit truth-telling (i.e., bidding the true valuation is an equilibrium).

It is important for our analysis that the public takes an interest in the decisions being taken. Otherwise decision makers may well reach a more beneficial outcome by colluding. As argued in the introduction and for the situations described there, it is a natural assumption that players find it in their interest to participate in our auctions. Nevertheless, for our first game, we spell out the equilibrium for all symmetric outside options. For the second game, we only derive the full-participation case and the corresponding truthful bidding result.

3 Example

Consider an auction where the highest bidder wins and every participant pays this bid. Consider player $i \in \mathcal{N}$, with $n = 2$, and index the opponent by $j = 3 - i$. (In this example we assume that the players' equilibrium bidding function $\beta(\theta_i)$ is monotonic.) Being of type $\theta_i \in [0, 1]$, player i chooses her bid b such as to

$$\max_b \int_0^{\beta^{-1}(b)} (\theta_i - b) f(\theta_j) d\theta_j - \int_{\beta^{-1}(b)}^1 \beta(\theta_j) f(\theta_j) d\theta_j \tag{1}$$

subject to a participation constraint which we assume to hold. Taking the derivative wrt b, we obtain the foc

$$f(\beta^{-1}(b))(\theta_i + \beta(\beta^{-1}(b)) - b) \frac{d\beta^{-1}(b)}{db} - \int_0^{\beta^{-1}(b)} f(\theta_j)\, d\theta_j = 0 \tag{2}$$

[6] In the interpretation of (most of) our examples, this is the value the individual attaches to public approval.

which, after invoking symmetry, $\beta^{-1}(b) = \theta_i$ and $\frac{d\beta^{-1}(b)}{db} = \frac{1}{\beta'(\theta_i)}$, reduces to

$$\beta'(\theta) = \frac{f(\theta)}{F(\theta)}\theta. \tag{3}$$

By integrating up both sides over $[0, \theta_i]$, using the boundary condition $\beta(0) = 0$, we obtain the bidding function[7]

$$\int_0^{\theta_i} \beta'(s)ds = \int_0^{\theta_i} \frac{f(s)s}{F(s)}ds \iff \beta(\theta_i) = \int_0^{\theta_i} \frac{f(s)s}{F(s)}ds, \tag{4}$$

which, for the special case of Uniform valuations, $F(\theta) = \theta$, implies truth-telling, $\beta(\theta_i) = \theta_i$.

4 Analysis

4.1 Everyone pays (a fraction of) the largest bid

In the following, we analyse the all-pay auction game where all bidders pay the same constant fraction $\alpha > 0$ of the largest bid. The symmetric equilibrium of this auction can be characterised as follows. For sufficiently low outside options, there is full participation and bidders employ monotonic bidding functions, for sufficiently high outside options there is no participation, while for an intermediate range of outside options, bidders participate and the bid is monotonic only if bidders' valuations exceed a threshold value $\hat{\theta}$. The function $\Pi(\hat{\theta}, \theta_i)$ is the expected equilibrium auction payoff of a bidder with valuation $\theta_i > \hat{\theta}$ where outside options are such that only players with valuations above $\hat{\theta}$ participate. All proofs can be found in the appendix.

Proposition 1. *Consider the uniform-price all-pay auction, with symmetric outside options, $\bar{u} \in (-\infty, \infty)$, where all bidders pay the fraction $\alpha > 0$ of the highest bid. Bidders issuing positive bids participate in the auction while a player who bids zero receives her outside option. The auction has a symmetric pure-strategy equilibrium as follows. Define*

$$\Pi(\hat{\theta}, \theta_i) = 1 - (n-1)\int_{\hat{\theta}}^{1} \frac{f(t)t}{F(t)}dt - \int_{\theta_i}^{1} F^{n-1}(t)dt, \tag{5}$$

where $\hat{\theta} = \hat{\theta}(\bar{u})$ is the lowest valuation bidder who participates in the auction. The above payoff depends on the outside option and is implicitly defined by $\bar{u} = \Pi(\hat{\theta}, \hat{\theta})$. The equilibrium bidding

[7] Given full participation, this boundary condition is intuitive: If the valuation is zero, then winning has no upside. Thus, the zero type would focus on minimising payments: By making the lowest feasible bid, the bidder minimises the price in the event of winning. This does not hold if there is a nonnegative outside option, in which case a low-value player might prefer not to participate. Also note that (4) might not exist for all distributions. See the discussion of Proposition 1.

function is

$$\beta(\theta_i, \bar{u}) = \begin{cases} \frac{n-1}{\alpha}\int_0^{\theta_i} \frac{sf(s)}{F(s)}ds & \text{if } \bar{u} \leq \Pi(0,0) \\ \frac{n-1}{\alpha}\int_{\hat{\theta}}^{\theta_i} \frac{sf(s)}{F(s)}ds & \text{if } \Pi(0,0) < \bar{u} < 1 \text{ and } \theta_i > \hat{\theta} \\ 0 & \text{if } \bar{u} \geq 1 \text{ or } \Pi(0,0) < \bar{u} < 1 \text{ and } \theta_i \leq \hat{\theta}. \end{cases} \tag{6}$$

Note that (6) might not exist for all distributions. As long as the inverse elasticity of the distribution $f(s)s/F(s)$ converges for $s \to 0$, however, the equilibrium exists. Perhaps surprisingly, for certain parameters, this symmetric equilibrium exhibits truthful bidding. For example, under the uniform distribution and sufficiently small outside options (i.e., $\alpha = n-1$ and $\bar{u} \leq -(n-1)^2/n$), all players participate and bid their true valuations.

Lemma 1 (Truthful bidding)**.** *If $F(\theta) = \theta^{\alpha/(n-1)}$, and $\bar{u} \leq -\alpha^2/(1+\alpha)$, then in the symmetric equilibrium, all players bid their valuations, $\beta(\theta_i) = \theta_i$.*

It is easy to show that this truth-telling result extends to the case of asymmetric outside options as long as it is common knowledge that all outside options are below the level given in Lemma 1. Moreover, it is as easy to see that the equilibrium might exhibit overbidding by all players. For example, if $F(\theta) = \theta^k$, where $k > \alpha/(n-1)$ and $\bar{u}$ sufficiently small, we have $\beta(\theta_i) = (k(n-1)/\alpha)\theta_i > \theta_i$.

4.2 The price as a function of all bids

We now turn to the second set of applications, modelled as an all-pay-all auction where all bidders pay the sum of all bids. Again, we analyse a slightly more general payment rule in order to generalise our truthful bidding result in Lemma 2. In order to simplify the presentation, we ignore the participation issue in this subsection (see the introduction for a discussion). Thus, although in principle we could follow the same route as in the previous subsection and explicitly model outside options, here all players make nonnegative bids and fully participate by assumption.

Proposition 2. *Consider an auction where every bidder pays a function of all bids, $\alpha \sum_{j=1}^{n} (b_j)^k$, with $\alpha, k > 0$, where b_j is bidder j's bid. Given full participation, the auction has a symmetric equilibrium with bidding functions*

$$\beta(\theta) = \left(\frac{1}{\alpha} \int_0^{\theta} \left(F^{n-1}(s)\right)' s ds \right)^{\frac{1}{k}}. \tag{7}$$

As in the previous subsection, we obtain a truthful bidding result here as well. As an example, consider the uniform distribution. There, the parameters $k = n$ and $\alpha = (n-1)/n$ implement truth-telling.

Lemma 2 (Truthful bidding)**.** *If $F(\theta) = \theta^t$, with $t > 0$, and the auction parameters are $k = t(n-1)+1$ and $\alpha = (k-1)/k$, then the equilibrium exhibits truth-telling $\beta(\theta) = \theta$.*

Again, it is not hard to see that the bidding equilibrium in Proposition 2 as well as the truth-telling result hold for any (a)symmetric outside options as long as it is common knowledge that all outside options are sufficiently unattractive to ensure full participation. As in the previous model, equilibrium overbidding is easily obtained, e.g. by choosing α sufficiently small for given n. The proof is similar to that of Lemma 2 above: Just replace the equality sign in (22) by "$<$" and maintain $k = t(n-1)+1$. Then overbidding is obtained for $\alpha < t(n-1)/(t(n-1)+1)$. The rhs of that inequality approaches 1 for large n.

5 Conclusion

We employ uniform-price versions of the well-studied all-pay auction model under symmetric independent private valuations to study situations where all players pay either (a function of) the winner's bid or (a function of) the sum of all bids. We find that in the symmetric equilibria of the games we study, all players—with the possible exception of the winner—make a loss, depending on the distribution of values and the number of participants.

We find that both specifications of the auction admit truthful bidding in equilibrium for certain distributions of valuations. Our study of the first set of applications is motivated by decision making problems where public opinion forces players to compete on the reduction of own income. The second setting applies to problems where the player with the most stamina succeeds and every participant suffers the participants' combined effort costs.

There is a well known argument that the availability of excessive rents, e.g. due to bonus payments in the financial sector, attract too much talent from a welfare perspective. A policy implication of our findings is that, if a reduction of the decision makers' income increases welfare, then an open debate and a reward of the most aggressive proposal—for instance, through an increased probability of reelection—might achieve a welfare improvement even though the decision is a result of independent utility maximisation by the participants. This, of course, speaks in favour of transparency and public scrutiny of decision making processes.

Appendix

Proof of Proposition 1. If $\bar{u} \geq 1$, the outside option is superior to the auction, for all players. Thus, $\beta(\theta_i, \bar{u}) = 0$, see (6). In the following, we restrict attention to $\bar{u} \in (-\infty, 1)$.

Suppose there is a symmetric equilibrium where all bidders bid zero (and take their outside options) if their valuations are below the common threshold value $\hat{\theta}$. If their valuations exceed this value, then they bid according to a strictly monotonic bidding function. We first determine this bidding function, taking the threshold value $\hat{\theta} \in [0,1]$ as given. We start by considering bidder i's decision problem when her value exceeds the threshold value, $\theta_i > \hat{\theta}$. W.l.o.g. we only consider i's positive bids $\tilde{\beta}(z)$ for $z \in (\hat{\theta}, 1]$. Bidder i's expected profit is

$$\begin{aligned}
\pi_i(z,\theta_i) &= F^{(1:n-1)}(z)(\theta_i - \alpha\beta(z)) - (1 - F^{(1:n-1)}(z))\,\mathbb{E}[\alpha\beta(\Theta_{(1:n-1)}|\Theta_{(1:n-1)} > z] \\
&= F^{n-1}(z)(\theta_i - \alpha\beta(z)) - \int_z^1 \alpha\beta(s)dF^{n-1}(s); \\
\frac{\partial\pi_i(z,\theta_i)}{\partial z} &= (F^{n-1}(z))'(\theta_i - \alpha\beta(z)) - F^{n-1}(z)\alpha\beta'(z)) + \alpha\beta(z)(F^{n-1}(z))' = 0 \\
\iff \quad \beta'(z) &= \frac{1}{\alpha}\tfrac{(F^{n-1}(z))'}{F^{n-1}(z)}\theta_i = \frac{(n-1)}{\alpha}\frac{F(z)^{n-2}(z)f(z)}{F^{n-1}(z)}\theta_i = \frac{(n-1)}{\alpha}\frac{f(z)}{F(z)}\theta_i.
\end{aligned} \tag{8}$$

In symmetric equilibrium, $z = \theta_i$. Thus, with boundary condition $\beta(\hat{\theta}) = 0$,

$$\beta(\theta_i) = \frac{(n-1)}{\alpha}\int_{\hat{\theta}}^{\theta_i} \frac{f(s)}{F(s)} s ds. \tag{9}$$

This corresponds to the terms in the first two cases of (6) (with $\hat{\theta} = 0$ in the first case). The candidate (6) says that if i's value is below $\hat{\theta}$ then i should bid zero. We now turn attention to i's expected profit if she follows (9) and $\theta_i > \hat{\theta}$.

$$\begin{aligned}
\pi_i(\hat{\theta},\theta_i) &= F^{n-1}(\theta_i)(\theta_i - \alpha\beta(\theta_i)) - \int_{\theta_i}^1 \alpha\beta(s)dF^{n-1}(s) \\
&= F^{n-1}(\theta_i)\left(\theta_i - (n-1)\int_{\hat{\theta}}^{\theta_i}\frac{f(t)t}{F(t)}dt\right) - \underbrace{\int_{\theta_i}^1 (n-1)\int_{\hat{\theta}}^s \frac{f(t)t}{F(t)}dt(F^{n-1}(s))'ds}_{=:(A)}.
\end{aligned} \tag{10}$$

Interchanging the order of integration, the last term can be written as

$$\begin{aligned}
(A) &= \int_{\hat{\theta}}^{\theta_i}\int_{\theta_i}^1 (F^{n-1}(s))'ds(n-1)\frac{f(t)t}{F(t)}dt + \int_{\theta_i}^1\int_t^1 (F^{n-1}(s))'ds(n-1)\frac{f(t)t}{F(t)}dt \\
&= \int_{\hat{\theta}}^{\theta_i}(1 - F^{n-1}(\theta_i))(n-1)\frac{f(t)t}{F(t)}dt + \int_{\theta_i}^1 (1 - F^{n-1}(t))(n-1)\frac{f(t)t}{F(t)}dt.
\end{aligned} \tag{11}$$

After reinserting (A) into (10) and straightforward simplification, we obtain

$$\pi_i(\hat{\theta},\theta_i) = F^{n-1}(\theta_i)\theta_i - (n-1)\int_{\hat{\theta}}^1 \frac{f(t)t}{F(t)}dt + \int_{\theta_i}^1 \underbrace{F^{n-1}(t)(n-1)\frac{f(t)t}{F(t)}}_{=(F^{n-1}(t))'t}dt. \tag{12}$$

Integrating the last term by parts, we get

$$\begin{aligned}
\pi_i(\hat{\theta},\theta_i) &= F^{n-1}(\theta_i)\theta_i - (n-1)\int_{\hat{\theta}}^1 \frac{f(t)t}{F(t)}dt + 1 - F^{n-1}(\theta_i)\theta_i - \int_{\theta_i}^1 F^{n-1}(t)dt \\
&= 1 - (n-1)\int_{\hat{\theta}}^1 \frac{f(t)t}{F(t)}dt - \int_{\theta_i}^1 F^{n-1}(t)dt,
\end{aligned} \tag{13}$$

which equals $\Pi(\hat{\theta},\theta_i)$ in (5). Thus, (9) is an equilibrium only if $\Pi(\hat{\theta},\theta_i)$ is superior to the outside option for all valuations $\theta \in [\hat{\theta}, 1]$. Since $\Pi(\hat{\theta},\theta_i)$ is strictly increasing in the valuation θ_i (only the last term in $\Pi(\hat{\theta},\theta_i)$ is a function of the valuation), it must hold that $\Pi(\hat{\theta},\hat{\theta}) \geq \bar{u}$ in order for (9) to be an equilibrium bid. Moreover, $\Pi(\hat{\theta},\hat{\theta})$ must not exceed the outside option since otherwise positive bids are profitable even for some valuations below $\hat{\theta}$. Thus, the equilibrium threshold value must satisfy $\Pi(\hat{\theta},\hat{\theta}) = \bar{u}$.

Note that

$$\Pi(\hat{\theta},\hat{\theta}) = 1 - \int_{\hat{\theta}}^{1} (F^{n-1}(t))' t dt \tag{14}$$

is increasing in $\hat{\theta}$ (recall that $\hat{\theta} \in [0,1]$). Thus, by $\Pi(\hat{\theta},\hat{\theta}) = \bar{u}$, the threshold value is increasing in the value of the outside option. In turn, for sufficiently low outside option, the threshold value is zero. The critical outside option is implicitly given by $\Pi(0,0) = \bar{u}$. Thus, for $\bar{u} \leq \Pi(0,0)$, the threshold value is zero and there is full participation in the auction for all players and valuations.

It remains to discuss 'intermediate' outside options, defined by $\Pi(0,0) \leq \bar{u} < 1$. For these outside options the equilibrium threshold value is in the range $\hat{\theta} \in (0,1)$. Thus, bidders with valuations $\theta_i \leq \hat{\theta}$ take their outside options, while all higher valuations $\theta_i > \hat{\theta}$ participate in the auction with bids according to (9). Thus, we have completed the derivation of the equilibrium candidate. It remains to be shown that (9) does not only satisfy the foc of i's best response problem but that it is indeed a maximiser for $\theta_i > \hat{\theta}$.

Similar to the above, consider $\theta, z \in (\hat{\theta}, 1]$. Start from (8) and insert (9) cancelling out α:

$$\pi_i(z,\theta) = F^{n-1}(z)\left(\theta - (n-1)\int_{\hat{\theta}}^{z} \frac{f(k)k}{F(k)} dk\right) - \underbrace{\int_{z}^{1} (n-1)\int_{\hat{\theta}}^{s} \frac{f(k)k}{F(k)} dk (F^{n-1}(s))' ds}_{=:(A)} \tag{15}$$

Interchanging the order of integration, the term (A) can be written as:

$$(A) = (n-1)\Bigg(\int_{\hat{\theta}}^{z} \frac{f(k)k}{F(k)} \underbrace{\int_{z}^{1} (F^{n-1}(s))' ds}_{=1-F^{n-1}(z)} dk + \int_{z}^{1} \frac{f(k)k}{F(k)} \underbrace{\int_{k}^{1} (F^{n-1}(s))' ds}_{=1-F^{n-1}(k)} dk\Bigg) \tag{16}$$

Now, reinsert into (15), cancel out the terms $\pm\, F^{n-1}(z)(n-1)\displaystyle\int_{\hat{\theta}}^{z} \frac{f(k)k}{F(k)} dk$, and get

$$\begin{aligned}
\pi_i(z,\theta) &= \underbrace{F^{n-1}(z)\theta}_{=:(B)} - (n-1)\left(\int_{\hat{\theta}}^{z} \frac{f(k)k}{F(k)} dk + \int_{z}^{1} \frac{f(k)k}{F(k)} (1 - F^{n-1}(k)) dk\right) \\
&= \underbrace{\theta - \int_{z}^{1} (F^{n-1}(k))' \theta dk}_{=(B)} - (n-1)\int_{\hat{\theta}}^{1} \frac{f(k)k}{F(k)} dk + \underbrace{(n-1)\int_{z}^{1} \frac{f(k)k}{F(k)} F^{n-1}(k)) dk}_{\int_z^1 (F^{n-1}(k))' k dk} \qquad (17) \\
&= \theta - (n-1)\int_{\hat{\theta}}^{1} \frac{f(k)k}{F(k)} dk + \underbrace{\int_{z}^{1} (F^{n-1}(k))' (k-\theta) dk}_{=:(C)}.
\end{aligned}$$

Note, that only (C) depends on z. Thus,

$$\pi_i(\theta,\theta) - \pi_i(z,\theta) = \int_\theta^z (F^{n-1}(k))'(k-\theta)dk. \tag{18}$$

Since this is strictly positive for all $z \neq \theta$, where $z, \theta \in [0,1]$, this completes the proof. □

Proof of Lemma 1. With $F(s) = s^{\alpha/(n-1)}$, we have $sf(s)/F(s) = \alpha/(n-1)$. Then the first case of (6) straightforwardly simplifies to $\beta(\theta_i, \bar{u}) = \theta_i$. Similarly, replace $tf(t)/F(t)$ in (5) by $\alpha/(n-1)$, simplify, and solve $\bar{u} = \Pi(0,0)$. □

Proof of Proposition 2. Obviously, (7) is strictly monotonic increasing. Suppose player i's rivals bid according to (7). W.l.o.g. we only consider bidder i's deviating bids $\beta(z)$ where $z \in [0,1]$. Bidder i's expected payoff is (recall that $F^{(1:n-1)} = F^{n-1}$)

$$\begin{aligned} \pi_i(z,\theta) &= F^{n-1}(z)\theta - \alpha\beta^k(z) - \alpha(n-1)\,\mathbb{E}[\beta(\Theta)^k], \\ \frac{\partial \pi_i(z,\theta)}{\partial z} &= (F^{n-1}(z))'\theta - (\alpha\beta^k(z))'. \end{aligned} \tag{19}$$

Applying the symmetric equilibrium condition, $z = \theta$ and inserting the candidate, the foc is satisfied

$$(F^{n-1}(\theta))'\theta - \frac{\partial}{\partial\theta}\left(\int_0^\theta \left(F^{n-1}(s)\right)' s ds\right) = 0. \tag{20}$$

As a proof of sufficiency, we show that $\pi_i(\theta,\theta) > \pi_i(z,\theta)$ for all $z \in [0,1]$, $z \neq \theta$. Inserting the candidate in (19), we get

$$\begin{aligned} \pi_i(\theta,\theta) - \pi_i(z,\theta) &= \theta(F^{n-1}(\theta) - F^{n-1}(z)) - \int_z^\theta \left(F^{n-1}(s)\right)' s ds \\ &= \theta \int_z^\theta \left(F^{n-1}(s)\right)' ds - \int_z^\theta \left(F^{n-1}(s)\right)' s ds \\ &= \int_z^\theta \left(F^{n-1}(s)\right)' (\theta - s) ds > 0. \end{aligned} \tag{21}$$

□

Proof of Lemma 2. Inserting $F(s) = s^t$ into the rhs of (7) and setting the term equal to θ (truthful bidding), we get

$$\theta = \left(\frac{1}{\alpha}\int_0^\theta \left(s^{t(n-1)}\right)' s ds\right)^{\frac{1}{k}} \;\Rightarrow\; \alpha\theta^k = \frac{t(n-1)}{t(n-1)+1}\theta^{t(n-1)+1}. \tag{22}$$

This equation is satisfied if we insert the proposed $\alpha = (k-1)/k$ and $k = t(n-1)+1$. □

References

Amann, E., and W. Leininger (1996): "Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case," *Games and Economic Behavior*, 14, 1–18.

Baye, M. R., D. Kovenock, and C. G. de Vries (2005): "Comparative Analysis of Litigation Systems: An Auction-Theoretic Approach," *The Economic Journal*, 115(505), 583–601.

Frank, R. H., and P. J. Cook (1995): *The Winner-Take-All Society*. Penguin, New York.

Goeree, J., and J. Turner (2000): "All-pay-all auctions," *University of Virginia*, Mimeo.

Güth, W., and E. van Damme (1986): "A Comparison of Pricing Rules for Auctions and for Division Games," *Social Choice and Welfare*, 3, 177–98.

Hopkins, E., and T. Kornienko (2007): "Cross and Double Cross: Comparative Statics in First Price and All Pay Auctions," *B.E. Journal of Theoretical Economics*, 7, Article 19.

Kaplan, T., I. Luski, A. Sela, and D. Wettstein (2002): "All-Pay Auctions with Variable Rewards," *Journal of Industrial Economics*, 50(4), 417–30.

Klose, B., and D. Kovenock (2011a): "The All-Pay Auction with Complete Information and Identity-Dependent Externalities," *University of Zürich, Discussion paper*.

——— (2011b): "Extremism drives out moderation," *University of Zürich, Discussion paper*.

Konrad, K. (2008): *Strategy and Dynamics in Contests*. Oxford University Press, Oxford.

Krishna, V., and J. Morgan (1997): "An Analysis of the War of Attrition and the All-Pay Auction," *Journal of Economic Theory*, 70(2), 343–62.

Maasland, E., and S. Onderstal (2007): "Auctions with Financial Externalities," *Economic Theory*, 32, 551–74.