MODELING THE ECONOMIC INTERACTION OF AGENTS WITH DIVERSE ABILITIES TO RECOGNIZE EQUILIBRIUM PATTERNS

Michele Piccione
London School of Economics

Ariel Rubinstein
Tel Aviv University and
Princeton University

Abstract
We model differences among agents in their ability to recognize temporal patterns of prices. Using the concept of DeBruijn sequences in two dynamic models of markets, we demonstrate the existence of equilibria in which prices fluctuate in a pattern that is independent of the fundamentals and that can be recognized only by the more competent agents. (JEL: C7, D4, S477)

1. Introduction

In standard economic modeling, diversity among agents reflects differences either in preferences or in the information they possess. Agents may not have identical information about the fundamentals or about the knowledge of other agents. In this paper, we wish to explore a different type of asymmetry among agents. In everyday language, these asymmetries are expressed in statements such as “A understands the market better than B.” Such statements reflect the fact that agents differ in their ability to understand market behavior even if they possess the same raw information.

Introducing these features into a model is not an easy task. In this paper, we suggest a simple and naive modeling device that is applied to several elementary economic frameworks. A common characteristic of the ensuing models is that differences in agents’ abilities to process equilibrium prices allow for the existence of complicated equilibria in which an agent’s performance depends on his ability to recognize dynamic patterns. Specifically, we will construct dynamic models of markets in which prices fluctuate in a pattern that is independent of the fundamentals and that can be recognized only by the more competent agents.

We wish to emphasize that this paper is only an exploratory exercise and we make no pretense as to the realism of the models’ predictions.

Acknowledgments: We wish to thank Noga Alon, Tuvie Etzion, and Gil Kalai for their advice on combinatorics and Eric Maskin, Abraham Neyman, and Mike Woodford for their comments.
E-mail addresses: Piccione: m.piccione@lse.ac.uk; Rubinstein: rariel@post.tau.ac.il

2. Modeling Ability to Recognize Patterns

Our modeling device is related to a mathematical observation that is well-known in the field of combinatorics. Consider the infinite sequence $\{a^t\}$ of the *letters* 0 and 1, consisting of the repetition of the eight-tuple 0, 0, 0, 1, 0, 1, 1, 1. We are interested in the following two properties of the sequence:

1. At each point in the sequence, knowledge of the previous 3 letters is sufficient to correctly predict the next letter.
2. At each point in the sequence, knowledge of the previous 2 letters provides no information about the next letter.

If an agent's performance depends on his ability to correctly predict the next element in the sequence on the basis of the last k observations, an agent with $k \geq 3$ can make perfect predictions whereas an agent with $k = 2$ can do no better than to correctly guess the next letter 50 percent of the time.

Formally, consider a set Λ of cardinality L, an *alphabet*, whose elements are called *letters*. Let Λ^∞ be the set of all infinite sequences of letters. Given an infinite sequence $\{a^t\}$, we refer to any sequence $(a^{t+1}, \ldots, a^{t+m})$ as a *word of length m*. For any t, the word $h = (a^1, \ldots, a^{t-1})$ is the *history* at period t and we say that the element a^t *follows h*. The word $(a^{t-m}, \ldots, a^{t-1})$ is the *m-end* of h. The sequence is generated by the function g: $\Lambda^k \rightarrow \Lambda$ (referred to as a *generating rule of order k*) if $a^t = g(a^{t-k}, \ldots, a^{t-1})$ for all t. A sequence is *cyclical* if it consists of the infinite repetition of the same m-tuple for some m.

A *DeBruijn sequence of order k* is an infinite sequence of letters (a^t) that satisfies the following two properties:

(A) It has a generating function g: $\Lambda^k \rightarrow \Lambda$ of order k.

(B) If $t_1 < t_2 < \cdots < t_L$ is such that t_{j+1} is the minimal $t > t_j$ for which the $k - 1$–end of the history at t_{j+1} is the same as at t_1, then $\{a^{t_j}\}_{j=1}^{L=\Lambda}$.

It is easy to see that the two conditions imply that any k–word is the k–end of some history and that the sequence is cyclical.[1] Hence, a DeBruijn sequence of order k has the property that any k–word is always followed by the same letter, whereas any $k - 1$–word is followed by any $a \in \Lambda$ with frequency $1/L$. Facing such a sequence, an agent having recall of the previous k letters can in principle, construct a rule that correctly forecasts the next letter. However, any rule based on the previous $k - 1$ letters does not allow the agent to correctly forecast the sequence more than $1/L$ of the time.

1. Let $(b^1, \ldots, b^k)$ be an arbitrary word of length k. If $(a^1, \ldots, a^k)$ is a word in the sequence, then $(a^2, \ldots, a^k, b^1)$ is also a word in the sequence and, by repeating this argument k times, it follows that $(b^1, \ldots, b^k)$ is also a word in the sequence.

Example: Let $\Lambda = \{0, 1\}$. The infinite repetition of the k-tuple below generates the following DeBruijn sequences:

For $k = 2$, (0, 0, 1, 1).
For $k = 3$, (0, 0, 0, 1, 0, 1, 1, 1) as well as (0, 0, 0, 1, 1, 1, 0, 1).
For $k = 4$, (0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1), etc.

The existence of a DeBruijn sequence for any order and any alphabet of finite cardinality is well known[2] in the combinatorics literature (see, for example, Bondy and Murty (1976), pp. 181–182). The proof is presented here for completeness.

PROPOSITION 1. *(DeBruijn) For any finite alphabet Λ and any k, there is a DeBruijn sequence of order k.*

PROOF: Consider the directed graph $(V, \rightarrow)$ where V, the set of nodes, consists of all L^{k-1} words of length $k - 1$ and, for any node $(a_1, \ldots, a_{k-1})$ and any $a \in \Lambda$, $(a_1, \ldots, a_{k-1}) \rightarrow (a_2, \ldots, a_{k-1}, a)$. The graph is connected and has the property that there is an equal number of arrows in and out of each node. This is a necessary and sufficient condition for the existence of the "Euler cycle" (a path passing through every arrow once). The length of such a path is L^k. Let (b^t) be a sequence such that the t^{th} arrow in the Euler cycle is $(a_1, \ldots, a_{k-1}) \rightarrow (a_2, \ldots, a_{k-1}, b^t)$. The sequence (b^t) satisfies our claim. Indeed, note that knowledge of the k– end of the sequence at period t is equivalent to knowledge of the t^{th} arrow which implies knowledge of the next arrow that determines b^{t+1}. Knowledge of the $k - 1$ end of the sequence at period t is equivalent to the knowledge that the t^{th} arrow connects to the node $(b^{t-k+1}, \ldots, b^{k-1})$ except that there are L possible arrows originating from this node. ■

DeBruijn sequences of order k allow only uniform beliefs among agents with depth of memory smaller than k. However, one can use Proposition 1 to show that other beliefs can be generated as well.

PROPOSITION 2. *Let Λ be an alphabet and let $\alpha(\cdot)$ be a probability measure on Λ such that, for any $a \in \Lambda$, $\alpha(a)$ is a positive rational number. Then, for any k, there is an infinite cyclical sequence of letters such that all k–words appear in the sequence and any $k - 1$–word $(x_1, \ldots, x_{k-1})$, is followed by a with limiting frequency $\alpha(a)$.*

PROOF: Let $\alpha(a) = m(a)/m$ where all $m(a)$ are natural numbers. Define an auxiliary alphabet $\Lambda' = \{(a, l) | \, a \in \Lambda \text{ and } 1 \leq l \leq m(a)\}$ having cardinality m. By Proposition 1, there is a cyclical sequence $\{x^t\}$ of elements of Λ' such that, given any $k - 1$–word, the "next" letter is equally likely to be any letter in Λ'.

2. We would like to thank Noga Alon for referring us to this literature.

Let $\{y^t\}$ be the sequence of letters in Λ such that $y^t = a$ if and only if $x^t = (a, l)$ for some l. This sequence is cyclical, all the k–words appear, and the limiting frequency by which a letter $a \in \Lambda$ will follow a word $(y_1, \ldots, y_{k-1})$ is equal to $\alpha(a)$ by construction. ■

Note that we do not characterize the minimal order of the generating rule needed to generate the sequences that are described in Proposition 2. In general, the minimal order will exceed k.

In the following analysis, an agent will face a deterministic cyclical sequence of letters that represent entities such as prices or nature's actions. The agent will have a stationary instantaneous utility function $u(a, x)$, where a is his action and x is the next element in the sequence. The agent's ability will be characterized by a natural number k. Observing the sequence, the agent will be able to compute the limiting frequencies with which any word of length k is followed by a specific letter. Our behavioral assumption is that, given any word of length k, he chooses an action that optimizes his instantaneous expected utility where his beliefs are the long-run frequencies.

Strictly speaking, the number k is a bound on the memory of the agent. However, we prefer a broader interpretation in which k is a measure of his sophistication. An agent who observes a sequence of letters is able to grasp its generating rule if it is of order k or lower. Standard questions in IQ tests provide a useful analogy for this interpretation. A finite sequence of 0s and 1s is presented and the subject must determine its continuation. A subject with $k = 1$, for example, is able to uncover the pattern of a stationary or alternating sequence, whereas an agent with $k = 2$ is able to uncover the pattern of a sequence such as 0, 0, 1, 0, 0, 1.

Although we do not model the process by which the agent forms his beliefs and do not consider the agent's intertemporal preference, our analysis does extend to the case of an agent with bounded recall who chooses strategies in order to maximize the limit of the averages of his instantaneous utility or the discounted sum for a sufficiently high discount factor.

3. A Model of Price Discrimination

In this section, we demonstrate a potential application of the modeling device proposed above in a simple economic context. Consider a monopolist who produces one good and wishes to discriminate among consumers with different reservation values. We will demonstrate that if the consumers differ in their ability to recognize temporal price patterns, the monopolist can sometimes increase his profits beyond the level he would have achieved if no such differentiation had existed.

Suppose that the monopolist produces a perishable good in discrete quantities. Time is discrete and the monopolist is operating in a repetitive fixed environment. The monopolist chooses and commits in advance to an infinite sequence of prices $\{p^t\}$. His aim is to maximize the limit of the average profits (or the discounted sum with a sufficiently high discount factor). The monopolist can produce the good promptly in every period at a cost equal to zero (adding the cost variable makes the condition in the following proposition easier to satisfy).

The market has two consumers, 1 and 2. Consumer i has a reservation value υ_i, interpreted as the cost of obtaining the good instantaneously by other means. Consumer i obtains a utility higher than υ_i from consuming the good. We assume that $\upsilon_1 < \upsilon_2$.

The price for period t is posted only at the beginning of the period. However a consumer must decide at the end of period $t - 1$ (that is, after p^{t-1} is posted but before p^t is posted) whether to enter this market and pay a cost equal to ε, where $\upsilon_1 > \varepsilon > 0$. The need to make a decision in advance might be due, for example, to the need to make a specific complementary investment. If he enters the market in period t, he learns about p^t and purchases the good if and only if p^t does not exceed his reservation value.

In order to make the decision as to whether to enter the market in period t, a consumer has to form a belief about the price he will be charged. Consumer i is characterized by a number k_i. Given an infinite price sequence $\{p^t\}$, consumer i can base his inference at the end of period $t - 1$ only on the last k_i prices $(p^{t-k_i}, \ldots, p^{t-1})$. We assume that the probability that consumer i assigns to the event in which a price p follows the k_i prices $(p^{t-k_i}, \ldots, p^{t-1})$ is the long-run frequency with which p follows the k_i–word $(p^{t-k_i}, \ldots, p^{t-1})$ along the sequence $\{p^t\}$. In every period, a consumer maximizes his expected surplus.

In the following proposition we will show that if $k_1 \leq k_2$, the monopolist's optimal policy is a constant price, that is, the monopolist cannot use the limitations and differences in ability to recognize price patterns in order to extract a greater surplus from consumers. However, if $k_1 > k_2$, we identify conditions on the reservation values and the entry cost that allow the monopolist to increase his profits using a nonstationary price sequence. The pattern of prices will be recognized correctly by consumer 1 but not by consumer 2. Note that a negative correlation between υ_i and k_i, which implies that a more sophisticated consumer has better outside options, is not unreasonable.

PROPOSITION 3. *(a) If $k_1 \leq k_2$, then the optimal cyclical price sequence is constant. (b) If $k_1 > k_2$, $2\upsilon_1 - \upsilon_2 < \varepsilon$, and $\upsilon_2 - \upsilon_1 < \varepsilon$, a DeBruijn sequence P^* of order k_1, consisting of the prices $p_L = \upsilon_1 - \varepsilon$ and $p_H = 2\upsilon_2 - \upsilon_1 - \varepsilon$, is strictly better for the monopolist than any constant price sequence.*

PROOF: Take a cyclical sequence of prices $\{p^t\}$. Let $T = \{t | t \geq k_2 + 1\}$. For every finite sequence of prices z, let $T(z) = \{t \in T|$ the history $(p^1, \ldots, p^{t-1})$ ends with the word $z\}$. The set $I_i = \{T(x)| \, x$ is a vector of k_i prices$\}$ is a partition of T. The information available to agent i is identical in all periods in the same cell of I_i. Thus, one can think of I_i as the information partition of agent i. The behavior of consumer i in any cell of I_i is constant. I_2 is a partition of T that refines I_1. Consider a cell of I_1 where consumer 1 enters the market. The average price in this cell cannot exceed $\upsilon_1 - \varepsilon$. Thus, the average profit in such a cell cannot exceed $2(\upsilon_1 - \varepsilon)$. Consider a cell of I_1 where agent 1 does not enter the market. In any of the subcells in I_2, (recall that I_2 refines I_1) the average price cannot exceed $(\upsilon_2 - \varepsilon)$. Thus, the average profit of the monopolist cannot exceed $max\{2(\upsilon_1 - \varepsilon), (\upsilon_2 - \varepsilon)\}$. Hence, a constant price policy charging either $\upsilon_1 - \varepsilon$ or $\upsilon_2 - \varepsilon$ achieves this bound.

Consider a DeBruijn sequence P^* of order $k_2 + 1$, consisting of the prices p_L and p_H. Consumer 2, being less sophisticated, believes that the price is either p_L or p_H with equal probability. Since $(p_L + p_H)/2 = (\upsilon_2 - \epsilon)$, he always enters the market and, since $p_H \leq \upsilon_2$, (by $\upsilon_2 - \upsilon_1 < \varepsilon$), he buys the good at both prices. Consumer 1 recognizes when the price is low and only then enters the market. Thus, the average profit that P^* yields is $(\upsilon_2 - \epsilon) + (\upsilon_1 - \epsilon)/2$. Since by hypothesis $2\upsilon_1 - \upsilon_2 < \epsilon$, the optimal constant price sequence equals $\upsilon_2 - \varepsilon$ and thus the claim follows.

Note that, as in other models, the ability of the monopolist to price discriminate may enhance the welfare of all participants in the market. In particular, under condition (b), the pricing policy P^* makes the monopolist better off leaving both consumers as well off as under the optimal constant price policy.

The pricing policy P^* is not necessarily an optimal cyclical policy. The monopolist can raise his profits by increasing the high price up to υ_2 and reducing its frequency. Consider, for example, the case where $\upsilon_1 = 4$, $\upsilon_2 = 6$, and $\varepsilon = 3$. The number $\alpha = 0.4$ satisfies $\alpha\upsilon_2 + (1 - \alpha)(\upsilon_1 - \epsilon) = \upsilon_2 - \epsilon$. If the monopolist were able to generate a price sequence such that consumer 1 would always correctly predict the price while consumer 2 would always believe that the high price appears with probability 0.4, the monopolist would achieve a profit equal to $\upsilon_2 - \epsilon + (1 - \alpha)(\upsilon_1 - \epsilon)$, which is greater than $(\upsilon_2 - \epsilon) + (\upsilon_1 - \epsilon)/2$, the profit he would obtain using P^*. As shown in Proposition 2, there is a cyclical sequence of the prices $\upsilon_1 - \epsilon$ and $\upsilon_2 - \epsilon$ such that consumer 2 believes that the high price is charged with probability 0.4. If k_1 is sufficiently large, consumer 1 will correctly predict the price. However, as already noted, we do not have an exact expression for determining how large k_1 should be for a given k_2.

Bibliographic comment: In several previous models of price discrimination, uncertainty unrelated to economic fundamentals plays a key role. In Salop (1977), price dispersion allows the producer of a single good to price discriminate among consumers. Agents obtain price offers through search. Differences in search costs are correlated with willingness to pay. The monopolist spreads the spectrum of prices so that consumers with higher search costs find it optimal to buy the good at high prices.

In Rubinstein (1993), the monopolist, who is, for example, a supplier of a service, wants to limit the number of consumers accepting his price offer in some states of nature. A price offer is a vector of numbers and consumers differ in their ability to distinguish among the price offers. The monopolist's optimal strategy is to complicate the price offers so that only the more sophisticated agents will purchase the good when he wishes to limit demand.

4. A Market Example

We proceed with another "toy model" that demonstrates the theoretical possibility that differences in the ability of agents to understand price patterns allows for the existence of equilibrium fluctuations that are unrelated to fundamentals and that benefit the more competent agents.

Consider a market with one indivisible good, labor, and money. The market operates in periods $t = 1, 2, \ldots$.

On the supply side of the indivisible good there are three producers. Each has a technology that produces one (and only one) unit of the good per period. Producer $P0$ can produce one unit costlessly and has a reservation price of $w - 2$ units of money for selling the unit produced. Each of the other two producers, $P1$ and $P2$, needs to employ one worker. Labor is used in indivisible units and wages are fixed at the level w. Producers $P1$ and $P2$ must decide at the beginning of each period, before that period's price is determined, whether to produce the good or not. The objective of $P1$ and $P2$ is to maximize their expected money profits.

On the demand side of the indivisible good, one buyer wishes to buy at most two units of the good. In each period he is endowed with $w + 2$ units of money. He evaluates one unit of the good as equivalent to $w - 2$ units of money and an additional second unit as equivalent to $w + 4$ units of money.

The buyer has two units of labor. He is always willing to work for the fixed wage w (per unit of work). He purchases 0, 1, or 2 units of the good to maximize his surplus. No transfer of money or goods from period to period is feasible.

A candidate for an equilibrium consists of the following components for every period t:

- The price of the indivisible good in units of money.
- The decision of $P1$ and $P2$ as to whether to hire a worker and produce the good.
- The decision of $P0$ as to whether to produce a unit of the good.
- The decision of the consumer as to how many units to buy.

In equilibrium, the following conditions must hold in every period t:

- Each of the producers $P1$ and $P2$ maximizes his expected profit given his beliefs about the price at beginning of period t.
- The market for the good clears, that is, the number of units produced is equal to the number of units demanded by the consumer.
- The producer $P0$ and the consumer (given his budget constraint) make decisions consistent with their reservation values.

Thus, this is a sticky price model in which the wage is fixed and the labor market is not required to clear.

In any stationary equilibrium, the profits of the producers $P1$ and $P2$ are equal to zero. If the price is below w, $P1$ and $P2$ will be idle. In this case, the only equilibrium price for the good is $w - 2$. If the price is above w, then each of the three producers will supply one unit. Hence, the market for the good cannot clear since the buyer wishes to buy at most two units.

Now assume that $P1$ and $P2$ differ in their depths of memory, that is, $k_{P2} > k_{P1}$. Suppose that $w > 6$ and consider a DeBruijn sequence of prices of order $k_{P1} + 1$, with the alphabet consisting of two prices, $w + 1$ and $w - 2$.

$P1$ cannot predict the price and his expectations are that the price is equally likely to be $w + 1$ or $w - 2$. He will not produce the good since, on average, he will incur a loss if he does. Producer $P2$ enters the market when he anticipates (correctly) that the price is $w + 1$. Thus, in periods where the price is $w + 1$, the total supply is 2 and the consumer has an income $(2w + 2)$ sufficient to purchase two units of the good. In periods where the price is low, only $P0$ produces the good and the buyer is able to buy only one unit ($w > 6$ implies that $w + 2 < 2(w - 2)$).

In this equilibrium, the consumer has the resources to purchase more than one unit of the good only when the price is high. In “booms,” two units are produced and sold to the consumer in exchange for his wages and his initial wealth. In “recessions,” production is not profitable for $P1$ and $P2$ and, despite the low price, the consumer does not have sufficient resources to purchase more than one unit of the good. The profits of producer $P2$ are equal on average to 0.5, exceeding the zero profits of the stationary price equilibrium.

If the producers have identical abilities, an equilibrium would imply for $P1$ and $P2$ either “no trade” or “trade with zero profits”: $P1$ and $P2$ would be unable to extract from the consumer any part of his initial endowment of money. However, in the above nonstationary equilibrium, the more “sophisticated” seller extracts some of the buyer’s surplus despite the excess of suppliers in the market! The sophisticated producer can do so because of his unique ability to recognize the pattern of prices.

This toy model has no pretension of being realistic, nor do we claim that fluctuations in the economy are contrived by economic agents. We only wish to demonstrate the possibility that when agents differ in their ability to understand patterns, prices can fluctuate endogenously in a way that is perceived by some agents as random and by others as deterministic. The latter agents can then benefit relative to their situation in a simple stationary equilibrium.

5. A Comment on Strategies with Bounded Recall

Let us review the deliberations of the decision makers in the previous sections. In each period, a decision maker chooses an action whose payoff depends on the realization of a parameter determined independently of his choice. We refer to the determination of this parameter as nature’s move. For concreteness, we focus on the case in which nature chooses a cyclical sequence $\{x^t\}$ consisting of 0s and 1s, and in every period the decision maker chooses either 0 or 1 with the objective of guessing x^t correctly. A decision maker characterized by a number k first learns the frequencies of the successors of any word of k letters and then chooses the more likely letter given the frequencies that he correctly infers. Hence, his strategy is a function of the last k observations $(x^{t-k}, \ldots, x^{t-1})$ of nature's moves. Because the sequence $\{x^t\}$ faced by the decision maker is determined independently of his choices, this specification of the strategies seems to be a natural one.

Consider now an alternative approach to this decision problem. Suppose that the decision maker is not sure about the causality between his actions and the sequence $\{x^t\}$ and does not exclude the possibility that this sequence responds, in some fashion, to his own choices. To test the claim that his own past actions may be a better predictor than nature's past moves, he employs a strategy that selects his current guess as a function of his own past k guesses. One would expect that no strategy of this type could do better than all the strategies of the former type. The decision maker's guesses do not affect nature's moves and thus there is no reason why they should be more informative than nature's moves, especially as the latter are part of nature's informational base.

This intuition is not generally true. Again, we use DeBruijn sequences to illustrate this point. Consider, for example, the case in which $k = 1$ and nature's move is a DeBruijn sequence of order 3 such as the infinite repetition of (0, 0, 0, 1, 0, 1, 1, 1). Any guessing strategy that depends only on nature's last move will result in a 50 percent success rate. In contrast, if the decision maker chooses a rule that uses his own past actions, he can generate the sequence (0, 1, 0, 1, . . .) by reversing his own last guess. This strategy yields a success rate of 75 percent. If somehow the decision maker chooses to alternate 0 and 1 independently of nature's moves, then he will discover that after his choice of 0 it is more likely that nature chooses 1 and after his choice of 1 it is more likely that nature chooses 0. His behavior in each period is a consistent response to this inference.

Such behavior, however, does not seem to follow a natural procedure of deliberation. A decision maker who conditions his action on his last k choices must also choose an initial condition. Indeed, the sequence (1, 0, 1, 0, ...) is also produced by the same generating rule but guesses nature's move correctly only ¼ of the time. In contrast, in the original specification of the notion of strategy, the initial history is determined uniquely by nature.

We find this observation somewhat puzzling. A decision maker might find knowledge of his past behavior more valuable than knowledge of nature's moves and might erroneously conclude, that he has the power to affect nature's moves.

The standard definition of a strategy in game theory calls for yet another type of strategy: the decision maker's action depends on both nature's past actions and his own. With perfect recall, this definition of a strategy includes redundant components, since a player can make do with conditioning his actions only on nature's moves. This inclusion of redundancies in the definition of a strategy in game theory is discussed in Rubinstein (1991).

Whereas it might be surprising that a guessing strategy based on the decision maker's own actions is better than all strategies based on nature's moves, it is not surprising that the decision maker can improve his performance further if he conditions his action on both his past actions and nature's past moves. Recalling one's own actions in the past is a way to enhance recall of nature's moves.

Suppose, for example, that $k = 3$ and that nature generates the DeBruijn sequence of order 4,

$$0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, \ldots .$$

By definition, any guessing strategy based on the last three moves of nature will yield only a 50 percent chance of success. However, if the decision maker's guess depends on his own last three guesses and has the form of the DeBruijn sequence of order 3 with cycle 1, 0, 0, 0, 1, 0, 1, 1 . . . , the decision maker improves his rate of success[3] to ¾. If the decision maker can base his guess on nature's last three choices as well as on his own last three guesses, he can generate the sequence

$$0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, \ldots ,$$

which predicts nature's move correctly ⅞ of the time. In general, a decision maker characterized by k and facing a DeBruijn sequence of order $k_n = k + 1$ must have a frequency of mistakes[4] of at least $1/(2k + 1)$.

Comment on the literature: A branch of the literature on repeated games has studied the effects of differences in players' abilities to generate strategies on the set of equilibria. Two types of constraints on players' strategies have received special attention. In Neyman (1985) and Ben Porath (1993) players use

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3. It is easy to see that the rate of mistakes must be at least 3⁄16. At any time that nature moves through the 16-element cycle, there must be one mistake in the first four periods (otherwise the decision maker will follow the constant 0) and, similarly, one mistake in the last four periods. If there is no mistake in periods 5–12, the sequence must continue with either 1010 or 0110 having two mistakes during periods 13–16.

4. To see this, note that any strategy will induce a cycle of pairs of decision maker's actions and nature's move of length $L2^{k+1}$ for some natural number L (recall that the length of a cycle of a DeBruijn sequence of order $k + 1$ is 2^{k+1}). If an agent makes m mistakes in the cycle, then there are at most $(k + 1)m$ periods t in the cycle for which at least one mistake is made in one of the periods $(t - k, \ldots , t)$. If $(k+1)m < [L2^{(k+1)} - 1]/2$, by a counting argument there must be two periods, t and t', and two $(k + 1)$-tuples, $(x^1, \ldots , x^k, 0)$ and $(x^1, \ldots , x^k, 1)$, such that $(a^{t-k}, \ldots , a^t) = (x^1, \ldots , x^k, 0)$, $(a^{t'-k}, \ldots , a^{t'}) = (x^1, \ldots , x^k, 1)$, and the agent guesses correctly in the periods $t - k, \ldots , t$ and $t' - k, \ldots , t'$. However, if the decision maker guesses correctly the last k periods, he holds the same memory history at periods t and t'; thus, he cannot predict correctly nature's move in both periods. Hence, $(k + 1)m \geq [L2^{(k+1)} - 1]/2$. Obviously, it must be also true that $2(k + 1)m \geq L2^{k+1}$ and thus, the frequency of mistakes is at least $1/[2(k+1)]$.

finite automate of bounded size. In Lehrer (1988) players have bounds on the number of periods they can recall from the past. In both cases, the action of a player in a particular period can depend on his own past actions.

This literature has mostly focused on the conditions under which a "misleader" can mislead a "guesser" who observes the misleader's sequence of actions. Neyman's (1996) results imply, in particular, that a misleader can generate deterministic, n-periodic sequences that any automaton of size m, where $m \log m = o(n)$, will mismatch in almost 50 percent of the periods. Gossner and Hernandez (2001) show that there exists a constant c such that for any m satisfying $m \log m \geq cn$, some automata of the guesser with m states will be able to match almost any n-periodic sequence of the misleader, with a proportion going to 1 as n goes to infinity. Previously, Sabourian (1998) used DeBruijn sequences to characterize the set of equilibria of repeated games with bounded memory.

Gilboa and Schmeidler (1994) and Lehrer (1994) also notice that differences in memory sometimes allow longer-memory players to correlate their actions in a way that is concealed from the other players.

The considerations presented in this section may be relevant to the discussion of the meaning of a strategy in repeated games with bounded recall. The analysis of Nash equilibrium in repeated games with perfect recall does not depend on whether a player can condition his action on his own past actions. However, it seems that the set of Nash equilibria of a repeated game with bounded recall and differences in the players' depths of memory might vary depending on whether the histories include the players' own actions.

6. Discussion

Our aim has been to construct a formal tool for modeling differences among agents in their ability to recognize temporal patterns of prices. We applied this tool to two economic models and showed that price fluctuations that are independent of economic fundamentals can emerge in equilibrium.

We wish to emphasize the difference between this paper's approach and standard models of asymmetric information. In the latter, conventional models, agents differ in the information that they possess about the fundamentals in the economy. In our model, the asymmetry of information is not exogenous and depends on the equilibrium price sequence.

A related approach would be to allow prices to depend on "sunspots" that are observed only by some agents. Our paper shows that fluctuations in prices and diverse limitations to agents' abilities serve a similar role.

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