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Valuation Equilibrium With Clubs*

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ABSTRACT

This paper considers model worlds in which there is a continuum of individuals who form finite-sized associations to undertake joint activities. We show how, through a suitable choice of commodity space, restrictions on the composition of feasible groups can be incorporated into the specification of the consumption and production sets of the economy. We also show that if there are a finite number of types, then the classical results from the competitive analysis of convex finite-agent economies can be reinterpreted to apply.

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1. Introduction

Economists are interested in situations that involve people voluntarily forming associations in order to undertake joint activities. Important examples of such associations include households, firms, and communities. Buchanan (1965) has labeled these associations *clubs.* In this paper we extend the general competitive framework to a large class of environments where clubs can form.[1] What distinguishes our competitive analysis of these environments is our choice of commodity space. This choice allows us to fit a large class of club and matching environments into the *Theory of Value* (Debreu 1959) framework.

The features of the class of environments that we consider are the following: First, there are finitely many types of individuals and a large number of individuals of each type. Second, clubs are small relative to the number of individuals of any given type. Third, individuals maximize the expected value of a continuous utility function. Fourth, individuals have access to a randomizing device upon which they can condition their trades.

For a competitive analysis of this class of environments, a *commodity space* is needed that meets certain requirements. First, the space must be linear. Second, a firm (or consumer) can determine whether or not it can produce (or consume) a particular point in the commodity space without any knowledge of what any other firm or person is doing; that is, the firm's technology sets and consumers' consumption sets are subsets of the commodity space. Third, the feasibility of plans for all people and firms, along with the resource balance constraint, implies that the allocation is feasible.

The key requirement for our commodity space is the second one, since in club environments there are typically membership restrictions on group formation. To deal with this we index the commodity vector by consumer types as well as by characteristics of the potential clubs. With this indexation, consistency of membership composition is determined by restrictions on feasible plans. For example, a household club might consist of one male and one female and might be characterized by its consumption level of the club good. In this case, the consumption sets restrict persons to consuming only club memberships of their own

[1] We do not survey this large literature, but instead leave that to three recent surveys: Roth and Sotomayor (1990) survey the matching, or two-person—club, literature. Wooders (1992) and Scotchmer (1994) survey the club literature.

sex, while production sets impose the constraint that the number of male and female club memberships must be equal for each club consumption level. These restrictions permit this environment to be represented as an economy in the *Theory of Value* sense.

Because there can be a continuum of commodities and people can consume a finite quantity of an individual good, we follow Mas-Colell (1975) in using a commodity space that consists of signed measures. But we follow Prescott and Townsend (1984a, 1984b), rather than Mas-Colell, in interpreting the commodity vector from the viewpoint of the household as being a probability measure. This is done to ensure that all gains from trade are exhausted. Without randomization, preferences would not be convex, since club membership is discrete. Hence there could exist mutually beneficial gambles. (See Rogerson 1988.)

Randomization can be introduced into the Arrow-Debreu event-contingent analysis by introducing sunspots. (See Shell and Wright, 1993.) The reason why we adopt the lottery approach rather than the sunspot approach is that with lotteries the analysis is simpler. One simplification arises because there is no need to keep track of people's names. Another simplification arises because preferences become convex; this in turn permits attention to be restricted to type-identical allocations. An implication of this is that all the results for finite-agent convex economies hold for our large economies.

This commodity space does require that a rich and artificial menu of commodities be traded. However, just as with the Arrow-Debreu equilibrium, there are equivalent decentralizations that correspond more closely to the actual commodities that we see exchanged. To make this point, we establish the equivalence between lottery equilibria and the equilibria of an alternative two-stage decentralization. Under this decentralization, people gamble at the first stage and match or form clubs in a deterministic fashion at the second stage.

The paper is organized as follows. The first section shows how a simple, prototypical matching example can be represented as an economy in the *Theory of Value* sense. We then extend this example to a club model in which the club is characterized, in part, by the number of its members; hence, congestion can arise. In the third section, the analysis is generalized to permit multi-club membership, and the welfare theorems, core equivalence, and existence for the case in which there are a finite number of club types are established. Problems of extending the results to a continuum of club types are discussed. The fourth section establishes an equivalence between this decentralization and an alternative decentralization that entails gambling at the first stage and no randomization of allocations at the second stage.

2. A Simple Matching Example

The purpose of this section is to show that the simple model of household formation in Cole, Mailath, and Postlewaite (1993) can be represented as an economy in the sense of Debreu.

A. The Environment

There are two types of people, which we refer to as male or female and denote by $s \in S = \{f, m\}$. When referring to a person of type s, we use s' to denote the person's opposite sex. There is a continuum of each type, and the measure of people having type s is $\lambda^s > 0$. An individual is *matched* if paired with a member of the opposite sex. Matched individuals can consume both a pair-specific public or club good and, individually, the personal consumption good. Unmatched individuals can consume only the personal consumption good. A person of type s is endowed with ω^s of the personal consumption good. There exists a technology for converting units of the personal consumption good into half as many units of the club good.

Let A and B denote the Euclidean spaces of personal and club consumption levels respectively, and let $C = A \times B$ denote the space of consumptions. We assume that individuals of type s have preferences defined upon the compact set $C^s \subset C$. In this section, we take $C^s \subseteq [0, d]^2$, where d is sufficiently large that no feasible allocation exists for which all of any type have personal or club consumption level d or greater.

Preferences are identical for all people and are given by

$$E\{u(a, b)\},$$

where $u : C^s \rightarrow \Re$ is a continuous function.

B. Representation as an Economy

We need a linear commodity space with the property that the feasible actions of a firm or a consumer are a subset of this space. A feasible allocation of the economy is a specification of actions for each agent that are individually feasible and mutually consistent. Further, this consistency requirement takes the form of a resource balance condition, which depends only on the sum over the actions (commodity vectors) of producers and consumers and on the aggregate endowment.

The challenge posed by matching models is that, besides the usual need to specify production and consumption activities, we also need to specify the allocation of individuals to different pairs (or, more generally, to groups). We handle this by indexing the commodity vector by (a, b, s), which permits us to distinguish between male and female consumption. Hence the commodity vector specifies the quantity of each of the (a, b, s) commodities, where a commodity consists of three things: private consumption, club consumption, and a sex designation.

The *commodity space* is $L = \mathcal{M}(C \times S)$, where $\mathcal{M}(\cdot)$ denotes the space of signed measures on the Borel-sigma algebra of a set. The implicit topology on $C \times S$ is the product topology generated by the Euclidean metric on C and the discrete metric on S.

With this commodity space we can impose restrictions on group formation by a combination of restrictions on the technology and consumption sets. For our simple example, we impose the restrictions that the production of male and female commodities with club level b must be equal for every $b > 0$. We restrict the consumers to choosing a consumption vector that puts positive weight only on commodities with the appropriate sex designation. Without lotteries, the commodity vector chosen by an individual consumer puts weight one on one and only one commodity. With lotteries, the restriction is that the total probability weight sum to one.

Technology Set: A production plan, denoted by $y \in L$, is feasible if it satisfies the following two requirements.

First, the combined number of units of the private good used in producing units of the club good or given out as personal consumption must not exceed the number of units of the private good taken in, or

$$\int (b + a) \, dy \leq 0. \tag{1}$$

Note that $\int b \, dy$ is the total amount of the personal consumption good used up in the production of the club good, and that (1) implies the free disposal of a.

Second, the matching restriction on male and female commodities for each $b > 0$ must be satisfied, or

$$y(A \times D \times \{f\}) = y(A \times D \times \{m\}) \text{ for all } D \in \mathcal{B}(B \backslash \{0\}), \tag{2}$$

where $\mathcal{B}(\cdot)$ denotes the Borel-sigma algebra of a set.

The production set of a firm is given by

$$Y = \{y \in L : \text{ (1) and (2) hold}\}.$$

Note that the technology set is a convex cone. This implies that the aggregate technology set is Y as well.

Consumption Sets and Utility Functions: We follow Prescott and Townsend (1984a, 1984b) in interpreting components of a person's plan, $x \in L$, as being a probability measure. To rule out randomization would be tantamount to ruling out certain event-contingent commodities. Indeed, as was noted in Rogerson (1988), indivisibilities with respect to individual decisions typically result in there being gains to randomized consumptions. Component (a, b, s) of x is the probability or likelihood of having personal consumption a and being a type s member of a club with club consumption level b. Clearly, the probability of an individual of type s being a type $s' \neq s$ must be zero. Hence a person of sex s has the consumption set

$$X^s = \{x \in L_+ : x(C^s \times \{s\}) = \ x(C \times S) \leq 1\}$$

and the utility function, $U^s : X^s \to \Re$, is

$$U^s(x) = \int u(c)\, dx + (1 - \int dx) u(0, 0).$$

There are several things to note here. First, with these consumption sets an individual of type s must put weight one or less on his or her own sex-designated consumptions and must put weight zero on the other sex-designated consumptions. Note that $x(C^s \times \{s\}) = x(C \times S)$

implies that $x(C \times \{s'\}) = 0$. When $x(C^s \times \{s\}) < 1$, the residual weight is put on the zero-consumption commodity bundle $(0, 0)$. This will turn out to be convenient later when we discuss the existence of an equilibrium. Second, since X^s is convex and U^s is linear, preferences are convex. Third, with the weak-star topology, X^s is compact (since $C \times \{s\}$ is a compact metric space) and U^s is continuous (since $u(c)$ is continuous).

Endowments: We treat unmatched individuals as members of a singleton club. Consistent with this, we interpret endowments to be singleton clubs; that is, we assume that each individual is endowed with a club that has zero units of b and ω^s units of a. Formally, the endowment vector of a type s consumer is the commodity vector $e^s \in L$, where $e^s(D) = 1$ if $(\omega^s, 0, s) \in D$, and zero otherwise.

C. Valuation Equilibrium

A *type-identical valuation equilibrium* with a price system that has a dot product representation is a measurable function $p : C \times S \to \Re$ and a type-identical allocation $\{x^f, x^m, y\}$ that satisfy the following:

(i) x^s maximizes $U^s(x)$ subject to $\{x \in X^s : \int p \, dx \leq \int p \, de^s\}$ for all s.

(ii) y maximizes $\int p \, dy$ subject to $y \in Y$.

(iii) $\sum_s \lambda^s (x^s - e^s) = y$.

In the section on core equivalence, a more general definition of a competitive equilibrium is presented. However, as the following proposition makes clear, our current restriction to a type-identical competitive equilibrium is innocuous.

PROPOSITION 1. *Given any equilibrium, there is an equivalent type-identical equilibrium, where by equivalent we mean that the y and p are the same and that the type-average commodity vectors are the same.*

The proposition follows from the convexity of preferences. (See Debreu and Scarf 1963 for a formal proof.) The outline of a proof is as follows. Since only the type-average consumption vector enters the resource balance constraint, and since consumption sets are convex, if we give each individual his or her type average while keeping y the same, this allocation is also feasible. With p the same, the convexity of individuals' consumption sets and the linearity of their utility functions u^s imply that a type's average consumption vector solves the maximization problem for that type. Finally, since y and p are the same, profit maximization holds.

REMARK 1. We do not need a law of large numbers result for a continuum of identical and independently distributed random variables in our specification of condition (iii), the resource balance condition, because we are not assuming that the lotteries are independent. We simply assume that the lotteries are identically distributed for individuals of the same type. Implicitly, there can be thought to be one grand lottery that determines which individuals of a given type will receive which consumption.

3. Clubs With Multiple Members and Congestion

One of the classical issues in the clubs literature is congestion effects. Here we show how our decentralization can be extended to the case where there is a variety of possible membership compositions and where individuals care about the composition of the membership of their club.

A. The Environment

We assume that clubs come in different combinations of males and females. Let N be a bounded subset of $I \times I$, where I denotes the nonnegative integers. We assume that singleton clubs are possible; that is, $\{(1,0),(0,1)\} \subset N$. The membership composition of a club will be denoted by $n = (n_f, n_m) \in N$, where n_s is the number of members of sex s. The production of a single club involves a specification as to the input of the private good, a, used in the production of the club good, the level of the club good, b, and the composition of the membership, n. The set of clubs that can be produced is given by the compact set $T \subset A \times B \times N$.

REMARK 2. In our prior simple matching example, $N = \{(1,0),(0,1),(1,1)\}$ and the set of feasible clubs is

$$T = \{(a,b,n) \in A \times (B \backslash \{0\}) \times \{(1,1)\} : a = 2b\} \cup \{A \times \{0\} \times \{(0,1),(1,0)\}\}.$$

Consumers have preferences not only over private and club consumption, but also over the membership composition; hence, the space of possible consumptions is now $C^s \subseteq C = A \times B \times N$, and preferences are given by $u^s : C^s \to \Re$. Note that with this specification of preferences the club consumption being offered to an individual is not b but (b, n). For example, in consuming swimming it is not just the size of the pool that matters, but the size of the pool along with the number of people in it. In the lexicon of the club literature this is a model of *differentiated crowding,* since we do not restrict individuals to caring only about the total number of people in the pool.

B. Representation as an Economy

Since the commodities that can be produced are determined by the set of feasible clubs, it will be convenient to distinguish between the distribution of clubs, which we denote by δ, and the distribution of commodities, which we will continue to denote by y. We will determine the feasible commodity measures that can be produced by requiring that they be consistent with some feasible distribution of clubs. The distribution of commodities is an element of the commodity space $L = \mathcal{M}(C \times S)$.

The aggregate production set in terms of commodities offered and received is

$$Y = \Big\{ y \in L : \ \exists\, \delta \in \mathcal{M}_+(A \times B \times N) \text{ for which}$$

(i) $\delta(T) = \delta(A \times B \times N)$,

(ii) $\int a\, dy + \int a\, d\delta \leq 0$, and

(iii) $y(A \times D \times \{s\}) = \int_{A \times D} n_s\, d\delta$ for all $D \subseteq B \times N \Big\}$.

Constraint (i) is that only feasible clubs are produced, that is, that the support of δ is

T. Constraint (ii) is that the total amount of the private consumption good given out as personal consumption, minus the amount taken in from consumers, plus the amount used up in producing the club goods (which is $\int a\, d\delta$), must be less than or equal to zero. Constraint (iii) is that the number of consumers of sex s memberships in clubs with characteristics (b, n) produced must be consistent with the number of clubs operated.

The definitions of consumption sets, preferences, and the price system are unchanged, given our new definitions of C and C^s. For females, the endowment point $e^f(D) = 1$ if $(\omega^f, 0, (1, 0), f) \in D$ and is zero otherwise, while for males the endowment point $e^m(D) = 1$ if $(\omega^m, 0, (0, 1), m) \in D$ and is zero otherwise.

Remark 3. The club consumption good here is (b, n), but henceforth we will take b to be a vector that contains all the aspects of club type that are relevant to consumers.

4. A More General Class of Economies With Clubs

The method of decentralization that we have considered is applicable to a richer class of environments than we have thus far considered. Here we indicate how one can accommodate a much wider range of models that have a greater variety of types and in which a more extensive set of joint activities are undertaken. We can also allow for multiple club membership; for example, an individual can be a member of both a household club and a production club.

A. The Economy

Set of People: There are a finite number of types of people indexed by $i \in I$. The measure of type i is $\lambda^i > 0$. We say the distinction between two types of people is *basic* if some club technology does not treat them as perfect substitutes. In the previous section, the sex distinction is basic but the endowment of the private good is not. Type i's basic type is $s^i \in S$. Note that the dimension of S is, at most, I.

Commodities: The space of possible consumptions of the exchangeable goods is a finite dimensional Euclidean space, which we denote by A. There is a finite set of categories of clubs indexed by $j = 1, ..., J$. Each person must be a member of one and only one club in each category. Since we allow for a single member club that does nothing in each category, this membership requirement does not restrict the set of feasible allocations. We denote the space of possible club consumption vectors in category j by B_j, which, like A, is a finite dimensional Euclidean space. We define the space of consumption to be $C = A \times B_1 \times ... \times B_J$. The commodity space is $L = \mathcal{M}(C \times S)$, the space of signed measures on the Borel-sigma algebra of $C \times S$.

Consumption Sets, Preferences, and Endowments: A person of type i has preferences over probability measures on the Borel-sigma algebra of C^i, a compact measurable subset of C that is ordered by the expected value of the continuous function $u^i(c)$ defined on C^i. We assume that $0 \in C^i$. Type i's consumption set is

$$X^i = \{x \in L_+ : x(C^i \times \{s^i\}) = x(C \times S) \leq 1\}.$$

Type i's utility function is $U^i : X^i \rightarrow \Re$, where

$$U^i(x) = \int u^i(c)\, dx + (1 - \int dx)\, u^i(0).$$

An individual is endowed with $\omega^i \in C$. The endowment point of a type i person is denoted by $e^i \in L$, with e^i being the measure that assigns measure one to sets containing $(\omega^i, s^i) \in C \times S$ and that assigns measure 0 to all other sets.

Club Technologies: A club type of category $j = 1, ..., J$ is defined by its vector of exchangeable commodities a, the vector of club goods b_j, and its membership n. Its membership vector n specifies the integer number of each basic type s in the club. The set of possible memberships is the finite set N. Thus a club type in category j is denoted by $t_j \in A \times B_j \times N$. The set of possible category j club types is given by the compact set $T_j \subseteq A \times B_j \times N$, for $j = 1, ..., J$.

Aggregate Production Set: A commodity vector y belongs to Y if, for each category $j = 1, ..., J$, there is a distribution δ_j on T_j consistent with y. Formally, the aggregate production set is

$$Y = \Big\{ y \in L : \exists\, \delta_j \in \mathcal{M}_+(A \times B_j \times N) \text{ for } j = 1, 2, ..., J \text{ for which}$$

(i) $\delta_j(T_j) = \delta_j(A \times B_j \times N)$,

(ii) $\int a\, dy + \sum_j \int a\, d\delta_j \leq 0$, and

(iii) $\int_{A \times D \times N} n_s\, d\delta_j = y(A \times B_1 \times ... \times D \times ... \times B_J \times \{s\})$ for all $D \in \mathcal{B}(B_j),\ j,\ s\Big\}$,

where $\mathcal{B}(B_j)$ denotes the Borel subsets of B_j.

Constraint (i) is that only technologically feasible clubs are operated. Constraint (ii) is that the net input of the exchangeable goods a plus the amount of a used up in the production of the distribution of clubs is less than or equal to zero. Constraint (iii) is that the measure of people of type s who are receiving club consumption $b_j \in D$ in the output measure must be consistent with the distribution of clubs in category j. The aggregate production technology is a convex cone.

Price System and Competitive Equilibrium: A price system for our economy is a linear functional $p : L \to \Re$, which has representation

$$p \cdot x = \int p(c, s)\, dx$$

where $p : C \times S \to \Re$ is measurable.

A *type-identical competitive equilibrium* for our economy is a price system p and an allocation $[\{x^i\}, y]$ satisfying the following:

(i) For all i, x^i maximizes $U^i(x)$ subject to $x \in X^i$ and $p \cdot x \leq p \cdot e^i$.

(ii) y maximizes $p \cdot y$ subject to $y \in Y$.

(iii) $\sum_i \lambda^i(x^i - e^i) = y$.

REMARK 4. The set of *feasible allocations*, that is, the set $\{[\{x^i\}, y] : x^i \in X^i$ all i, $y \in Y$ and $\sum_i \lambda^i(x^i - e^i) = y\}$, is compact, since the consumption sets are compact.

B. The Welfare Theorems and Existence

The first welfare theorem of Debreu (1954) applies here, since preferences are convex, provided no type is satiated in equilibrium.

It is somewhat more complicated to establish the second welfare theorem. If L is finite dimensional and/or Y has an interior point, then Theorem 2 in Debreu (1954) can be applied to conclude that every Pareto optimum can be supported as a quasi-competitive equilibrium with transfers. If this is not the case, then one may be able to proceed to establish the second welfare theorem for an economy with a larger production set that has an interior, but which has the same set of Pareto optima and competitive equilibrium as our economy.

The second welfare theorem guarantees the existence of a quasi-competitive equilibrium in which the price system is a continuous linear functional on the commodity space. This does not imply that this price system has a dot product representation when the commodity space is infinite dimensional. To establish the existence of a competitive equilibrium with transfers and with a price system that has a dot product representation, we verify that assumptions 15.8′ and 15.9′ in Stokey and Lucas (1989, p. 468) hold and we apply their Theorem 15.9. Assumption 15.8′ requires that if $x \in X^i$, the modification of x obtained by setting the x equal to zero on some measurable subset of $C \times S$ is also in X^i. Assumption 15.9′ is essentially that preferences are weak-star continuous, which is the case given expected utility maximization and the fact that the utility functions, u^i, are continuous functions defined on a compact set for all $i \in I$.

If C is finite, then standard existence arguments may apply, where the commodity space L is just the Euclidean space with dimension equal to the number of commodities.

REMARK 5. In his discussion of the classical theorem of the existence of a competitive equilibrium, McKenzie (1981) lists the following conditions as being sufficient:

(i) The consumption sets are closed and convex.

(ii) The preference ordering is continuous and quasi-concave.

(iii) The aggregate production set is a closed convex cone.

(iv) $X^i + \{-e^i\} \cap Y$ is not empty for all i.

(v) $Y \cap L_+ = \{0\}$.

(vi) There is a common point in the relative interiors of Y and $X = \sum_i \lambda^i (X^i + \{-e^i\})$.

(vii) For any two nonempty, disjoint partitions of I, I_1, and I_2, and for $x_{I_1} = y - x_{I_2}$, where $x_{I_1} \in \sum_{i \in I_1} \lambda^i (X^i + \{-e^i\})$, $x_{I_2} \in \sum_{i \in I_2} \lambda^i (X^i + \{-e^i\})$, and where $y \in Y$, there exists a $w \in \sum_{i \in I_2} \lambda^i (X^i + \{-e^i\})$ such that $x'_{I_1} + x_{I_2} + w \in Y$ and such that x'_{I_1} can be decomposed into an allocation for I_1 that is weakly preferred by all members of I_1 and is strictly preferred by at least one member-type of I_1.

As has already been noted, conditions (i)–(iii) are satisfied. Conditions (iv)–(vii) are satisfied for our matching example if the set of commodities is suitably restricted. The commodities are $(A \times B \times S) \backslash ((0,0,f) \cup (0,0,m))$. The reason these two points are excluded is that these commodities can be produced costlessly. Condition (iv) is satisfied because each individual can always consume his or her endowment. Condition (v) is satisfied because no commodity can be produced without some inputs. The following argument establishes that condition (vi) is satisfied. For all s, if $x^s(c,s) = \epsilon/\lambda^s$ and if $x^s(c,s') = 0$ for all c, then for any positive ϵ that is sufficiently small, $x^s \in X^s$. Hence the point $y = \sum_s (x^s - e^s)\lambda^s$ is in the relative interior of $\sum_s (X^s - \{e^s\})\lambda^s$. Since $y(c, \{f\}) = y(c, \{m\})$, condition (2) is satisfied. Finally, if R is the total amount of resources needed to produce measure one of every type of commodity, then for any positive ϵ that is sufficiently small, $\epsilon R < \sum_s \lambda^s \omega^s$ and condition (1) is satisfied. Thus y is in the relative interior of Y. Condition (vii), which ensures that all agents have positive income, is trivially satisfied, since more personal consumption is preferred to less personal consumption in any feasible allocation.

If C is not finite, the commodity space is infinite dimensional and it is more difficult to establish the existence of a competitive equilibrium. A standard method of proof is to show that the limit of a sequence of equilibria for finite dimensional economies exists and is an equilibrium for the infinite dimensional limit economy. An outline of this approach, as applied to our more general class of economies, is as follows. Restrict the aggregate production set to elements whose support belongs to finite set $C_r \subseteq C$, where all points in C lie in a $1/r$-neighborhood of at least one point in C_r. For the economy $[\{X^i, u^i, e^i\}, Y_r]$, the commodity space $L_r = \mathcal{M}(C_r \times S)$ is finite dimensional. The competitive equilibrium $[\{x_r^i\}, y_r, p_r]$ for the rth economy can be used to find an equilibrium for the economy with the general commodity space and the restricted production set. Given this equilibrium, the problem is to find a convergent subsequence with the property that the limit is an equilibrium for our economy with the unrestricted production set Y.

The key part of the proof is to find a uniformly convergent subsequence of $\{p_r\}$. This ensures that the limits of the sequences x_r^i are maximal, given the price system $p = \lim_{r\to\infty} p_r$, where r now indexes elements of a subsequence for which allocations converge weak-star and prices converge uniformly. For our economy, if the $u^i(c)$ have uniformly bounded slopes, then a sequence of competitive equilibria exists with the desired property. First, we normalize prices so that the sup $|p_r(c, s)| = 1$. Next, we let $q_r(c, s)$ be the maximal demand reservation price over all i. Note that the q_r functions are defined on all of $C = \cup_i C^i$. Note also that $[\{x\}, y_r, q_r]$ is a competitive equilibrium for the economy with the commodity space $L = \mathcal{M}(C \times S)$ and the aggregate technology set Y_r. The functions $q_r(c, s)$ will have uniformly bounded slopes, given that the underlying utility function $u^i(c)$ has a uniformly bounded slope. This completes an outline of the extension to economies in which C is a compact metric space.

C. Equivalence of Core and Equilibrium Allocations

In this section we show that the standard arguments establishing the equivalence of the set of core and equilibrium allocations hold with a lottery commodity space. (See Hildenbrand and Kirman 1988 for an exposition of the standard arguments in exchange economies.)

To specify a core allocation, we must allow for allocations in which people of a given type do not receive the same commodity point. Hence, we alter our definition of an allocation for type i individuals in this section; it will now be a measure defined on our commodity space L that specifies the distribution of consumption vectors which are consumed by these individuals.

Having specified an allocation more generally, we show that any core allocation must satisfy equal welfare among people of the same type. Next, we show that any equilibrium allocation is in the core. Finally, we show that, given a core allocation, there exists a price system that supports it as a quasi-competitive equilibrium, and we discuss the conditions under which a quasi-competitive equilibrium is a competitive equilibrium.

A general definition of an *allocation* is a set of measures $\{z^i\}$, where $z^i : \mathcal{B}(L) \to R_+$. An allocation is feasible if

(i) for all i, the support of z^i is a subset of X^i, and

(ii) $y(z) \equiv \sum_i \int (x - e^i)\, dz^i \in y$.

A feasible allocation $\{z^i\}$ is *blocked* by the feasible allocation $\{\tilde{z}^i\}$ if

(i) for all i, $\tilde{z}^i(L) \le z^i(L)$, and

(ii) for all $(i, \bar{u})$,

$$\tilde{z}(\{x : u^i(x) \le \bar{u}\}) \le z(\{x : u^i(x) \le \bar{u}\}),$$

with strict inequality for some $(i,\, \bar{u})$.

A *core allocation* is a feasible allocation that cannot be blocked.

Implicitly, the members of the blocking allocation are those receiving the lowest utility under $\{z^i\}$. Everyone in the blocking allocation receives larger utility under $\{\tilde{z}^i\}$, with some positive measure receiving strictly larger utility.

PROPOSITION 2. *If $\{z^i\}$ is a core allocation, then all people of the same type receive the same welfare level.*

PROOF. If utilities were not equal for almost all people of some type, then some positive measure of that type must receive utility strictly less than the average. Pick $\epsilon > 0$ sufficiently small that

$$z^i\left(\{x : u^i(x) \le \bar{u}^i\}\right) \ge \epsilon\lambda^i \text{ for all } i,$$

where $\bar{u}^i$ denote the type i average utility under z. Then, for the population $\beta^i = \varepsilon\lambda^i$ for all i, the allocation $\{\tilde{z}^i\}$—which puts mass β^i on the average commodity point under z, $\int x dz^i$, and zero elsewhere—is feasible and blocks z. ■

A *competitive equilibrium* is a triplet $[\{z^i\}, y, p]$ satisfying the following:

(i) For each $i, z^i(L) = \lambda^i$.

(ii) The support of $z^i \subseteq argmax\{U^i(x) : x \in X^i$ and $p \cdot x \leq p \cdot e^i\}$ for each i.

(iii) $y \in argmax_Y \; p \cdot y$.

(iv) $\sum_I \int (x - e^i) \, dz^i = y$.

Given a type-identical competitive equilibrium $[\{x^i\}, y, p]$, we can construct its associated general competitive equilibrium $[\{z^i\}, y, p]$ in the obvious manner: by having z^i put weight λ^i on sets containing x^i and put weight zero on sets not containing x^i.

For a *quasi-competitive equilibrium,* condition (ii) in the definition of a competitive equilibrium is replaced by the following:

(ii′) For each i, if x' is in the support of z^i, then for any $x \in X^i, U^i(x) \geq U^i(x')$ implies that $p \cdot x \geq p \cdot x'$.

Proposition 3. *If* $[\{z^i\}, y, p]$ *is a competitive equilibrium with no type satiated, then its allocation* $\{z^i\}$ *is a core allocation.*

Proof. We will show that the existence of a $\tilde{z}$ which blocks z leads to a contradiction. All points in the support of z^i must yield the same utility for a member of type i, since all members of type i have the same maximization problem. Points that yield higher utility must be more expensive, that is, if $\bar{u}^i$ is the type i utility under $\{z^i\}$, then $u^i(x) > \bar{u}^i$ implies that $p \cdot x > p \cdot e^i$. Further, the convexity of preferences, along with the existence of a point in X^i that yields utility exceeding $\bar{u}^i$, implies that if $u^i(x) \geq \bar{u}^i$, then $p \cdot x \geq p \cdot e^i$. Therefore, if $\{\tilde{z}^i\}$ blocks $\{z^i\}$, then, for every i,

$$p \cdot \int (x - e^i) d\tilde{z}^i \geq 0$$

with strict inequality for at least one i. If we sum over i, this implies that $p \cdot y\,(\tilde{z}) > 0$, while the feasibility of $\{\tilde{z}^i\}$ implies that $y(\tilde{z}) \in Y$. But given that y is a cone, equilibrium profits must be zero; hence, we have a contradiction. ■

To show the existence of a price system which supports a core allocation as a quasi-competitive equilibrium, we apply the Hahn-Banach theorem. However, just as with the second welfare theorem, the potential emptiness of the interior of the production set when the commodity space is infinite dimensional precludes this theorem's direct application. There are two obvious resolutions to this problem: (i) to assume that C is finite and, hence, L is finite dimensional, or (ii) to construct an alternative production set, Y', which includes Y and for which the set of Pareto optimal allocations is the same, regardless of the distribution of the population (the λ's). The condition in approach (ii) is stronger than the condition in the second welfare theorem because we need to ensure that, given any blocking allocation that is feasible for the production set Y', there is at least as good an allocation that is feasible for Y; hence, the expansion in the set of possible blocking allocations under Y' has not shrunk the core. We pursue approach (i) below. Approach (ii) can be applied in our prototype economy by incorporating a sufficiently expensive method of producing singleton clubs with positive levels of the club good.

PROPOSITION 4. *If C is finite, then for any core allocation z in which at least one type is not satiated, there exist p and y such that $[z, y, p]$ is a quasi-competitive equilibrium.*

PROOF. The proof is by contradiction. With $y = y(z)$, the resource balance condition is satisfied. Now we need only find a supporting price system. We show that such a price system exists as follows. First, Proposition 2 states that, for a core allocation, there is a utility level for each type which we denote by $\bar{u}^i$. Next we define

$$X_{\geq} = \Big\{ x \in L : \exists\{0 \leq \theta^i \leq \lambda^i, x^i \in X^i\} \text{ for which } u^i(x^i) \geq \bar{u}^i \text{ all } i,$$

$$\text{with strict inequality for some } i, \text{ and } x = \sum \theta^i (x^i - e^i) \Big\}.$$

The set $X_{\geq}$ is nonempty, since at least one type is not satiated. $X_{\geq} \cap Y = \phi$, since z is a core allocation. We now show that $X_{\geq}$ is convex. Let $\bar{x}$, $\hat{x}$ be any elements belonging to $X_{\geq}$, and let $\{\bar{\theta}^i, \bar{x}^i\}$ and $\{\hat{\theta}^i, \hat{x}^i\}$ be a set of associated weights and consumption vectors for $\bar{x}$ and $\hat{x}$ in the definition of $X_{\geq}$. For $\alpha \in [0,1]$, let $\theta^i = \alpha\bar{\theta}^i + (1-\alpha)\hat{\theta}^i$. Let $x^i = (\alpha\bar{\theta}^i\bar{x}^i + (1-\alpha)\hat{\theta}^i\hat{x}^i)/\theta^i$ if $\theta^i > 0$, and let $x^i = \bar{x}^i$ otherwise. These $\{\theta^i, x^i\}$ satisfy the conditions for $\alpha\bar{x} + (1-\alpha)\hat{x}$ to belong to $X_{\geq}$. Thus $X_{\geq}$ is convex.

Given that $X_{\geq}$ and Y are nonempty and convex, that $X_{\geq} \cap Y = \phi$, and that C, being finite, implies that L is finite dimensional, the Hahn-Banach theorem guarantees the existence of a nontrivial hyperplane that separates $X_{\geq}$ and Y, which we can represent by $p \in L$.

Next we show that if x^i is in the support of z^i, then $\lambda^i(x^i - e^i)$ belongs to the closure of $X_{\geq}$. Let $x_n = \lambda^i(1-1/n)(x^i - e^i) + (1/n)(x^j - e^j)$, where (j, x^j) is such that $x^j \in X^j$ and $u^j(x^j) > \bar{u}^j$. For sufficiently large n, $x_n \in X_{\geq}$. Furthermore, $\lim_{n\to\infty} x_n = \lambda(x^i - e^i)$. Thus the result follows.

The fact that set Y is a cone implies that points in the separating hyperplane must have value 0. Thus $p \cdot x \geq 0$ for all x in the closure of $X_{\geq}$, and $p \cdot y \leq 0$ for all $y \in Y$. An implication of these results is that $p \cdot y(z) \leq 0 \leq \sum_i \int p \cdot (x - e^i)dz^i$. But $y(z) = \sum_i \int (x - e^i)dz^i$. Thus $p \cdot y(z) = 0$, and $y(z)$ is profit maximizing. Similarly, for x^i in the support of z^i, $p \cdot \lambda^i(x^i - e^i) = 0$, which implies that $p \cdot (x^i - e^i) = 0$. This in turn implies that x^i minimizes expenditure subject to $u^i(x^i) \geq \bar{u}^i$. ■

Just as in the second welfare theorem, the existence of a cheaper point than the support of z^i in i's budget set for each i, along with the convexity of preferences, imply that a quasi-competitive equilibrium is a competitive equilibrium.

5. Equivalence of Lottery and Value-Gamble Equilibria

The type of decentralizations (commodity space and price system) that we have thus far considered have proven convenient from the standpoint of competitive analysis. However, as was noted earlier, they do require that a rich menu of artificial commodities be traded. In this section we want to consider an alternative decentralization in which a somewhat more standard menu of commodities is traded, and we want to show that the equilibrium allocations are equivalent. The decentralization that we will consider here is a two-stage process with gambles over value transfers in the first stage and a no-lottery competitive equilibrium in the second stage. To distinguish between these two decentralizations, we will label our original economy as the *lottery economy* and the economy with the two-stage decentralization (which we consider here) as the *gambling economy.*

A. The Simple Matching Example

To anticipate the formal notation for the gambling economy, it is useful to set out the choice problem of a person of type s in our simple matching example.

In the first stage, a type s individual picks a probability distribution over wealth transfers, the expected value of which is equal to zero, and then, conditional on his or her realized wealth level, the individual picks a second-stage consumption level of the private good and of his or her sex-specific club good. If we denote the feasible wealth transfers by W and the person's marginal distribution over wealth transfers by g_w, then in the first stage that person is picking $g_w \in \mathcal{M}_+(W)$ such that $g_w(W) = 1$ and $\int w \, dg_w \leq 0$.

In the second stage, the commodity space, the price system, and the production set of the gambling economy are the same as in the lottery economy. Individuals now differ not only with respect to their type but also with respect to their first-stage wealth transfers. The problem of an individual of type s with wealth transfer w is to

$$\max_c u^s(c), \text{ subject to } p(c, s) \leq p \cdot e^s + w.$$

Since individuals' second-stage consumption sets are no longer convex, we cannot, without loss of generality, assume that all individuals of the same type$\times$wealth make the same consumption choice. Therefore, we describe the choices of individuals of type s and wealth w by the distribution $g^{iw} \in M(C \times S)$.

B. Definition of an Equilibrium in the Gambling Economy

The second-stage commodity space, production sets, and price system are unchanged from the lottery economy. The set of possible value transfers in the first stage is denoted by $W = \Re$. To avoid measurability issues, we will denote a type's *gambling allocation* by the single measure $g^i \in \mathcal{M}_+(W \times C \times S)$, where the g_w above is simply the marginal over wealth, and where g^{iw} is type i's choices over consumptions conditional on w. Therefore, a type i

individual's consumption set is

$$G^i \equiv \{g \in \mathcal{M}_+(W \times C \times S) : g(W \times C^s \times \{s\}) = g(W \times C \times S) = 1\ \}.$$

The preferences of agent i over G^i are given by $\int u^i(c)\, dg$.

A *gambling equilibrium* is defined as a gambling allocation $[\{g^i\}, y]$ and a price system p that satisfy the following:

(i) g^i maximizes $\int u^i(c)\, dg$ over G^i, subject to

(1) $\int w\, dg \leq 0$ and

(2) $c \in argmax\{u^i(c) : p(c,s) \leq p \cdot e^i + w\}$, if (w, c, s) is in the support of g^i.

(ii) y maximizes $p \cdot y$, subject to $y \in Y$.

(iii) $\sum_i \lambda^i [\int g^i(dw \times D) - e^i(D)] = y(D)$ for all $D \in \mathcal{B}(C \times S)$.

We will refer to an equilibrium in our lottery economy as being *equivalent* to an equilibrium in the gambling economy if the price system, production vector, and ex ante distributions over consumption for each type are equal (that is, if $x^i(\cdot, \cdot) = \int g^i(dw, \cdot, \cdot)$ for all i). This implies that the ex ante welfare levels of people of a given type are also the same. The following proposition implies that the sets of lottery and gambling equilibria are equivalent:

PROPOSITION 5. *Any lottery equilibrium is equivalent to a gambling equilibrium and vice versa.*

PROOF. The proof involves constructing two mappings, θ and ϕ, which are such that, if the gambling allocation $[\{g^i\}, y]$ and the price system p are a gambling equilibrium, then the lottery allocation $[\{\theta(g^i)\}, y]$ and the price system p are an equivalent lottery equilibrium, and if $[\{x^i\}, y]$ and the price system p are a lottery equilibrium, then the gambling allocation $[\{\phi(x^i, p, e^i)\}, y]$ and p are an equivalent gambling equilibrium.

We define $\theta(g)$ for $g \in \sum G^i$ to be

$$\theta(g)(D) = \int g(dw \times D) \text{ for all } D \in \mathcal{B}(C \times S).$$

The map θ is simply the ex ante distribution over consumption implied by g.

We define $\phi(x, p, e)$ for $x \in L$, $p(c, s)$ measurable and for $e \in L$ by the requirement that

$$\phi(x^i, p, e^i)(D) = x^i(\{(c, s) : (p(c, s) - p \cdot e^i, c, s) \in D\}) \text{ for all } D \in \mathcal{B}(W \times C \times S).$$

The map ϕ is a slightly more complicated object than θ, since we implicitly map into a higher dimensional space. The wealth transfer required to buy commodity (c, s) is $w = p(c, s) - p \cdot e^i$. Loosely speaking, we can select g^i so that $g^i(p(c, s) - p \cdot e^i, c, s) = x^i(c, s)$.

We now want to establish a key property of these mappings. If g^i is within i's consumption and budget sets in the gambling economy, then $\theta(g^i)$ is within i's consumption and budget sets in the lottery economy. Similarly, if x^i is within i's consumption and budget sets in the lottery economy, then $\phi(x^i, p, e^i)$ is within i's consumption and budget sets in the gambling economy. To see this, recall that type i's budget constraint for the lottery equilibrium is

$$p \cdot x^i - p \cdot e^i \leq 0 \tag{3}$$

while type i's budget constraints for the gambling equilibrium are

$$\int w \, dg^i \leq 0 \tag{4}$$

and

$$p(c, s) \leq w + p \cdot e^i \text{ for all } (w, c) \text{ in the support of } g^i_s. \tag{5}$$

To see that $\theta(g^i)$ satisfies individual i's budget constraint in the lottery economy, note that

$$0 \geq \int w \, dg^i \geq \int (p(c, s) - p \cdot e^i) \, dg^i = \int (p(c, s) - p \cdot e^i) \, d\theta(g^i).$$

To see that condition (3) of the lottery economy implies that condition (4) and condition (5) of the gambling economy are satisfied by $\phi(x^i, p, e^i)$, note that, by construction,

$$\int w\, d\phi(x^i, p, e^i) = \int (p(c, s) - p \cdot e^i)\ dx^i = p \cdot x^i - p \cdot e^i \leq 0.$$

Thus the corresponding gambling elements satisfy their budget constraints if the set of lottery elements satisfies its budget constraints with equality.

It is apparent that this method of mapping one type of equilibrium into the other type preserves the feasibility and profit maximization conditions of the respective definitions of equilibrium, since the ex ante distributions of consumptions for each type and for the production vector are the same. Further, utility levels are the same, given that the distributions of consumptions are the same for each type i. What remains is to show that if x^i is a solution to type i's problem in the lottery economy, then its associated element under the mapping ϕ is a solution to i's problem in the gambling economy, and vice versa for g^i and θ.

The proof that these mappings preserve optimality is a proof by contradiction: Assume that g^i is a solution to type i's problem in the gambling economy. Assume also that there exists an $\hat{x}^i$ which is within type i's budget set in the lottery economy and is preferred to $\theta(g^i)$. Then, since $\phi(\hat{x}^i, p, e^i)$ also satisfies type i's gambling economy wealth and second-period budget constraints, individual i could have done better by choosing $\phi(\hat{x}^i, p, e^i)$. This contradicts our assumption that g^i is optimal for i. Similarly, we can establish by contradiction that if x^i is a solution to type i's problem in the lottery economy, then $\phi(x^i, p, e^i)$ is a solution to type i's problem in the gambling economy. ■

Remark 6. This result makes it clear that the contribution of lotteries in the lottery economy is simply to allow individuals to engage in implicit wealth gambles. Individuals have an incentive to gamble because their indirect utility function over wealth is not concave everywhere, but rather has convex regions induced by the underlying indivisibility with regard to club memberships.

Remark 7. If local nonsatiation holds for all equilibria, then the budget constraints hold with equality and the map between equilibria of the lottery and the gambling decentralizations is a bijection.

References

Buchanan, J. M. (1965): "An Economic Theory of Clubs," *Economica,* 32, 1–14.

Cole, H. L., Mailath, G. J., and Postlewaite, A. (1993): "Bargaining and Investment in a Two-Sided Matching Model," Mimeo.

Debreu, G. (1954): "Valuation Equilibrium and Pareto Optimum," *Proceedings of the National Academy of Sciences of the U.S.A.,* 40, 588–92.

Debreu, G. (1959): *Theory of Value: An Axiomatic Analysis of Economic Equilibrium.* Cowles Foundation for Research at Yale University, Monograph 17. New Haven: Yale University Press.

Debreu, G. and Scarf, H. (1963): "A Limit Theorem on the Core of an Economy," *International Economic Review,* 4, 235–46.

Hildenbrand, W. and Kirman, A. P. (1988): *Equilibrium Analysis: Variations on Themes by Edgeworth and Walras.* Amsterdam: North-Holland.

Mas-Colell, A. (1975): "A Model of Equilibrium With Differentiated Commodities," *Journal of Mathematical Economics,* 2, 263–95.

McKenzie, L. W. (1981): "The Classical Theorem of Existence of Competitive Equilibrium," *Econometrica,* 49, 819–41.

Prescott, E. C. and Townsend, R. M. (1984a): "Pareto Optima and Competitive Equilibria With Adverse Selection and Moral Hazard," *Econometrica,* 52, 21–45.

____________ (1984b): "General Competitive Analysis in an Economy With Private Information," *International Economic Review,* 25, 1–20.

Rogerson, R. (1988): "Indivisible Labor, Lotteries and Equilibrium," *Journal of Monetary Economics,* 21, 3–16.

Roth, A. E. and Sotomayor, M. A. O. (1990): *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis.* Cambridge: Cambridge University Press.

Scotchmer, S. (1994): "On Price-Taking Equilibria in Club Economies With Nonanonymous Crowding," Mimeo.

Shell, K. and Wright, R. (1993): "Indivisibilities, Lotteries, and Sunspot Equilibria," *Economic Theory,* 3, 1–17.

Stokey, N. L. and Lucas, R. E. (1989): *Recursive Methods in Economic Dynamics.* Cambridge, MA: Harvard University Press.

Wooders, M. H. (1994): "Large Games and Economies With Effective Small Groups," forthcoming in *Game-Theoretic Methods in General Equilibrium Analysis,* ed. J.–F. Mertens and S. Sorin. Boston: Kluwer Academic Press.