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ARROW-DEBREU AND THE CLASSICAL AND NEOCLASSICAL ECONOMICS

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RESUMO

Este artigo discute a noção de que o modelo de equilíbrio geral de Arrow-Debreu é uma formulação rigorosa da teoria econômica neoclássica e que, por contraste, as teorias marxista e sraffiana são incompatíveis com o mesmo. Ele demonstra que as hipóteses de Arrow-Debreu em relação ao conjunto das possibilidades de produção e à maximização de lucros são suficientes para se determinar os preços de equilíbrio, que, assim, não dependem das preferências dos consumidores. Os preços de equilíbrio de Arrow-Debreu são similares aos valores-trabalho de Marx, uma vez que são proporcionais ao tempo de trabalho, e os preços dos fatores determinam a distribuição de renda, mas não os preços das mercadorias. Em lugar de estarem relacionados à quantidade de capital, os lucros também são proporcionais à quantidade de trabalho, fazendo com que o capital possua diferentes preços no mesmo ponto do tempo e no mesmo mercado, idéia dificilmente conciliável com a hipótese de livre concorrência. Se a noção de preços de equilíbrio é modificada de forma a fazer com que o capital seja remunerado pela mesma taxa em todos os setores da economia, a hipótese de retornos decrescentes de escala assegura que os preços sejam uma função crescente da demanda e, como conseqüência, que sejam determinados pelo jogo entre oferta e demanda. Contudo, em nenhuma hipótese se pode assegurar uma relação inversa entre a quantidade de capital e sua taxa de remuneração, como requer a lei dos rendimentos decrescentes.

Palavras-chave: Modelo de Arrow-Debreu; teoria do equilíbrio geral; economia marxista; economia sraffiana; controvérsia do capital.

ABSTRACT

This article challenges the notion that the modern general equilibrium theory of Arrow-Debreu is a rigorous formulation of neoclassical economics and that, by contrast, Sraffian and Marxian economics are not compatible with it. It shows that the standard Arrow-Debreu assumptions regarding the production sets and profit maximization are sufficient to determine equilibrium prices, which then do not depend on consumers' preferences. Arrow-Debreu equilibrium prices are similar to Marxian labor values since they are proportional to labor time and factor prices are variables that determine the distribution of income but not commodity prices. Instead of being related to the quantity of capital, profits are also proportional to the quantity of labor, causing capital to have different prices at the same point in time and at the same market, which is hardly compatible with the hypothesis of free competition. If the notion of equilibrium prices is modified as to make capital to be rewarded at the same rate in all sectors of the economy, the hypothesis of decreasing returns to scale ensures that competitive prices are an increasing function of demand and, as a consequence, they can be viewed as a product of the interaction between supply and demand. However, in any case there is no inverse relationship between the quantity of capital and its rate of rewards, as requires the neoclassical law of diminishing returns.

Keywords: Arrow-Debreu model, general equilibrium theory; Marxian economics; Sraffian economics; capital controversy

JLE Classification: B51; C62; D50.

SUMÁRIO

1. INTRODUCTION AND OBJECTIVES

The conventional wisdom states that the modern equilibrium theory of Arrow-Debreu or McKenzie is a rigorous formulation of neoclassical economics, and that it is incompatible with Sraffian and Marxian economics.

The objective of this article is to address the relationship between the modern general equilibrium theory and the neoclassical, Sraffian and Marxian schools of economic thought. To this end, the following discussion looks into the nature and the implications of the Arrow-Debreu model, specifying it so as to make it possible to compare it with neoclassical, Sraffian and Marxian theories. Special attention is given to the level of abstraction, which is much higher in the general equilibrium theory. For the sake of simplicity, McKenzie's version of the general equilibrium theory is not discussed.

The term "neoclassical" was first used by Veblen in 1900 to characterize Marshall and Marshallian economics, and it became synonymous with the supply and demand marginalist economic theory after World War II. Its unifying core is comprised of five elements:

(i) methodological individualism;

(ii) the belief that supply and demand are universal laws that explain all economic phenomena;

(iii) the marginal utility theory of demand, developed initially by S. Jevon, C. Menger and L. Walras, in the 1870s;

(iv) the marginal productivity theory of distribution, which was first presented in its modern form around 1890 by J. B. Clark; A. Marshall, F. Y. Edgeworth, and P. H. Wicksteed; and

(v) constant returns to scale, which is required to assure that if all productive agents are rewarded in accord with their marginal products, then the total net product will be exactly exhausted.

Neoclassical economists usually believe that the first four elements are common to the modern general equilibrium economic theory developed by Arrow, Debreu, McKenzie and others. The mathematical consistency of the general equilibrium theory is therefore also applicable to neoclassical economics, which, as a consequence is immune to any major criticism, like those inspired by Sraffa's *Production of Commodities by Means of Commodities*.

Marxian economics is based on the labor theory of value, which Marx regards as the key to understanding all market economies, including capitalist economies. In a market economy where workers own the means of production, normal prices are proportional to the quantity of labor embodied in the commodities. Applying the concept of labor value to the capitalist economy yields**:** i) the concept of wage, which is the value of the commodities necessary for the workers to reproduce their labor power; and ii) the concept of surplus value, which is the difference between value added and wage costs. The surplus value is the ultimate source of profits, interest, ground rents and state taxes. Although competition makes normal prices and profits proportional to the quantity of capital in the capitalist economy, prices and profits are still governed by the law of value.

Based on the idea of long run equilibrium, Sraffian economics arrive at the following tenets: (i) normal prices are determined by technology and by the real wage, but not by demand; (ii) for a given technology, there is an inverse relationship between the profit rate and the wage rate; (iii) the quantity of capital cannot be determined independently of prices and distribution; and (iv) there is generally no inverse relation between the quantity of a productive factor and the rate of its rewards.

The discussion below is divided into seven parts. Section 2 presents the assumptions of the Arrow-Debreu model of general equilibrium with respect to production and entrepreneurial behaviour. Section 3 shows that those assumptions are sufficient to determine all equilibrium prices. Section 4 discusses the economic meaning of Arrow-Debreu equilibrium prices, drawing a parallel between them and the Marxian labor values. It is shown that Arrow-Debreu equilibrium prices yield different rates of return on capital. In Section 5, the Arrow-Debreu concept of equilibrium prices is modified to make it compatible with free competition. The results show that competitive prices are independent of consumers' preferences only under the condition of constant returns to scale, and that there is no basis for the neoclassical Law of Diminishing Returns. Section 6 contains a brief note on the relationship between returns to scale and Sraffian economics. Section 7 presents the conclusions.

2. PRODUCTION SETS AND PROFIT MAXIMIZATION

The standard Arrow-Debreu theory of production and profit maximization can be described as follows: consider an economy with n commodities and m producers. The commodity space is thus an n-dimensional space, denoted by R^n. Each producer **f** is endowed with a technology, denoted by $\mathbf{Y}_f$, which lies in R^n and which constitutes the set of feasible plans. A producer formulates a production plan $\mathbf{y}_f$ that is feasible, which is expressed as $\mathbf{y}_f \in \mathbf{Y}_f$. A production plan is a specification of all inputs and outputs that are related by a given technology; outputs are represented by positive numbers, inputs by negative numbers. For simplicity, the production plan is assumed to be a single production, which means that each $\mathbf{y}_f$ has only one positive entry and all others are non-positive. The set $\mathbf{Y} = \Sigma_f \mathbf{Y}_f$ is the total production set and it describes the production possibilities of the whole economy.

For the production set of producer f, $\mathbf{Y}_f$, it is further assumed that

(1) $\mathbf{Y}_f$ is closed;

(2) $\mathbf{Y}_f$ is bounded;

(3) $\mathbf{0} \in \mathbf{Y}_f$;

(4) $\mathbf{Y}_f$ is strictly convex;

(5) $(-\Omega) \subset \mathbf{Y}_f$, where Ω is the nonnegative cone of R^n;

(6) $\mathbf{Y}_f \cap \Omega \subset \{\mathbf{0}\}$;

(7) $\mathbf{Y}_f \cap (-\mathbf{Y}_f) \subset \{\mathbf{0}\}$;

(8) $\mathbf{Y}_f \cap \mathbf{Y}_k \subset (-\Omega)$, where $j \neq k$.

Closeness means that a production plan is feasible if a sequence of feasible production plans converge toward it. Boundedness refers to the notion that resources are limited, which constrains the production possibilities of each producer. Assumption (3) incorporates the possibility of inaction to the model, and strict convexity implies decreasing returns to scale. Assumptions (5), (6), and (7) mean, respectively, free disposal, no free lunch (no commodity can be produced without the use of inputs), and irreversibility of the production processes. Assumption (8) prohibits joint production

For the total production set, the following assumptions are made: the possibility of inaction; free disposal; no free lunch**,** and boundedness:

(9) $\mathbf{0} \in \mathbf{Y}$;

(10) $(-\Omega) \subset \mathbf{Y}$;

(11) $\mathbf{Y} \cap \Omega \subset \{\mathbf{0}\}$;

(12) $\mathbf{e} + \mathbf{Y} \geq \mathbf{0}$;

where $\mathbf{e} \geq \mathbf{0}$ is the vector of (limited) initial endowments.

Since the commodity space has finite dimension, assumptions (1) and (2) ensure that the producer set, $\mathbf{Y}_f$, is compact. Because the sum of n compact convex sets in R^n is compact and convex, and considering that compactness may be decomposed into closeness and boundedness, we can say that:

(13) **Y** is closed;

(14) **Y** is bounded;

(15) **Y** is strictly convex.

Finally, closeness, convexity and free disposal imply that feasible total production is one where no output is larger and no output is smaller (in absolute value):

(16) $(\mathbf{Y} - \Omega) \subset \mathbf{Y}$.

It is assumed that the producer chooses from the set of feasible plans, $\mathbf{Y}_j$; those plans that maximize his profit, defined as $\mathbf{p}\ \mathbf{y}_f$, where **p** is the row vector of prices. The hypothesis of perfect competition ensures that **p** is taken as given. Producer **f** must then face the problem of choosing $\mathbf{y}_f$ from $\mathbf{Y}_f$ as to maximize $\mathbf{p}\ \mathbf{y}_f$, subject to a feasibility constraint given by $\mathbf{y}_f$. Thanks to the Weierstrass theorem, which states that in finite-dimensional spaces a continuous function defined on a closed and bounded (compact) set has a maximum, this problem has a solution – an equilibrium production of the producer relative to **p**. Note that if $\mathbf{y}_f^*$ is a maximizer and $\mathbf{p} \neq \mathbf{0}$, the price vector is orthogonal to the production set such that the production set $\mathbf{Y}_f$ is contained in the closed half-space below the closed supporting hyperplane **H** that is tangent to it at $\mathbf{y}_f^*$, with normal **p**.

Strict convexity allows the profit function $\pi(\mathbf{p})$ of firm **f** to be defined as follows:

(17) $\pi(\mathbf{p}) = \text{Max}\ \mathbf{p}\ \mathbf{y}_f,\ \mathbf{y}_f \in \mathbf{Y}_f$.

Because $\mathbf{Y}_f$ is closed and bounded and has full dimensionality (thanks to free disposal), $\pi(\mathbf{p})$ is a continuous strictly convex function over the set of prices and can be defined as the supply function of producer **f**, $\mathbf{Y}_f(\mathbf{p})$, as follows:

(18) $\mathbf{Y}_f(\mathbf{p}) = \{\mathbf{y}_f / \mathbf{p}\, \mathbf{y}_f = \pi(\mathbf{p}), \mathbf{y}_f \in \mathbf{Y}_f\}$

which is also continuous.

Considering that the total supply function is defined as $\Sigma_f \mathbf{Y}_f(\mathbf{p})$ and the total profit function as $\Sigma_f \pi_f(\mathbf{p})$, for a given $\mathbf{p}$, $\mathbf{y} = \Sigma_f \mathbf{y}_f$ maximizes total profits on $\mathbf{Y} = \Sigma_f \mathbf{Y}_f$ if and only if each $\mathbf{y}_f$ maximizes profit on $\mathbf{Y}_f$. Both the total profits and total supply functions are continuous, and the total profits function is strictly convex. When $\mathbf{p} \neq \mathbf{0}$, the price vector is normal to the total production set $\mathbf{Y}$, which is contained in the closed half-space below the hyperplane that is tangent to it at the equilibrium production point $\mathbf{y}^*$.

3. DETERMINATION OF EQUILIBRIUM PRICES

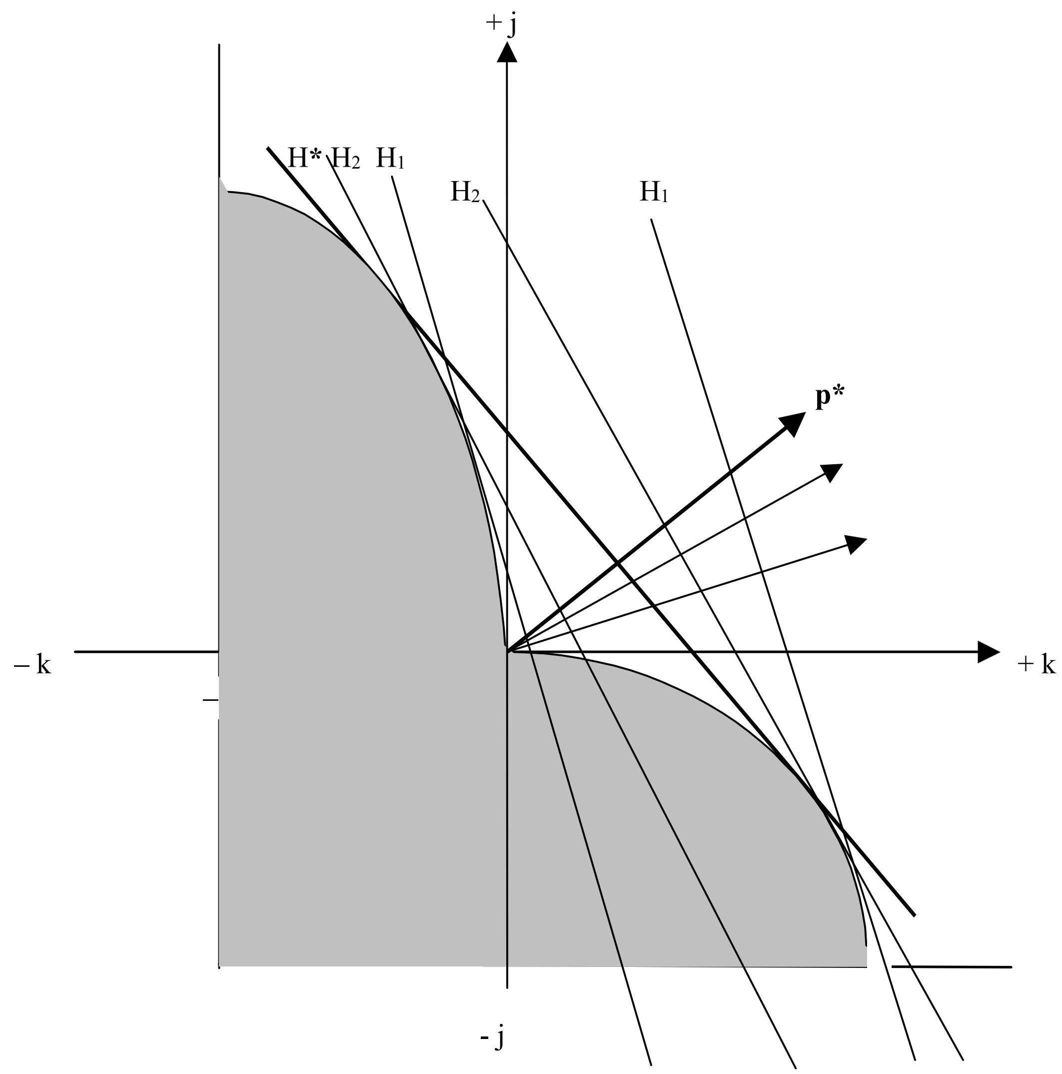
From the theory of production and profit maximization developed in section 2, it can be shown that, given the total production set **Y** and the vector of initial endowments **e**, there is only one price vector $\mathbf{p}^*$, called equilibrium price vector, that satisfies the profit maximization condition for all producers. The assumption of single production assures that the theory will be sufficient to show that one equilibrium price vector will apply to all commodities.

Let $\mathbf{Y}_j$ be the total production set of commodity j, called activity set j. If $\mathbf{p} = \mathbf{0}$, the equilibrium production of j, $\mathbf{y}_j^*$ will obviously be zero so that $\mathbf{y}^* = \mathbf{0}$ will be the unique solution for the profit maximization problem for all producers in all activities. For any $\mathbf{p} \neq \mathbf{0}$ and $\mathbf{p} \neq \infty$, the strict convexity of $\mathbf{Y}_j$ guarantees that there is always an equilibrium production vector $\mathbf{y}_j^*$ that satisfies the equilibrium condition under the assumption that the vector of initial endowments **e** is large enough. So there exists a vector $\mathbf{p}^*$ such that $\mathbf{p}^*\, \mathbf{y}_j^* = 0$ for all j. Because the hyperplane H that is tangent to $\mathbf{Y}_j^*$ at $\mathbf{y}_j^*$ is unique, so is its normal $\mathbf{p}^*$.

Another way to arrive at the same result is by using the Hahn-Banach theorem, which in its simplest form states that given a convex set $\mathbf{Y}_j$ with no empty interior and a point $\mathbf{y}_{qk}$ not in the interior of $\mathbf{Y}_j$, there is a closed hyperplane H_j containing $\mathbf{y}_{qk}$ but disjoint from the interior of $\mathbf{Y}_j$. If $\mathbf{y}_{qk}$ is a boundary point of $\mathbf{Y}_j$, then the strict convexity of this set ensures that H_j is a supporting hyperplane of $\mathbf{Y}_j$ and therefore associated with the unique price vector $\mathbf{p}_j$ normal to H_j at $\mathbf{y}_{qk} \in \mathbf{y}_k \in \mathbf{Y}_j$. For the same reasons, if $\mathbf{y}_{qk} \in \mathbf{y}_k \in \mathbf{Y}_k \neq \mathbf{Y}_j$, which is not in the interior of $\mathbf{Y}_k$, there is also a closed hyperplane H_k that is tangent to $\mathbf{Y}_k$ at $\mathbf{y}_{qk}$, and it is associated with the unique price vector $\mathbf{p}_k$ normal to H_k. Because only one hyperplane is tangent to a strictly convex set at any of its boundary points, then the whole question hinges on the existence of such a hyperplane H tangent to both $\mathbf{Y}_j$ and $\mathbf{Y}_k$. Convexity once more is helpful because, assuming once more that the vector of initial endowments **e** is large enough, it guarantees that every hyperplane associated with a non-negative but finite price vector is tangent to each activity set. This concludes the proof for a commodity space of dimension n = 2. Now the statement is assumed to be true for a commodity space of dimension m with m – 1 commodities and we proved it for m commodities.

Now, assume a commodity space of dimension m. If the activity sets $\mathbf{y}_1, \ldots, \mathbf{y}_{m-1}, \mathbf{y}_m$ have the properties listed in section 2, there is a unique equilibrium price vector $\mathbf{p}_m^* = [p_1, \ldots, p_{m-1}, p_m]$ and a unique equilibrium production matrix $[\mathbf{y}_1^* \ldots \mathbf{y}_{m-1}^* \; 0]$ such that $\mathbf{p}_m^*$ is normal to the hyperplane tangent to $[\mathbf{y}_1^* \ldots \mathbf{y}_{m-1}^* \; 0]$ for each price of commodity m, $\mathbf{p_m}$, which is non-negative but finite. Since the convexity of the activity set $\mathbf{Y} = [\mathbf{y}_1 \ldots \mathbf{y}_{m-1} \ldots \mathbf{y}_m]$ ensures that there is a unique supporting hyperplane for any price vector at $\mathbf{y}_m \in \mathbf{Y}$, and p_m varies from zero to any finite desirable value, it can be concluded that there exists a unique price vector, $\mathbf{p^*}$ and a corresponding unique production matrix $\mathbf{Y^*} = [\mathbf{y}_j^* \ldots \mathbf{y}_{m-1}^* \; \mathbf{y}_m^*]$ such that $\mathbf{p^*}$ is normal to the hyperplane tangent to $\mathbf{Y^*}$. Figure 1 below provides an intuitive idea of the mathematical proof for this conclusion:

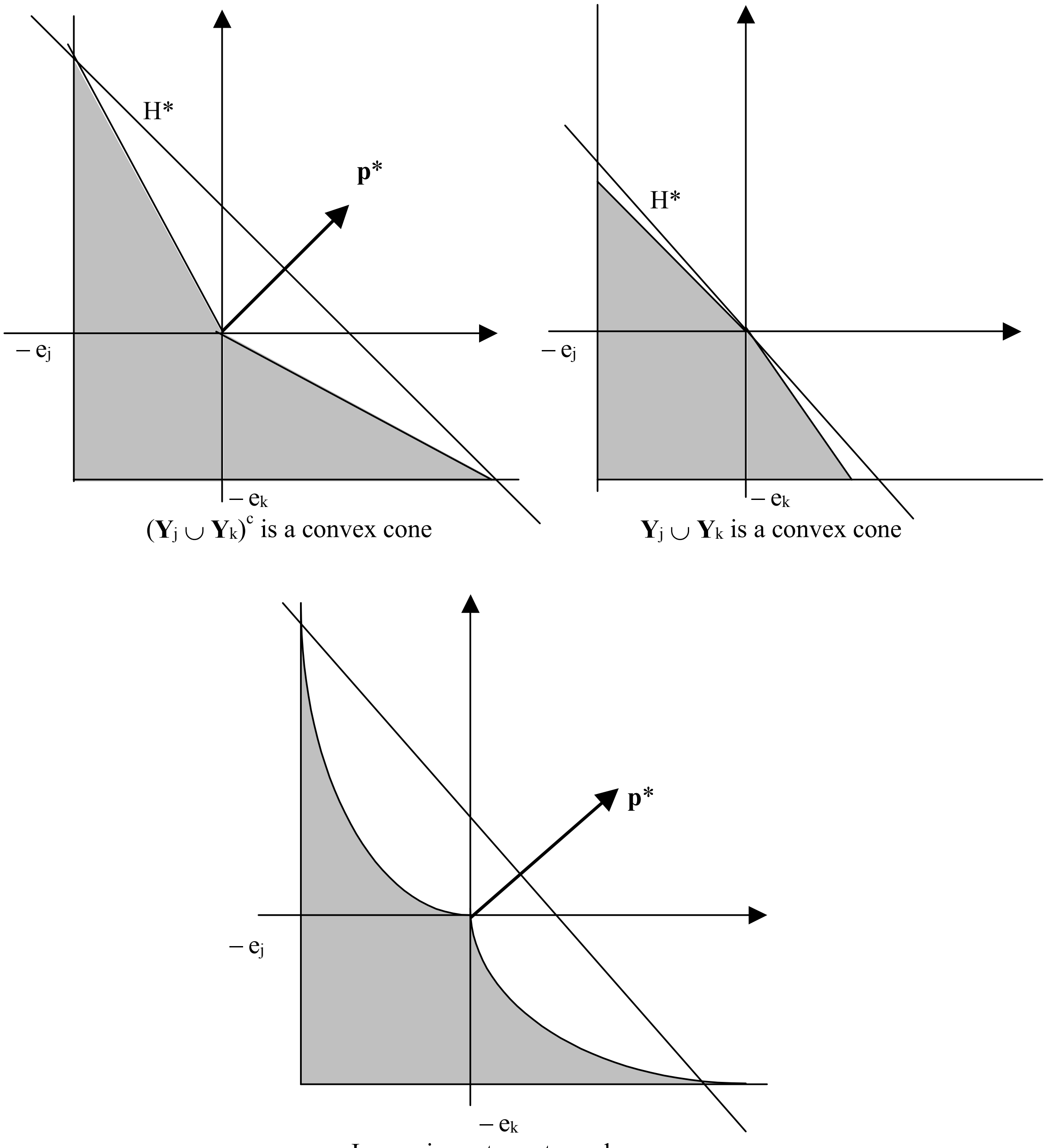
Figure 1
Equilibrium Prices with Two Commodities



When the relative price of commodity j, $\mathbf{p_{jk}}$ is zero, the price vector coincides with the commodity k axis and the supporting hyperplane of set $\mathbf{Y}_j$ coincides with the commodity j axis. At this point, the production of commodity j, $\mathbf{p_{jk}}$ is zero. The rise of $\mathbf{p_{jk}}$ triggers the fall of the slope of the supporting hyperplane of set $\mathbf{Y}_j$ and causes its supply to increase. Because the price of commodity k is the inverse of the price of commodity j ($\mathbf{p_{kj}} = 1/\mathbf{p_{jk}}$), the rise of the latter means the fall of the former as well as the decrease of the slope of supporting hyperplane of set $\mathbf{Y}_k$. Due to the convexity of sets $\mathbf{Y}_j$ and $\mathbf{Y}_k$ and the continuity of their boundaries, there will always be a hyperplane that is tangent to both. In other words, there will exist a unique price vector $\mathbf{p^*} = [p_{jk}\ 1]$ or $[1\ p_{kj}] > \mathbf{0}$ and a production matrix $\mathbf{Y^*} = [\mathbf{y}_j^*\ \mathbf{y}_k^*] > \mathbf{0}$ such that $\mathbf{p^*}$ is normal to the hyperplane tangent to $\mathbf{Y^*}$, $\mathbf{H^*}$.

It should be stressed that if the complement of $\cup_j\ \mathbf{Y}_j$ is a convex cone, the solution of the maximization problem (condition 17) is a hyperplane that touches the production sets on their extreme points. In such cases, the solution will be determined by the availability of resources. This is also the case with increasing returns to scale. Only if $\cup_j\ \mathbf{Y}_j$ is a convex cone is there no price vector that solves the maximization problem for a positive $\mathbf{Y}$ (Figure 2). When $\cup_j\ \mathbf{Y}_j$ or its complement is a convex cone, constant returns to scale prevail.

Figure 2
Profit Maximixation Under Constant and Increasing Returns to Scale



$(\mathbf{Y}_j \cup \mathbf{Y}_k)^c$ is a convex cone

$\mathbf{Y}_j \cup \mathbf{Y}_k$ is a convex cone

Increasing returns to scale

4. PRODUCTION FACTOR REWARDS AND THE LABOR THEORY OF VALUE

Due to the strict convexity of the production set and the possibility of inaction, the origin **0** is not in any supporting hyperplane. It follows that if $\mathbf{y}_j$* is a maximizer and $\mathbf{p} > \mathbf{0}$, profits are always positive in each activity:

$$(19)\quad \mathbf{p}^* [\mathbf{y}_1^* \dots \mathbf{y}_n^*] = \mathbf{p}^* \mathbf{Y}^* > \mathbf{0}$$

Apparently, inequality (19) contradicts the neoclassical tenet that economic profits are null in equilibrium such that the total net product is exactly equal to the weighted sum of factor rewards. However, a closer examination of the question makes clear that primary factor rewards are not included in the product **p* Y***. Although labor and land are required inputs for the production of any commodity, they do not appear as positive entries in $\mathbf{y}_j$ because they are not produced; rather, they are primary inputs. Therefore, there is no activity set for labor and land, and their prices do not enter the price vector **p**. The same is valid for capital, which is not a produced commodity in the sense defined herein, but nevertheless has the value of a bundle of commodities.

In other words, **p* Y*** represents the economic surplus or value added (income) by production and thus is exactly equal to the weighted sum of factor rewards. To make room for productive factors, quantities used in production should be included as negative entries in the production set, the vector of endowments **e** should take into account factor endowments, and the price vector should be extended in order to include factor rewards. For the sake of consistency, it is assumed that no commodity can be produced without the use of some amount of labor, land, and capital. This is equivalent to the hypothesis of no free lunch regarding commodity inputs. Condition (19) must then be discarded to make room for the neoclassical equilibrium condition, which can be expressed as:

$$(20)\quad [\mathbf{p}^*\ \mathbf{w}]\ \mathbf{Y}^* = \mathbf{0}$$

where $\mathbf{w} \geq \mathbf{0}$ is the row vector of factor prices, which must have at least two positive components. If $\mathbf{w} = \mathbf{0}$, the inequality holds once again, and we have $[\mathbf{p}^*\ \mathbf{w}]\ \mathbf{Y}^* > \mathbf{0}$. It should be emphasized, however, that as shown in section 2, both the equilibrium price vector **p*** and the equilibrium production matrix **Y*** do not depend on **w**, but only on the production set **Y**. In other words, factor prices are variables that determine the distribution of income but not commodity prices.

Now if the production activity vector $\mathbf{y}_j$ is interpreted not in terms of net product but as including the use of commodity j as an input (which is a negative entry), the hypothesis of single production allows the reordering of the production matrix **Y*** so that its first n rows compose a diagonal positive matrix <**Ψ***> and the next n rows form a matrix **X*** of commodity inputs. The remaining rows correspond to the different types of labor and land used as inputs to produce <**Ψ***>. If it is then assumed, for the sake of simplicity, that all workers have the same preferences and there is only one kind of homogeneous labor, there will be only one wage rate w, which will be equal to the value of the wage basket, **d**. With these assumptions, we have:

$$(21)\quad w = \mathbf{p}^* \mathbf{d}$$

If then it is assumed that land is a free resource, identity (20) may be rewritten as:

(22) $\mathbf{p}^*[<\Psi^*> \ \mathbf{X}^* \ \mathbf{d} \ \lambda^*] = \pi^*$

where λ is the row vector of labor used in the production process and π is the row vector of capital rewards.

Identity (22) shows that in the Arrow-Debreu model of general equilibrium, capital rewards are not proportional to the amount of this factor (i.e. capital), regardless of the concept of capital that is adopted. If we consider a model of circulation capital, we have:

(23) $\pi = -<\mathbf{r}> \mathbf{p}^* \ [\mathbf{X}^* \ \mathbf{d} \ \lambda^*]$

where $<\mathbf{r}>$ is a diagonal matrix of rates of return. Note that, in general, $i_1 \neq i_2 \neq \ldots \neq i_n$.

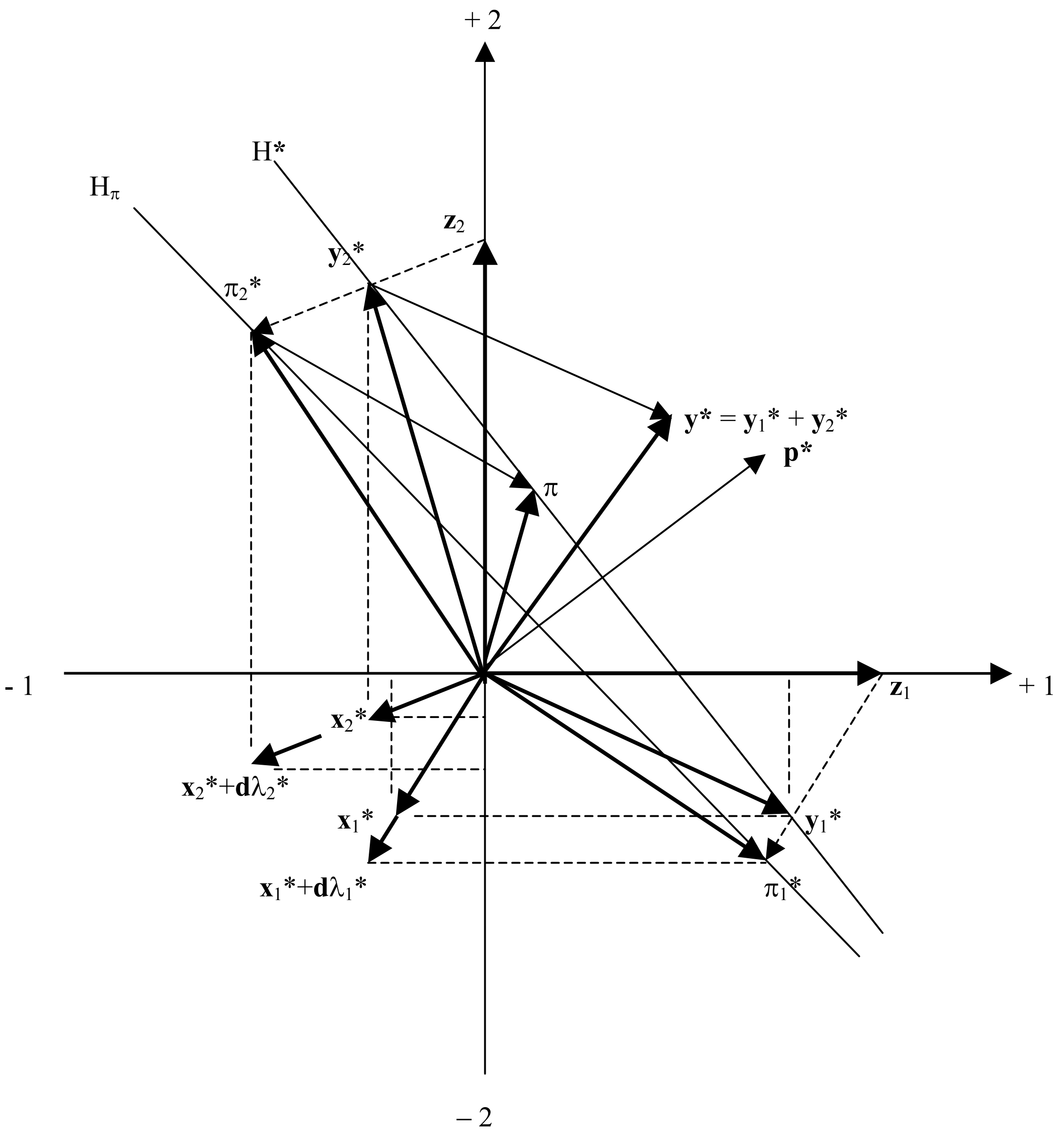
It is worthwhile to highlight that Debreu (1956, pp. 33-35) admits different rates of return, but only when considering two or more different points of time or locations. The reasoning outlined above shows that the rates of return are different at the same point of time and at the same place (market). It is difficult to find an economic justification for this.[1]

The relations between inputs, outputs, commodity prices and factor prices for a two commodity economy are analyzed graphically in Figure 3. In it, $\mathbf{z}_1^*$ and $\mathbf{z}_2^*$ are the gross production equilibrium vectors, $\mathbf{y}_1^*$ and $\mathbf{y}_2^*$ are the net production equilibrium vectors, and $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ are the corresponding input vectors. The factors $\mathbf{y}_1^*$ and $\mathbf{y}_2^*$ determine the hyperplane H*, and thus, the normal equilibrium price vector, $\mathbf{p}^*$. If the wage bundle is $\mathbf{d}$, the real wage costs are $-(\mathbf{x}_1^*+\mathbf{d}\ \lambda_1^*)$ and $-(\mathbf{x}_2^*+\mathbf{d}\lambda_2^*)$, where λ_j^* is the quantity of labor used in the production of $\mathbf{y}_j^*$.[2] When labor costs are introduced, we get the vectors of “surplus” (production net of commodity and real wage inputs), $\pi_1^* = \mathbf{y}_1 + \mathbf{d}\ \lambda_1$ and $\pi_2^* = \mathbf{y}_2 + \mathbf{d}\ \lambda_2$. By adding π_1^* and π_2^*, we arrive at the vector of capital rewards, π^*. Since the hyperplane determined by π_1^* and π_2^*, H_π, has the same slope of hyperplane H* by sheer accident, $\mathbf{p}^*$ is not necessarily orthogonal to it, which means that prices in general are not proportional to costs. In a model of circulating capital, this is equivalent to say that the rates of return normally differ from each other.

[1] See also Hahn (1982).

[2] It should not be forgotten that input vectors have negative signals.

Figure 3
Equilibrium Prices and Factor Rewards



Strange as it might seem, it is not difficult to show that the analysis developed above can also be found in Book I of Marx's *Capital*. Indeed, when prices are proportional to labor time, we have

$$\mathbf{p}[\langle\Psi\rangle\, \mathbf{X}] = -\lambda = -\ \mathbf{p}\ \mathbf{d}\ \lambda + \ \pi$$

which yields identity (22). This is a direct consequence of the fact that the price vector **p*** is orthogonal to hyperplane H*, which is tangent to all activity in set $\mathbf{y}_j$*.

Because labor costs, prices, and total costs are proportional to labor time, so are profits. As a consequence, Marx's theory of surplus value works and capital rewards are proportional to wage costs:

$$\pi = -\,\varepsilon\ \mathbf{p}\ \mathbf{d}\ \lambda$$

where ε (the rate of surplus value) is positive whenever $\mathbf{p}\langle\Psi\rangle \geq -\,\mathbf{p}\ [\mathbf{X} + \mathbf{d}\ \lambda]$.[3]

[3] The sign ≥ means that the strict inequality prevails in at least one case.

5. COMPETITIVE PRICES, PREFERENCES AND THE LAW OF DIMINISHING RETURNS

Section 4 shows that Arrow-Debreu's equilibrium condition is quite unsatisfactory because it implies that a productive factor (capital) has different prices, which is contrary to the free competition hypothesis. From a Neoclassical perspective, there are also other problems besides the rejection of constant returns to scale. One of these problems is that demand has no role in the determination of both prices and quantities, and commodity prices do not depend on factor prices; as a result, there cannot be any law of diminishing returns. For both Marxians and Sraffians, the simultaneous determination of both prices and quantities violates the classical dichotomy (Garegnani, 1984), and the independence of commodity prices from distribution contradicts their theories of equilibrium prices.

In a model of circulating capital equilibrium, prices that are compatible with free competition must obey the following condition:

(24) $\mathbf{p}^*[\rho\, I + [\mathbf{X} + \mathbf{d}\, \lambda]<\Psi>^{-1}] = \mathbf{p}^*[\rho\, I - [\mathbf{A} + \mathbf{d}\, \mathbf{a_0}]] = \mathbf{0}$

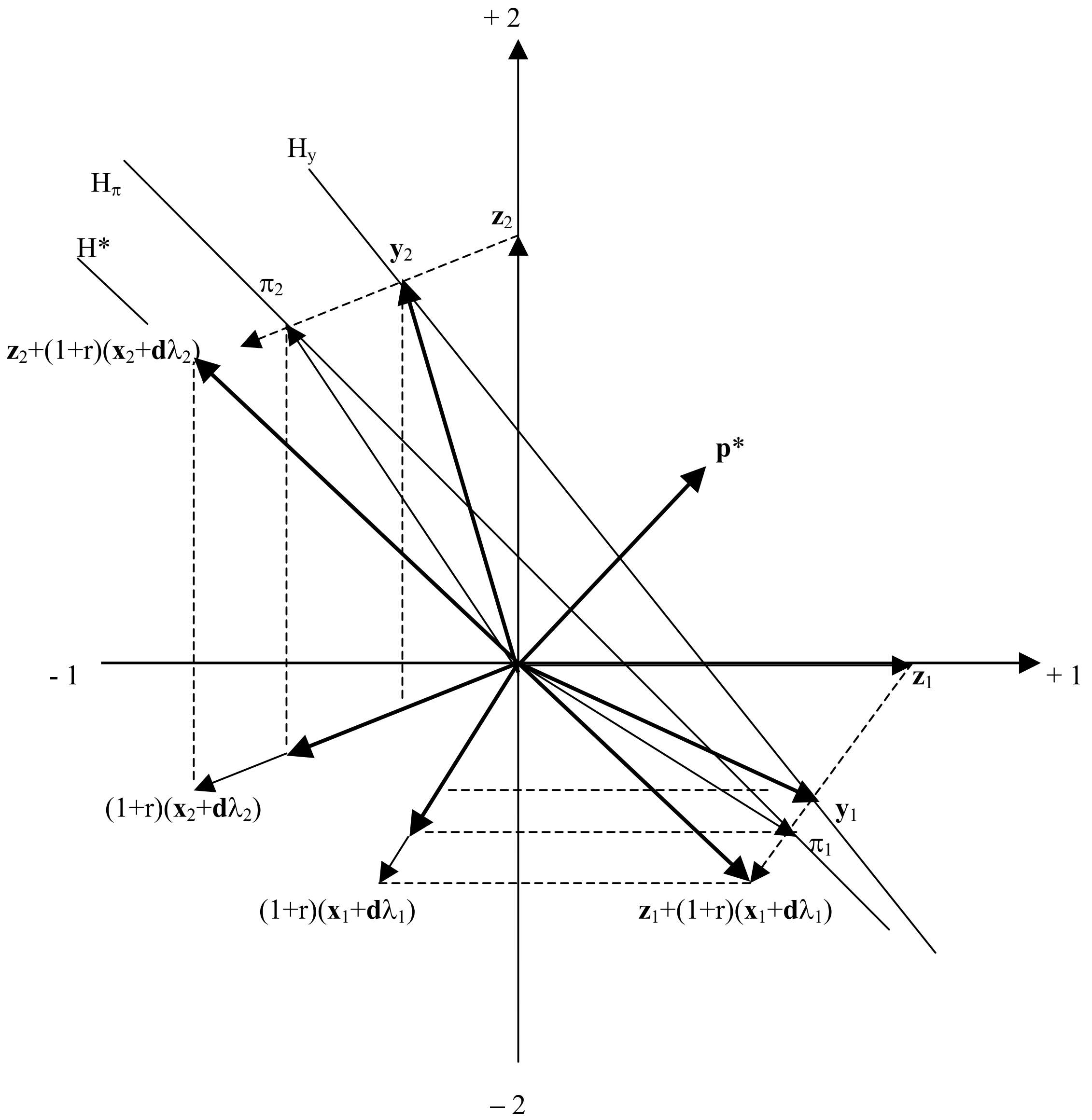
where **A** is a non-negative matrix of input-output coefficients. This differs from Leontief's matrix because its entries are not fixed and the sum of its rows is not necessarily less than one; $\mathbf{a_0}$ is a positive row vector of labor coefficients that are not fixed either; and ρ is the spectral radius of matrix $-[\mathbf{X}+\mathbf{d}\lambda]<\Psi>^{-1} = [\mathbf{A}+\mathbf{d}\mathbf{a_0}]$, which is related to the uniform rate of return by:

(25) $\rho = 1/(1 + r)$

Using the Perron-Frobenius Theorem, it can be shown that both competitive prices and the rate of return are positive if the system is productive, i.e., whenever net production is greater than workers' consumption $<\Psi>\mathbf{X} \geq -\, \mathbf{d}\, \lambda$.

Figure 4 shows the geometry of competitive prices: profits proportional to costs – $r[\mathbf{x}_j+\mathbf{d}\lambda_j]$ – are added to the input vectors such that the total net product is exactly exhausted, eliminating any surplus. Note that the hyperplanes H_y and H* are not, in general, parallel to one another, which means that the vector of competitive prices **p*** is not normal to the production sets, $\mathbf{y}_j$s.

Figure 4
Competitive Prices and Factor Rewards



With competitive prices, the system is set free from the rigidities of the former definition of equilibrium, according to which neither prices nor produced quantities depend on demand, and hence they do not depend on consumers' preferences either. Although it is clear from (24) and (25) that technology and the real wage are sufficient to determine competitive prices, the hypothesis of decreasing returns to scale guarantees that preferences have a role in price determination because it implies that the technical coefficients depend on the level of production. In the case of Figure 4, for instance, any increase in the production of commodity 2 puts the supporting hyperplane H_y closer to the origin **0**, thereby reducing the "surplus". If the real wage remains unchanged, the hyperplane H_y gets closer to the origin as well, causing H* and thus the normal price vector **p*** to rotate counterclockwise. The limit of this process is given by that point where all surplus goes to the workers and profits are zero.[4] Workers' preferences also affect competitive prices through the composition of the real wage, but in this case there is no possible way to establish any systematic relationship between prices and preferences.

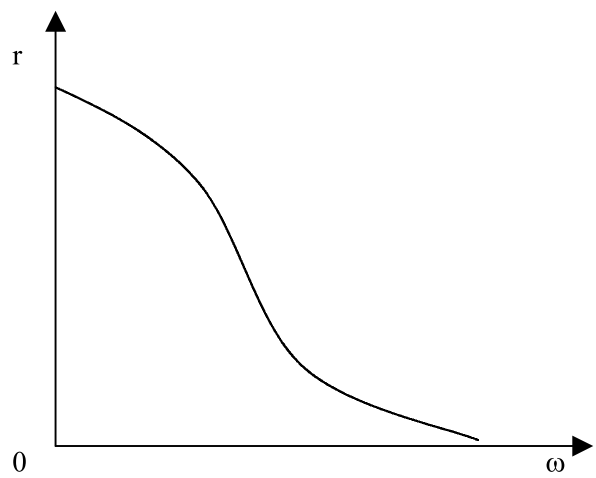
In short, like in the neoclassical theory, competitive prices are an increasing function of demand. As a consequence, competitive prices result from the interaction between supply and demand. However, there is no inverse relationship between the quantity of a productive factor and its rate of rewards. The proof is given as follows:

Suppose that the real wage may vary, but its composition is fixed. In this case, the nominal wage may be expressed as:

(21a) $w = \omega \, \mathbf{p} \, \mathbf{d}$

where $\omega \geq 0$ indicates the level of the real wage and **d** its (fixed) composition. For a given production **y**, we obtain the maximum real wage ω_{MAX}, which makes $\rho = 1$ in (24) and the maximum rate of profits = r_{MAX}, which is the spectral radius of matrix **A**. Varying ω from 0 to ω_{MAX}, we get the corresponding wage-profit curve, which is monotonically decreasing due to the fact that ρ is an increasing function of the elements of matrix $[\mathbf{A}+\omega \mathbf{d}\mathbf{a}_0]$. Since the convexity of the production set does not imply that the wage-profit rate be convex from below, it can assume any form. As a result, the wage-profit frontier, which is the envelope of the wage-profit curves associated with all feasible production vectors, can assume any form, like the one presented in Figure 5.

Figure 5
Wage Profit Frontier



[4] With decreasing returns to scale – $\mathbf{X}<\square>^{-1} = \mathbf{A}$ grows with production. Since □ is an increasing function of the elements of matrix $\mathbf{A}+\mathbf{d}\mathbf{a}_0$, it becomes greater with the expansion of the scale of production. When □ becomes unitary, profits are zero, and production will not react to increases in demand.

Because the slope of the wage-profit curve is determined by the labor/capital ratio, there is no inverse monotonic relation between the rate of return on capital and the quantity of this productive factor or between the real wages and the quantity of labor. In other words, the neoclassical law of diminishing return does not hold even if the production set has all the desired properties.

6. RETURNS TO SCALE AND SRAFFIAN ECONOMICS

If it is assumed that labor is post-paid, condition (24) yields to the equilibrium condition of the Sraffian price model:

(26) $\mathbf{p}^*[\rho\, I + \mathbf{X}\langle\Psi\rangle^{-1}] = \mathbf{p}^*[\rho\, I - \mathbf{A}] = \mathbf{0}$

which prevails not only under decreasing returns to scale (assumption 4), but with constant returns (where $\mathbf{Y}_f$ is convex but not strictly convex) and increasing returns of scale (where $\mathbf{Y}_f$ is not convex). In this sense, the Sraffian theory is more broadly applicable than the general equilibrium theory. It works even when $\cup_j \mathbf{Y}_j$ is a convex cone.

However, as shown in section 5, competitive prices depend on demand if returns vary with the scale of production. Under decreasing returns of scale, prices are an increasing function of production up to the point where profits become null. Under decreasing returns, prices fall and profits rise when production expands. This prevails even with Sraffian prices.

7. CONCLUSIONS

From the above discussion it can be concluded that in the standard Arrow-Debreu model the assumptions regarding the production sets and profit maximization are sufficient to determine equilibrium prices, which then do not depend on consumers' preferences. Arrow-Debreu equilibrium prices are similar to Marxian labor values since they are proportional to labor time and factor prices are variables that determine the distribution of income but not commodity prices. Instead of being related to the quantity of capital, profits are also proportional to the quantity of labor, causing capital to have different prices at the same point in time and at the same market, which is hardly compatible with the hypothesis of free competition.

If Arrow-Debreu equilibrium prices are modified as to incorporate the concept of competition, the hypothesis of decreasing returns to scale ensure that competitive prices are an increasing function of demand and, as a consequence, they can be viewed as a product of the interaction between supply and demand. However, in any case there is no inverse relationship between the quantity of capital and its rate of rewards, as requires the neoclassical law of diminishing returns.

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