

Applications to risk theory of a Monte Carlo multiple integration method

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Abstract

Evaluation of multiple integrals is a commonly encountered problem in risk theory, specially in ruin probability. Using Monte Carlo simulation we obtain an unbiased and consistent point estimator, and also confidence intervals as approximations of a special case of multiple integral frequently used in risk theory. The variance reduction achieved compared to straight simulation and some specific properties make this approach interesting when approximating ruin probabilities.

Keywords: Monte Carlo multiple integration; Variance reduction; Convolutions; Ruin probability

1. Introduction

Let us define the discrete stochastic process $\{S_t\}$

$$S_t = \sum_{i=1}^t Z_i, \quad Z_i \geq 0,$$

where Z_i , $i = 1, \dots, t$, are independent random variables with p.d.fs. $g_i(x)$, c.d.fs. $G_i(x)$ and $E\{Z_i\} = \mu_i < \infty$.

Let us define the following function:

$$\begin{aligned} \int_{\mathcal{R}(\mathbf{X})} \varphi(\mathbf{S}) = H_t(\mathbf{X}) &= P\{S_j \leq x_j, j = 1, \dots, t\} \\ &= \int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} G_t(x_t - s_{t-1}) g_{t-1}(s_{t-1} - s_{t-2}) \cdots g_2(s_2 - s_1) g_1(s_1) ds_1 \cdots ds_{t-1} \end{aligned} \quad (1.1)$$

where $\mathbf{X} = (x_1, x_2, \dots, x_t)$ and $\mathbf{S} = (s_1, s_2, \dots, s_{t-1})$.

Expression (1.1) considers the probability that the paths of stochastic process $\{S_t\}$ will be bounded by vector \mathbf{X} .

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Evaluating multiple integrals is one of the classical problems of numerical analysis, principal among the methods designed for this purpose, we highlight quadrature formulas, equidistributed sequences and Monte Carlo (Fishman, 1996, Chapter 2):

1.1. Quadrature formulas

The most commonly encountered multivariable quadrature formulas are direct extensions of quadrature formulas of the one-dimensional case. The number of evaluations of the function to integrate is n^t where n is the number of evaluation points for one-dimensional case and t is the dimension of the multiple integral, making this method very messy when t is not small because the number of evaluations is exponentially increased. Under certain conditions (Bahvalov, 1959; and Haber, 1970; Fishman, 1996) the absolute error of the approximation is $O(n^{-j/t})$, for some $j \geq 1$.

1.2. Equidistributed points

In this case the absolute error is $O(\delta n^{-1}(\ln(n))^t)$ under certain conditions (Haber, 1970; Niederreiter, 1978, 1992; Fishman, 1996).

1.3. Monte Carlo methods

The convergence of the approximation is $O(n^{-1/2})$ provided that $\int_{\mathfrak{R}^t} \varphi^2(\mathbf{X}) < \infty$.

Each approach has advantages and limitations. The convergence of deterministic methods seems clearly better (specially if the dimension of the integral t is not large) when they can be applied. Nevertheless, the applicability matters make Monte Carlo techniques competitive because the verification of the conditions we cited for deterministic methods becomes very difficult as t increases or very restrictive for function $\varphi(\mathbf{X})$.

Reproducing (Fishman, 1996, Chapter 2), we can argue that Monte Carlo methods can be applied considerably more broadly to functions that merely satisfy $\int_{\mathfrak{R}^t} \varphi^2(\mathbf{X}) < \infty$. Also, the Monte Carlo error depends on φ only through this integral, and in no way on the continuity and variational properties of φ . Another interesting fact is that Monte Carlo methods allow one to estimate error from generated data, whereas one needs to rely in considerably more global measures of error when employing deterministic techniques. Finally, the Monte Carlo convergence is always $O(n^{-1/2})$ regardless of the dimension of the integral t , this is an interesting aspect when t is large because in the deterministic methods this convergence worsens as t increases.

2. Simple unbiased estimator

We will introduce the estimator:

$$H_t(\mathbf{X}) \simeq \mathcal{H}_t(\mathbf{X}) = G_t(x_t - S_{t-1})G_{t-1}(x_{t-1} - S_{t-2}) \cdots G_2(x_2 - S_1)G_1(x_1), \quad (2.1)$$

where S_i , $i = 1, \dots, t$, are random numbers generated using the following density functions:

$$\begin{aligned} S_1 &\longrightarrow \mathcal{D}_1(s_1) = \frac{g_1(s_1)}{G_1(x_1)}, & s_1 &\in [0, x_1], \\ S_j &\longrightarrow \mathcal{D}_j(s_j) \frac{g_j(s_j - S_{j-1})}{G_j(x_j - s_{j-1})}, & s_j &\in [S_{j-1}, x_j], \quad j > 1. \end{aligned} \quad (2.2)$$

Theorem 1. *The estimator $\mathcal{H}_t(\mathbf{X})$ is unbiased.*

Proof. The expected value of the estimator $\mathcal{H}_t(\mathbf{X})$ can be expressed:

$$\begin{aligned}
E\{\mathcal{H}_t(\mathbf{X})\} &= \int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} G_t(x_t - s_{t-1}) G_{t-1}(x_{t-1} - s_{t-2}) \cdots G_2(x_2 - s_1) G_1(x_1) \\
&\quad \times \cdots \frac{g_1(s_1)}{G_1(x_1)} \frac{g_2(s_2 - s_1)}{G_2(x_2 - s_1)} \cdots \frac{g_{t-1}(s_{t-1} - s_{t-2})}{G_{t-1}(x_{t-1} - s_{t-2})} ds_1 \cdots ds_{t-1} \\
&= \int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} G_t(x_t - s_{t-1}) g_{t-1}(s_{t-1} - s_{t-2}) \cdots g_2(s_2 - s_1) g_1(s_1) ds_1 \cdots ds_{t-1} \\
&= H_t(\mathbf{X}) \quad \square
\end{aligned}$$

3. Variance of the simple estimator

The variance of the estimator will be studied using the following two theorems:

Theorem 2. *The variance of the estimator $\mathcal{H}_t(\mathbf{X})$ has an upper bound:*

$$\text{Var}\{\mathcal{H}_t(\mathbf{X})\} \leq \left[\prod_{i=1}^t G_i(x_i) H_t(\mathbf{X}) - [H_t(\mathbf{X})]^2 \right].$$

Proof. Due to the fact that $\mathcal{H}_t(\mathbf{X})$ is an unbiased estimator,

$$\text{Var}\{\mathcal{H}_t(\mathbf{X})\} = E\{(\mathcal{H}_t(\mathbf{X}))^2\} - [H_t(\mathbf{X})]^2. \quad (3.1)$$

Let us study the former expected value:

$$\begin{aligned}
E\{(\mathcal{H}_t(\mathbf{X}))^2\} &= \int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} G_t^2(x_t - s_{t-1}) G_{t-1}^2(x_{t-1} - s_{t-2}) \\
&\quad \times \cdots G_2^2(x_2 - s_1) G_1^2(x_1) \frac{g_1(s_1)}{G_1(x_1)} \frac{g_2(s_2 - s_1)}{G_2(x_2 - s_1)} \cdots \frac{g_{t-1}(s_{t-1} - s_{t-2})}{G_{t-1}(x_{t-1} - s_{t-2})} ds_1 \cdots ds_{t-1} \\
&\leq G_t(x_t) G_{t-1}(x_{t-1}) \cdots G_2(x_2) G_1(x_1) \\
&\quad \times \int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} G_t(x_t - s_{t-1}) G_{t-1}(x_{t-1} - s_{t-2}) \cdots G_2(x_2 - s_1) \\
&\quad \times G_1 \left(\frac{g_1(s_1)}{G_1(x_1)} \frac{g_2(s_2 - s_1)}{G_2(x_2 - s_1)} x_1 \right) \cdots \frac{g_{t-1}(s_{t-1} - s_{t-2})}{G_{t-1}(x_{t-1} - s_{t-2})} ds_1 \cdots ds_{t-1}, \quad (3.2)
\end{aligned}$$

because

$$\begin{aligned}
&\text{Max}[G_t(x_t - s_{t-1}) G_{t-1}(x_{t-1} - s_{t-2}) \cdots G_2(x_2 - s_1) G_1(x_1)] \\
&= G_t(x_t) G_{t-1}(x_{t-1}) \cdots G_2(x_2) G_1(x_1),
\end{aligned}$$

and $G_i(y)$ $i = 1, \dots, t$, are distribution functions and then non-decreasing. Substituting (3.2) into (3.1) we get the statement of the theorem. \square

Theorem 3. If $H_t(\mathbf{X}) < 1$ (nontrivial case) then the variance of the estimator $\mathcal{H}_t(\mathbf{X})$ is less than the variance of the direct simulation of the value of $H_t(\mathbf{X})$:

$$\text{Var}\{\mathcal{H}_t(\mathbf{X})\} < H_t(\mathbf{X}) - [H_t(\mathbf{X})]^2$$

Proof.

$$\begin{aligned} H_t(\mathbf{X}) &= \int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} G_t(x_t - s_{t-1}) G_{t-1}(x_{t-1} - s_{t-2}) \cdots G_2(x_2 - s_1) \\ &\quad \times G_1(x_1) \frac{g_1(s_1)}{G_1(x_1)} \frac{g_2(s_2 - s_1)}{G_2(x_2 - s_1)} \cdots \frac{g_{t-1}(s_{t-1} - s_{t-2})}{G_{t-1}(x_{t-1} - s_{t-2})} ds_1 \cdots ds_{t-1} \end{aligned}$$

It is obvious that

$$\begin{aligned} &[G_t(x_t - s_{t-1}) G_{t-1}(x_{t-1} - s_{t-2}) \cdots G_2(x_2 - s_1) G_1(x_1)]^2 \\ &\leq G_t(x_t - s_{t-1}) G_{t-1}(x_{t-1} - s_{t-2}) \cdots G_2(x_2 - s_1) G_1(x_1) \end{aligned}$$

because $G_i(x)$ are distribution functions.

If $H_t(\mathbf{X}) < 1$ then we always can find at least one vector $(s_1^*, s_2^*, \dots, s_{t-1}^*)$ for which

$$G_t(x_t - s_{t-1}^*) G_{t-1}(x_{t-1} - s_{t-2}^*) \cdots G_2(x_2 - s_1^*) G_1(x_1) < 1$$

and

$$\int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} \frac{g_1(s_1)}{G_1(x_1)} \frac{g_2(s_2 - s_1)}{G_2(x_2 - s_1)} \cdots \frac{g_{t-1}(s_{t-1} - s_{t-2})}{G_{t-1}(x_{t-1} - s_{t-2})} ds_1 \cdots ds = 1.$$

Then at least for $(s_1^*, s_2^*, \dots, s_{t-1}^*)$:

$$\begin{aligned} &[G_t(x_t - s_{t-1}^*) G_{t-1}(x_{t-1} - s_{t-2}^*) \cdots G_2(x_2 - s_1^*) G_1(x_1)]^2 \\ &< [G_t(x_t - s_{t-1}^*) G_{t-1}(x_{t-1} - s_{t-2}^*) \cdots G_2(x_2 - s_1^*) G_1(x_1)] \end{aligned}$$

and

$$\begin{aligned} E\{(\mathcal{H}_t(\mathbf{X}))^2\} &= \int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} G_t^2(x_t - s_{t-1}) G_{t-1}^2(x_{t-1} - s_{t-2}) \cdots G_2^2(x_2 - s_1) \\ &\quad \times G_1^2(x_1) \frac{g_1(s_1)}{G_1(x_1)} \frac{g_2(s_2 - s_1)}{G_2(x_2 - s_1)} \cdots \frac{g_{t-1}(s_{t-1} - s_{t-2})}{G_{t-1}(x_{t-1} - s_{t-2})} ds_1 \cdots ds_{t-1} \\ &< \int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} G_t(x_t - s_{t-1}) G_{t-1}(x_{t-1} - s_{t-2}) \cdots G_2(x_2 - s_1) G_1(x_1) \\ &\quad \times \frac{g_1(s_1)}{G_1(x_1)} \frac{g_2(s_2 - s_1)}{G_2(x_2 - s_1)} \cdots \frac{g_{t-1}(s_{t-1} - s_{t-2})}{G_{t-1}(x_{t-1} - s_{t-2})} ds_1 \cdots ds_{t-1} = H_t(\mathbf{X}). \end{aligned}$$

Finally

$$\text{Var}\{\mathcal{H}_t(\mathbf{X})\} = E\{(\mathcal{H}_t(\mathbf{X}))^2\} - [H_t(\mathbf{X})]^2 < H_t(\mathbf{X}) - [H_t(\mathbf{X})]^2. \quad \square$$

4. Sample mean estimator

Let us now define this new estimator as a sample mean of $\mathcal{H}_t(\mathbf{X})$:

$$\begin{aligned} \mathfrak{N}_t(\mathbf{X}, n) &= \frac{\sum_{i=1}^n \mathcal{H}_t^i(\mathbf{X})}{n} \\ &= \frac{\sum_{i=1}^n G_t(x_t - S_{t-1}^i) G_{t-1}(x_{t-1} - S_{t-2}^i) \cdots G_2(x_2 - S_1^i) G_1(x_1)}{n} \end{aligned} \quad (4.1)$$

using (2.1), where S_j^i are random numbers generated from the p.d.fs. (2.2) for $i = 1, \dots, n$ and $j = 1, \dots, t-1$:

$$\begin{aligned} S_1^i &\longrightarrow \mathcal{D}_1^i(s_1^i) = \frac{g_1(s_1^i)}{G_1(x_1)}, \quad s_1^i \in [0, x_1], \\ S_j^i &\longrightarrow \mathcal{D}_j^i(s_j^i) = \frac{g_j(s_j^i - S_{j-1}^i)}{G_j(x_j - S_{j-1}^i)}, \quad s_j^i \in [S_{j-1}^i, x_j], \quad j > 1 \end{aligned}$$

and $\{\mathcal{H}_t^i(\mathbf{X})\}_{i=1}^n$ is a sample of independent estimators.

As a sample mean of an unbiased estimator, $\mathfrak{N}_t(\mathbf{X}, n)$ is also unbiased and consistent with variance bounds (in the nontrivial case):

$$\begin{aligned} \text{Var}\{\mathfrak{N}_t(\mathbf{X}, n)\} &= \frac{\text{Var}\{\mathcal{H}_t(\mathbf{X})\}}{n} < \frac{H_t(\mathbf{X}) - [H_t(\mathbf{X})]^2}{n} \\ \text{Var}\{\mathfrak{N}_t(\mathbf{X}, n)\} &\leq \frac{\prod_{i=1}^t G_i(x_i) H_t(\mathbf{X}) - [H_t(\mathbf{X})]^2}{n}, \end{aligned} \quad (4.2)$$

under fairly general conditions that include $\int_{\mathfrak{N}} \phi^4(\mathbf{X}) < \infty$ (Fishman, 1996):

$$\lim_{n \rightarrow \infty} \mathfrak{N}_t(\mathbf{X}, n) \longrightarrow N \left[H_t(\mathbf{X}), \sqrt{\frac{\text{Var}\{\mathcal{H}_t(\mathbf{X})\}}{n}} \right] \quad (4.3)$$

and the interval estimation with a confidence level $1 - \alpha$ is:

$$\left[\mathfrak{N}_t(\mathbf{X}, n) \mp \phi(1 - \alpha) \sqrt{\frac{\text{Var}\{\mathcal{H}_t(\mathbf{X})\}}{n}} \right]. \quad (4.4)$$

We can use an estimator of the variance of the $\mathcal{H}_t(\mathbf{X})$:

$$\text{Var}\{\mathcal{H}_t(\mathbf{X})\} \simeq k = \frac{1}{n-1} \left(\sum_{i=1}^n (\mathcal{H}_t^i(\mathbf{X}))^2 - n \sum_{i=1}^n \mathcal{H}_t^i(\mathbf{X}) \right) \quad (4.5)$$

as recommended in Fishman (1996, p. 68), k is a strongly consistent estimator of $\text{Var}\{\mathcal{H}_t(\mathbf{X})\}$.

Then an asymptotically valid confidence interval can be

$$\left[\mathfrak{N}_t(\mathbf{X}, n) \mp \phi(1 - \alpha) \sqrt{\frac{k}{n}} \right]. \quad (4.6)$$

We can also avoid the use of an estimator for the variance substituting the result of Theorem 2 into (4.4) and get a broader confidence interval:

$$\left[\mathfrak{N}_t(\mathbf{X}, n) \mp \phi(1 - \alpha) \sqrt{\frac{\tau}{n}} \right], \quad (4.7)$$

where

$$\tau = \prod_{i=1}^t G_i(x_i) \mathfrak{N}_t(\mathbf{X}) - [\mathfrak{N}_t(\mathbf{X})]^2.$$

5. Some comments about the use of the estimator $\mathfrak{N}_t(\mathbf{X})$

The first aspect that should be considered is the true variance reduction, proved in Theorem 3, that could be achieved using the simple estimator $\mathcal{H}_t(\mathbf{X})$ compared with the straight simulation. The result of Theorem 2 is certainly an upper bound for the variance of the estimator, but when the values of vector $\mathbf{X} = (x_1, x_2, \dots, x_t)$ are large the true reduction of the variance will stay hidden. For this last reason we consider more proper the use of the estimated variance (4.5) in order to assess the true variance reduction achieved.

One of the main features of this method is the fact that if we increase the dimension t , we do not have to start again the simulation process as should be done in direct simulation. When we get the estimator from (4.1) and store these pairs of values:

$$(\mathcal{H}_t^i(\mathbf{X}), S_{t-1}^i), \quad i = 1, \dots, n,$$

then

$$\begin{aligned} \mathfrak{N}_{t+1}(\mathbf{X}, n) &= \frac{\sum_{i=1}^n \mathcal{H}_{t+1}^i(\mathbf{X})}{n} \\ &= \left(\frac{1}{n}\right) \sum_{i=1}^n G_{t+1}(x_{t+1} - S_t^i) G_t(x_t - S_{t-1}^i) G_{t-1}(x_{t-1} - S_{t-2}^i) \cdots G_2(x_2 - S_1^i) G_1(x_1) \\ &= \frac{\sum_{i=1}^n G_{t+1}(x_{t+1} - S_t^i) \mathcal{H}_t^i(\mathbf{X})}{n} \end{aligned}$$

where using (2.2) :

$$S_t^i \longrightarrow \mathcal{D}_t^i(s_t) = \frac{g_t(s_t^i - S_{t-1}^i)}{G_t(x_t - S_{t-1}^i)}, \quad s_t^i \in [S_{t-1}^i, x_t], \quad i = 1, \dots, n$$

and $\mathbf{X} = (x_1, x_2, \dots, x_t, x_{t+1})$.

This last result means that increasing one unit the dimension of the multiple integral only implies generating n random numbers more, and the total amount of random numbers required is $n(t-1)$, where t is the dimension considered. The saving of number of steps – random numbers in our case – becomes even more obvious when we need to evaluate the integral (1.1) $H_t(\mathbf{X})$ for $t = 1, 2, \dots, k$, one by one up to a certain integer k , as the convolutions in the solution of a renewal equation, evaluation of compound processes, solving Fredholm equations of the second kind using Neumann series, in these cases the total amount of steps still remains $n(t-1)$.

There is another fact that could make this method appealing, if we want to approximate $H_t(\mathbf{X})$ for different values of the last component of the vector $\mathbf{X}(x_t)$, it is not necessary to start another simulation again. We only need to evaluate again the values $G_t(x_t - S_{t-1}^i)$, $i = 1, \dots, n$, for the new x_t .

6. Applications to risk theory

The integral of expression (1.1) is frequently found in risk theory. Two examples could be the n -fold convolutions of compound processes that model the total claims or the ultimate non-ruin probability and discrete time

Table 1

Confidence intervals for ruin probability. $\alpha = 0.01$; $\Psi(U) = 1 - \sum_{t=0}^{\infty} \{\theta/(1+\theta)\} \{1/(1+\theta)\}^t G^{*t}(U)$; and Pareto Claim size $F(x) = 1 - (\mu/(\mu+x))^{\mu+1}$, $\mu = 1$

Security loading	Initial reserves(U)	Lower limit	Upper limit	Sum terms
$\theta = 0.10$	20	0.4942	0.5006	38
	100	0.1631	0.1695	89
	500	0.0244	0.0258	189
	1000	0.0111	0.0122	226
$\theta = 0.25$	20	0.2415	0.2468	35
	100	0.0505	0.0531	70
	500	0.0085	0.0087	100
	1000	0.0041	0.0044	103
$\theta = 0.50$	20	0.1177	0.1208	30
	100	0.0225	0.0234	50
	500	0.0041	0.0043	57
	1000	0.0021	0.0021	57
$\theta = 0.75$	20	0.0751	0.0770	27
	100	0.0143	0.0147	39
	500	0.0027	0.0028	41
	1000	0.0014	0.0014	42
$\theta = 1.00$	20	0.0540	0.0552	24
	100	0.0021	0.0110	32
	500	0.0021	0.0021	34
	1000	0.0011	0.0011	34

non-ruin probability. Nevertheless, we will consider the ruin probability for numerical illustrations because it can be considered a most interesting practical information.

Confidence intervals were obtained using (4.6) with significance level $\alpha = 0.01$ and $n = 5000$.

6.1. Compound processes

Compound processes are a very appropriate example to implement the advantages of the estimator $\mathfrak{N}_r(\mathbf{X}, n)$ described in Section 5.

The infinite sums were calculated up to a certain number of terms (lim) for which the rest of the terms of the sums were smaller than 10^{-10} . Then the total amount of random number used to get all the convolutions up to lim will be $n(lim - 1)$, as it was stated in Section 5.

The advantage compared with straight simulation is outstanding for two reasons. First the saving of random numbers in the case of straight simulation would be $(lim - 1)lim/2n$ and second the reduction in the variance guaranteed by Theorem 3. In the cases studied, this approach could be considered reliable, even compared with deterministic numerical methods, because the number of steps – random numbers in our case – is not very large (see Tables 1–3).

6.1.1. Ultimate non-ruin probability in the classical case

The probability of ultimate survival could be written as a compound process:

$$\Phi(U) = \sum_{t=0}^{\infty} \frac{\theta}{1+\theta} \left(\frac{1}{1+\theta} \right)^t G^{*t}(U) \quad (6.1)$$

Table 2

Confidence intervals for ruin probability: $\alpha = 0.01$; $\Psi(U) = 1 - \sum_{t=0}^{\infty} \{\theta/(1+\theta)\} \{1/(1+\theta)\}^t G^{*t}(U)$; and exponential claim size. $F(x) = 1 - e^{-(1/p_1)x}$, $p_1 = 1$

Security loading	Initial reserves (U)	Lower limit	Upper limit	Sum terms
$\theta = 0.10$	1	0.8296	0.8306	13
	5	0.5749	0.5819	21
	10	0.3615	0.3696	28
	80	0.0006	0.0007	115
	1	0.6543	0.6559	12
$\theta = 0.25$	5	0.2904	0.2981	20
	10	0.1058	0.1115	30
	80	0.0000	0.0000	93
$\theta = 0.50$	1	0.4768	0.4784	11
	5	0.1242	0.1297	20
	10	0.0228	0.0252	27
	80	0.0000	0.0000	57
$\theta = 0.75$	1	0.6270	0.6284	11
	5	0.9323	0.9358	19
	10	0.9915	0.9926	25
	80	0.9999	0.9999	42
$\theta = 1.00$	1	0.3026	0.3039	11
	5	0.0400	0.0425	18
	10	0.0031	0.0037	23
	80	0.0000	0.0000	34

Table 3

Confidence intervals for total claims Poisson Compound process. $\alpha = 0.01$; $T(x) = \sum_{t=0}^{\infty} e^{-\lambda} \lambda^t / t! F^{*t}(x)$, $x \geq 0$, and exponential claim size. $F(x) = 1 - e^{-(1/p_1)x}$

Parameters	x	Number of terms	Lower limit	Upper limit
$\lambda = 1, p_1 = 20$	1	6	0.386040	0.386049
	10	9	0.529791	0.530424
	100	14	0.975472	0.977482
	150	14	0.995754	0.996594
$\lambda = 2, p_1 = 10$	1	7	0.162362	0.162413
	10	12	0.392771	0.395827
	100	18	0.995111	0.996465
	150	19	0.999437	0.999834
$\lambda = 2, p_1 = 20$	1	7	0.148860	0.148873
	10	10	0.268473	0.269497
	100	16	0.912056	0.918233
	150	17	0.978257	0.981244

(see for example Panjer and Willmot, 1992, Theorem 11.4.5) where $G^{*t}(U)$ is the t -fold convolution of a random variable with p.d.f.:

$$g(x) = \frac{1 - F(x)}{p_1}$$

and $F(x)$ is the c.d.f. of the claim size, $E\{X\} = p_1$ and θ the security loading.

The t -fold convolution $G^{*t}(U)$ could be expressed in terms of (1.1) and approximated using (4.1) and (4.6)

$$G^{*t}(U) = H_t(\mathbf{X}) \simeq \frac{\sum_{i=1}^n G(U - S_{i-1}^i)G(U - S_{i-2}^i) \cdots G(U - S_1^i)G(U)}{n},$$

where $\mathbf{X} = \{U, U, \dots, U\}$ and $G_i(x) = G(x)$, $i = 1, 2, \dots, n$.

In Tables 1 and 2 this approach was tested for ruin probabilities, respectively, for exponential ($p_1 = 1$) and Pareto ($F(x) = 1 - (\mu/(\mu + x))^{\mu+1}$, $\mu = 1$) claim sizes and different values of θ and the initial reserves. The random numbers were generated by the inverse method from (2.2).

6.1.2. Total claims compound process

The total claims are frequently modeled using compound processes:

$$T(x) = \sum_{t=0}^{\infty} P_t F^{*t}(x), \quad x \geq 0,$$

where:

- P_t : Probability of finding exactly t claims in the period considered.
- $F^{*t}(x)$: c.d.f. of the t -fold convolution of the claim size distribution.

As the t -fold convolutions are multiple integrals of the type (1.1), the estimator (4.1) could be used as an approximation.

Table 3 shows some results for a Poisson Compound process and exponential claims with different values for the expected number of claims (λ) and the expected claim size (p_1).

6.2. Finite time survival probability

We will consider now the survival probability for finite (t) and discrete time used in Usábel (1995a,b).

Let us define the discrete stochastic process of the discount accumulated claims in the present moment $\{S_t\}$:

$$S_t = \sum_{i=1}^t Z_i = \sum_{i=1}^t d_i Y_i,$$

where:

- $y_i \geq 0$, $i = 1, \dots, t$, are the total claims of the i th period with p.d.f. $f_i(x)$, c.d.f. $F_i(x)$ and $E\{Y_i\} = \mu_i$, assuming they are uniformly distributed over the period considered, they will be paid in the mid-point of this period.
- $d_i = (1/(1 + I_1))^{-1/2} \prod_{j=2}^i (1/(1 + I_j))$, $j = 1, \dots, i$, the discount factor to the present moment for the amounts of the i th period and I_j the rate of interest of the j th period.

The c.d.fs. $G_i(x)$ and p.d.fs. $g_i(x)$ could be expressed, respectively,

$$G_i(x) = P\{d_i Y_i \leq x\} = P\left\{Y_i \leq \frac{x}{d_i}\right\} = F_i\left(\frac{x}{d_i}\right),$$

$$g_i(x) = f_i\left(\frac{x}{d_i}\right) \frac{1}{d_i}, \tag{6.2}$$

and $E\{X\} = d_i \mu_i$.

Then the probability of non-ruin for $t - 1$ periods and that the accumulated discount claims up to t are less than x is:

$$\begin{aligned}
 & P\{S_i \leq R_0 + P_i, i = 1, \dots, t - 1; S_t \leq x\} \\
 &= \int_0^{x_1} \int_{s_1}^{x_2} \cdots \int_{s_{t-2}}^{x_{t-1}} G_t(x_t - s_{t-1}) g_{t-1}(s_{t-1} - s_{t-2}) \cdots \\
 & \quad \times g_2(s_2 - s_1) g_1(s_1) ds_1 \cdots ds_{t-1}
 \end{aligned} \tag{6.3}$$

(see for example Bühlmann, 1970, p. 137, for a similar expression for a non-financial model ($I = 0$))

where:

– R_0 : Initial reserves.

– $P_i = \sum_{j=1}^i d_j(1 + \theta_j)\mu_j$: Accumulated discount premiums up to the i th period.

It is clear that (6.3) is a multiple integral of the kind of (1.1), where $\mathbf{X} = \{R_0 + P_1, \dots, R_0 + P_{t-1}, x\}$ and will be approximated using (4.1):

$$\begin{aligned}
 & P\{S_i \leq R_0 + P_i, i = 1, \dots, t - 1; S_t \leq x\} \\
 & \simeq \mathfrak{K}_t(\mathbf{X}, n) = \left(\frac{1}{n}\right) \sum_{i=1}^n G_i(x - S_{t-1}^i) G_{t-1}(R_0 + P_{t-1} - S_{t-2}^i) \\
 & \quad \times \cdots G_2(R_0 + P_2 - S_1^i) G_1(R_0 + P_1)
 \end{aligned}$$

Table 4

Confidence intervals for discrete time ruin probability. $\alpha = 0.01$; time span $t = 10$; Poisson number of claims ($\lambda = 1$); exponential claim size $p_1 = 1$; variance reduction percentage (VRP) = [(direct simulation variance – method variance)/direct simulation variance] $\times 100$; and classical case $I = 0$

Initial reserves	Security loading	VRP	Lower limit	Upper limit
$U = 0$	0.05	97	0.7115	0.7159
	0.10	97	0.6850	0.6894
	0.15	97	0.6583	0.6626
	0.20	97	0.6330	0.6376
	0.25	97	0.6062	0.6108
	0.30	97	0.5812	0.5859
	1.00	98	0.2987	0.3020
$U = 20$	0.05	94	0.5622	0.5688
	0.10	94	0.5343	0.5410
	0.15	95	0.5051	0.5116
	0.20	95	0.4759	0.4823
	0.25	95	0.4476	0.4540
	0.30	95	0.4249	0.4312
	1.00	97	0.1848	0.1885
$U = 200$	0.05	78	0.0267	0.0313
	0.10	73	0.0258	0.0305
	0.15	75	0.0220	0.0262
	0.20	80	0.0174	0.0208
	0.25	77	0.0157	0.0192
	0.30	78	0.0137	0.0169
	1.00	86	0.0018	0.0026

Table 5

Confidence intervals for discrete time ruin probability: $\alpha = 0.01$; time span $t = 100$; Poisson number of claims ($\lambda = 1$); exponential claim size $p_1 = 1$; variance reduction percentage VRP = [(direct simulation variance – method variance)/direct simulation variance] $\times 100$; and classical case $I = 0$

Initial reserves	Security loading	VRP	Lower limit	Upper limit
$U = 0$	0.05	90	0.8779	0.8836
	0.10	91	0.8365	0.8428
	0.15	92	0.7888	0.7954
	0.20	92	0.7447	0.7517
	0.25	93	0.7018	0.7088
	0.30	93	0.6601	0.6670
$U = 20$	0.05	84	0.8028	0.8118
	0.10	84	0.7491	0.7588
	0.15	86	0.6889	0.6988
	0.20	87	0.6252	0.6351
	0.25	88	0.5739	0.5837
	0.30	89	0.5250	0.5344
$U = 200$	0.05	55	0.3129	0.3307
	0.10	53	0.2343	0.2510
	0.15	53	0.1688	0.1836
	0.20	53	0.1259	0.1388
	0.25	55	0.0826	0.0932

and from (2.2) and (6.2):

$$S_1^i \longrightarrow \mathcal{D}_1^i(s_1^i) = \frac{g_1(s_1^i)}{G_1(R_0 + P_1)} = \frac{f_1(s_1^i/d_1)(1/d_1)}{F_1((R_0 + P_1)/d_1)},$$

$$s_1^i \in [0, R_0 + P_1],$$

$$S_j^i \longrightarrow \mathcal{D}_j^i(s_j^i) = \frac{g_j(s_j^i - S_{j-1}^i)}{G_j(x_j - S_{j-1}^i)} = \frac{(f_j(s_j^i - S_{j-1}^i)/d_j)(1/d_j)}{F_j((R_0 + P_j - S_{j-1}^i)/d_j)}$$

$$s_j^i \in [S_{j-1}^i, R_0 + P_j], \quad j > 1.$$

One frequently used model in risk theory is a simplification of the former one, where

$$F_i(x) = F(x), \quad \theta_i = \theta, \quad I_i = I = 0 \quad \forall i \quad (6.4)$$

as in the paper of De Vylder and Goovaerts (1988). This probability of non-ruin could be expressed with (1.1) $H_t(\mathbf{X})$ where

$$\mathbf{X} = (R_0 + c, R_0 + 2c, \dots, R_0 + tc), \quad c = \mu(1 + \theta).$$

In Tables 4 and 5, we obtained confidence intervals of the ruin probability for different initial reserves and security loadings.

The values of the variance reduction using this approach compared with the straight simulation are shown in the column VRP (variance reduction percentage) of the tables:

$$\text{VRP} = \frac{\text{direct simulation variance} - \text{method variance}}{\text{direct simulation variance}} \times 100.$$

The amount of random numbers is $5000(t - 1)$, $n = 5000$, and were generated from (2.2) using the cutpoint method (Bratley et al., 1987; Fishman, 1996).

Table 6

Confidence intervals for discrete time ruin probability. $\alpha = 0.01$; time span $t = 10$; Poisson number of claims ($\lambda = 1$); exponential claim size $p_1 = 1$; variance reduction percentage $VRP = [(direct\ simulation\ variance - method\ variance) / direct\ simulation\ variance] \times 100$; and rate of interest $I = 0.06$

Initial reserves	Security loading	VRP	Lower limit	Upper limit
$U = 0$	0.05	97	0.6751	0.6797
	0.10	97	0.6485	0.6532
	0.15	97	0.6218	0.6266
	0.20	97	0.5938	0.5986
	0.25	97	0.5698	0.5745
	0.30	97	0.5438	0.5485
	1.00	98	0.2820	0.2851
$U = 20$	0.05	94	0.5024	0.5095
	0.10	94	0.4749	0.4819
	0.15	94	0.4478	0.4547
	0.20	94	0.4190	0.4256
	0.25	94	0.3991	0.4057
	0.30	95	0.3777	0.3841
	1.00	97	0.1665	0.1700
$U = 200$	0.05	70	0.0093	0.0125
	0.10	77	0.0067	0.0090
	0.15	75	0.0060	0.0084
	0.20	75	0.0052	0.0074
	0.25	82	0.0038	0.0054

In the context and the assumptions of the paper by DeVnyder and Goovaerts (1988) (6.4), our approach cannot compete with the method designed by these two authors, but when the model is generalized and the deterministic methods (often based in restrictive assumptions as stationarity) are very hard to find, this approach could certainly be interesting to consider instead of the straight simulation.

As an illustration, ruin probabilities in Table 4 were calculated introducing a constant rate of interest $I = 0.06$ in the non-financial model (6.4) and time span 10.

It is important to state the significant percentage of reduction of the variance (53–98%) added to the fact that we can get the severity of the ruin without restarting the simulation again (as explained in the last paragraph of Section 5).

7. Concluding comments

The discrete time ruin probability and the distribution of the severity of ruin are very difficult to obtain when the model considered includes different rates of interest, varying annual distributions of the total claims and security loadings for each period. The direct simulation is sometimes the only method available to approximate these probabilities.

The variance reduction achieved with estimator $\mathfrak{N}_t(\mathbf{X}, n)$ (4.1) compared with direct simulation was proved in Theorem 3 and tested with examples (Tables 4–6). The significant percentage of reduction of the variance (53–98%) add to the fact that we can get the severity of the ruin without restarting the simulation again (as explained in the last paragraph of Section 5) make this method clearly better than the straight simulation.

Regarding computational aspects, each ruin probability considered in Tables 4–6 (time span $t = 10$ and $t = 100$) was obtained in fractions of a second using a Pentium 120 processor and Turbo Pascal 6.0. Extension to increasing time horizon is still tractable because the total figure of random numbers will always be $n(t - 1)$, linearly increasing with time.

Finally, in the context of discrete time, this method also allows increasing the number of periods considered without restarting the simulations again.

Due to the last property of estimator $\mathfrak{N}_t(\mathbf{X}, n)$ the method was also used in approximating t -fold convolutions in the context of Compound processes (Tables 1–3) obtaining narrow confidence intervals for ruin probabilities and total claims distributions.

When considering Compound processes (Tables 1–3) the computing times will increase again linearly depending on the number of terms of the sum involved in the calculations. For instance, in the worst case considered in Table 1, evaluating 226 terms of the sum of the Pollaczec–Khinchine formula, the computing time is about 5 s.

Numerical illustrations were performed with significant level $\alpha = 0.01$ using (4.6):

$$\left[\mathfrak{N}_t(\mathbf{X}, n) \mp \phi(1 - \alpha) \sqrt{\frac{k}{n}} \right].$$

We should mention that the nature or properties of our estimator are not affected when considering very small values of parameter α .

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