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Optimal Partial Harvesting Schedule for Aquaculture Operations

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Abstract *When growth is density dependent, partial harvest of the standing stock of cultured species (fish or shrimp) over the course of the growing season (*i.e.*, partial harvesting) would decrease competition and thereby increase individual growth rates and total yield. Existing studies in optimal harvest management of aquaculture operations, however, have not provided a rigorous framework for determining "discrete" partial harvesting (*i.e.*, partially harvest the cultured species at several discrete points until the final harvest). In this paper, we develop a partial harvesting model that is capable of addressing discrete partial harvesting and other partial harvesting using impulsive control theory. We derive necessary conditions of the efficient partial harvesting scheme for a single production cycle. We also present a numerical example to illustrate how partial harvesting can improve the profitability of an aquaculture enterprise compared to single-batch harvesting and gradual thinning. The study results indicate that well-designed partial harvesting schemes can enhance the profitability of aquaculture operations.*

Key words Partial harvesting, impulsive control theory, aquaculture.

JEL Classification Codes C61, Q22.

Introduction

Partial harvest of the standing stock of cultured species (fish or shrimp) over the course of the growing season (hereafter refer as "partial harvesting") would decrease competition and thereby increase individual growth rates and total yield (Allen *et al.* 1984; Brummett 2002). Paessun and Allison (1984) and Watten (1992) observed that partial harvesting could increase the productivity of a tilapia and trout farming system, respectively. Moss, Otoshi, and Leung (2005) tested partial harvesting in a super-intensive recirculating shrimp production system. Their results indicate that a well-managed partial harvesting schedule could improve the overall productivity and profitability of shrimp mariculture in the presence of density-dependent growth. With affordable new techniques and machines nowadays, partial harvesting could become a potential avenue to utilize efficiently the growout capacity of ponds, tanks, cages, or raceways in aquaculture operations, especially in the semi-intensive and intensive farms with the eventual goal of enhancing profitability. The basic

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premise of partial harvesting is that growth would be impeded when carrying capacity is reached or simply growth would be depressed with increasing biomass (density). As a result, partial harvest of the standing stock of cultured species would increase overall productivity. The paradigm of partial harvesting has been well documented in the literature (Allen *et al*. 1984; Hannesson 1986; Sendak, Brissette, and Frank 2003; Burgess, Robinson, and Wetzel 2005), but we found that existing harvest management models are still rather restrictive in nature and generally not suitable for practical on-farm applications. Further, no practical model that is able to address the "discrete" partial harvesting strategy consisting of several discrete harvests throughout the growout process can be found in the literature. Therefore, we attempt to develop an optimal harvest model that is capable of addressing "discrete" partial harvesting for a single cohort of farmed fish (or shrimp and other cultured species) represented by uniform, density-dependent growth.

Literature Review

Economic analysis of harvesting management in aquaculture is not new. Studies on optimal harvesting management of aquaculture focus on investigating the efficient harvesting policies under various biological and economic conditions (Bjorndal, Lane, and Weintraub 2004). Most of them considered only single-batch harvesting; *i.e*., one harvest for the entire production cycle. Several distinct but related harvesting strategies involve multiple harvests during a production cycle. They are selective harvesting (Bjorndal 1988), sequential rearing (Paessun and Allison 1984; Watten 1992), gradual thinning (Hannesson 1986), gradual culling (Heaps 1993, 1995), and graded harvesting (Summerfelt *et al*. 1993; Forsberg 1996, 1999). Selective harvesting, sequential rearing, and graded harvesting are strategies to be used when individual growth is different among a single cohort of cultured species or there are multiple cohorts. Gradual thinning (or culling) is designed for a single cohort of cultured species with uniform, density-dependent growth. Hannesson (1986) developed a continuous time optimal control model to investigate the emergence of gradual thinning; *i.e*., continuous thinning (until the final harvest) after a period of uninterrupted growth, for a single year class of fish in the presence of density-dependent growth. He proved that with density dependency, the farmer could use gradual thinning to enhance the returns from a cohort of cultured fish, especially when the effect of dependency is strong. Hannesson modeled the thinning decision as an artificial mortality rate controlled by the farmer to maximize the present value of harvesting activities (*i.e*., $N = -(M + F)N$, where N, F, M are the number of remaining fish, thinning rate, and natural mortality rate, respectively). This specification, however, only permits Hannesson's optimal control model to handle "continuous" gradual thinning. This is because in order to solve for the optimal control path in the conventional optimal control framework, the state variable (*i.e*., the population (or the number) of remaining fish in the pond) has to be piecewise differentiable and continuous in time (Chiang 1984). In other words, the population curve permitted in Hannesson's model has to be a continuous curve in time as illustrated in figure 1. Discontinuous population curves, such as the one illustrated in figure 2, are inadmissible in Hannesson's model. Notice in figure 2 that there are jumps in the state variable whenever a harvest occurs. We label the harvesting policy associated with a discontinuous population curve, such as the one in figure 2, as "discrete" partial harvesting. Discrete partial harvesting means that thinning takes place at several discrete points until the final harvest. From the practical point of view, discrete partial harvesting is likely to be a more realistic harvesting policy. Arnason (1992) introduced the feed regime into harvesting management. He argued that if fish

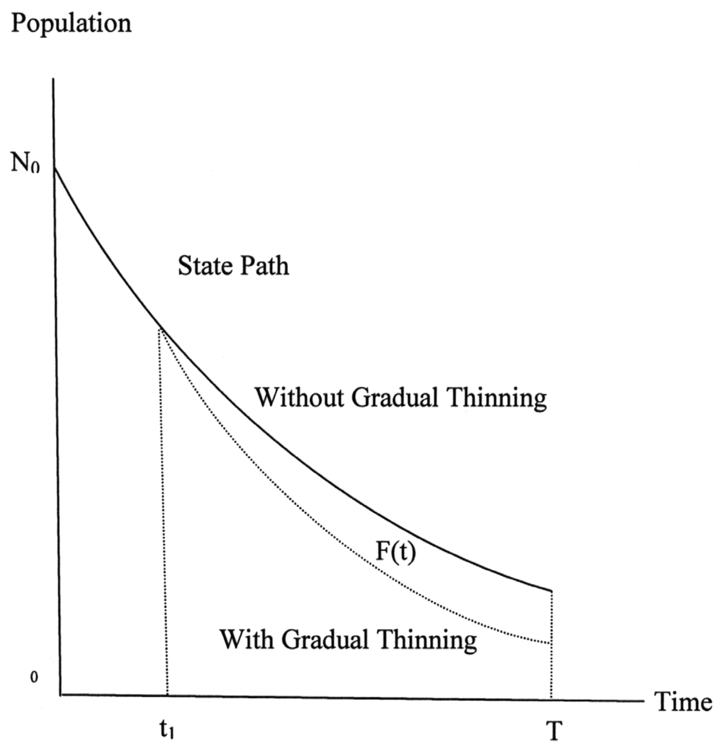

Figure 1. Fish Population Curve (Gradual Thinning)

growth does not depend directly on density, it is always optimal to harvest the entire cohort at one point in time. On the other hand, with density-dependent growth, some thinning of the fish cohort prior to the final harvest would normally be optimal, assuming the size and growth of fish are identical within the cohort. Heaps (1995) further extended Arnason's model and derived a combined optimal feeding-thinning schedule for a single year class of fish under the same optimal control framework as Hannesson's model. His results again indicated that if fish growth is density dependent, the optimal harvest policy would involve a period of no culling followed by a period in which there is a continuous culling up to a final harvest where all the remaining fish are harvested; *i.e.*, gradual culling. Gradual culling is equivalent to gradual thinning in Hannesson's model. From the above brief review, it is clear that existing optimal harvesting models of farmed fish either have not considered or cannot evaluate the "discrete" partial harvesting strategy for a single year class of fish when growth is density dependent. In addition, the integer linear programming decision model developed by Shaftel and Wilson (1990) proposed "discrete" partial harvesting as the optimal harvesting policy in a very naive way. They presumed that there is a threshold level of biomass, which could affect significantly shrimp growth and survival, so the farmer would remove a portion of the shrimp stock to maintain the biomass level under the threshold before the final harvest. The resulting "discrete" partial harvesting is enforced by the specification of a threshold *per se*. Consequently, their model cannot serve as a general analytical framework of harvesting management. We have to resort to an innovative optimal control framework that can admit jumps in state variables.

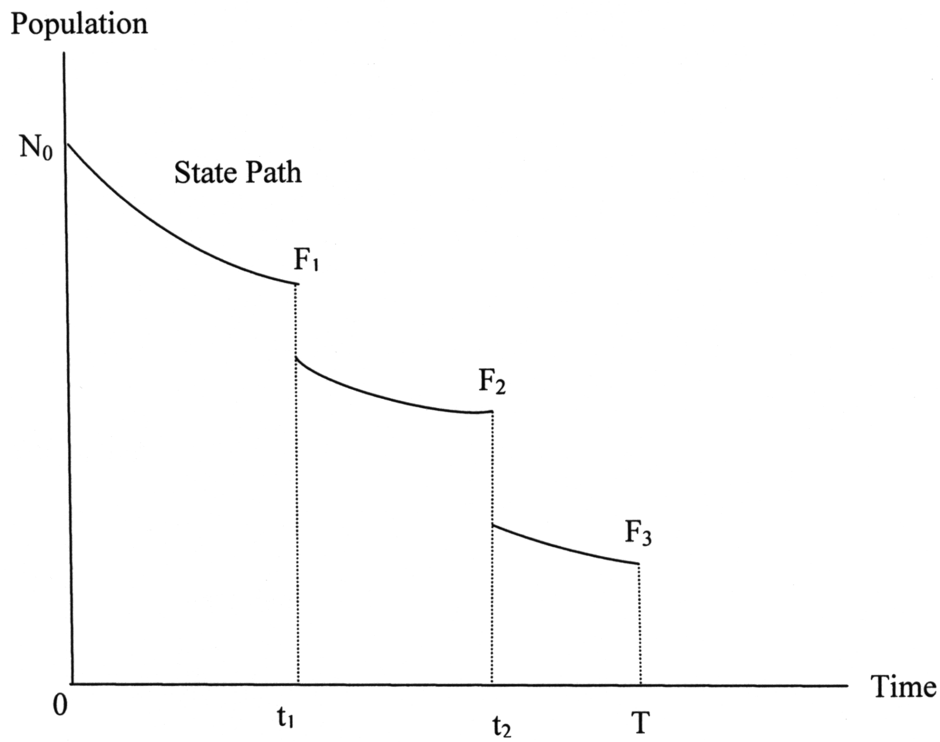

Figure 2. Fish Population Curve (Discrete Partial Harvesting)

The appropriate method to construct optimal control problems with jumps in the state variables is impulsive control theory. Impulsive control is a control paradigm based on impulsive differential equations, which allows for jumps in the state variables (Yang 2001). Seierstad (1981) first derived the fundamental necessary and sufficient conditions for the solution of basic impulsive control problems. Since then, research continues to extend impulsive control systems by incorporating features such as free ending time, stochastic variables, state constraints, and periodic motions (Barles 1985; Yong and Zhang 1992; Miller 1993; Yang 2001; Arutyunov and Pereira 2000; Arutyunov, Karamzin, and Pereira 2003). Impulsive control theory has been extensively applied in many areas, including electrical engineering, mechanical engineering, medicine, and biological sciences, using different names such as jump control and impulse control (Rogovchenko 1997). Using impulsive control theory, the planner of a farming operation can determine the desired total number of harvests and the magnitude and timing of each harvest. Hence, there are no *a priori* assumptions about the nature of harvest strategy (*i.e.*, single-batch harvesting, gradual thinning, or discrete partial harvesting). For instance, (continuous) gradual thinning could lead to the case where the total number of harvests becomes infinite. Single-batch harvesting is the case where the desired total number of harvests is one.

To date, only a few applications of impulsive control theory can be found in forestry and fishery. Touza-Montero and Termansen (2001) applied the impulsive control approach to derive the optimal rotation time for the clear-cutting strategy in timber management. Baumeister and Leitao (2004) derived the first-order necessary conditions of optimality for efficient exploitation of an open-sea fishery by modeling harvesting as an impulsive depression in fish populations. The present study is the first to apply impulsive control theory to solve a practical farming problem.

The Model

We use impulsive control theory to set up a partial harvesting model for a single cohort of cultured species with uniform, density-dependent growth. We use shrimp culture as an illustrative example. Nonetheless, we can apply the derived results and conclusions to other cultured species as well.

Let $w(t)$ and $n(t)$ represent the average weight and number of remaining shrimp in the growout pond, respectively. The biological growth process of shrimp stock can be described by the following two differential equations:

$$\dot{n}(t) = -mn(t), \qquad n(0) = n^0 \tag{1}$$

$$\dot{w}(t) = g\big[f(t), w(t), n(t), t\big], \quad w(0) = w^0, \tag{2}$$

where m is the constant instantaneous rate of natural mortality of shrimp. Equation (1) is the simplest and most common functional form in which mortality is depicted in the literature. Equation (2) describes the growth of an individual shrimp, which is enhanced by feeding $[f(t)]$ and affected by density $[w(t) \cdot n(t)]$ and age (t). w^0 and n^0 are the average stocking weight and stocking number of juveniles, respectively. We ignore the variation of weights among different shrimp in this basic model.

Define $[0, T]$ as the planning horizon under our investigation. Let τ_j denotes the timing of a decision period when the j-th harvest occurs ($\tau_j \in [0, T]$). Changes in the average weight and number of remaining shrimp due to a particular harvest at τ_j are described by the following impulsive differential equations:

$$w(\tau_j^+) - w(\tau_j^-) = 0 \tag{3}$$

$$n(\tau_j^+) - n(\tau_j^-) = -v_j n(\tau_j^-), \tag{4}$$

where τ_j^+ denotes the instant just after the harvest occurs at the τ_j^{th} decision period, and τ_j^- denotes the instant just before the harvest. v_j denotes the amount (in percentage terms) of shrimp stock that is harvested at the τ_j^{th} period.

The reward (revenue) function associated with a typical harvest, v_j, can be described as:

$$R\Big\{p[w(t), t], w(t), n(t), v_j, c, h\Big\} = \Big\{p[w(\tau_j)] - c\Big\}w(\tau_j)n(\tau_j^-)v_j - h, \tag{5}$$

where $p[w(\tau_j)]$ is the price of shrimp (*e.g.*, in \$/kg), which is weight-dependent;[1] c is the variable cost of harvesting a unit of shrimp stock (*e.g.*, in \$/kg); and h is the quasi-fixed cost associated with each harvest.

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[1] Weight-dependent or size-dependent price in the shrimp market is a very important relationship. Many studies have been devoted to address this issue. For example, Mistiaen and Strand (1998) demonstrated that a piecewise-continuous price schedule would generate stepwise-nonlinear responses of management behavior to changes in exogenous parameters.

On the other hand, the cost of feeding can be expressed as follows:

$$FC = \int_0^T sf(t)w(t)n(t)dt, \tag{6}$$

where $f(t)$ denotes the amount of feed in terms of percentage of the prevailing biomass $[w(t) \cdot n(t)]$. s is the cost per unit of feed (*e.g.*, in \$/kg). For ease of exposition, we ignore other fixed costs and variable costs of shrimp culture in the present analysis.

Assume that there are k harvests over the production cycle. The timing and magnitude of these harvests can be denoted as $\tau_1,\ldots,\tau_k$ and $v_1,\ldots,v_k$, respectively. The objective of the farmer (or the planner) is to maximize the net revenue from the harvesting and related feeding activities. Because of the relatively short planning horizon involved in our problem, we ignore the discount factor. Hence, the optimization problem can be formally described as follows:

$$\underset{f(t),k,v_1\ldots v_k,\tau_1\ldots\tau_k}{Max} = \sum_{j=1}^{k}\left[\left\{p\left[w(\tau_j)\right] - c\right\}w(\tau_j)n(\tau_j^-)v_j - h\right] - \int_0^T sf(t)w(t)n(t)dt \tag{7}$$

$$\begin{array}{lll}
w(\tau_j^+) - w(\tau_j^-) = 0 & j = 1,\ldots,k & \\
n(\tau_j^+) - n(\tau_j^-) = -v_j n(\tau_j^-) & j = 1,\ldots,k & \\
\dot{n}(t) = -mn(t) & n(0) = n^0 & n(T^+) = 0 \\
\dot{w}(t) = g\left[f(t), w(t), n(t), t\right] & w(0) = w^0 & w(T^+) > w^0 \ . \\
v_j \in [0,1] & j = 1,\ldots,k &
\end{array}$$

The Hamiltonian, $H(t)$, of problem (7) is defined by:

$$H(t) = -sf(t)w(t)n(t) - \lambda_1(t)mn(t) + \lambda_2(t)g\left[f(t), w(t), n(t), t\right]. \tag{8}$$

Let $[n^*(t), w^*(t), f^*(t), \tau_1^*,\ldots,\tau_k^*, v_1^*,\ldots,v_k^*]$ be an admissible collection which solves the above maximization problem (7). Then there exists two piecewise continuous functions, $\lambda_1(t)$ and $\lambda_2(t)$, according to the maximum principle of impulsive control problems developed by Seierstad and Sydsater (1987). $\lambda_1(t)$ is the shadow value of n (*in situ* value of one shrimp) at time t. $\lambda_2(t)$ is the shadow value of w (*in situ* value of one unit of shrimp weight), *e.g.*, \$/kg.

Following Seierstad and Sydsater (1987), necessary conditions for the solution of problem (7) can be described as follows:

For all non-jump points of $[n^*(t), w^*(t)]$ (*i.e.*, the no-harvest period), $H[n^*(t), w^*(t), f(t), \lambda_1(t), \lambda_2(t), t] \leq H[n^*(t), w^*(t), f^*(t), \lambda_1(t), \lambda_2(t), t]$. This gives:

$$\lambda_2(t)\left\{g\left[f(t),w^*(t),n^*(t),t\right] - g\left[f^*(t),w^*(t),n^*(t),t\right]\right\} \leq sw^*(t)n^*(t)\left[f(t) - f^*(t)\right] \tag{9}$$

for all $f(t) \in (0, \infty)$, where the right-hand side of equation (9) represents the cost of deviating from optimal feeding (*i.e.*, excess feed cost due to overfeeding or savings on feed cost due to underfeeding); the left-hand side of equation (9) represents the change in the potential benefits of shrimp stock due to change in shrimp growth caused by the deviation of optimal feeding (*i.e.*, extra weight gain due to overfeeding or insufficient weight gain due to underfeeding). Equation (9) indicates that an optimized feeding regime would make the cost of any deviation greater than its associated return (benefit) such that any deviation is not worthwhile.

The co-state variables, $\lambda_1(t)$ and $\lambda_2(t)$, are continuously differentiable at the no-harvest period and satisfy the following conditions:

$$\dot{\lambda}_1(t) = -\frac{\partial H^*(t)}{\partial n(t)} = sf^*(t)w^*(t) + \lambda_1(t)m - \lambda_2(t)g_n\left[f^*(t), w^*(t), n^*(t), t\right] \tag{10}$$

$$\dot{\lambda}_2(t) = -\frac{\partial H^*(t)}{\partial w(t)} = sf^*(t)n^*(t) - \lambda_2(t)g_w\left[f^*(t), w^*(t), n^*(t), t\right]. \tag{11}$$

Equation (10) shows that the change in the shadow value of n (*in situ* value of one shrimp) is the sum of the added feeding cost, the cost associated with shrimp mortality, and the marginal cost due to the negative effect on growth. Equation (11) shows that the change in the shadow value of w (*in situ* value of one unit of shrimp weight) is the sum of the added feeding cost and the marginal cost due to the negative effect on growth.

Furthermore, for the no-harvest period, we have:

$$\left\{p\left[w^*(t)\right] - c\right\}w^*(t) \leq \lambda_1(t). \tag{12}$$

Equation (12) shows that during the no-harvest period the net market value of a piece of shrimp by harvesting it at time t, $[\{p[w^*(t)] - c\}w^*(t)]$, will be smaller than its shadow value $[\lambda_1(t)]$; *i.e.*, the potential net revenue that can be obtained by leaving it to grow and to be harvested at a later date. Under an optimized harvest schedule, the no-harvest period is defined as the time when the net benefit (market value minus variable harvest cost) of harvesting one piece of shrimp immediately is smaller than the potential net benefit of harvesting that piece of shrimp at a later date (or the benefit of not harvesting at that moment).

In our application, the entire shrimp standing stock will be harvested at the end of the planning horizon, T. Hence, the transversality conditions for the two co-state variables can be expressed as:

$$\lambda_1(T^+) = \text{Free} \tag{13}$$

$$\lambda_2(T^+) = 0. \tag{14}$$

Now, we move to the conditions describing the jump points where a harvest occurs (*i.e.*, $\tau_1^*,\ldots,\tau_k^*$). First, we have:

$$\lambda_1(\tau_j^{*+}) - \lambda_1(\tau_j^{*-}) = -\left\{p\left[w^*(\tau_j^*)\right] - c\right\}w^*(\tau_j^*)v_j^* + \lambda_1(\tau_j^{*+})v_j^* \quad \text{for } j = 1,\ldots,k \tag{15}$$

$$\lambda_2(\tau_j^{*+}) - \lambda_2(\tau_j^{*-}) = -n^*(\tau_j^{*-})v_j^*\left\{p\left[w^*(\tau_j^*)\right] - c + p_w\left[w^*(\tau_j^*)\right]w^*(\tau_j^*)\right\} \quad \text{for } j = 1,\ldots,k. \tag{16}$$

Moreover, at all harvest points τ_j^*, we have:

$$\left\{p\left[w^*(\tau_j^*)\right] - c\right\}w^*(\tau_j^*) = \lambda_1(\tau_j^{*+}), \quad \text{for } j = 1,\ldots,k. \tag{17}$$

Equation (17) indicates that at the moment when harvest occurs, the *in situ* value of shrimp equals the net value of harvested shrimp. Equation (17) is the well-known necessary condition for optimal harvesting in fisheries economics (Anderson 1977; Clark 1990). Equation (17) also describes "the law of indifference" in optimal control theory: an optimal harvest schedule would make it indifferent between harvesting one more piece of shrimp right now and leaving that piece of shrimp to grow for future harvest. Therefore, $\lambda_1(t)$ can be interpreted as the partial (shadow) value foregone by harvesting a piece [or $w(t)$ in terms of its average weight] of shrimp at time t instead of leaving it to grow for harvest at a later date, while ignoring the current effect of morality. In other words, it is the partial opportunity cost of harvesting a single shrimp at time t or the partial potential benefit of harvesting a single shrimp at a later date, ignoring the effect of morality at that instant. Subsequently, $\lambda_1(t)/w(t)$ measures the partial value foregone by harvesting a unit of shrimp biomass at time t, while ignoring the effect of mortality at that instant. Similarly, $\lambda_2(t)/w(t)$ measures the partial value forgone by harvesting a unit of shrimp biomass at time t, while ignoring the effect of growth at that instant. Because $\lambda_1(t)$ and $\lambda_2(t)$, respectively, ignore the effect of mortality and growth, $\lambda_1(t)/w(t)$ will not equal $\lambda_2(t)/n(t)$. In fact, it is $\lambda_1(t)\lambda_2(t)/w(t)n(t)$ that represents the total potential benefit of a unit of shrimp biomass at time t by leaving it to grow further, taking into consideration the effect of morality and growth together.

Substituting equation (17) into equation (15) gives $\lambda_1(\tau_j^{*+})=\lambda_1(\tau_j^{*-})$. Therefore, the co-state variable, $\lambda_1(t)$, is continuous even at the jump points. This is because at the instant of harvest there is no change in shrimp weight such that the partial shadow value of a piece of shrimp would remain the same.

To explore the properties of equation (16), we will consider the special case where shrimp price (p) is constant and variable cost of harvesting (c) is zero. Equation (16) now can be rewritten as $\lambda_2(\tau_j^{*+})=\lambda_2(\tau_j^{*-})-pn^*(\tau_j^{*-})v_j^*$. It shows that when the effect of growth is ignored, the change in the partial potential value of $n(t)$ units of shrimp biomass due to a particular harvest would equal the market value associated with the number of shrimp harvested. It is clear that when the effect of future growth is ignored, the change in the current value of the shrimp standing stock $[\lambda_2(\tau_j^{*+})w(\tau_j^{*+})-\lambda_2(\tau_j^{*-})w(\tau_j^{*+})]$ due to the decreased number of shrimp $[n^*(\tau_j^{*-})v_j^*]$ would be equal to the market value of harvested shrimp $[pw^*(\tau_j^{*-})n^*(\tau_j^{*-})v_j^*]$. The term $p_w[w^*(\tau_j^*)]w^*(\tau_j^*)$ in equation (16) captures the effect of weight-dependent price.

The above necessary conditions [equation (9) to equation (17)] adopted directly from Seierstad and Sydsater's maximum principle can help us single out one or a few solution candidates. For our particular problem, since the partial derivatives of the cost, revenue, and growth functions with respect to time (t) exist and are continuous, we have two more conditions to help us locate the optimal harvest policy. For the no-harvest period, we have $[\partial H^*(t)/\partial f(t)] = 0$, or:

$$sw^*(t) = \frac{\lambda_2(t)}{n^*(t)}\, g_f\left[f^*(t), w^*(t), n^*(t), t\right]. \tag{18}$$

Equation (18) indicates that the optimal feeding trajectory for the no-harvest period is determined when the marginal benefit of feeding due to its effect on shrimp growth $\{[\lambda_2(t)/n^*(t)]g_f[f^*(t), w^*(t), n^*(t), t]\}$ equals the corresponding marginal cost of feeding [$sw^*(t)$]. Condition (18) is essentially condition (12) presented in a more rigorous fashion.

For the jump points where a harvest occurs, we have:

$$H(\tau_j^{*+}) - H(\tau_j^{*-}) - \frac{\partial R^*(\cdot)}{\partial \tau} - \lambda_1(\tau_j^{*+})\frac{\partial\left[-n^*(\tau_j^{*-})v_j^*\right]}{\partial \tau} = \begin{cases} \geq 0, if \ \ \tau_j^* = 0 \\ = 0, if \ \ \tau_j^* \in (0, T). \\ \leq 0, if \ \ \tau_j^* = T \end{cases} \quad (19)$$

For the internal harvest point, $\tau_j^* \in (0, T)$, substituting equations (3), (4), (15), and (16) into equation (19) and rearranging the terms gives:

$$\begin{aligned} &\left\{p\left[w^*(\tau_j^*)\right] - c\right\}mn^*(\tau_j^{*-})w^*(\tau_j^*)v_j^* + sw^*(\tau_j^*)n^*(\tau_j^{*+})[f^*(\tau_j^{*-}) - (1 - v_j^*)f^*(\tau_j^{*+})] \quad (20) \\ &= \lambda_2(\tau_j^{*+})g\left[f^*(\tau_j^{*+}), w^*(\tau_j^*), n^*(\tau_j^{*+}), \tau_j^*\right] - \lambda_2(t_j^{*-})g\left[f^*(\tau_j^{*-}), w^*(\tau_j^*), n^*(\tau_j^{*-}), \tau_j^*\right]. \end{aligned}$$

To explore the properties of condition (20), we consider the special case where the feed regime (f^*), mortality (m), and variable cost of harvesting (c) are ignored and shrimp price (p) is constant. Condition (20) now can be rewritten as:

$$\lambda_2(\tau_j^{*+})g\left[w^*(\tau_j^*), n^*(\tau_j^{*+}), \tau_j^*\right]w^*(\tau_j^{*+}) = \lambda_2(\tau_j^{*-})g\left[w^*(\tau_j^*), n^*(\tau_j^{*-}), \tau_j^*\right]w^*(\tau_j^{*-}), \quad (21)$$

where the left-hand side of equation (21) represents the increase in the potential value of the remaining shrimp standing stock immediately after harvest. The right-hand side of equation (21) represents the possible increase in the value of the entire shrimp standing stock before (or without) harvest. From equation (16), we know that $\lambda_2(\tau_j^{*+}) < \lambda_2(\tau_j^{*-})$. Because of the enhanced shrimp growth caused by reduced density, we have $g[n^*(\tau_j^{*+}), ...] > g[n^*(\tau_j^{*-}), ...]$. Consequently, with the enhanced shrimp growth, the return from the decreased number of shrimp after the harvest would remain the same as the return from the entire shrimp standing stock before the harvest. This essentially states that the current value of a capital stock could remain unchanged if the rate of return increases due to a reduction in the amount of capital stock. The terms $\{p[w^*(\tau_j^*)]-c\}mn^*(\tau_j^{*-})w^*(\tau_j^*)v_j^*$ and $sw^*(\tau_j^*)n^*(\tau_j^{*+})[f^*(\tau_j^{*-})-(1-v_j^*)f^*(\tau_j^{*+})$ in equation (20) reflect the effect of mortality and feeding on the marginal value of the shrimp standing stock, respectively. Therefore, condition (20) indeed defines that an optimal partial harvest would render the current value of the shrimp standing stock unchanged before and after the harvest. In addition, the reward (revenue) associated with a harvest at τ_j^* must outweigh the cost associated with it, or:

$$\left\{p\left[w^*(\tau_j^*)\right] - c\right\}w^*(\tau_j^*)n^*(\tau_j^{*-})v_j^* \geq h. \quad (22)$$

Think of the standing stock of shrimp as a capital stock. The current value of this capital stock is determined by the average rate of return during the planning horizon. The rate of return of the shrimp standing stock is determined by shrimp growth plus price appreciation due to larger shrimp minus mortality, which will be so high initially that it is normally worth holding it at the beginning of the planning horizon. However, as shrimp grows, carrying capacity of the facility will be reached sooner or later due to increased biomass at some point in time, for instance, at time

τ_i^*. As a result, the relative rate of return of holding the capital stock falls. In that case, if the average rate of return during the rest of planning horizon $[(\tau_j^*, T)]$ can be improved by harvesting a part of the capital stock, it is possible that the current value of decreased capital stock would remain the same as the current value of the entire capital stock. In consequence, it is profitable to undertake a harvest at τ_j^* when conditions (20) and (22) are satisfied. In addition, if harvesting is not free, the revenue generated from harvesting part of the capital stock must be able to outweigh the harvesting cost. Thus, gradual thinning, which suggests harvesting an infinitesimal number of shrimp at every infinitesimal point throughout the harvest period normally is not optimal when there is a quasi-fixed cost associated with each harvest; *i.e.*, $h > 0$. On the other hand, if the rate of return cannot be influenced by reducing capital stock (*i.e.*, density-independent growth), it is always not worth undertaking a harvest during the planning horizon. This suggests that the entire shrimp standing stock should be harvested at the end of the planning horizon or at the moment when the rate of return becomes zero; *i.e.*, single-batch harvesting.

A variety of harvesting policies can be explored under the above framework by controlling the total number of harvests (k) and the magnitude of each harvest (v). For instance, by specifying $k = 1$, the resulting policy becomes single-batch harvesting. By specifying that $v_1 = v_2 \ldots = v_{k-1}$ (except the last harvest, $v_k = 1$) equals some constant, the solution is for a constant proportional harvesting policy. Therefore, the optimal partial harvesting model developed here provides a consistent framework for the comparison of different harvesting policies and can be tailored to reflect specific harvesting conditions of various farming systems in practice.

A Numerical Example

In this section, we provide a numerical example to illustrate the significance of partial harvesting. The example is portrayed as follows:

$$\dot{n}(t) = -0.03n(t), \quad n(0) = 40{,}000 \tag{23}$$

$$\dot{w}(t) = 3.5 - 0.00001w(t)n(t), \quad w(0) = 1(g) \tag{24}$$

$$R(t) = (p - c)w(\tau_j)n(\tau_j)v_j - h, \quad p = \$5/kg, \quad c = 0, \quad h = \$100, \tag{25}$$

where shrimp price is constant and equals \$5/kg. The feed regime is entirely ignored in this simple example. Variable cost of harvesting, c, is also ignored. The quasi-fixed cost of harvesting (per harvest), h, equals \$100/harvest. This example is simple and only used to illustrate the potential of partial harvesting in aquaculture. Solutions of three alternate harvest schemes: single-batch harvesting, partial harvesting, and gradual thinning, are obtained. For the gradual thinning scenario, the quasi-fixed cost of harvesting, h, is assumed to be zero. For the purpose of comparison, the time of final harvest is set to be week 13.2 when net revenue is maximal for the strategy of single-batch harvesting.[2]

[2] To compare the harvesting strategies for a single production cycle, we have to set a uniform ending time for final harvest. It does not make sense to compare returns from production cycles with different lengths.

To obtain the solutions of the above example, we use MATLAB and its toolbox DIDO based on a special computational algorithm of reparameterization proposed by Liu *et al*. (1998) for impulsive control problems. The results are presented in table 1.

From table 1, we can see that overall revenue could be enhanced by allowing more harvests during the course of growout compared to single-batch harvesting. Without considering harvesting cost, h, gradual thinning is the most efficient harvest scheme. However, a positive, quasi-fixed harvesting cost (h) would make gradual thinning the worst scheme, because the costs associated with an infinite number of thinning would easily offset any possible revenue. In fact, this is the exact reason why gradual thinning is impractical. With a positive harvesting cost (h), discrete partial harvesting will become the most efficient. Figure 3 illustrates how the marginal incremental revenue (increased revenue) and marginal incremental cost (increased harvesting cost) change as the number of harvests increases. The marginal incremental revenue decreases as the number of harvests increases. Thus, the optimal number of harvests must be finite when the marginal cost of harvesting (h) is positive. The results in figure 3 indicate that the optimal number of total harvests is determined by quasi-fixed harvesting costs associated with each harvest. As the quasi-fixed harvesting cost increases, the optimal number of harvests decreases. For the extreme case of zero cost ($h = 0$), gradual thinning is the most efficient. But for the practical situation where the quasi-fixed cost of harvesting is positive ($h > 0$), an optimal partial harvesting schedule must involve only a finite number of harvests. Furthermore, from the necessary conditions derived above, we can see that the optimal harvest time does not directly depend on the quasi-fixed cost of harvesting (h). Once the total number of harvests is determined, the optimal harvest time for every partial harvest can be uniquely obtained. This implies that the quasi-fixed cost of harvesting (h) only influences the time of harvest indirectly.

As for our example, the best partial harvesting scheme is comprised of one partial harvest at week 5.3 and a final harvest at week 13.2 (figure 4). We can see that growth of shrimp is seriously impeded in the later stage of single-batch harvesting when density becomes too high. With the help of partial harvest, the growth rate of the remaining shrimp could be relatively high due to reduced density.

Although the above comparative results illustrate that partial harvesting could enhance profitability, it does not imply that any partial harvest scheme could outperform single-batch harvesting. For instance, let a partial harvest occur at week 3,

Table 1
Results of the Numerical Example

	Single-Batch Harvesting	Partial Harvesting		Gradual Thinning	
Harvesting Strategy	1st	1st	2nd	Start Week	End Week
Time of harvest (weeks)	13.2	5.3	13.2	2.9	13.2
Number of animals harvested	27,240	19,790	11,307		
Average weight at harvest (*g*)	11.6	8.5	20.6	6.4	23.2
Revenue per harvest ($)	1,576	838	1,179		
Total revenue ($)	1,576	2,017	2,129		
Harvesting cost per harvest ($)	100	100	100		
Total net revenue ($)	1,476	1,817	2,129		
Change in total net revenue		23%			

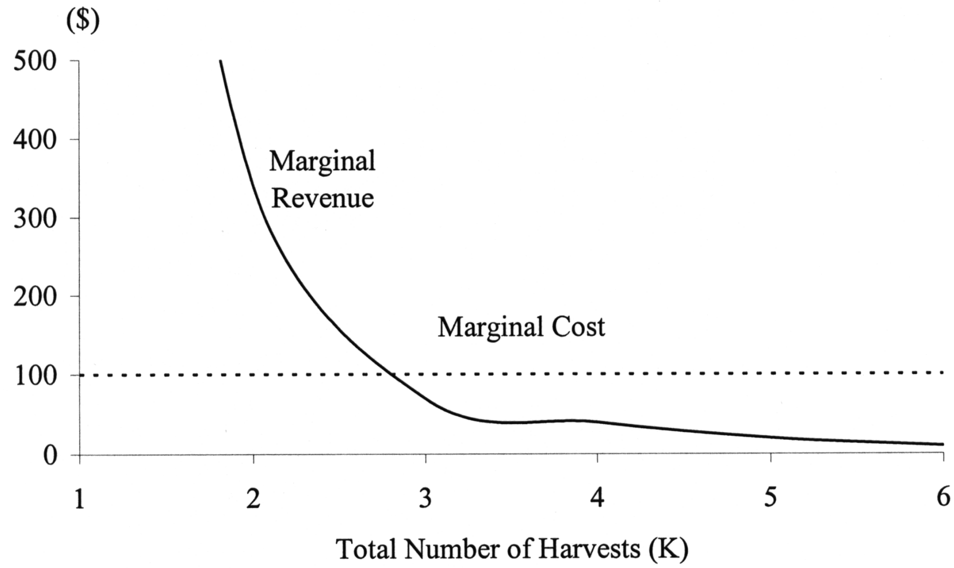

Figure 3. Optimal Number of Harvests

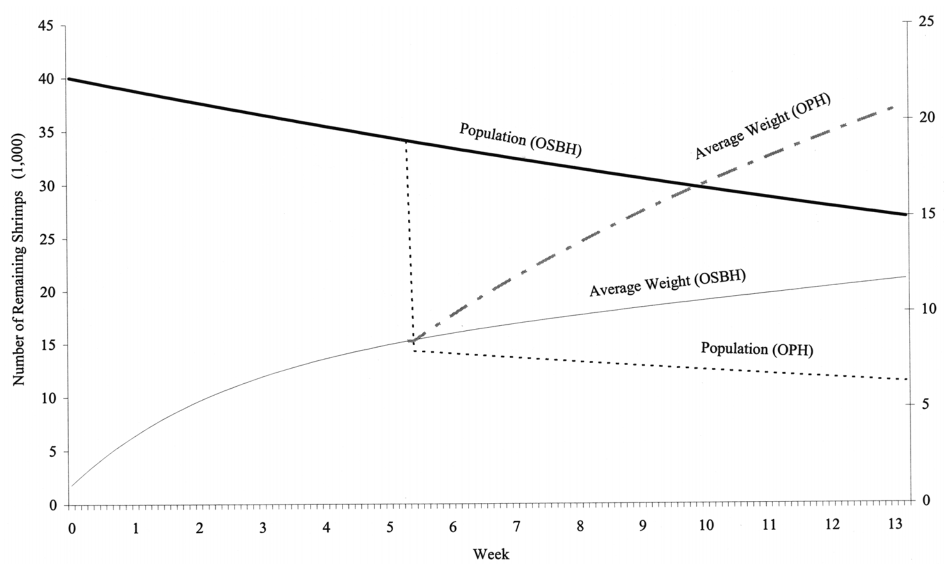

Figure 4. Optimal Partial Harvesting (OPH) vs. Optimal Single-batch Harvesting (OSBH)

with a 90% harvest of the shrimp stock, and let the final harvest occurs at week 13.2. The net revenue associated with this partial harvest schedule is \$1,365, which is smaller than that of a single-batch harvest schedule (\$1,476). Hence, a bad partial harvesting strategy could be worse than single-batch harvesting. This suggests that using partial harvesting to enhance profitability can be tricky. It points to the importance of having a reliable management tool, such as the one developed in this study in order for aqua-farmers to realize the benefits of partial harvesting.

Conclusion

We argue that harvest activities would suddenly alter the status of aquaculture systems so that impulsive control theory is appropriate for constructing harvesting models. Using the impulsive control method, we develop an optimal partial harvesting model for shrimp culture as an illustration and derive the necessary conditions for an efficient harvesting scheme. The model is constructed in a rather general way so that single-batch harvesting and gradual thinning would emerge as two special cases where the number of harvests equals one and infinity, respectively. The analytical results indicate that in the presence of density-dependent growth, partial harvesting could outperform single-batch harvesting. We also present a numerical example to illustrate how well-managed, discrete partial harvesting can outperform single-batch harvesting and thus enhance profitability. The numerical results also suggest that an economic model of partial harvesting is rather vital in using partial harvesting as an avenue of improving profitability.

This paper is mainly an application of an existing theory to a new field. The present analysis can be extended in a number of ways. First, specific cost information, such as fixed maintenance costs and variable operating costs other than the ones considered here can be introduced. Second, appropriate functional forms of growth and price functions can be specified so as to extract more insights of the harvesting decision. Finally, the present model is just for a single-cycle operation. Efforts can also be made to extend it for continuous-production operation involving multiple cycles. These topics represent interesting avenues for future research.

References

Allen, P.G., L.W. Botsford, A.M. Schurr, and W.E. Johnston. 1984. *Bioeconomics of Aquaculture*. New York, NY: Elsevier Science Publishers.

Anderson, L.G. 1977. *The Economics of Fisheries Management*. Baltimore, MD: Johns Hopkins University Press.

Arnason, R. 1992. Optimal Feeding Schedules and Harvesting Time in Aquaculture. *Marine Resource Economics* 7:15–35.

Arutyunov, A., and F.L. Pereira. 2000. A Maximum Principle for a Free Time Impulsive Control Problem. *Proceedings of the 10th Mediterranean Conference on Control and Automation*, Lisbon, Portugal. http://med.ee.nd.edu/MED10/pdf/492.pdf

Arutyunov, A., D. Karamzin, and F. Pereira. 2003. Nondegenerate Necessary Conditions for Optimal Impulsive Control Problems with State Constraints. *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, U.S.A. http://ieeexplore.ieee.org/iel5/8969/28478/01272566.pdf?tp=&arnumber=1272566&isnumber=28478

Barles, G. 1985. Deterministic Impulse Control Problems. *SIAM Journal on Control and Optimization* 23:419–32.

Baumeister, J., and A. Leitao. 2004. Optimal Exploitation of Renewable Resource Stocks: Necessary Conditions. *Optimal Control Applications and Methods* 25:19–50. http://www3.interscience.wiley.com/cgi-bin/abstract/107632682/ABSTRACT

Bjorndal, T. 1988. Optimal Harvesting of Farmed Fish. *Marine Resource Economics* 5:139–59.

Bjorndal, T., D.E. Lane, and A. Weintraub. 2004. Operational Research Models and the Management of Fisheries and Aquaculture: A Review. *European Journal of Operational Research* 156:533–40.

Brummett, R.E. 2002. Comparison of African Tilapia Partial Harvesting Systems. *Aquaculture* 214:103–14.

Burgess, D., C. Robinson, and S.Wetzel. 2005. Eastern White Pine Response to Release 30 Years after Partial Harvesting in Pine Mixed Wood Forests. *Forest Ecology and Management* 209:117–29.

Chiang, A.C. 1984. *Fundamental Methods of Mathematical Economics. S*ingapore: McGraw-Hill.

Clark, C.W. 1990. *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*. New York, NY: John Wiley & Sons.

Forsberg, O.I. 1996. Optimal Stocking and Harvesting of Size-Structured Farmed Fish: A Multi-Period Linear Programming Approach. *Mathematics and Computer in Simulations* 42:299–305.

___. 1999. Optimal Harvesting of Farmed Atlantic Salmon at Two Cohort Management Strategies and Different Harvest Operation Restrictions. *Aquaculture Economics and Management* 3:143–58.

Hannesson, R. 1986. Optimal Thinning of a Year-Class with Density-Dependent Growth. *Canadian Journal of Fisheries and Aquatic Sciences* 43:889–92.

Heaps, T. 1993. The Optimal Feeding of Farmed Fish. *Marine Resource Economics* 8: 89–99.

___. 1995. Density Dependent Growth and the Culling of Farmed Fish. *Marine Resource Economics* 10:285–98.

Liu, Y., K.L. Teo, L.S. Jennings, and S. Wang. 1998. On a Class of Optimal Control Problems with State Jumps. *Journal of Optimization Theory and Applications* 98:65–82.

Miller, B.M. 1993. Method of Discontinuous Time Change in Problems of Control for Impulse and Discrete Continuous System. *Automation and Remote Control* 54:1727–50.

Mistiaen, J.A., and I. Strand. 1998. Optimal Feeding and Harvest Time for Fish with Weight-Dependent Prices. *Marine Resource Economics* 13:231–46.

Moss, S.M., C.A. Otoshi, and P.S. Leung. 2005. Optimizing Strategies for Growing Larger *L. vannamei*. *Global Aquaculture Advocate* 8:68–9.

Paessun, M.A., and R. Allison. 1984. Maximizing Tilapia Production in Recirculation Systems by Sequential Rearing. *Aquaculture* 42:185–89.

Rogovchenko, Y.V. 1997. Impulsive Evolution Systems: Main Results and New Trends. *Dynamics of Continuous, Discrete, and Impulsive Systems* 3:57–88.

Seierstad, A. 1981. Necessary Conditions and Sufficient Conditions for Optimal Control with Jumps in the State Variables. Memorandum from the Institute of Economics, University of Oslo, Norway, June 15.

Seierstad, A., and K. Sydsaeter. 1987. *Optimal Control Theory with Economic Applications*. New York, NY: Elsevier Science Publishers.

Sendak, P.E., J.C. Brissette, and R.M. Frank. 2003. Silviculture Affects Composition, Growth, and Yield in Mixed Northern Conifers: 40-Year Results from the Penobscot Experimental Forest. *Canadian Journal of Forest Research* 33: 2116–28.

Shaftel, L.T., and M.B. Wilson. 1990. A Mixed-Integer Linear Programming Decision Model for Aquaculture. *Managerial and Decision Economics* 11:31–8.

Spaargaren, D.H. 1999. Optimal Harvest Size in Shrimp Cultures. *Crustaceana* 72:297–306.

Summerfelt, S.T., J.A. Hankins, S.R. Summerfelt, and J.M. Heinen. 1993. Modeling Continuous Culture with Periodic Stocking and Selective Harvesting to Measure the Effect on Productivity and Biomass Capacity to Fish Culture Systems. *Techniques for Modern Aquaculture*, J.K. Wang, ed., pp. 581–95. *Proceedings of the Aquacultural Engineering Conference*, Spokane, Washington, 21-23 June.

Touza-Montero, J., and M. Termansen. 2001. The Faustmann Model as a Special Case. *Paper presented at the workshop on Conservation and Sustainable Development: Comparative Perspectives*, Yale University, New Haven, CT. http://research.yale.edu/CCR/environment/papers/touze-montero_termansen.pdf

Watten, B.J. 1992. Modeling the Effects of Sequential Rearing on the Potential Production of Controlled Environment Fish-Culture Systems. *Aquacultural Engineering* 11:33–46.

Yang, T. 2001. Impulsive Control Theory. *Lecture Notes in Control and Information Sciences*. Vol. 272, 348 pp. Heidelberg: Springer-Verlag.

Yong, J., and P. Zhang. 1992. Necessary Conditions of Optimal Impulse Control for Distributed Parameter Systems. *Bulletin of the Australian Mathematical Society* 45:305–26.