Stochastic Efficiency Analysis Using Multiple Utility Functions

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Abstract

Evaluating the risk of a particular decision depends on the risk aversion of the decision maker related to the underlying utility function. The objective of this paper is to use stochastic efficiency with respect to a function (SERF) to compare the ranking of risky alternatives using alternative utility functional forms.

Key words: stochastic efficiency with respect to a function, certainty equivalent, utility function, risk aversion

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Risk assessment requires coming to grips with both probabilities and preferences for outcomes held by the decision maker. Chances of bad versus good outcomes can only be evaluated and compared knowing the decision maker's relative preferences for such outcomes. In the context of the subjective expected utility (SEU) hypothesis (Anderson, Dillon, Hardaker 1977: 66-69), the decision maker's utility function for outcomes is needed to assess risky alternatives.

The shape of the utility function reflects an individual's attitude towards risk. Several attempts have been made to elicit such utility functions from relevant decision makers in order to put the SEU hypothesis to work in the analysis of risky alternatives in agriculture. Usually the results have been rather unconvincing (King and Robison 1984; Anderson and Hardaker 2003).

Partly to avoid the need to elicit a specific single-valued utility function, methods under the heading of stochastic dominance or efficiency criteria have been developed. Hadar and Russell (1969) and Hanoch and Levy (1969) presented the concepts of first-degree stochastic dominance (FSD) and second-degree stochastic dominance (SSD). FSD is used to partition alternatives for decision makers who prefer more wealth to less and have absolute risk aversion with respect to wealth, $r_a(w)$, where $-\infty < r_a(w) < \infty$. SSD requires the additional assumption that decision makers are not risk preferring, i.e. that absolute risk aversion bounds are $0 < r_a(w) < \infty$. In empirical work it is often found that these two forms of analysis are not discriminating enough to yield useful results, meaning that the efficient set, or the alternative(s) that represent the preferred choice within a given range of risk aversion, can still be too large to be easily manageable (King and Robison 1981, 1984).

An alternative to FSD and SSD is Meyer's (1977) stochastic dominance with respect to a function (SDRF). For SDRF the absolute risk aversion bounds are reduced to $r_L(w) \le r_a(w) \le r_U(w)$, and ranking of risky scenarios is defined for all decision makers whose absolute risk aversion function lies anywhere between lower and upper bounds $r_L(w)$ and $r_U(w)$, respectively. The method has stronger discriminatory power than FSD and SSD, because of the introduced tighter risk aversion bounds; however, SDRF often results in ambiguous rankings that suggest that rankings change between the lower and upper bounds.

SERF: An Alternative Procedure

A simpler method of analysis based on the same assumptions about risk attitudes as SDRF, is illustrated by Richardson, Schumann, and Feldman (2001) and expounded on by Hardaker, et al. (2004). The method, named stochastic efficiency with respect to a function (SERF) using risk aversion bounds, works by identifying utility efficient alternatives for ranges of risk attitudes, not by finding (a subset of) dominated alternatives. SERF partitions alternatives in terms of certainty equivalents as a selected measure of risk aversion is varied over a defined range.

The SERF method includes all the advantages of SDRF yet is more transparent, is easier to implement, and has a stronger discriminating power. These seem to be powerful advantages that suggest that SERF could extend the power of risk efficiency analysis in the SEU framework to practical applications in business and policy decision-making. This method can be an attempt to partially ameliorate the pitfalls of SEU pointed out by Rabin (2000). The SERF method does not attempt to pinpoint risk aversion levels elicited by experimentation or estimation to categorize alternatives. Rather, it takes risk aversion levels as given and presents a class of rankings based on categories of decision makers within ranges of risk aversion.

SERF has been demonstrated for two utility functions commonly used for analytical purposes, the negative exponential and power utility functions. The objective of this paper is to use SERF to compare the ranking of risky alternatives using alternative utility functional forms, including composite utilities. The relative importance of choices of utility functions, or classes of utility functions, for groups of decision makers will be considered.

Methods

Primary assumptions involved in expected utility analyses are those relating to the distribution of the data and the utility function of the decision maker. Differing assumptions regarding these can greatly impact decision analysis and efficient sets under alternative risk aversion levels.

The SERF procedure can be carried out for many specified parametric distributions. For the purpose of this study, the assumption of nonparametric or discrete uniform distributions for the alternatives will be used. This assumption simplifies the assignment of probabilities to the individual data values, and, in large samples, approaches the true underlying distributions. The drawback of this assumption is that data are limited to observed values and may be unduly truncated at the observed extremes. Equal probability weighting is also problematic in the presence of extreme outliers when moment applications are necessarily utilized since there are no parameters to estimate. Regardless, calculations of certainty equivalents for each alternative are relatively straightforward.

The additional assumption made is in regards to the utility function. For this study, several utility functions are considered which are characterized by their respective risk aversion coefficients (RACs), or more specifically, their relative risk aversion coefficients (RRACs) represented as r_r. Absolute risk aversion coefficients (ARACs) are represented as r_a. The SERF procedure assumes a continuous range of RACs over which to evaluate efficient sets of alternatives. Since RACs are dependent on the random wealth variable through the utility relationship, expected RACs are used in the SERF procedure. The expected RACs are based on the same distributional assumptions made regarding the certainty equivalents.

Utility functions of interest for this analysis will include the constant absolute risk aversion (CARA) negative exponential, constant relative risk aversion (CRRA) power, parameterized restrictions to constant or decreasing absolute risk aversion expo-power (Saha 1993), and decreasing absolute risk aversion (DARA) log utility functions. Additional information will be presented using the quadratic, exponent, and a hyperbolic absolute risk aversion (HARA) type utility function. We restrict ourselves to the class of utility functions typically used in classical SEU analyses that exhibit concavity in the range of risk aversion. The specific functional forms of the utility functions of interest are

(1) Negative exponential: $U(w) = -\exp(-r_a w)$,

(2) Power: $U(w) = \frac{1}{1-r_r} w^{1-r_r}$,

(3) Expo-Power: $U(w) = -\exp(\beta w^{\alpha}),\ \alpha \le 1,\ \alpha\beta > 0$,

(4) Log: $U(w) = \ln(w+c),\ c \ge 0$,

(5) Quadratic: $U(w) = aw - \frac{b}{2} w^2$,

(6) Exponent: $U(w) = (w+c)^p,\ 0 < p < 1,\ c \ge 0$,

(7) HARA: $$U(w) = \frac{\gamma}{1-\gamma}\left(a + \frac{b}{\gamma}w\right)^{1-\gamma},$$

where w represents a random wealth variable and the remaining variables are parameters specific to each function.

Given an empirical sample, some plausible assumptions about possible efficient choices involved in a SERF analysis can intuitively be made. Using one of many of the commonly used utility functions as the assumed utility of the decision maker, it seems plausible that a risk neutral decision maker would choose the mean value of a series and thus rank alternatives based on the ordered average values. This assumption can be used in SDD and SERF procedures.

Other assumptions under the previous conditions can be made. The certainty equivalent for a given series should not take on values outside the bounds of the observed minimum and maximum order statistics. Clearly, if there is no information about the true bounds of the random series, then a decision maker would not choose an indifference fixed value that is less than the minimum. Similarly, but not as obvious, a risk-loving decision maker should not choose a value that is above the maximum observation as an indifference value. This assumes a type of behaviour that the decision maker would not continuously seek higher indifference values than the random outcome can produce.

The previous assumptions on the bounds and characteristics of an empirical certainty equivalent are not robust under many combinations of utility functions and assumed distributions. An empirical sample assumes a truncated distribution when often the variables in risk analysis can take on a virtually infinite range on the real line. Additionally, a mean for a small sample may be a poor estimate of the true mean for many distributions. Many utility functions do not retain the aforementioned characteristics due to functional form. These represent behaviours that are not well estimated with nonparametric sample data. It is useful to be cautious of these limitations when implementing a SERF analysis.

Numerical Analysis

For the purpose of illustration, empirical samples for five alternative hypothetical net returns, labelled A-E, will be compared with a SERF procedure for alternative utility functions. These data are given in Appendix Table A1. None of the alternatives exhibit FDD. Each alternative has a sample mean of 100. Alternatives A and B have sample standard deviations of 25 and alternatives C, D, and E have standard deviations of 30. These data were chosen in such a way as to make mean-based ranking ambiguous and mean-variance analysis generally ineffective at sorting out preferences. Many situations involving data for utility-based decision-making will not be as intricate as presented here. A sample cumulative distribution function of the data is shown in Figure 1.

Each sample contains eleven values, which is a size that is typically considered too small to evaluate distributional assumptions. It would be difficult to elicit an unknown utility function for this type of sample, and imposing specific distributional assumptions may bias the premise of utility estimation. Basing an ordering of alternatives on a specific elicited utility function could produce non-unique rankings, especially if the particular RAC is near a boundary point in the efficient set and has a large variance due to the small sample size.

For the SERF analysis, initial wealth is assumed to be zero. The wealth exponent parameter for the expo-power function is set at 0.4 to represent decreasing absolute risk aversion. For a given RRAC level, any remaining parameters for the utility functions are iteratively approximated. There are several methods to determine relevant bounds for stochastic efficiency analyses (see McCarl and Bessler 1989), but in this analysis, bounds can be determined based on the outermost crossing points of the alternatives. A relative risk aversion range of –8 to 8 was used to compare the results across all of the utility functions of interest.

The SERF charts of these results are given in Figures 2-5. The lines on the chart are the empirical certainty equivalent values as a function of risk aversion for each alternative. The overall efficient set for a specified range of risk aversion is the set of uppermost values, which can be one alternative or a combination of alternatives. Sub-efficient sets are the uppermost values when considering a subset of alternatives. A relative risk premium between two alternatives is seen as the vertical distance between those alternatives for a given RRAC.

The overall efficient set for risk neutral and risk averse decision makers with a negative exponential utility function includes Alternative A in Figure 1. The overall efficient set includes all alternatives for all risk aversion levels. Most, if not all, of the changes of efficient sets are made within the previously stated risk aversion range of $-8 < r_r(w) < 8$ for all of the utility functions. Not only can the point of change of overall preferences be identified, but comparisons of subsets as well as degree of difference can also be elicited.

A comparison of overall efficient sets across utility functions is shown in Figure 6. The log utility function at this range of risk aversion is too narrow to pick up any perceptible preferences among the alternatives. The remaining utility functions rank efficient sets within approximately similar ranges of relative risk aversion. In general, Alternative B is preferred by extremely risk loving individuals and Alternative D is preferred by strongly risk loving individuals. Alternative C is preferred by nearly risk neutral decision makers under a power utility function. Risk neutral to strongly risk averse decision makers seem to prefer Alternative A.

Interestingly, Alternative E is preferred by both moderately risk loving and extremely risk averse decision makers. This can be explained by tail-myopic behaviour. Referring back to Figure 1, it can been seen that Alternative B is preferred by some risk averse individuals because the lower tail is truncated and the minimum observed value is greater than that of any of the other alternatives. This alternative is preferred by moderately risk loving decision makers because the empirical probability of observing the maximum value is 45%. In addition, this maximum value is in the approximate 10 and 20% upper tails of Alternatives C and D, respectively.

It should be noted that the rankings based on the utility functions other than the log utility are similar because these utility functions are in the same general class. Although the specific parameterizations represent CARA, DRRA, and DARA utilities, they can all be considered special cases of the expo-power utility function (see Saha 1993).

As an extension, several other utilities were considered using a SERF analysis. These included the quadratic, exponent, and HARA utility functions as previously specified. The range of parameterization of these particular functions makes precise specification of utility difficult without additional information. The rankings based on the DARA exponent utility approach the rankings of the log utility as $p \to 0$ and use strictly mean ranking if $p = 1$. The quadratic and HARA rankings depend on the relationship of the parameters specific to the utility function. More specifically, both of these can exhibit increasing absolute risk aversion (IARA), thereby essentially reversing the order of the rankings given by the expo-power type functions.

Composite Utility

In many cases, the specific form of the utility function, let alone the parameterization, is relatively unknown apart from general characteristics. Apart from having to choose a specific utility function, a composite of several utility functions can be used in the ranking process for a SERF analysis. Choosing utility functions with the assumption of concavity in the range of risk aversion and weighting them to create a composite ranking can be useful to analyze decision maker's choices under quasi-risk aversion conditions. An example of a composite ranking using the four utility functions discussed previously can be seen in Figure 7. This type of ranking can be useful in a Baysian analysis of continuing choice ordering.

Conclusions

In the general spirit of risk analysis, the SERF procedure is a useful tool when many of the components necessary for a SEU analysis are unknown. We have discussed the key assumptions that drive this type of analysis and have used an example with a few commonly used utility functions to illustrate the results. When comparing several alternatives simultaneously, the overall efficient set can be similar across differing utility functions; however, the class of utility functions and individual parameterizations of each are fundamental when specific point risk aversion decision results are desired. The SERF method is preferable when ranges or categories of risk aversion are the desired indicators for an overall efficient set. There does not need to be a rigorous elicitation of risk preferences from decision makers to calibrate a choice outcome, nor do bounds on risk aversion need to be estimated a priori. A relatively full picture of the risk situation at hand can be made explicit with this procedure.

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Figures

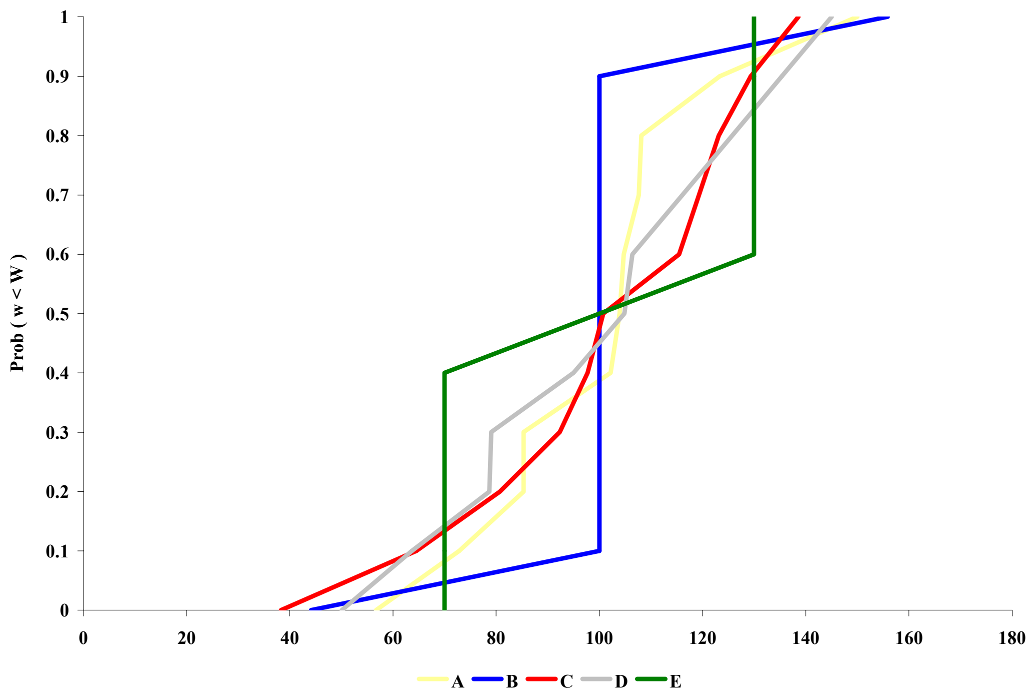

Figure 1. Cumulative distribution function of hypothetical net returns for SERF analysis

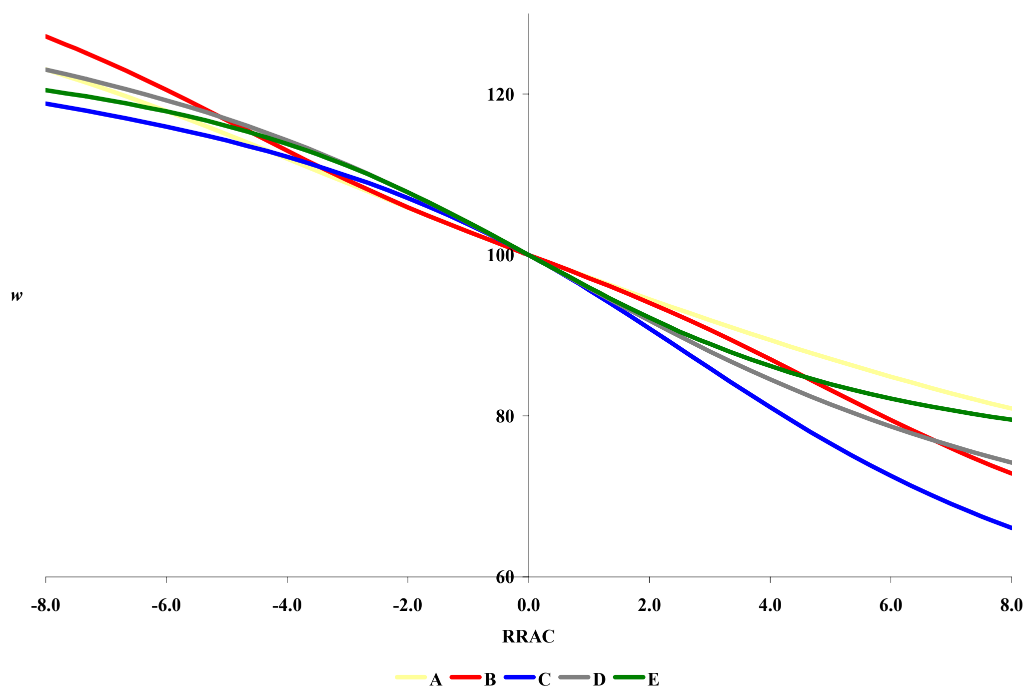

Figure 2. SERF chart for hypothetical net returns given a negative exponential utility function

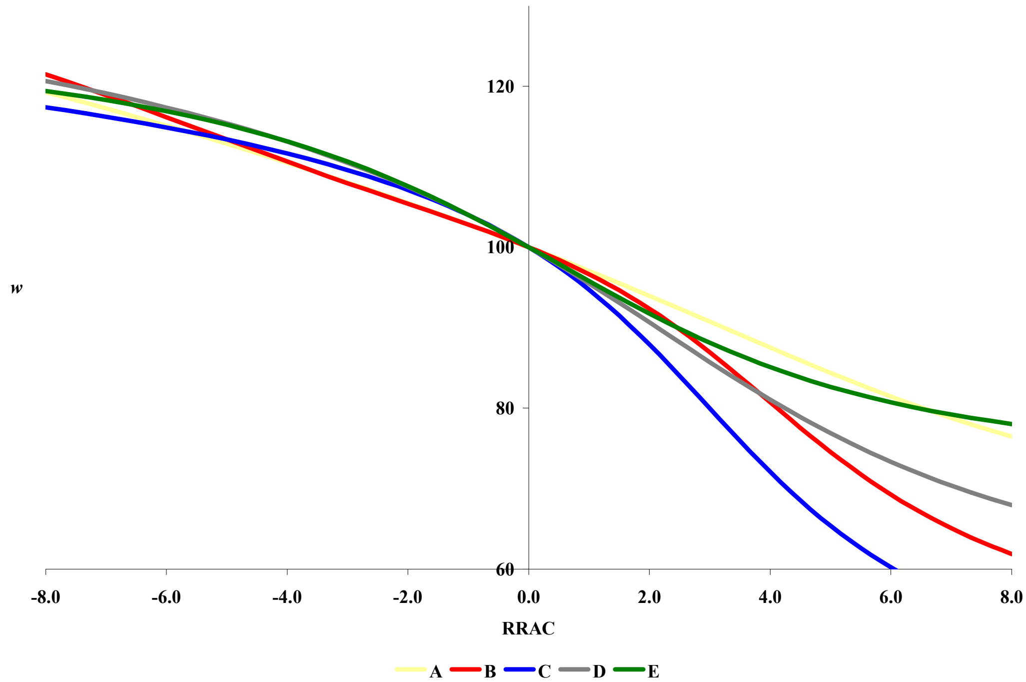

Figure 3. SERF chart for hypothetical net returns given a power utility function

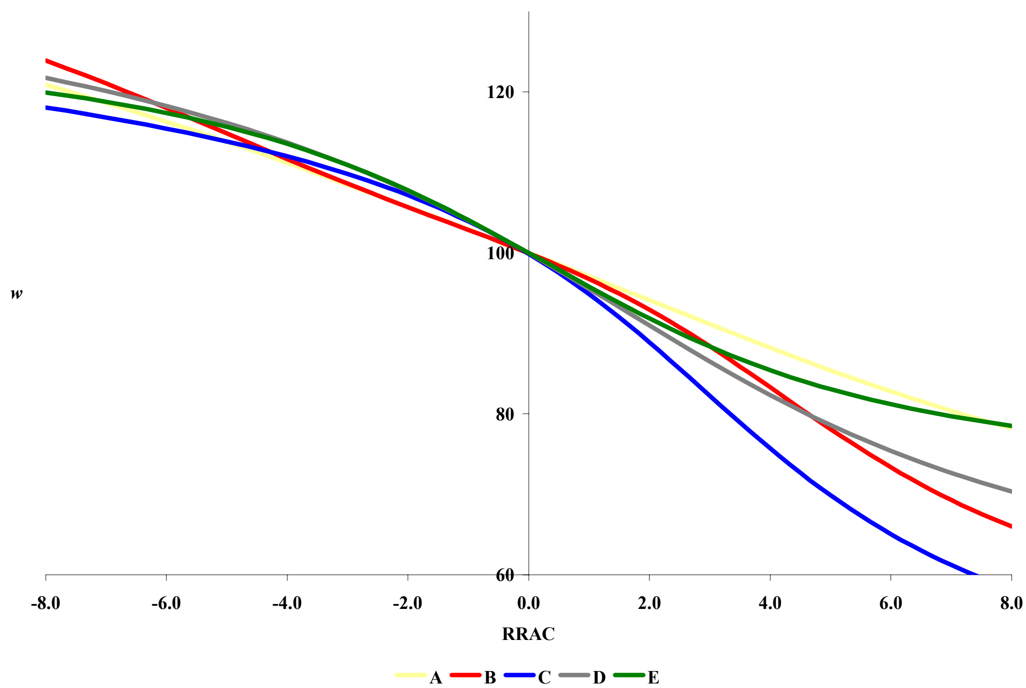

Figure 4. SERF chart for hypothetical net returns given an expo-power utility function

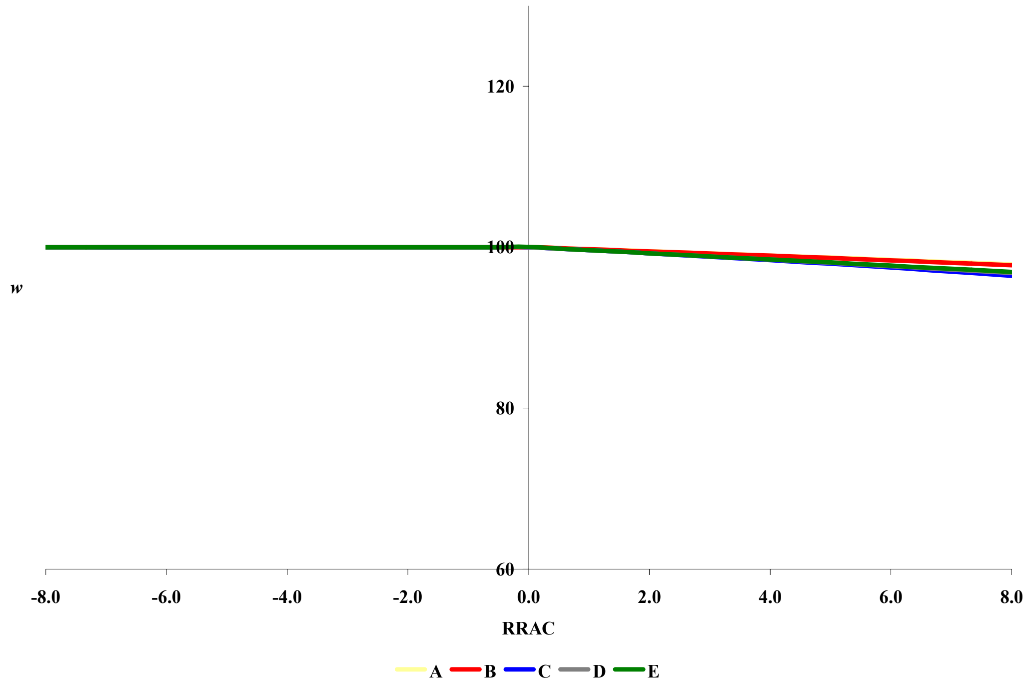

Figure 5. SERF chart for hypothetical net returns given a log utility function

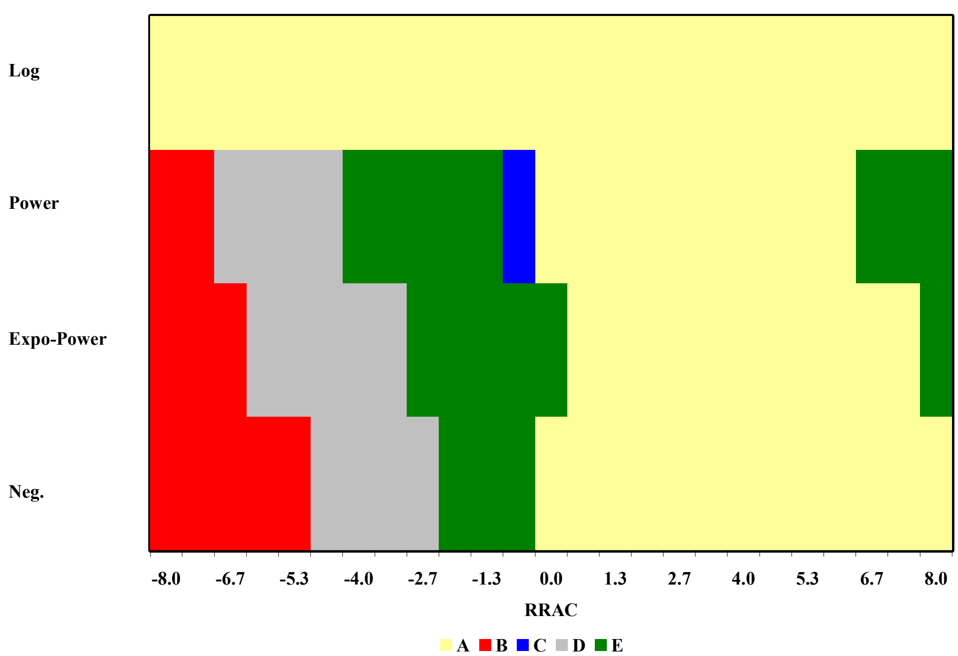

Figure 6. Comparison of SERF efficient sets across utility functions including negative exponential, power, expo-power, and log utilities

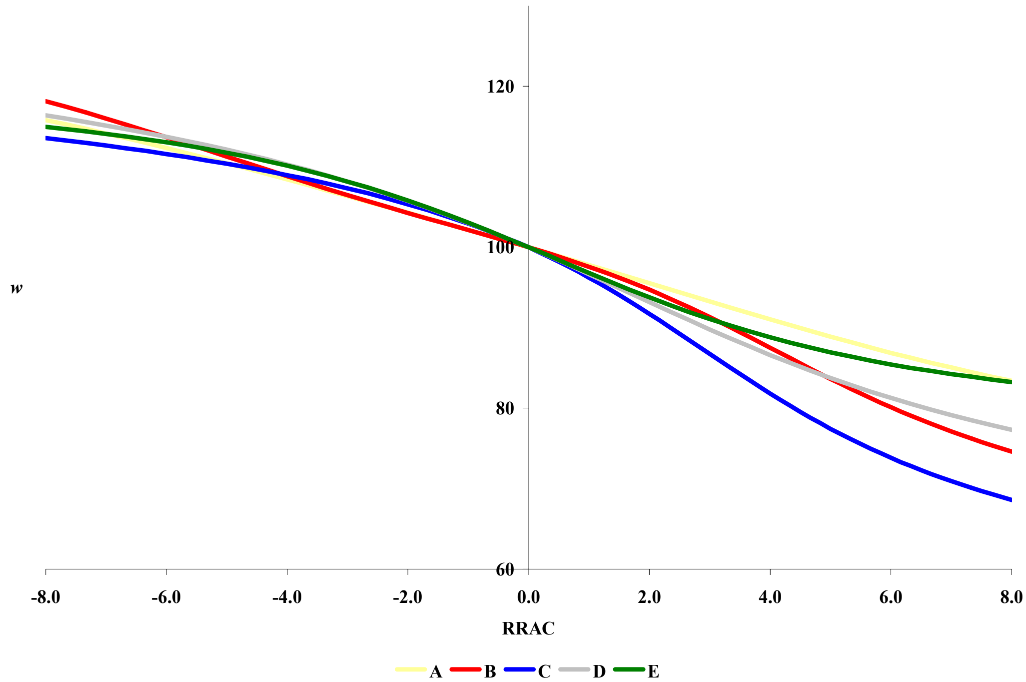

Figure 7. SERF chart for hypothetical net returns given an equally weighted composite utility function comprised of negative exponential, power, expo-power, and log utility functions

Appendix

Table A1. Pseudo Net Returns for SERF Analysis

Obs.	**A**	**B**	**C**	**D**	**E**
1	56.73	44.10	38.28	50.06	70.00
2	72.87	100.00	64.51	63.69	70.00
3	85.28	100.00	80.71	78.67	70.00
4	85.28	100.00	92.28	79.06	70.00
5	102.18	100.00	97.69	95.01	70.00
6	103.90	100.00	100.77	104.82	100.00
7	104.72	100.00	115.43	106.42	130.00
8	107.63	100.00	119.29	116.08	130.00
9	108.10	100.00	123.15	125.74	130.00
10	123.31	100.00	129.32	135.40	130.00
11	150.00	155.90	138.58	145.06	130.00

Table A2. SERF Table of Certainty Equivalents for Pseudo Net Returns Under an Expo-Power Utility Function and Discrete Uniform Distribution

RRAC	A	B	C	D	E
-8	120.86	123.90	118.07	121.74	119.93
-7.3333	119.42	122.03	117.27	120.67	119.19
-6.6667	117.89	120.06	116.39	119.51	118.35
-6.0000	116.30	118.01	115.44	118.25	117.38
-5.3333	114.65	115.90	114.40	116.87	116.28
-4.6667	112.94	113.77	113.26	115.36	115.00
-4.0000	111.18	111.66	111.99	113.71	113.54
-3.3333	109.38	109.60	110.57	111.89	111.86
-2.6667	107.55	107.61	108.98	109.90	109.94
-2.0000	105.70	105.68	107.16	107.71	107.77
-1.3333	103.82	103.79	105.08	105.33	105.36
-0.6667	101.92	101.91	102.68	102.74	102.75
0	100.00	99.96	99.89	99.97	100.01
0.6667	98.05	97.86	96.66	97.05	97.20
1.3333	96.09	95.54	92.97	94.03	94.45
2.0000	94.11	92.92	88.87	90.98	91.84
2.6667	92.12	89.98	84.48	87.97	89.44
3.3333	90.15	86.73	80.01	85.06	87.29
4.0000	88.21	83.29	75.69	82.32	85.41
4.6667	86.32	79.81	71.70	79.79	83.78
5.3333	84.50	76.45	68.15	77.47	82.38
6.0000	82.77	73.35	65.05	75.38	81.18
6.6667	81.14	70.56	62.39	73.51	80.15
7.3333	79.62	68.11	60.12	71.83	79.27
8	78.21	65.98	58.18	70.33	78.52