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## **Recursive Sustainability: Intertemporal Efficiency and Equity**

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## **Recursive Sustainability: Intertemporal Efficiency and Equity**

### *Abstract*

PV-optimality in a capital-resource economy can imply decreasing utility over some portion of the time horizon. Various criteria have been proposed to maintain intergenerational equity defined as nondeclining utility, but these have some limitations and problems. This paper proposes a new welfare criteria incorporating present value to maintain efficiency, and an equity function with convex costs on declining utility. This criterion is economically efficient, time-consistent and recursive. An extension of dynamic programming to multiple value functions is developed to solve this problem. Increasing the equity weight increasingly eliminates declining portions of utility time paths. Sustainability implies increasing consumption in the early time periods and some intermediate time periods relative to PV-optimality. A surprising result is that sustainability can actually result in increased resource usage in early time periods, followed later by higher levels of resource stocks compared to PV-optimality. The sustainability analysis shows that while conventional benefit-cost and valuation analysis contribute to efficiency, they do not necessarily induce sustainability due to incorrect dynamic GE prices. Similar comments apply to Green NNP analysis. The concepts and extended DP methods developed in this paper extend naturally to uncertainty and can also be applied to generalized consumer and social choice models beyond those typically considered in the literature.

*Keywords:* growth, environment, intergenerational equity, dynamic programming

## 1. Introduction

Sustainability is much discussed in policy arenas but a fundamental analytical base in environmental economics are lacking. Within the general environmental literature, some popular definitions such as the Bruntland Commission - while conceptually appealing - lack sufficient specificity to be the basis for quantitative policy analysis. Other definitions within the environmental policy and scientific literature define sustainability as maintaining the existing stock of natural and biological resources. However, this definition confuses means with ends, ignores substitution possibilities, and, taken literally, leads to nonsensical conclusions such as never using exhaustible resources or drawing down the stocks of renewable resources. Within the economic literature, there appears to be implicit agreement that sustainability involves a notion of intertemporal equity that future generations not be worse off than preceding generations. However, it would hardly make sense to have an equitable economy in this sense, but inefficient with productivity levels below what is possible. Thus we define sustainability as intertemporal efficiency formulated as Pareto-optimality, and intergenerational equity implied by non-declining utility. This definition is consistent with the majority of economic sustainability analyses which implicitly or explicitly include efficiency as well as equity. A more developed argument may be found in Stavins, Wagner, and Wagner (2003) and an axiomatic development from ethical principles in Asheim (2001).

The standard criteria for dynamic economic choice problems is present-value [Frederick et al (2002)]. However, in a capital-resource economy it is known that present-value optimality may (and often does) imply declining utility after some point [Dasgupta and Heal (1974, 1979), Pezzey and Withagen (1998)]. Several other criteria have been proposed [Heal (1998)], but as discussed later, the various criteria have significant limitations in one or more dimensions. Some of the

difficulties include extreme but unnecessary conservatism, incomplete ranking, imposition of all-or-nothing constraints, definitions which may allow intermediate generations to be better off than both preceding and succeeding generations, and time inconsistency. The paper first synthesizes several major strands of the available literature. While there is considerable development and several informative reviews [Toman et al (1995), Heal (1998)], the literature can appear disjoint and scattered. The results are recast in a common DP framework, and some numerical examples are developed in contrast to the almost exclusive reliance on analytical derivations.

The prime original contribution of the paper is the development and analysis of a new welfare criterion for identifying sustainable intertemporal allocations. Present-value is retained to guarantee intertemporal efficiency. An additional equity criterion is proposed with increasing penalty for downward movements in intergenerational utility. The combination of the two criteria implies a recursive welfare ordering which yields efficient, time-consistent paths for consumption. A generalization of standard dynamic programming methods to multiple value functions is developed to solve this problem. This procedure successfully identifies sustainable time paths for economic growth. A very surprising result is that sustainability may imply increased resource usage in the early time periods. Policy instruments, implications for achieving sustainability, and extensions to uncertainty are also discussed. The extended DP methods developed in this paper to solve the sustainability problem may also allow for generalized dynamic consumer and social choice models.

## **2. Bioeconomic Model.**

Economic growth with natural resources and constant technology and population are considered. Consumers live for a single period, have aggregate utility  $u(c_t)$  where  $c_t$  is consumption, and supply a unit of aggregate labor per year. Output is a function of resource flows, capital services,

and labor. Under the assumption of constant returns to scale, inelastic labor supply implies output  $q_t = f(k_t, r_t)$  where  $k_t$  is the capital stock,  $r_t$  = resource flows, and the production function  $f$  exhibits the neoclassical properties of diminishing marginal productivity in capital and natural resources. The capital stock evolves according to

$$k_{t+1} = (1 - \delta)k_t + q_t - c_t \quad (1)$$

where  $0 < \delta < 1$  is the depreciation rate. Investment  $q_t - c_t$  is constrained to be non-negative.

An energy-type natural resource is considered. There is an initial stock representing fossil-fuels, and a renewable component for solar power and associated flows (wind, hydropower). The latter is available in a constant amount every year as an endowment to households analogous to labor supply. Natural resource flows and the stock equation of motion are

$$r_t = \omega + z_t \quad s_{t+1} = s_t - z_t \quad (2)$$

respectively, where  $s_t$  is the exhaustible resource stock,  $z_t$  is extraction, and  $\omega$  is the renewable flow. Extractions are constrained by  $z_t \leq s_t$ . This is similar to a model in Dasgupta and Heal (1974, 1979).

Computational simulations are used to further explore model properties beyond what can be accomplished analytically [Judd (1998)]. Computations assume isoelastic utility  $u(c) = c^\rho / \rho$  and Cobb-Douglas production  $q = k^{\beta_k} r^{\beta_r}$ . Parameter values for baseline simulations are in Table 1, and initial stocks are  $k_1 = .2$  and  $s_1 = 5$  representing an economy starting out with a full complement of natural resources and essentially no physical capital. This is intended to represent the life-history of an economy interacting with the environment over long time spans. Whether or not market economies will achieve sustainability is an open question. Mourmouras (1991) demonstrates an OLG economy with declining utility even though that is not a technological necessity. In other instances,

however, market economies might automatically achieve sustainability [Toman et al (1995)]. Here we will concentrate on the normative question of sustainability criteria and policy in the adverse case.

Efficiency is defined here as Pareto-optimality. Fixing the states in periods  $t$  and  $t+2$  and period  $t+1$  utility (consumption), period  $t$  extractions and investment maximize period  $t$  consumption. This results in

$$f_r(k_{t+1}, r_{t+1}) = [1 - \delta + f_k(k_{t+1}, r_{t+1})] f_r(k_t, r_t) \quad (3)$$

assuming an interior solution which can be rearranged to demonstrate equal rates of return to natural resource and capital stocks. In a competitive economy with output price as a numeraire, the resource marginal product is its relative price, while  $f_k(k_{t+1}, r_{t+1}) - \delta$  is the capital rate of return, so (3) is just Hotelling's rule and hence a necessary condition for Pareto optimality. An Edgeworth box diagram provides a graphical interpretation. In figure 1 (generated from the baseline data and state values as noted), indifference curves for generation  $t$  are concave to the origin and decreasing in a left to right direction, while indifference curves for generation  $t+1$  are just the opposite. An efficiency locus is plotted along with the tangent indifference curves for one point. Geometrically, Hotelling's rule (3) is a necessary condition for being on the efficiency locus; location is dependent on the specific criterion being used. Sustainability is interpreted here as also being on the efficiency locus with period  $t$  utility no larger than period  $t+1$  utility.

The primary focus of the paper is welfare optima, but a competitive equilibrium supporting the optimum is also of interest. Theoretical analysis of efficient allocations and a supporting competitive price system is given in Mitra(1978) for a capital-resource economy similar to that here. Let  $\{p, p_{kr}, p_{re}, p_{kr}, p_{st}\}$  denote prices for output, capital rental, extractions, and capital and natural

resource stocks respectively. Consumers maximize  $u(c_t) - p_t c_t$  implying that

$$p_t = u'(c_t) \quad (4)$$

for equilibrium output prices. Profit-maximization and constant returns to scale imply

$$p_{rt} = p f_r(k_t, r_t) \quad p_{kr,t} = p f_k(k_t, r_t) \quad w_t = p f_l(k_t, r_t) - p_{rt} r_t - p_{kr,t} k_t \quad (5)$$

for resource flow prices, capital services prices, and wages respectively, and where  $f_x$  is marginal product of input  $x$ . Investment profits are  $(p_{k,t+1} - p_t) i_t$ , while extraction profits are  $(p_{rt} - p_{s,t+1}) z_t$  as

extractions now imply less stock to be sold in the future. These imply that

$$p_{k,t+1} = p_t \quad p_{s,t+1} = p_{rt} \quad (6)$$

must be satisfied for an equilibrium to exist when  $i_t > 0$  and  $z_t > 0$ . For equilibrium we must also have equalized rates of return in both stock markets. Collectively these observations result in

$$p_{s,t+1} = (p_t [1 - \delta + f_k(k_t, r_t)] p_{st}) / p_{t-1} \quad (7)$$

where the expression multiplying the current resource stock price is just one plus the rate of return in the capital market (i.e. Hotelling's rule).

In principle these prices are unobservable since they depend on marginal utility. Prices can be scaled so that they are observable [Weitzman (2003)]; however, this is unnecessary here since only prices relative to  $p_t$  are considered. Another point is that in viewing (4) - (7), if we take the states  $k_t$  and  $s_t$  as given, then in each time period there are seven relations in eight unknowns (6 prices and 2 decision variables), implying indeterminacy. This is consistent with the fact that there are an infinity of efficient allocations in this model with an infinity of associated competitive equilibria. The welfare criteria to be investigated will select a specific allocation which closes the above system thereby resulting in a well-defined competitive equilibrium for the selected allocation.

### 3. PV-optimality and economic efficiency

Present-value (PV) optimality maximizes  $\sum_{t=1}^{\infty} \alpha^{t-1} u(c_t)$  subject to the bioeconomic relations

defined previously, and where  $\alpha$  is the discount factor. Bellman's equation for this problem is

$$V[k,s] = \text{Max } u(c) + \alpha V[g(k,s,c,z)] \quad (8)$$

where  $V$  is the optimal value function, the maximization is over consumption  $c$  and resource extractions  $z$  subject to the resource and investment constraints, and  $g$  is the (vector) equation of motion for capital and the natural resource. PV-optimality generates time paths which are recursive, time-consistent, and economically efficient defined as Pareto-optimality (PO). Choice of the discount factor will select alternate members of the PO set; however, the constant discount rate means that some members will not be selected, these would require time-varying discount factors.

Euler conditions can be derived from the dynamic programming problem (8). For clarity only the interior part of the solution (positive investment and resource extractions) is considered. The first-order conditions for period  $t$  include partial derivatives for  $V(k_{t+1}, s_{t+1})$ . These can be estimated using the envelope theorem and first-order conditions evaluated in period  $t+1$ . Substituting into the period  $t$  first-order conditions yields after some re-arranging

$$u'(c_t) = \alpha u'(c_{t+1}) [1 - \delta + f_k(k_{t+1}, r_{t+1})] \quad (9)$$

$$f_r(k_{t+1}, r_{t+1}) = [1 - \delta + f_k(k_{t+1}, r_{t+1})] f_r(k_t, r_t) \quad (10)$$

where the expression in brackets represents the gross rate of return on capital taking consumption as the numeraire good. Equation (9) is the standard Ramsey condition from economic growth theory. It indicates that investment is carried out to the point where the marginal utility of consumption in period  $t$  equals the discounted marginal value of lost consumption in period  $t+1$ , where the reduction in period  $t+1$  consumption equals the reduction in production due to the smaller capital stock, plus



an amount necessary to restore the capital stock to its previous level for succeeding generations.

Equation (10) is Hotelling's rule expressed in terms of relative resource prices. As previously noted, the geometric interpretation of Hotelling's rule in the Edgeworth box diagram (figure 1) is as a condition for being on the efficiency locus. Ramsey's rule can be geometrically interpreted as locating a specific allocation along the efficiency locus. The steady-state can be readily derived from either the first-order and envelope conditions implied by Bellman's equation, or from the Euler conditions. Using the latter, we see immediately from (9) that the steady-state capital stock is given by  $(1-\alpha)/\alpha = f_k(k, \omega) - \delta$  which is determined solely by the production parameters and the discount factor. The other variables can be determined readily from the other relations. Thus the steady-state capital stock goes up as the discount rate decreases for example.

Dasgupta and Heal (1974, 1979) conjecture that PV-optimality is single-peaked for an exhaustible resource: that is, utility may initially increase at first, but then is inevitably declining after some point. Pezzey and Withagen (1998) provide rigorous proofs for particular functional forms, while Toman et al (1995) further summarize the PV-optimality literature and the extent to which that criterion leads to sustainable outcomes. Computational results are presented here as a basis for the equity analysis that follows.<sup>1</sup> Quantity time-series are in figures 2(a-b) starting from a large initial resource and are consistent with the qualitative behavior predicted by theoretical analyses. The capital stock, consumption and utility first increase, then eventually decrease until the initial resource stock is depleted, after which a steady-state occurs. Defining output and input prices by (4-5) and letting  $p_{k,t+1} = \alpha V_k(k_{t+1}, s_{t+1}]$  and  $p_{s,t+1} = \alpha V_s(k_{t+1}, s_{t+1}]$ , it is then easily checked from the first-order conditions and Euler conditions that the remaining equilibrium price conditions in (6) and (7) are also

satisfied. In figure 4(c), capital rental prices start high reflecting initial scarcity, and then fall to the steady-state, while resource prices do just the opposite reflecting their initially-large stocks. Overshooting consistent with that observed for the quantity variables is also evident.

The implications of these results are striking and perhaps under-appreciated. Perfect foresight economies under standard assumptions may exhibit overshooting such that some generations have a level of well-being that exceeds generations after them and likely before as well. While the capital-resource economy considered here is simple, the results have considerable empirical validity. Brander and Taylor (1998) develop a Ricardo-Malthusian model of Easter Island which exhibits overshooting where the population level eventually exceeds the production capacity of the environment with eventual decline in both population and natural resources. Diamond (1997) reviews the history of human settlements and conquests over long time and spatial scales, and argues that new settlements and technology often resulted in a rapidly increasing economy followed later by eventual decline.

Experiments were also done for changes in the discount rate. Increasing the discount rate shifts the various quantity curves down, but still maintains the unsustainable consumption levels during the intermediate periods. Decreasing the discount rate to very small levels (.001) results in almost monotone convergence to the steady-state (which is the Golden rule level), thus exhibiting sustainability. The difficulty is that there are an infinity of efficient paths exhibiting non-declining utility. This measure only selects a single one, thus preventing the legitimate societal choice of how much to leave future generations, not just that they be at least as well off.

#### **4. Sustainability criteria and Green GNP**

Early normative economic studies [e.g. Solow (1974)] emphasize Rawlsian criteria and constant consumption paths, but these are extremely conservative in that they may lock future

consumption and utility levels into the same very low levels experienced by initial generations even though significant growth is possible and might well occur otherwise. The classic Hartwick (1977) study showed that investment just equal to natural resource rents implies sustainability under specific conditions. This result is not necessarily true in general [Toman et al (1995)]; however, it still provides considerable insight and intuition. Chichilnisky (1996) derives a sustainability criterion from axiomatic principles. Heal (1998) evaluates this welfare criterion utilizing logarithmic discounting and a *lim-sup* operator on future consumption. This ‘Green Golden Rule’ criteria successfully maintains higher levels of future well-being than PV-optimality, but is time-inconsistent, there is no obvious computational scheme for non-analytic, infinite-horizon problems, and intermediate generations could have higher levels of welfare than all other generations. Toman et al (1995) note that imposing non-declining utility as a hard constraint implies an unrealistic all-or-nothing quality.

Woodward (2000) develops an interesting and sophisticated approach relying in part on Asheim (1988). He proposes that the value function in a dynamic programming model of the economy or resource in question be considered as the welfare function for each generation. This incorporates both the generation’s own utility as well as altruism on their part for future generations. Woodward (2000) then proposes and solves the problem of maximizing this welfare function subject to the constraint that this measure of welfare be non-declining over time.<sup>2</sup> A possible concern with this criterion is that it does not rule out the possibility of declining own utility of individual generations. As with Woodward (2000), the approach developed here relies on dynamic programming but will directly incorporate the idea of nondeclining own-utility.

Closely related to the question of sustainability criteria and policies is monitoring economic performance for economic growth and sustainability. Conventional measures of aggregate economic

performance typically ignore degradation of natural resource stocks, thus likely overstating actual welfare. A substantial literature on ‘Green GNP’ has risen in response to this issue [Asheim (1994), Lozada (1995), Maler (1991)]. For our purposes, a key insight is provided by Weitzman (1976) in the context of a capital-only model. He provides a rigorous justification for the idea that NNP is a measure of ‘sustainable’ consumption, where ‘sustainable’ consumption is defined as an equal annual payment with the same present value as the discounted sum of future consumption levels.

A similar result can be derived here. Consider a first-order approximation to the value function on the right-hand side of (8). Substituting into (8) and re-arranging yields

$$\theta V(k_t, s_t)/(1+\theta) = u(c_t) + [V_k \cdot (k_{t+1} - k_t) + V_s \cdot (s_{t+1} - s_t)]/(1+\theta) \quad (11)$$

where  $\theta$  is the discount rate. The left-hand side of (11) is an equal annual payment with the same present value as  $V(k, s)$ . The partial derivatives of  $V$  correspond to the co-state variables in an optimal control problem and are the shadow prices for capital and natural resource stocks in a competitive economy. The right-hand side is therefore NNP defined as current consumption plus net investment in capital and natural resource stocks. This shows that NNP in this economy measures Hicksian ‘sustainable consumption’ as developed by Weitzman (1976) and others.

Welfare measures are plotted in figure 3 for the capital-resource economy under PV-optimality. Sustainable utility increases to a maximum level, then decreases to a steady-state. When actual utility is below the sustainable amount, then both wealth and sustainable utility are increasing; the opposite occurs when actual utility exceeds the sustainable level. Does green NNP forecast future problems? Since green NNP rises initially despite a future downturn, the answer is that knowing green GNP only is not a sufficient indicator. However, it can be observed that actual utility increasing faster than sustainable utility, or, equivalently, a slowdown in the increase of asset buildup, might be

econometric indicators of future problems.

### 5. Intergenerational equity and welfare optimization.

A new welfare function is now defined for achieving efficiency and equity. This retains PV as one component to induce economically efficient (Pareto-optimal) time paths, while equity is conceptually defined as non-declining utility over time. More specifically, an equity function  $e(\cdot)$  is defined to penalize downward movements in utility over time at increasing marginal cost. Let  $x = u(c_t) - u(c_{t+1})$ . Then  $e(x) = 0$  for  $x \leq 0$ , and  $e(x) > 0$ , increasing and convex for  $x > 0$  with slope going to  $\infty$  and 0 respectively as  $x$  does the same. The welfare mapping for generation  $t$  is defined as

$$w_t = \sum_{\tau=t}^{\infty} \alpha^{\tau-t} [u(c_{\tau}) - p_e e(u_{\tau} - u_{\tau+1})] \quad (12)$$

where the first summation is discounted utility to induce efficiency,  $p_e$  is an equity “price” or weight, and the second term is discounted equity “cost” associated with downward movements in utility over time. Welfare is optimized by appropriate choice of resource extraction and capital investment.

In (12) consumption after period  $t$  appears in 3 different terms - own utility, and equity cost in the current and preceding period. Thus a first-order condition for  $c_{\tau}$ ,  $\tau > t$ , contains 3 terms with that variable; however, the first-order condition for generation  $\tau$  only contains 2 terms for the same variable, so the criterion as it stands is not time-consistent. Optimal consumption for generation  $\tau$  from the perspective of generation  $t$  is not necessarily the same as optimal consumption for generation  $\tau$  from its own perspective. To overcome this problem we follow Asheim (1988) and assume that each generation selects its own consumption and resource extractions to maximize its welfare function (12), conditioned on the values of the state variables it receives and *taking as given* the decision rules of future generations. Let  $h_{\tau}(k_{\tau}, s_{\tau})$  be the decision rule for generation  $\tau$ ; this gives its

choice of consumption and resource extractions as a function of the capital and natural resource stocks it inherits. If this is substituted in (12) for all generations after the current one, then the decision problem for each generation  $t$  is to select  $c_t$  and  $z_t$  which maximizes (12) given  $k_t$  and  $s_t$ .

This problem can be solved recursively. Imagine for openers a finite horizon. Then the last generation just maximizes its own utility conditional on the states which defines its own decision rule. The previous generation takes this rule as given and then maximizes (12) which determines its decision rule. Proceeding backwards in time we can postulate that the successive welfare functions and decision rules will eventually converge to limiting functions under standard conditions. The problem then is to identify these functions, the associated decision rules, and characterize the implied time paths. Thus, rather than considering the usual problem of a social planner's problem over time, we will actually be considering the more realistic problem of a welfare-generating process.

Let  $V(k,s)$  denote discounted welfare and  $U(k,s)$  denote current utility under an infinite horizon, both as functions of the capital and resource stocks, and both assuming optimality. These are the limit functions for the process outlined above and must satisfy the functional equation system

$$\begin{aligned}
 V[k,s] &= \text{Max } u(c) - p_e e[u(c)-U[g(k,s,c,z)]] + \alpha V[g(k,s,c,z)] \\
 U[k,s] &= u(c^*)
 \end{aligned}
 \tag{13}$$

where the maximization is over consumption  $c$  and resource extractions  $z$ ,  $g$  is the (vector) equation of motion for capital and the natural resource, and  $c^*$  in the second equation is optimal consumption as determined in the first equation. This represents a system of two functional equations in the two unknown value functions, and is an extension of the usual single Bellman equation.

The decisions and time paths defined by the above process are time-consistent and recursive by construction [Blackorby et al]. A later section shows that it can easily be extended to include

uncertainty, and it avoids the downsides of a “hard” nondeclining utility constraint as pointed out in Toman et al (1995). Are the generated time-paths efficient? The first issue is precisely what is meant by efficiency. If efficiency is defined in a wide-sense by the generational welfare functions  $w_t$ , then it is guaranteed by the standard arguments of dynamic programming and Bellman’s Principle of Optimality. More interesting is efficiency in the narrow sense used previously in the paper, that is, will the solution to this problem generate time-paths of generational utility  $u(c_t)$  which are not Pareto-dominated? Because of the equity term, this is not immediate as in the PV-optimality case. A Pareto-superior path -if one existed - would obviously have higher present value, but if it contained a downward-sloping portion then it might not be optimal according to the WGP.

An answer in the affirmative is given for a finite horizon, although intuitively the same reasoning would appear to hold more generally. Let  $\{z_t^*, c_t^*, k_t^*, s_t^*\}$  be a time path from the WGP. Now suppose that there is some path  $\{\hat{z}_p, \hat{c}_p, \hat{k}_p, \hat{s}_p\}$  which has at least as much utility in every time period and more in at least one period. If consumption is equal in all periods prior to the last period but greater in the last, then the contradiction to the optimality of the original path is immediate since the equity term does not appear in the last period and all generations would have higher welfare. Let us now suppose the higher consumption occurs prior to the last period. We next construct a time path  $\{\tilde{z}_p, \tilde{c}_p, \tilde{k}_p, \tilde{s}_p\}$  from  $\{\hat{z}_p, \hat{c}_p, \hat{k}_p, \hat{s}_p\}$  by keeping the same level of extractions in each period but reducing consumption to  $c_t^*$  in each period prior to the last. This new path has the same level of resource stocks as the postulated superior path, but a larger capital stock entering into the last period. This implies that consumption in the last period of the constructed path exceeds the postulated superior path and hence

the WGP optimum. This again contradicts the optimality of the original path since consumption is greater in the last period and equal prior to that, and hence all generations would view the constructed path as superior. We conclude that the WGP optimum is efficient in the narrow-sense of Pareto optimality for generational own-utility for a finite horizon.

The first order conditions implied by the Bellman system (13) are

$$\begin{aligned} u'(c_t) &= p_e e' [u(c_t) - U(k_{t+1}, s_{t+1})] [u'(c_t) + U_k(k_{t+1}, s_{t+1})] + \alpha V_k(k_{t+1}, s_{t+1}) \\ \alpha V_k(k_{t+1}, s_{t+1}) f_r(k_t, r_t) &= p_e e' [u(c_t) - U(k_{t+1}, s_{t+1})] [f_r(k_t, r_t) U_k(k_{t+1}, s_{t+1}) - U_s(k_{t+1}, s_{t+1})] \\ &\quad + \alpha V_s(k_{t+1}, s_{t+1}) \end{aligned} \quad (14)$$

for consumption and extractions respectively. The first-order condition for consumption indicates that consumption in the current time-period should be carried out to the point where the marginal benefit equals the marginal cost of equity and reduced future welfare from a smaller capital stock. For extractions, increased extractions in the current period leave a higher capital stock in the next period (left-hand side), but reduced future stocks (right-hand side); the marginal effects are equalized. The envelope theorem applied to (13) yields

$$V_k(k_{t+1}, s_{t+1}) = [1 - p_e e' [u(c_{t+1}) - U(k_{t+2}, s_{t+2})] u'(c_{t+1})] [1 - \delta + f_k(k_{t+1}, s_{t+1})] \quad (15a)$$

$$V_s(k_{t+1}, s_{t+1}) = [1 - p_e e' [u(c_{t+1}) - U(k_{t+2}, s_{t+2})] u'(c_{t+1})] f_r(k_{t+1}, r_{t+1}) \quad (15b)$$

after substituting in the period  $t+1$  first-order conditions.

It can be shown that the Euler conditions

$$\begin{aligned} u'(c_t) &= p_e e' [u(c_t) - u(c_{t+1})] [u'(c_t) + U_k(k_{t+1}, s_{t+1})] \\ &\quad + \alpha [1 - p_e e' [u(c_{t+1}) - u(c_{t+2})]] [1 - \delta + f_k(k_{t+1}, s_{t+1})] u'(c_{t+1}) \\ f_r(k_{t+1}, s_{t+1}) &= [1 - \delta + f_k(k_{t+1}, s_{t+1})] f_r(k_t, r_t) \end{aligned} \quad (16)$$

satisfy the first-order and envelope conditions in the interior. The second condition is Hotelling's rule confirming the earlier efficiency result, while the first condition is a modified Ramsey's rule. Here  $U_k(k_{t+1}, r_{t+1}) = u'(c_{t+1}) \partial c / \partial k$



so these Euler equations can be interpreted as two functional equations in the unknown decision rules for consumption and extractions. Alternately, if the time interval is small or stocks evolve slowly enough, then  $\partial c/\partial k \approx (c_{t+2} - c_t)/(k_{t+2} - k_t)$  as a finite-difference approximation, so this plus the equations of motion are a second-order difference equation system. As a check, if the equity price  $p_e = 0$ , then the generalized Euler conditions (16) reduce to the PV-optimal Euler conditions.

The system steady-state can be derived from the first-order and envelope conditions, and that steady-state is identical to that of PV-optimality. *Ex post* this is not surprising since utilities are identical in the steady-state implying that equity costs are zero. Given consumer and producer parameters, the discount factor determines the ending steady-state of the system, while the equity price  $p_e$  influences the shape of the path towards that steady-state. In particular, it works to eliminate non-sustainable consumption in the intermediate generations as will be demonstrated in the computations that follow. A final observation is that this system will not necessarily generate all possible sustainable paths. Introducing higher-order approximations for the equity function with additional tuning parameters would find new paths.

## **6. Generalized dynamic programming and computational sustainability.**

Computational solutions are obtained by extending dynamic programming methods to consider multiple value functions. In particular, successive approximations are used to solve this system. The value functions are approximated by bicubic splines which allow an exact fit to the estimated values over the state-space grid and are differentiable [Press et al (1992)]. The control optimization problems are solved using the MINOS solver system [Murtagh and Saunders (1998)]. Finite-difference approximations of the partial and cross-partial derivatives for each value function are estimated along with the level values of that value function over the state-space grid. These

derivatives and level values are used as input to specify the bicubic spline approximation of each value function in the succeeding iteration. The use of splines and derivative information helps to avoid shape-preservation issues associated with polynomial approximation of the value function and contributes to algorithm efficacy [Judd (1998)].

Time-series results for the sustainability model are contrasted with PV-optimality in figure 4. Successively increasing the equity price  $p_e$  increasingly flattens out the hump in the PV-optimality results for physical capital, consumption, and utility. The hump can be driven arbitrarily close to zero as the equity weight is made arbitrarily large. In contrast to PV-optimality, sustainability as defined here implies increased consumption in the early periods, less consumption in the first intermediate time period, and more consumption in a second intermediate time period, with eventual convergence to the same steady-state. In viewing the flow variables [Figure 4(b)], this is accomplished in part by evening out investment over some intermediate time period in comparison to PV-optimality. This then results in similar phenomena for output and consumption. However, resource flows are also altered with somewhat higher flows in the early periods, but lower flows in intervening periods before the steady-state. This is a counterintuitive and somewhat surprising result: due to the increased consumption in the early periods, the natural resource stock is actually less early on under sustainability. Only later does the natural resource stock become greater under sustainability; however, the resource stock eventually converges to the same value in both regimes.

Supporting competitive prices for the flow variables are calculated as before. Next define

$$\begin{aligned}
 p_{ks,t+1} &= p_e e^{[u(c_t) - U(k_{t+1}, s_{t+1})]} [u'(c_t) + U_k(k_{t+1}, s_{t+1})] + \alpha V_k(k_{t+1}, s_{t+1}) \\
 p_{s,t+1} &= \frac{p_t}{\alpha V_s(k_{t+1}, s_{t+1})} \\
 &= \frac{p_t}{\alpha V_s(k_{t+1}, s_{t+1}) - p_e e^{[u(c_t) - U(k_{t+1}, s_{t+1})]} [f_r(k_t, r_t) U_k(k_{t+1}, s_{t+1}) - U_s(k_{t+1}, s_{t+1})]}
 \end{aligned} \tag{17}$$

as stock prices giving the PV of a unit of capital and natural resource stocks respectively under competitive conditions. From the first-order conditions (14) these imply that the competitive price conditions (6) are satisfied. This and the second Euler condition (16, Hotelling's rule) imply that the final competitive price condition (7) is satisfied. Stock prices for the welfare optimum and PV-optimality are depicted in figure 4(c). Stock prices reflect present-values and since PV is lower under sustainability, then these prices are also lower in absolute magnitude under sustainability between the initial periods and the steady-state. However, the results are suggestive that, again aside from the initial periods and the steady-state, sustainability places a relatively higher valuation on natural resources, and a somewhat lower valuation on capital stocks, than does PV-optimality.

Green NNP measures can also be computed for the sustainable economy allocations. Define the value functions  $J_u(k_1, s_1) = \sum_{t=1}^{\infty} \alpha^t u(c_t)$  and  $J_e(k_1, s_1) = \sum_{t=1}^{\infty} \alpha^t e[u(c_t) - u(c_{t+1})]$  as the present value functions for utility and equity cost respectively under the sustainable (W-opt) decision rules, and note that these decompose the welfare optimum value function as  $V(k, s) = J_u(k, s) - J_e(k, s)$ . These value functions satisfy the Bellman equations  $J_u(k, s) = u(c) + \alpha J_u[g(k, s, c, z)]$  and  $J_e(k, s) = e[u(c) - u(c[g(k, s, c, z)])] + \alpha J_e[g(k, s, c, z)]$  respectively. Following the same procedures as before, we can then define and compute Green NNP as

$$\theta \alpha J_u(k_t, s_t) = u(c_t) + \alpha J_{uk}(k_{t+1}, s_{t+1})(k_{t+1} - k_t) + \alpha J_{us}(k_{t+1}, s_{t+1})(s_{t+1} - s_t) \quad (18)$$

where  $J_{uk}$  and  $J_{us}$  denote shadow values for capital and natural resource stocks respectively. Thus sustainable consumption is given by Green NNP as before provided that the correct shadow values

are used for the stock prices. Considering the decomposition of  $V$  into  $J_u$  and  $J_e$  and noting the supporting competitive prices defined in (17), it can be seen that in general the shadow values  $\alpha J_{uk}$  and  $\alpha J_{us}$  do not equal the competitive stock prices for capital and natural resources respectively. They reflect equity adjustments and, more generally, the use of a different criterion function than the PV criterion used in previous studies of Green GNP.<sup>3</sup>

The top diagram in Figure 5 tests Hartwick's rule, while the bottom diagram reports the various welfare measures. Initially the economy is investing on net at a greater rate than the rents on the depletable part of extractions (i.e. that part of extractions exceeding the renewable flows portion), and Green GNP/sustainable consumption is growing. After this point the opposite occurs until eventual convergence to a steady-state. Note that in this model, constant consumption implies net investment equal to exhaustible resource rents (which are zero), but that the converse does not hold true. The bottom diagram compares standard GNP to Green GNP. The former overstates the latter as expected; however, it does seem to indicate possible turning points in the economy.

## **7. Project evaluation and environmental valuation**

Economic policy analysis contributes to social welfare in part by valuing capital, infrastructure, public goods, and environmental services. The question arises whether micro-level/partial equilibrium policy analysis such as benefit-cost or valuation studies can contribute to sustainability beyond economic efficiency. This issue can be addressed using the dynamic general equilibrium model developed in the paper. The baseline economy for this exercise operates according to PV-optimality without the equity criterion, implying that it is economically efficient but not necessarily sustainable. Prices in this economy are then used to evaluate sustainability allocations.

To begin, first consider a one-year project which produces capital for future production of

goods and services from inputs of labor, energy, and capital services in the year of the project. While more general projects can be analyzed within the model framework, this will be completely sufficient for the purpose at hand. Given the current allocation and prices, the value of an additional unit of new capital produced ( $\Delta k_t=1$ ) is  $p_{k,t+1}=\alpha\partial v/\partial k$ , while the cost is  $(p_{kr,t}k_t+p_{rt}(\omega+z_t)+p_w)/q_t$  assuming that inputs are used in proportion to the current allocation.<sup>4</sup> In the PV-optimal economy, the first-order conditions (14) imply that capital is produced to the point where price equals marginal cost after recalling that consumption and capital stock prices are equal. Project evaluation in this economy would therefore find a zero present-value of net benefits from the proposed project of new (additional) capital production beyond the current investment level. Thus the level of investment in the analogous sustainable economy would not be supported by project evaluation in the PV-optimal economy except by accident. If the sustainable investment was greater (less) than the PV-opt investment level, then capital prices would be lower (higher) implying that project evaluation would find a negative (positive) return and would point toward reduced (increased) investment.<sup>5</sup>

As a numerical example, consider a state of the economy as depicted in table 2. The first column corresponds to the baseline economy and the second is the sustainable allocation evaluated according to the PV-optimal pricing rules. The first row gives extractions, investment and consumption for the PV-optimal and sustainable allocations respectively. At this state vector, sustainable extractions are smaller, investment is larger, and consumption is smaller. The second row is capital pricing. For the PV-optimal economy, capital price equals marginal cost as it should. However, a benefit-cost analysis of the sustainable level of investment would find that marginal cost exceeds marginal benefits, thus the sustainable level of investment would not be supported.

Valuation of unpriced environmental services is a major topic in environmental economics.

In the PV-optimal economy, the marginal present value of preservation benefits is  $p_{s,t+1} = \alpha \partial v / \partial s$  while the marginal cost to the economy is the value of lost current production or the resource price  $p_r = p f_r$  under competitive conditions. The first-order conditions for the PV-optimal economy imply that a valuation exercise would find marginal benefits equal to marginal costs so it would not pay to alter the current allocation under traditional benefit-cost principles. As with capital, this means that the sustainable level of resource extractions would not be evaluated favorably by micro-level policy analysis except by accident. If sustainable extractions were less (greater) than the baseline PV-optimal extractions, then the resource stock price would be higher (lower) and benefit-cost analysis would signal increased (decreased) extractions. The third row of table 2 reports an environmental valuation exercise for the numerical example. Again the PV-optimal case finds that the marginal value of preserving a unit of the resource just equals the marginal cost. For the sustainable allocation, however, the opportunity cost of resource stock preservation exceeds the marginal benefit so the sustainable level of preservation would not be indicated under PV-optimal pricing.

Micro-level project evaluation and environmental valuation clearly contribute to economic efficiency in the PV-optimal economy, but not necessarily sustainability. Moving from an inefficient point in the economy to an efficient point may be a move towards sustainability, especially in the case of previously unpriced environmental services. However, neither will necessarily achieve a sustainable outcome as defined by some given sustainability criterion [Howarth and Norgaard (1992)]. As seen from the above, the difficulty is that both of these methods rely on prices as determined in the PV-optimal economy. Achieving sustainability requires a reallocation over time supported by a different set of prices. While standard partial equilibrium benefit-cost analysis will

assist in achieving efficiency in the PV-optimal economy, it will not necessarily meet intergenerational equity goals. Note also that just introducing intergenerational equity into public sector investment and conservation benefit-cost analysis will also not suffice as the private sector will adjust in response.

## 8. Uncertainty and sustainability

Economic and environmental systems are typically characterized by substantial uncertainty, especially over time-scales likely relevant to sustainability analysis. This uncertainty might be over preferences, technology, resource availability, and environmental stock dynamics. Under uncertainty, a “hard” constraint for non-declining utility implies several difficulties including what objective to use, that is, is the “hard” constraint to be satisfied under all states of the world or only as an expected value, and second, satisfying the constraint for all states of the world, including unlikely extreme events, may require drastic measures at a minimum, or even be mathematically impossible. In contrast, the sustainability criterion developed here extends naturally to the uncertainty case.

To illustrate, suppose the resource stock is a subjective estimate of the total resource stock available to the economy over a time horizon of 2 time periods. The optimization problem is to maximize expected PV of welfare  $u(c_1) - p_e E[e^{u(c_1) - u(c_2)}] + \alpha E[u(c_2)]$  subject to  $c_2 = f(k_2, \omega + s_2)$ ,  $k_2 = (1 - \delta)k_1 + f(k_1, \omega + z_1) - c_1$ , and  $s_2 = s_1 - z_1 + \epsilon_1$  where  $\epsilon_1$  is a random resource shock with mean zero and the decision variables are  $z_1$  and  $c_1$ . The first-order conditions are

$$u'(c_1) = p_e E[e^{u(c_1) - u(c_2)} (u'(c_1) + u'(c_2) f_k(k_2, \omega + s_2))] + \alpha E[u'(c_2) f_k(k_2, \omega + s_2)] \quad (19)$$

$$E[u'(c_2) [f_k(k_2, \omega + s_2) f_r(k_1, \omega + z_1) - f_r(k_2, \omega + s_2)] (\alpha + p_e e^{u(c_1) - u(c_2)})] = 0 \quad (20)$$

for  $c_1$  and  $z_1$  respectively. A complete analysis of this system is not attempted; however, intuition can be gained by proceeding somewhat informally. In particular the resource shock takes on two

values  $\epsilon_{11} < 0$  and  $\epsilon_{12} > 0$ , and the first-order conditions are analyzed independently.

Consider first the PV-optimal economy ( $p_e = 0$ ) and suppose  $\{k_1, s_1\}$  are such that at  $\epsilon_{11}$ ,  $c_1 > c_2$ , while at  $\epsilon_{12}$ , the opposite  $c_1 < c_2$  is true. Now with this allocation if we introduce a positive equity price  $p_e$ , then the equity derivative on the right-hand side of (19) is positive. For the first-order condition to then hold, marginal utility would need to increase implying a smaller  $c_1$ .<sup>6</sup> In (20) the key expression is  $(f_k(k_2, \omega + s_2)f_r(k_1, \omega + z_1) - f_r(k_2, \omega + s_2))$  as the other terms are positive. In a deterministic two-period model this is just Hotelling's rule as a guarantee of Pareto-optimality as discussed earlier. It can be readily established here that this expression is negative (positive) for the negative (positive) resource shock. Introducing a positive equity price weights the negative value of this expression more heavily, making the left-hand side of (20) negative. Bringing this back to zero involves increasing the "Hotelling" expression which occurs by reducing  $z_1$ .

Table 3 gives a numerical example with the economy parameters of table 1,  $\epsilon_1 = \pm\sigma$ , and state values and equity price as noted. Taking  $p_e = 0 = \sigma$  as the baseline, it can be seen that increasing either the equity price or the resource shock standard deviation results in a more conservative decision rule in that both consumption and extractions in the first-period decline, leaving more for the future generation. The first result is due to increased concern for the future via the equity price while the second can be explained as risk aversion/intertemporal substitution effects. The most conservative decision results from the combination of a positive equity price and uncertainty.

## 9. Conclusions

Previous analytical economic research on sustainability has largely concentrated on deriving the implications for intergenerational equity of specific functional forms. The existing approaches can



lack generality, may be time-inconsistent, or imply an infinite cost of minor deviations from a non-declining utility path. The research reported here develops a recursive approach to sustainability economics which allows direct computation of sustainable time paths in a very general setting. This approach results in a system of Bellman equations and is solved by generalizing standard DP methods to include multiple value functions. Characterization results including Euler equations, steady-states and efficiency are derived for this generalized system.

Numerical results are given under several sets of parameter values and functional forms. Starting from a small capital stock and a large initial resource stock, PV-optimality implies initial increases in the capital stock, consumption and utility, followed by a decline to a steady-state. Under the sustainability criterion, these entities increase more smoothly to steady-state values. Natural resource stocks may actually be lower for initial time periods under sustainability in contrast to PV-optimality. This follows from the observation that sustainability implies a “spreading” of some of the intermediate-generation consumption under PV-optimality both backward and forward in time.

The generalized DP approach here may also be applicable to other problems of consumer and social choice. Frederick et al (JEL, 2002) point out that existing choice models do not account for the “shape” of consumption paths. The sustainability problem addressed here can be seen as a particular instance of doing just that. Using multiple value functions may also allow for a generalized approach to consumer decision-making under risk. In contrast to recursive utility where risk is measured via fluctuations in the value function (and hence present values), multiple value functions could be used to capture the moments of future utility as a function of the state variables, and these could then enter into the current decision-making criterion.

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## Footnotes

- 1 The solution to Bellman's equation (8) is obtained by a dynamic programming algorithm which is a special case of that described later in the paper.
2. Stavins, Wagner, and Wagner (2003) also propose this same problem but do not provide analysis.
3. However, in the pure sustainable economy with no declining utility, the equity adjustments drop out and it can be seen from (17) and the component value functions  $J_u$  and  $J_e$  that the stock shadow values in (18) will then equal the respective supporting competitive equilibrium stock prices.
- 4 This is optimal under the assumption that the project faces the same production function as the aggregate economy consistent with the constant returns to scale assumption.
5. A concave value function implies that capital stock prices will fall as the investment level increases.
6. This analysis is informal since both extractions and consumption adjust simultaneously to a change in the equity price; however, we are just considering each independently.

**Table 1. Empirical specification for the capital-resource economy.**

Utility  $u(c)=c^\rho/\rho$ , output  $f(k,r)=k^{\beta_k}r^{\beta_r}$ , discount factor  $\alpha=1/(1+\theta)$ .

<u>Parameter</u>	<u>Description</u>	<u>Value</u>
$\theta$	discount rate	.04
$\rho$	intertemporal substitution	.5
$\beta_k$	capital services in production	.5
$\beta_r$	production natural resources	.2
$\delta$	capital depreciation	.2
$\omega$	natural resource flows	.2
$k_1$	initial capital stock	.2
$s_1$	initial resource stock	5.

**Table 2. Project evaluation and environmental valuation for states  $k_t = 3$  and  $s_t = 5$ .**

<u>Variables</u>	<u>PV-optimum</u>	<u>Policy analysis</u>
$\{z_p, i_t, c_t\}$	{.108, .5336, .835}	{.0760, .6099, .7290}
$\{p_{ks,t+1}, \gamma_{kt}\}$	{1.0940, 1.0944}	{1.0789, 1.1712}
$\{p_{s,t+1}, p_{rt}\}$	{.7010, .7009}	{.9613, 1.1363}

---

Policy analysis evaluates the sustainability (W-opt,  $p_e=50$ ) allocation at PV-opt ( $p_e=0$ ) competitive prices. Capital cost  $\gamma_{kt} = (p_{kt}k_t + p_{rt}(\omega + z_t) + p_{it})/f(k_t, \omega + z_t)$ . All other parameters as in Table 1.

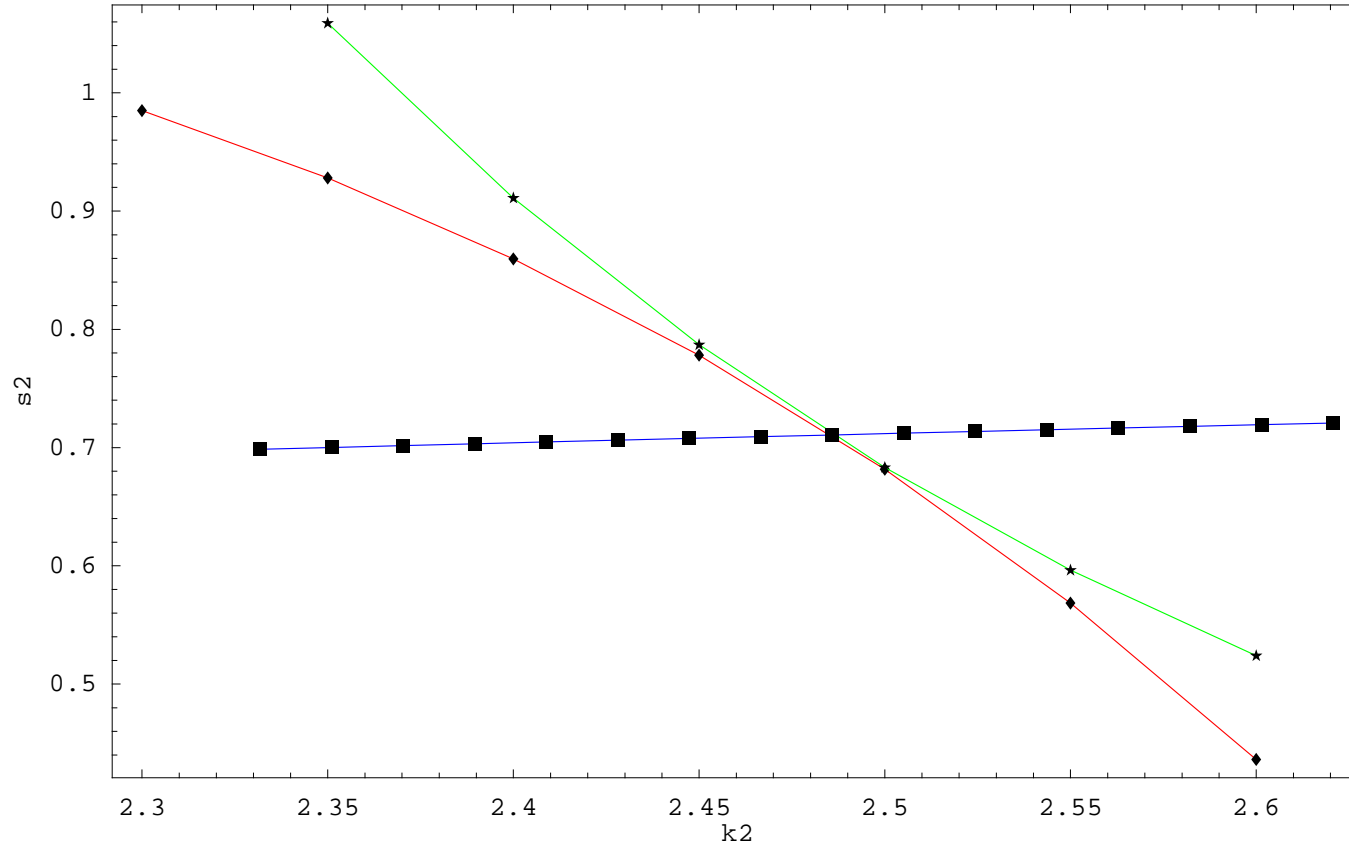
**Table 3. Equity and variance effects on extractions ( $z_1$ ) and consumption ( $c_1$ ) in the stochastic sustainability model (T=2).**

<u>Equity price</u>	<u><math>\sigma = 0</math></u>	<u><math>\sigma = .04</math></u>
$p_e = 0$	{0.940769, 0.536913}	{0.940463, 0.535713}
$p_e = 1$	{0.911807, 0.430251}	{0.911262, 0.428372}

Table entries are  $\{z_1, c_1\}$ . The parameter  $\sigma$  = standard deviation of natural resource shocks. The state values are  $k_1 = 1$  and  $s_1 = 1$ . All other parameters as in Table 1.

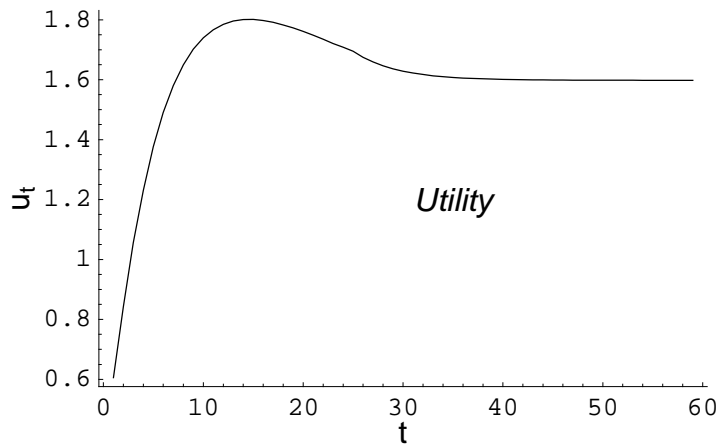
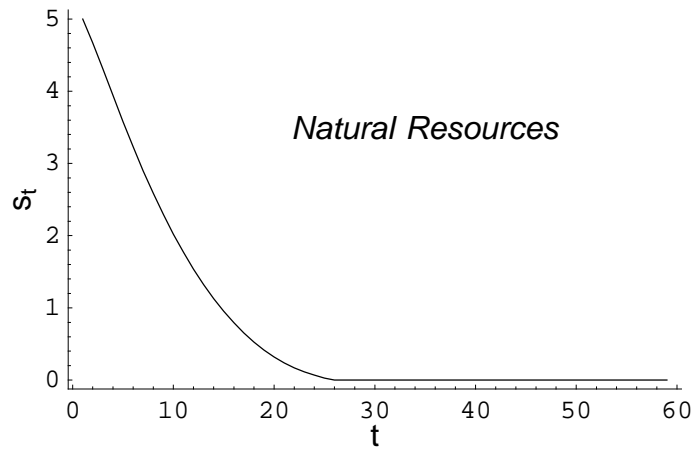
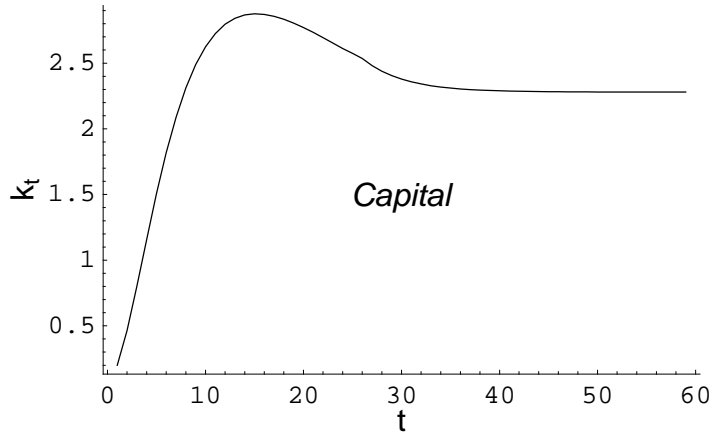
**Figure 1. Edgeworth box diagram: indifference curves for generation  $t$  and  $t+1$  and efficiency locus.**

$[k_1 = 2, s_1 = 1, k_3 = 2.6, s_3 = 4]$

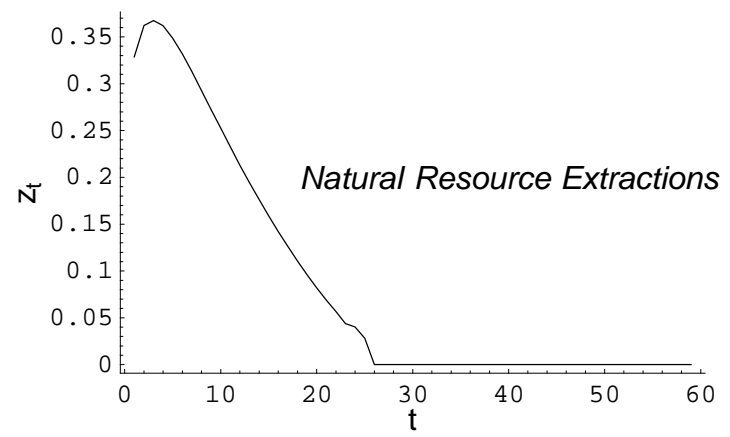
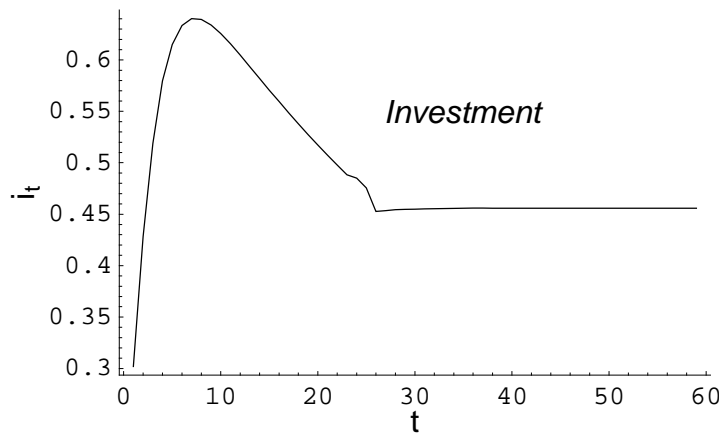
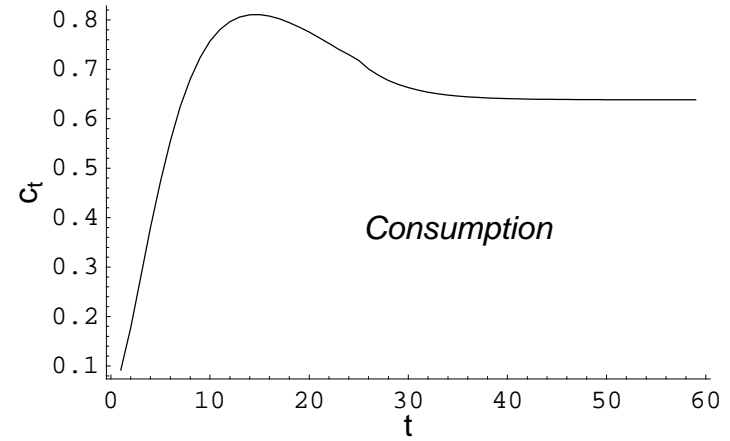
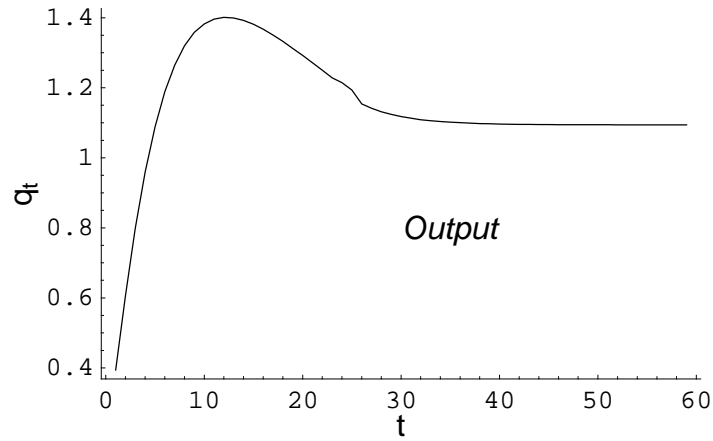




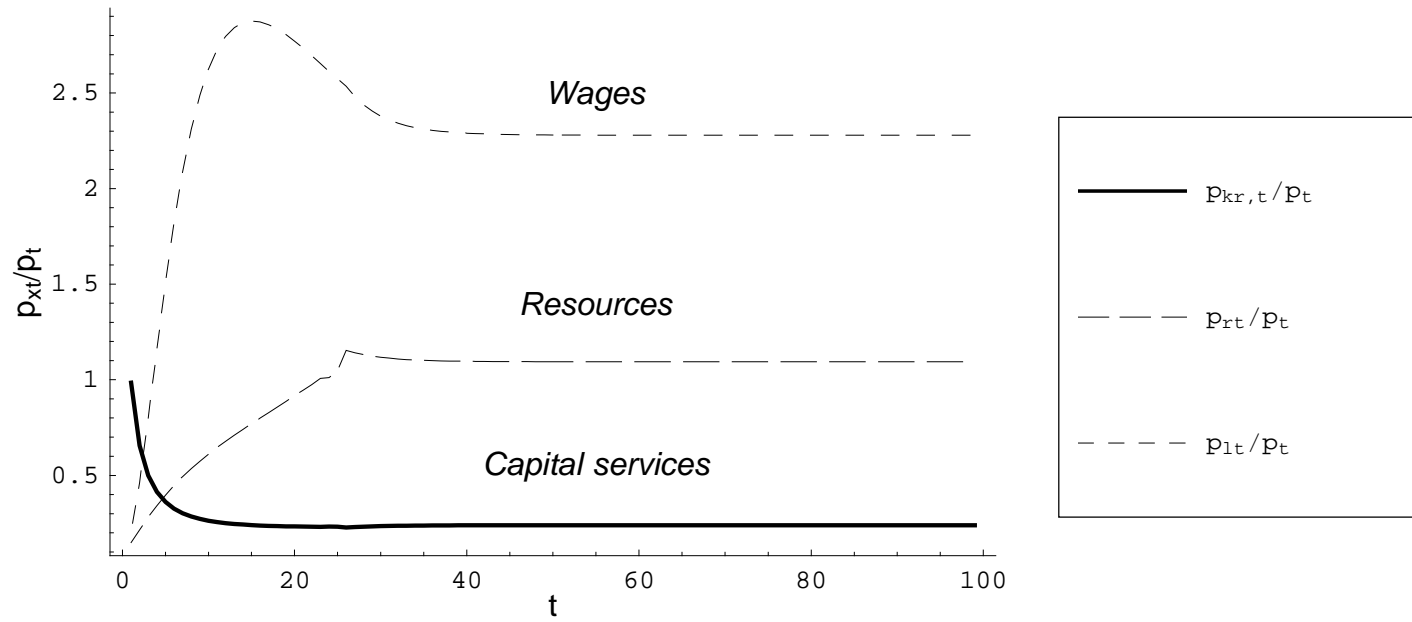
**Figure 2(a). PV-optimality time-series: capital (k), natural resource stocks (s), and utility (u)**



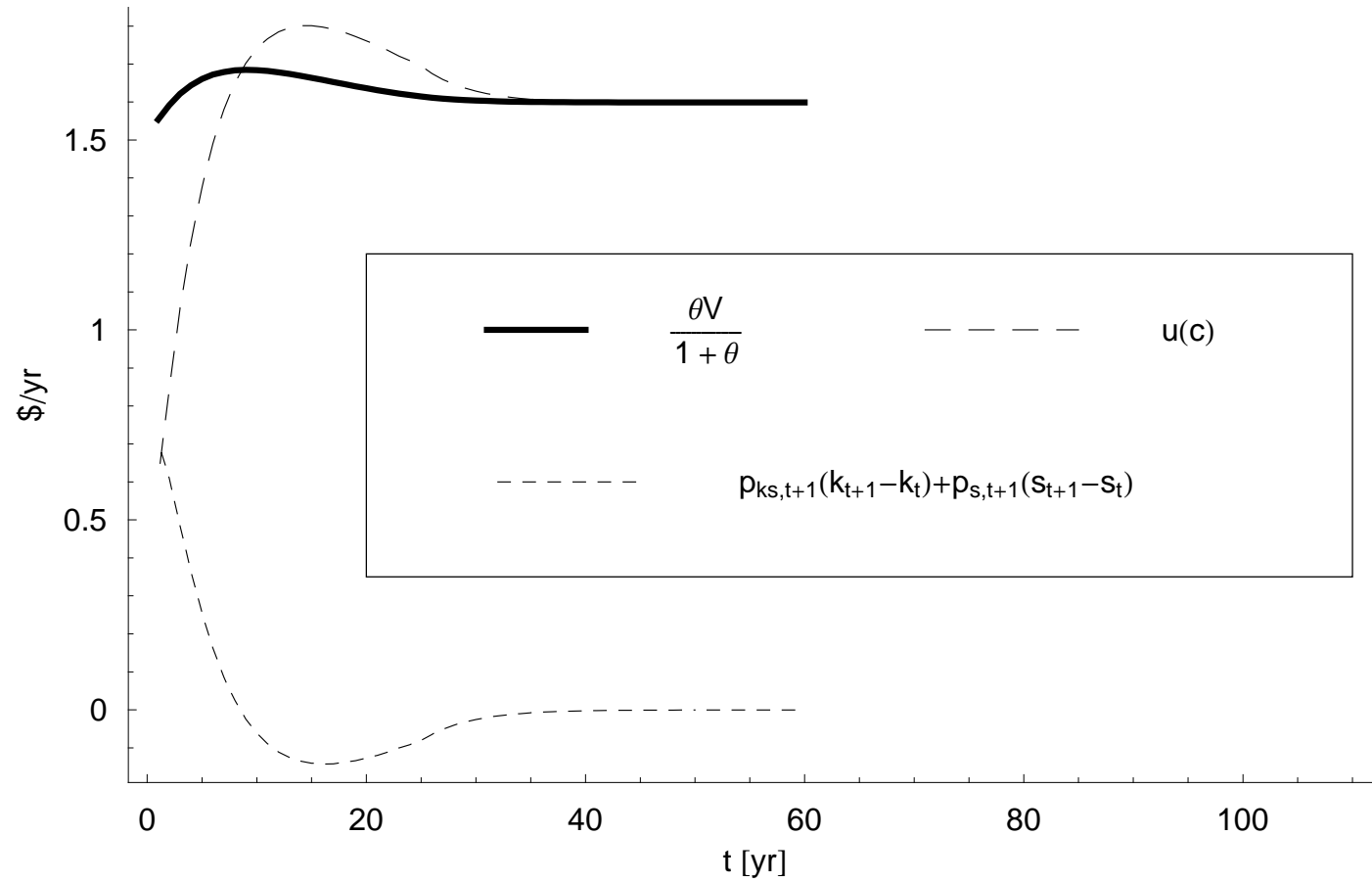
**Figure 2(b). PV-opt time-series: output ( $q_t$ ), consumption ( $c_t$ ), investment ( $i_t$ ), and resource extractions ( $z_t$ ).**



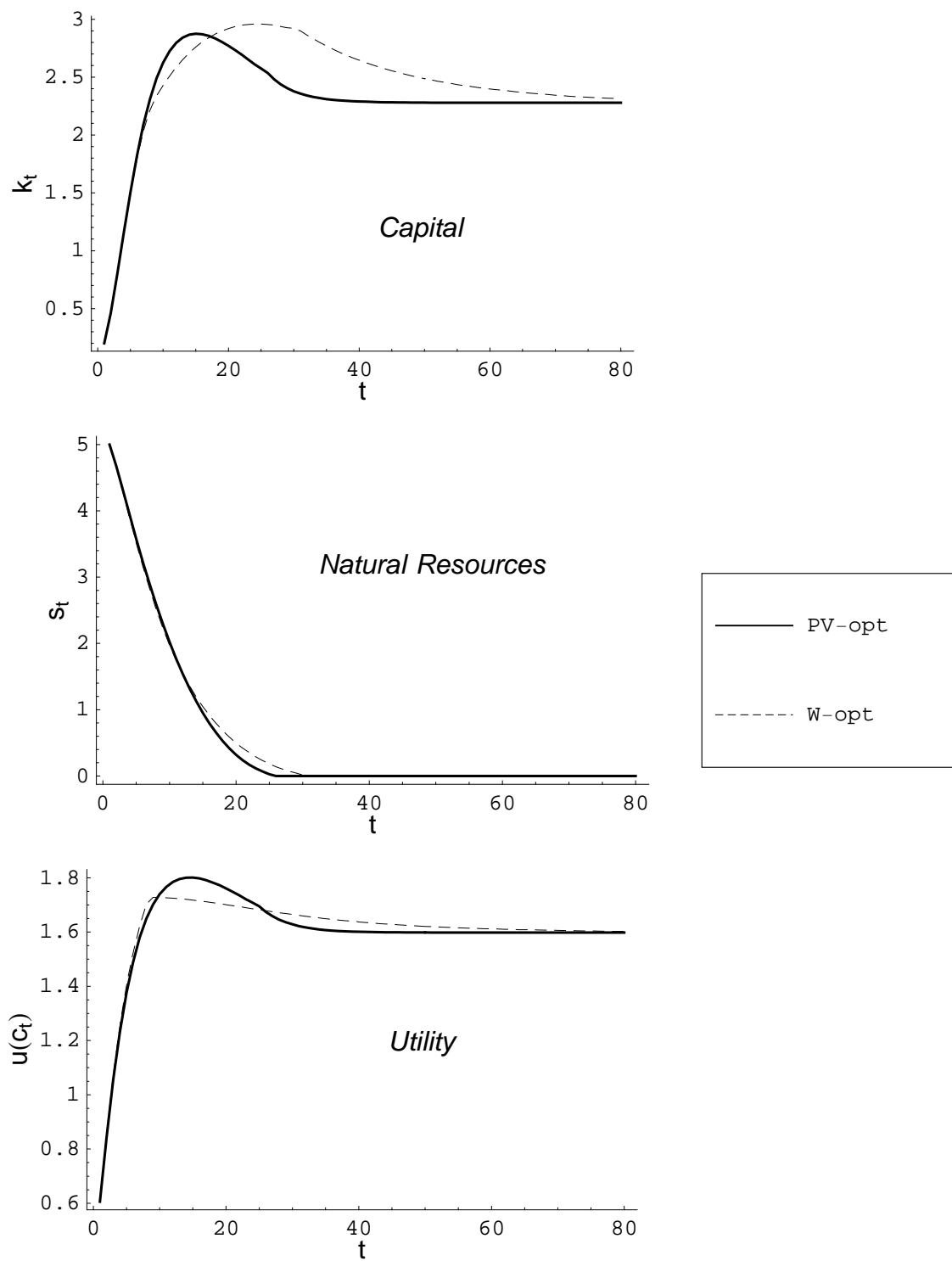
**Figure 2c. PV-optimality: relative prices for capital services, natural resources, and labor**



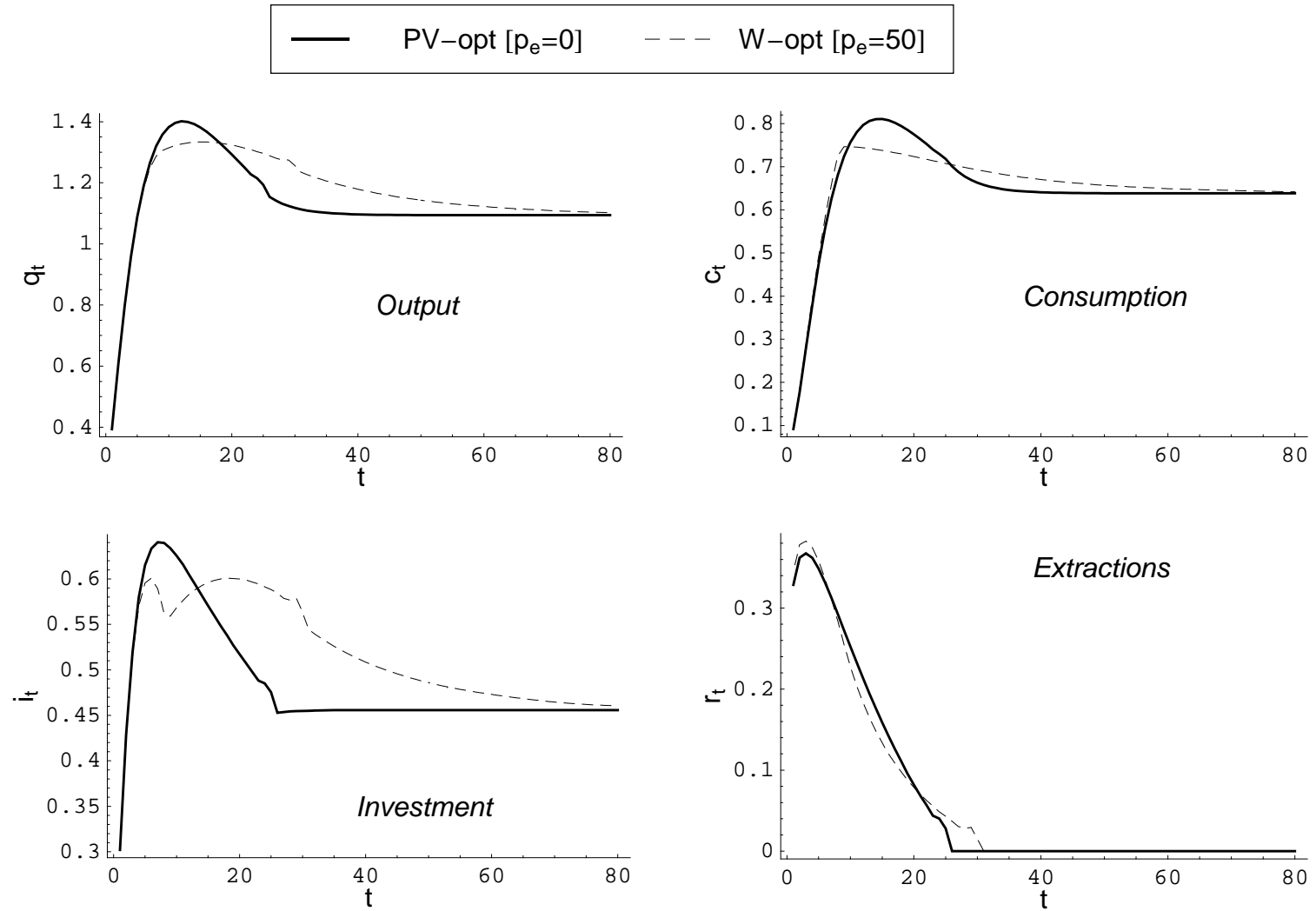
**Figure 3. Welfare measures under PV-optimality: sustainable utility =  $\theta V/(1+\theta)$ , current utility =  $u(c)$ ,  
net investment =  $p_{ks,t+1} (k_{t+1} - k_t) + p_{s,t+1} (s_{t+1} - s_t)$**



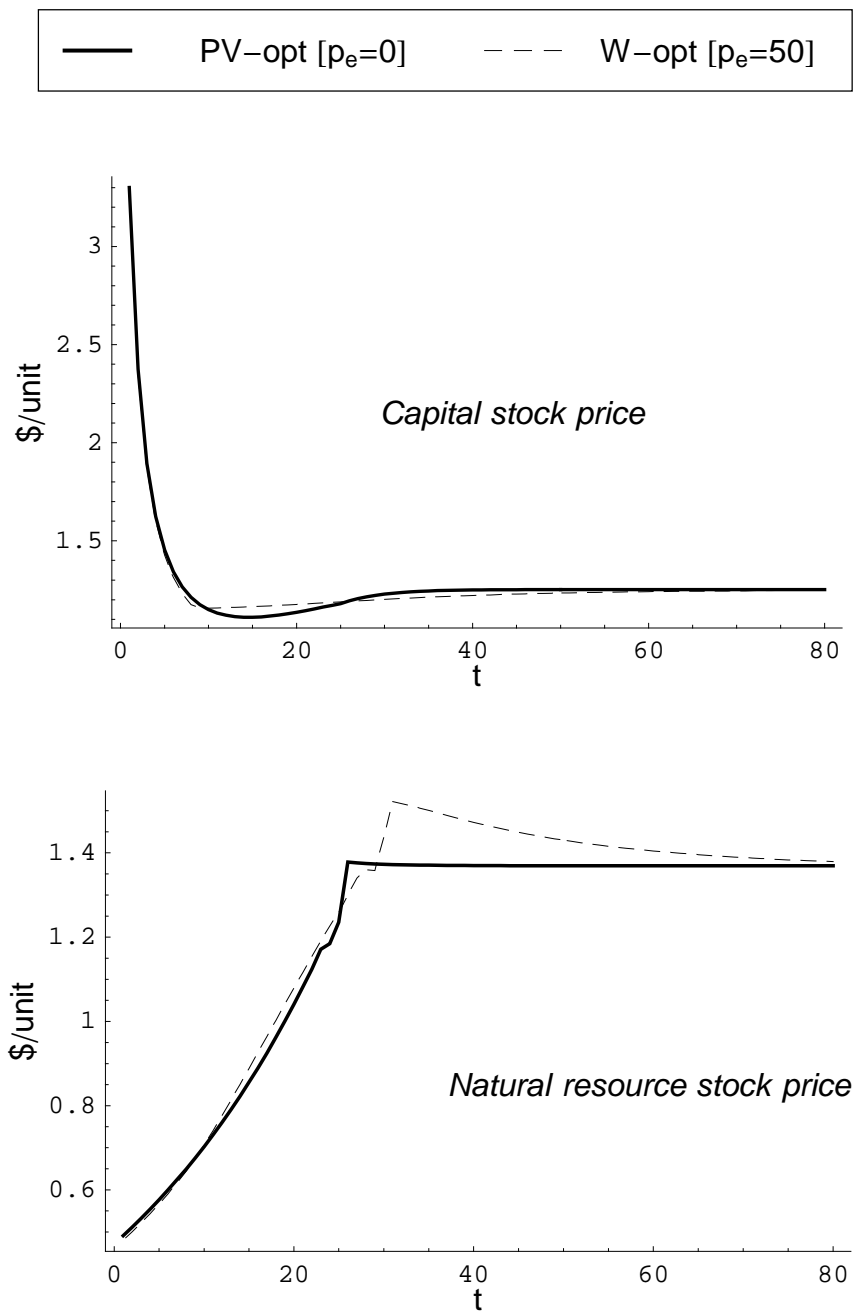
**Figure 4(a). PV-optimality vs Welfare optimum [ $p_e=50$ ]: Stocks and utility.**



**Figure 4(b). PV-optimality vs Welfare optimum [ $p_e=50$ ]: flow variables.**



**Figure 4(c). Stock prices: PV-optimality and welfare optimum [ $p_e=50$ ].**



**Figure 5. Welfare measures under sustainability: sustainable utility  $= \theta J / (1 + \theta)$ , current utility  $= u(c)$ , net investment  $= p_{ks,t+1} (k_{t+1} - k_t) + p_{s,t+1} (s_{t+1} - s_t)$**

