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Quantal Response Equilibria for Normal Form Games*

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We investigate the use of standard statistical models for quantal choice in a game theoretic setting. Players choose strategies based on relative expected utility and assume other players do so as well. We define a quantal response equilibrium (QRE) as a fixed point of this process and establish existence.

For a logit specification of the error structure, we show that as the error goes to zero, QRE approaches a subset of Nash equilibria and also implies a unique selection from the set of Nash equilibria in generic games. We fit the model to a variety of experimental data sets by using maximum likelihood estimation. *Journal of Economic Literature* Classification Numbers: C19, C44, C72, C92.

1. INTRODUCTION

We investigate the possibility of using standard statistical models for quantal choice in a game theoretic setting. Players choose among strategies in the normal form of a game based on their relative expected utility, but make choices based on a quantal choice model, and assume other players do so as well. For a given specification of the error structure, we define a quantal response equilibrium (QRE) as a fixed point of this process.

Under this process best response functions become probabilistic (at least from the point of view of an outside observer) rather than deterministic. Better responses are more likely to be observed than worse responses, but

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best responses are not played with certainty. The idea that players make infinitesimal errors underlies some of the refinement literature (Myerson, 1978; Selten, 1975). Introduction of noninfinitesimal errors has been specifically studied by Van Damme (1987, Chap. 4), Rosenthal (1989), and Beja (1992). Rosenthal assumes that the probability of adopting a particular strategy is linearly increasing in expected payoff. Beja, in contrast, assumes that players attempt to implement a "target" strategy but fail to do so perfectly. Recent work by El-Gamal *et al.* (1993), El-Gamal and Palfrey (1995, 1994), McKelvey and Palfrey (1992), Ma and Manove (1993), Chen (1994), and Schmidt (1992) also explores the equilibrium implications of error-prone decisionmaking in specific settings.

It is important to emphasize that this alternative approach does not abandon the notion of equilibrium, but instead replaces the perfectly rational expectations equilibrium embodied in Nash equilibrium with an imperfect, or noisy, rational expectations equilibrium. The equilibrium restriction in our model is captured by the assumption that players estimate expected payoffs in an unbiased way. That is, an estimate by player i about the expected payoff of action a_{ij} will on average equal the expected payoff of action a_{ij} calculated from the equilibrium probability distribution of other player's action choices, given that they are adopting estimated best responses. Thus players expectations are correct, on average.

This model is a natural extension of well-developed and commonly used statistical models of choice or quantal response that have a long tradition in statistical applications to biology, pharmacology, and the social sciences. Accordingly, we call an equilibrium of our model a quantal response equilibrium. The name is borrowed from the statistical literature on quantal choice/response models in which individual choices or responses are rational, but are based on latent variables (in our case a player's vector of estimated payoffs) that are not observed by the econometrician. The added complication is that the underlying latent variables assumed to govern the discrete responses are endogenous.

A valuable feature of this alternative approach to modeling equilibrium in games is that it provides a convenient statistical structure for estimation using either field data or experimental data. For a particular specification of the error structure, we compute the QRE as a function of the variance of the player estimation errors in several games that have been studied in laboratory experiments. We use these data to obtain maximum likelihood estimates of the error variance. This is possible because, in contrast to the traditional Nash equilibrium approach which makes strong deterministic predictions, this model makes statistical predictions.

We find that the statistical predictions of the QRE model depend in systematic ways on the precision of the players' estimates of the expected payoffs from different actions. Therefore, to the extent that we can find observable independent variables that *a priori* one would expect to be correlated with the precision of these estimates, one can make predictions about the effects of different experimental treatments that systematically vary these independent variables. An obvious candidate that we investigate here is *experience*. As a player gains experience playing a particular game and makes repeated observations about the actual payoffs received from different action choices, he/she can be expected to make more precise estimates of the expected payoffs from different strategies. This is only slightly different from the simple observation that, for an econometrician, standard errors of regression coefficients can be expected to decrease in the number of observations. We refer to this as *learning*.[1]

The rest of the paper consists of four sections. Section 2 lays out the formal structure and establishes existence of QRE in finite games. Section 3 specializes the QRE model to the case of logistic response, where the errors follow a log Weibull distribution. This is called the *Logit Equilibrium*. We establish several properties of the logit equilibrium correspondence and use these properties to define a generically unique selection of Nash equilibrium as the limit point (as the error variance goes to zero) to the unique connected equilibrium manifold defined by the graph of the equilibrium correspondence as a function of the estimation error. Section 4 compares QRE to other equilibrium concepts in traditional game theory and establishes a formal connection between our approach and traditional game theory by demonstrating an equivalence between a QRE of a game and a Bayesian equilibrium of an incomplete information version of the game. Section 5 presents the estimation of the model and the measurement of learning effects, using data from experimental games.

2. Quantal Response Equilibrium

Consider a finite n-person game in normal form: There is a set $N = \{1, \ldots, n\}$ of *players*, and for each player $i \in N$ a *strategy set* $S_i = \{s_{i1}, \ldots, s_{iJ_i}\}$ consisting of J_i pure strategies. For each $i \in N$, there is a *payoff function*, $u_i: S \rightarrow \mathbb{R}$, where $S = \prod_{i \in N} S_i$.

Let Δ_i be the set of probability measures on S_i. Elements of Δ_i are of the form $p_i: S_i \rightarrow \mathbb{R}$ where $\sum_{s_{ij} \in S_i} p_i(s_{ij}) = 1$, and $p_i(s_{ij}) \geq 0$ for all $s_{ij} \in S_i$. We use the notation $p_{ij} = p_i(s_{ij})$. So Δ_i is isomorphic to the J_i dimensional

[1] The term "learning" means different things to different people. El-Gamal has suggested that what we call learning is close to what some economists call "learning-by-doing." However, we do not model the detailed mechanics of learning as is done in some of the literature on repeated games, where learning is modeled as either by fully Bayesian updating or as a myopic but deterministic process such as fictitious play or Cournot dynamics.

simplex $\Delta_i = \{p_i = (p_{i1}, \ldots, p_{iJ_i}): \sum_j p_{ij} = 1, p_{ij} \geq 0\}$. We write $\Delta = \prod_{i \in N} \Delta_i$, and let $J = \sum_{i \in N} J_i$. We denote points in Δ by $p = (p_1, \ldots, p_n)$, where $p_i = (p_{i1}, \ldots, p_{iJ_i}) \in \Delta_i$. We use the abusive notation s_{ij} to denote the strategy $p_i \in \Delta_i$ with $p_{ij} = 1$. We use the shorthand notation $p = (p_i, p_{-i})$. Hence, the notation (s_{ij}, p_{-i}) represents the strategy where i adopts the pure strategy s_{ij}, and all other players adopt their components of p.

The payoff function is extended to have domain Δ by the rule $u_i(p) = \sum_{s \in S} p(s) u_i(s)$, where $p(s) = \prod_{i \in N} p_i(s_i)$. A vector $p = (p_1, \ldots, p_n) \in \Delta$ is a *Nash Equilibrium* if for all $i \in N$ and all $p_i' \in \Delta_i$, $u_i(p_i', p_{-i}) \leq u_i(p)$.

Write $X_i = \mathbb{R}^{J_i}$, to represent the space of possible payoffs for strategies that player i might adopt, and $X = \prod_{i=1}^n X_i$. We define the function $\bar{u}$: $\Delta \rightarrow X$ by

$$\bar{u}(p) = (\bar{u}_1(p), \ldots, \bar{u}_n(p)),$$

where

$$\bar{u}_{ij}(p) = u_i(s_{ij}, p_{-i}).$$

Next, we define quantal response equilibrium as a statistical version of Nash equilibrium where each player's utility for each action is subject to random error.[2] Specifically, for each i and each $j \in \{1, \ldots, J_i\}$, and for any $p \in \Delta$, define

$$\hat{u}_{ij}(p) = \bar{u}_{ij}(p) + \varepsilon_{ij}.$$

Player i's error vector, $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iJ_i})$, is distributed according to a joint distribution with density function $f_i(\varepsilon_i)$. The marginal distribution of f_i exists for each ε_{ij} and $E(\varepsilon_i) = 0$. We call $f = (f_1, \ldots, f_n)$ *admissible* if f_i satisfies the above properties for all i. Our behavioral assumption is that each player selects an action j such that $\hat{u}_{ij} \geq \hat{u}_{ik}$ $\forall k = 1, \ldots, J_i$. Given this decision rule (i chooses action j if $\hat{u}_{ij}$ is maximal[3]), then for any given $\bar{u}$ and f this implies a probability distribution over the observed actions of the players, induced by the probability distribution over the vector of observation errors, ε. Formally, for any $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$ with $\bar{u}_i \in \mathfrak{R}^{J_i}$ for each i, we define the *ij-response set* $R_{ij} \subseteq \mathbb{R}^{J_i}$ by

[2] One interpretation of this is that player i calculates the expected payoff, but makes calculation errors according to some random process. An alternative interpretation is that players calculate expected payoffs correctly but have an additive payoff disturbance associated with each available pure strategy. This latter interpretation is discussed in Section 5.

[3] Standard arguments show that results do not depend on how ties ($\hat{u}_{ij} = \hat{u}_{ik}$) are treated (Harsanyi, 1973).

$$R_{ij}(\bar{u}_i) = \{\varepsilon_i \in \mathfrak{R}^{J_i} \mid \bar{u}_{ij} + \varepsilon_{ij} \geq \bar{u}_{ik} + \varepsilon_{ik}\, \forall k = 1, \ldots, J_i\}.$$

Given p, each set $R_{ij}(\bar{u}_i(p))$ specifies the region of errors that will lead i to choose action j. Finally, let

$$\sigma_{ij}(\bar{u}_i) = \int_{R_{ij}(\bar{u}_i)} f(\varepsilon)\, d\varepsilon$$

equal the probability that player i will select strategy j given $\bar{u}$. We then define for any admissible f and game $\Gamma = (N, S, u)$ a *quantal response equilibrium* as a vector $\pi \in \Delta$ such that $\pi_{ij} = \int_{R_{ij}(\bar{u}_i)} f(\varepsilon)\, d\varepsilon$, where $\bar{u} = \bar{u}\,(\pi)$. Formally,

Definition 1. Let $\Gamma = (N, S, u)$ be a game in normal form, and let f be admissible. A *quantal response equilibrium* (QRE) is any $\pi \in \Delta$ such that for all $i, \in N$, $1 \leq j \leq J_i$,

$$\pi_{ij} = \sigma_{ij}(\bar{u}_i(\pi)).$$

We call $\sigma_i : \mathfrak{R}^{J_i} \rightarrow \Delta^{J_i}$ the *statistical reaction function* (or quantal response function) of player i. Several results about statistical reaction functions can be verified easily:

1. $\sigma \in \Delta$ is nonempty.
2. σ_i is continuous on $\mathfrak{R}^{J_i}$.
3. σ_{ij} is monotonically increasing in $\bar{u}_{ij}$.
4. If, for all i and for all $j, k = 1, \ldots, J_i$, ε_{ij} and ε_{ik} are i.i.d., then for all $\bar{u}$, for all i, and for all $j, k = 1, \ldots, J_i$,

$$\bar{u}_{ij} > \bar{u}_{ik} \Rightarrow \sigma_{ij}(\bar{u}) > \sigma_{ik}(\bar{u}).$$

The first two properties of σ imply Theorem 1.

Theorem 1. *For any Γ and for any admissible f, there exists a QRE.*

Proof. A QRE is a fixed point of $\sigma \circ \bar{u}$. Since the distribution of ε has a density, $\sigma \circ \bar{u}$ is continuous on Δ. By Brouwer's fixed point theorem, $\sigma \circ \bar{u}$ has a fixed point. ■

The third and fourth properties say that "better actions are more likely to be chosen than worse actions." Property three specifically compares the statistical best response function if one of i's expected payoffs, $\bar{u}_{ij}$, has changed and every other component of $\bar{u}_i$ has stayed the same. In this case, the region R_{ij} expands and each other $\bar{u}_{ik}$ weakly decreases. Note that this is not the same as saying that π changed in such a way that $\bar{u}_{ij}$ increased, since the change in π could, in principle, change the value of all components of $\bar{u}_i$. The fourth property states that σ orders the probability of different actions by their expected payoffs.

3. The Logit Equilibrium

In the rest of the paper, we study a particular parametric class of quantal response functions that has a tradition in the study of individual choice behavior (Luce, 1959). For any given $\lambda \geq 0$, the *logistic* quantal response function is defined, for $x_i \in \mathbb{R}^{J_i}$, by

$$\sigma_{ij}(x_i) = \frac{e^{\lambda x_{ij}}}{\sum_{k=1}^{J_i} e^{\lambda x_{ik}}}$$

and corresponds to optimal choice behavior[4] if f_i has an extreme value distribution, with cumulative density function $F_i(\varepsilon_{ij}) = e^{-e^{-\lambda \varepsilon_{ij} - \gamma}}$ and the ε_{ij}'s are independent. Therefore, if each player uses a logistic quantal response function, the corresponding QRE or *Logit Equilibrium* requires, for each i, j,

$$\pi_{ij} = \frac{e^{\lambda x_{ij}}}{\sum_{k=1}^{J_i} e^{\lambda x_{ik}}}$$

where $x_{ij} = \bar{u}_{ij}(\pi)$.

For the logistic response function, we can parameterize the set of possible response functions σ with the parameter λ, which is inversely related to the level of error: $\lambda = 0$ means that actions consist of all error, and $\lambda = \infty$ means that there is no error. We can then consider the set of Logit Equilibria as a function of λ. It is obvious that when $\lambda = 0$, there is a unique equilibrium at the centroid of the simplex. In other words, $\pi_{ik} = 1/J_i$ for all i, k. On the other hand, when $\lambda \to \infty$, the following result shows that the Logit Equilibria approach Nash equilibria of the underlying game.

We define the *Logit Equilibrium correspondence* to be the correspondence $\pi^* : \mathbb{R}_+ \Rightarrow 2^{\Delta}$ given by

$$\pi^*(\lambda) = \left\{ \pi \in \Delta : \pi_{ij} = \frac{e^{\lambda \bar{u}_{ij}(\pi)}}{\sum_{k=1}^{J_i} e^{\lambda \bar{u}_{ik}(\pi)}} \forall i, j \right\}.$$

[4] See, for example, McFadden (1976).

THEOREM 2. *Let σ be the logistic quantal response function. Let $\{\lambda_1, \lambda_2, \ldots\}$ be a sequence such that $\lim_{t\to\infty} \lambda_t = \infty$. Let $\{p_1, p_2, \ldots\}$ be a corresponding sequence wih $p_t \in \pi^*(\lambda_t)$ for all t, such that $\lim_{t\to\infty} p_t = p^*$. Then p^* is a Nash equilibrium.*

Proof. Assume p^* is not a Nash equilibrium. Then there is some player i and some pair of strategies, s_{ij} and s_{ik}, with $p^*(s_{ik}) > 0$, and $u_i(s_{ij}, p^*_{-i}) > u_i(s_{ik}, p^*_{-i})$. Equivalently, $\bar{u}_{ij}(p^*) > \bar{u}_{ik}(p^*)$. Since $\bar{u}$ is a continuous function, it follows that for sufficiently small ε there is a T such that for $t \geq T$, $\bar{u}_{ij}(p^t) > \bar{u}_{ik}(p^t) + \varepsilon$. But as $t \to \infty$, $\sigma_k(\bar{u}_i(p^t))/\sigma_j(\bar{u}(p^t)) \to 0$. Therefore $p^t(s_{ik}) \to 0$. But this contradicts $p^*(s_{ik}) > 0$. ■

The following theorem establishes several properties of the equilibrium correspondence. The proof is in the Appendix.

THEOREM 3. *For almost all games $\Gamma = (N, S, u)$.*

1. *$\pi^*(\lambda)$ is odd for almost all λ.*
2. *π^* is upper hemicontinuous.*[5]
3. *The graph of π^* contains a unique branch which starts at the centroid, for $\lambda = 0$, and converges to a unique Nash equilibrium, as λ goes to infinity.*

The third property is particularly interesting and is similar to properties of the "tracing procedure" of Harsanyi and Selten[6] (1988). The third property implies that we can define a unique selection from the set of Nash equilibrium by "tracing" the graph of the logit equilibrium correspondence beginning at the centroid of the strategy simplex (the unique solution when $\lambda = 0$) and continuing for larger and larger values of λ. We have already seen that all limit points of QREs as $\lambda \to \infty$ are Nash equilibria. Results in differential topology are used in the Appendix to show that for almost all games there is a unique selection as $\lambda \to \infty$. We call this Nash equilibrium the *Limiting Logit Equilibrium* of the game.

4. Relation to Other Equilibrium Notions

One may be tempted to conjecture Theorem 2 can be extended to prove that limit points of Logit Equilibria as λ grows will not only be Nash equilibria, but will also be trembling-hand perfect. But that is not true. Consider the game in Table I.

[5] This is always true, not just generically.

[6] These properties of the tracing procedure are proven rigorously in the work of Schanuel *et al.* (1991).

TABLE I
A GAME WITH A UNIQUE PERFECT EQUILIBRIUM AND A DIFFERENT UNIQUE LIMITING LOGIT EQUILIBRIUM
$A > 0, B > 0$

	L	M	R
U	1,1	0,0	1,1
M	0,0	0,0	0,B
D	1,1	A,0	1,1

This game has a unique perfect equilibrium (D, R), and the Nash equilibria consist of all mixtures between U and D for Player 1 and L and R for Player 2. The limit of Logit Equilibria selects $p = (.5, 0, .5)$, $q = (.5, 0, .5)$ as the unique limit point. Along the limit, for finite λ, $p_D \gg p_M$ and $q_R > q_L \gg q_M$ but as λ becomes large p_2 and q_2 converge to 0. So M is eliminated in the limit.[7]

Note that the Limiting Logit Equilibrium does not depend on the magnitudes of A and B. However, the Logit Equilibria for intermediate values of λ are quite sensitive to A and B. Figures 1 and 2 illustrate the logit Equilibrium graph as a function of λ for the cases of $A = B = 5$ and $A = B = 100$.

One might consider the fact that the limiting QREs are not always perfect equilibria to be a drawback of the QRE definition. Alternatively, it could be viewed as "independence of irrelevant alternative" property of the limiting QRE. For large values of λ, strategies that have sufficiently small probability in the QRE do not affect the play of the rest of the game.

A rational-choice justification of the logistic quantal response function, based on McFadden's (1973) random utility maximization model, leads to a connection with the literature on "purification" of Nash equilibria.[8] For any $x \in \mathbb{R}^m$, let the vector of expected utility payoffs to player i be $x + \varepsilon_i$, where $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{im})$ is a vector of draws from a distribution with commonly known density f. If ε_i is known to i but to no one else and i is maximizing expected payoffs given his/her information, then the ordering assumptions imply that i's statistical best response function to any x will

[7] In the game without strategy M it is obvious that the unique limit point of Logit Equilibria is (0.5, 0.5).

[8] See Harsanyi, 1973 in particular. This has its roots in the work of Dvoretsky *et al.* (1951). See also Radner and Rosenthal (1982).

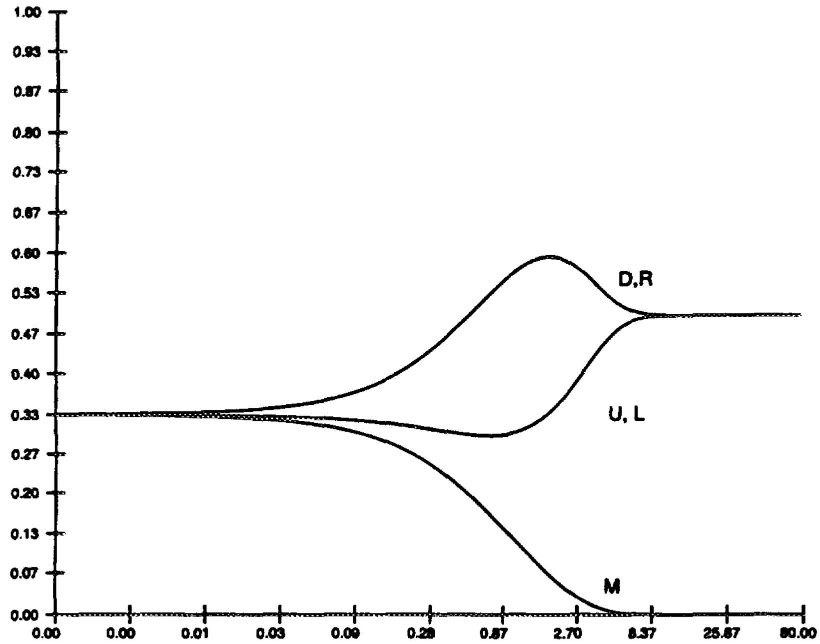

FIG. 1. QRE for game of Table 1, with $A = B = 5$.

be given by σ. The connection to the purification literature is through the Bayesian equilibrium of the game $\langle N, S, u \rangle$ where ε_i is viewed as a random disturbance to i's payoff vector (Harsanyi, 1973). Suppose that for each $s \in S$ each player i has a disturbance of ε_{ij} added to $u_i(s_{ij}, s_{-i})$ and that each ε_{ij} is independently and identically distributed according to f. Alternatively viewed, each player has a randomly determined "predisposition" for each of his/her different available strategies which takes the form of an extra term added to every payoff associated with that strategy. This is illustrated in Table II for a 3×3 game.

This differs only slightly from the Harsanyi (1973) setup which assumes a separate disturbance $\varepsilon_i(s)$ for i's payoff to each strategy profile, s, while we assume that this disturbance for i is the same for payoffs of all strategy profiles in which i uses the same strategy. That is, we assume $\varepsilon_i(s_i, s_{-i}) = \varepsilon_i(s_i, s'_{-i})$ for all i and for all $s_{-i}, s'_{-i} \in S_{-i}$. This violates Harsanyi's condition (1973, p. 5) that requires the existence of a density function for $\varepsilon(s)$. In spite of this, it is easy to see that the main results in Harsanyi (1973) are still true under the weaker assumption that for each i a density function exists for $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iJ_i})$. See Radner and Rosenthal (1982). This weaker assumption is met in our model.

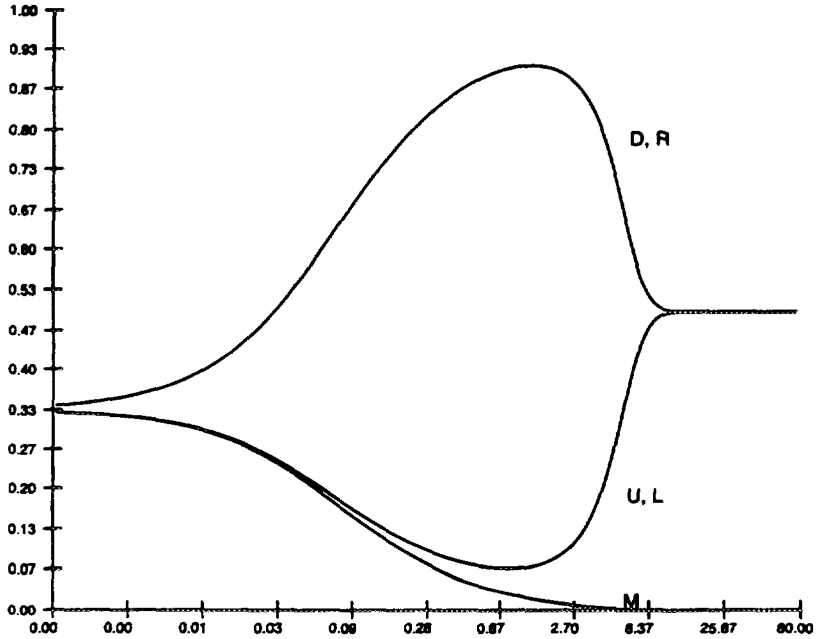

FIG. 2. QRE for game of Table 1, with $A = B = 100$.

TABLE II
ILLUSTRATION OF MODIFIED HARSANYI DISTURBED GAME FOR A 3×3 GAME

	s_{21}	s_{22}	s_{23}
s_{11}	$u^2_{11} + \varepsilon_{21}$ $u^1_{11} + \varepsilon_{11}$	$u^2_{12} + \varepsilon_{22}$ $u^1_{12} + \varepsilon_{11}$	$u^2_{13} + \varepsilon_{23}$ $u^1_{13} + \varepsilon_{11}$
s_{12}	$u^2_{21} + \varepsilon_{21}$ $u^1_{21} + \varepsilon_{12}$	$u^2_{22} + \varepsilon_{22}$ $u^1_{22} + \varepsilon_{12}$	$u^2_{23} + \varepsilon_{23}$ $u^1_{23} + \varepsilon_{12}$
s_{13}	$u^2_{31} + \varepsilon_{21}$ $u^1_{31} + \varepsilon_{13}$	$u^2_{32} + \varepsilon_{22}$ $u^1_{32} + \varepsilon_{13}$	$u^2_{32} + \varepsilon_{23}$ $u^1_{33} + \varepsilon_{13}$

Therefore, our model inherits the properties of Bayesian equilibrium in Harsanyi's disturbed game approach:

1. Best replies are "essentially unique" pure strategies.
2. Every equilibrium is "essentially strong" and in essentially pure strategies.
3. There exists an equilibrium.

The idea of smoothing out the best response correspondence by assuming that players might adopt inferior strategies with declining, but positive, probability is the main feature of the bounded-rationality equilibrium model studied in Rosenthal (1989). Rosenthal considers a linear version of the quantal response equilibrium model and analyzes the equilibrium correspondence as a function of the slope of the (linear) response function. His analysis produces equilibrium graphs of the sort we use in later sections to estimate the response parameter of our logit specification of the response function. He points out that there is also a connection between the statistical response function approach and the "control cost" model explored by Van Damme (1987, Chap. 4). The control cost model assumes that it is costly to implement strategies that deviate from a uniform distribution over available actions. Thus, in the control cost approach good strategies will be played more often than bad strategies, but bad strategies may still be used with positive probability.

The quantal response equilibrium is also related to Beja's (1992) imperfect equilibrium. The approach taken there is that each player has a "target" (mixed) strategy that he/she attempts to play but fails to implement that strategy perfectly. The target strategy maximizes expected payoff, given the probability distribution of strategies induced by the imperfect implementation of target strategies by the other players. This idea of "equilibrium" imperfect implementation of target strategies also appears in Chen (1994), El-Gamal *et al.* (1993), El-Gamal and Palfrey (1994, 1995), Ma and Manove (1993), McKelvey and Palfrey (1992), and Schmidt (1992).[9]

The QRE concept does not use the notion of target strategies but, like Beja (1992), does assume that the probability of implementing a particular strategy is increasing in the expected payoff of the strategy. Furthermore,

[9] There are a number of other papers that use explicit models of the error structure that can be interpreted as imperfect implementation. Logit and probit specifications of the errors are common (Palfrey and Rosenthal, 1991; Palfrey and Prisbrey, 1992; Harless and Camerer, 1992; Stahl and Wilson, 1993; and Anderson, 1993). However, most of these are nonequilibrium models in the sense that a player's choice strategy does not take account of other players' errors (or their own) and therefore is not an optimal response to the probability disribution of other players' actions. An exception is Zauner (1993) who uses a Harsanyi (1973) equilibrium model with independent normal errors to explain data from the centipede game (McKelvey and Palfrey, 1992).

these expected payoffs are calculated from the equilibrium distribution of joint strategies.

5. Data

In this section we explore how well the logistic version of QRE explains some features of data from past experiments on normal form games that are anomalous with respect to standard game theory. We focus on experiments involving two-person games with unique Nash equilibria where there are not outcomes Pareto preferred to the Nash equilibrium. This avoids games where there are supergame equilibria which achieve more than the Nash equilibrium for both players (for example, the prisoner's dilemma game).

The experiments that we analyze were run across a span of more than 30 years. In order to have some comparability across experiments, we express payoffs in terms of the expected monetary payoff in real (1982) dollars.

For each experiment, we calculate a maximum likelihood estimate of λ in the logistic version of the QRE and see how well the model fits the data.

Lieberman (1960)

Lieberman [1960] conducted experiments on the following two person zero sum game:

	B_1	B_2	B_3
A_1	15	0	−2
A_2	0	−15	−1
A_3	1	2	0

The payoffs represent payments, in 1960 pennies, from Player 2 to Player 1. For our estimates, the payoff matrix of the above game is multiplied by 3.373 to express the 1960 payoffs in 1982 pennies.

This game can be solved by iterated elimination of strictly dominated strategies. It has a unique Nash equilibrium at (A_3, B_3). In this experiment, Lieberman reports the choice frequencies as a function of time. Each subject participated in 200 plays of the game, with a single opponent.

The data, broken down into 20 experience levels of 10 periods each, as well as the QRE estimates for each experience level and the negative log

TABLE III
DATA AND ESTIMATES FOR LIEBERMAN (1960) EXPERIMENTS: $N = 300$ FOR EACH EXPERIENCE LEVEL

Periods	Actual Data				Predicted		λ	$-\mathfrak{L}^*$
	A_1	A_3	B_2	B_3	$\hat{A}_1$ $\hat{B}_2$	$\hat{A}_3$ $\hat{B}_3$		
1–10	0.260	0.720	0.300	0.667	0.277	0.696	0.176	212.0
11–20	0.167	0.806	0.227	0.760	0.196	0.781	0.252	177.0
21–30	0.113	0.880	0.160	0.833	0.138	0.838	0.329	134.3
31–40	0.093	0.887	0.120	0.853	0.106	0.869	0.390	134.4
41–50	0.060	0.907	0.073	0.907	0.066	0.906	0.500	109.5
51–60	0.060	0.873	0.120	0.860	0.087	0.886	0.435	144.7
61–70	0.060	0.853	0.113	0.867	0.083	0.890	0.448	152.7
71–80	0.060	0.907	0.047	0.933	0.054	0.916	0.547	98.9
81–90	0.047	0.893	0.067	0.920	0.056	0.915	0.542	112.3
91–100	0.027	0.920	0.080	0.907	0.053	0.918	0.553	105.6
101–120	0.053	0.907	0.047	0.933	0.051	0.920	0.564	99.5
111–120	0.027	0.920	0.047	0.933	0.037	0.932	0.635	94.2
121–130	0.040	0.927	0.040	0.920	0.040	0.929	0.616	97.1
131–140	0.033	0.927	0.047	0.953	0.040	0.929	0.616	80.2
141–150	0.053	0.913	0.060	0.900	0.056	0.915	0.542	112.3
151–160	0.053	0.900	0.053	0.920	0.052	0.919	0.558	109.3
161–170	0.027	0.946	0.060	0.927	0.045	0.925	0.592	83.4
171–180	0.053	0.900	0.033	0.927	0.042	0.927	0.604	107.1
181–190	0.027	0.933	0.020	0.973	0.023	0.946	0.737	67.0
191–200	0.040	0.920	0.047	0.933	0.044	0.926	0.598	93.7

likelihood ($-\mathfrak{L}^*$), are reported in Table III. The data and estimates from each period are also superimposed on the QRE graph in Fig. 3. The notable feature of the data is that during early rounds the row player overplays strategy A_1, and the column player overplays strategy B_2 relative to the Nash equilibrium prediction. Figure 3 shows the QRE for the Lieberman experiment as a function of λ. We see that the QRE has the feature that, for small values of λ, A_1 and B_2 are overplayed. The frequency of these strategies decreases as λ gets larger and Nash equilibrium is approached. If one hypothesizes that the amount of error individuals make decreases as they gain more experience with the game, then one would expect the time series to correspond to QRE solutions with gradually increasing λ. This is similar to what occurs in the Lieberman data and we also see that the maximum likelihood estimates for λ in Table III generally increase with the period number. The Nash model is easily rejected since it predicts A_1, A_2, B_1, and B_2 will never be used. The random model (i.e., constraining $\lambda = 0$) has $-\mathfrak{L}^* = 329$ and is rejected in every time period.

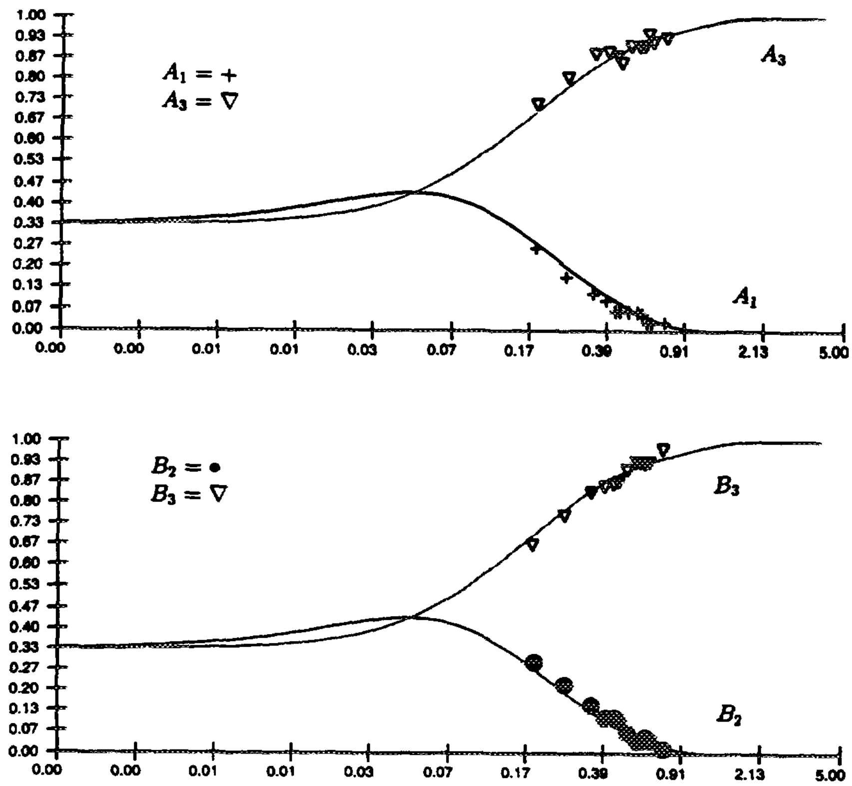

FIG. 3. QRE as a function of λ for Lieberman experiment.

O'Neill (1987)

O'Neill conducted experiments on the following two person zero sum normal form game:

	B_1	B_2	B_3	B_4
A_1	5	−5	−5	−5
A_2	−5	−5	5	5
A_3	−5	5	−5	5
A_4	−5	5	5	−5

The entries represent the payoff, in pennies, from Player 2 to Player 1. Each subject participated in 105 plays of the game. Details of the procedures

TABLE IV
DATA AND ESTIMATES FOR O'NEILL

	Number	Frequency	Rand	NE	QRE
A_1	949	0.362	0.250	0.400	0.360
A_2	579	0.221	0.250	0.200	0.213
A_3	565	0.215	0.250	0.200	0.213
A_4	532	0.203	0.250	0.200	0.213
B_1	1119	0.426	0.250	0.400	0.426
B_2	592	0.226	0.250	0.200	0.191
B_2	470	0.179	0.250	0.200	0.191
B_4	444	0.169	0.250	0.200	0.191
λ			0	∞	1.313
$-\mathcal{L}^*$			7278	7016	7004

can be found in O'Neill (1987). In our estimates, the payoffs are multiplied by 0.913 to express them in 1982 pennies.

The game has a unique Nash equilibrium at (.4, .2, .2, .2) for both players and has the feature that the equilibrium is invariant to the choice of utility function, since the payoffs of the game take on only two values. Table IV gives the aggregate data for the O'Neill experiments.

O'Neill interpreted the data as providing support for the minimax hypothesis[10] and did not view as important the finding that Player 1 underplayed strategy (A_1) while Player 2 overplayed strategy (B_1). He claims that "Players' average selecting frequencies for the moves . . . were almost exactly as predicted." However, in the quantal response equilibrium we predict systematic differences ($A_1 < B_1$). The other discrepancy from the theoretical prediction involves the overplay of strategy B_2 relative to B_3 and B_4. O'Neill attributes this to a flaw in the experimental design,[11] which seems quite plausible. Given the symmetry of payoffs with respect to the last three strategies, it is hard to image any other explanation. Figure 4 displays a plot of the QRE predictions of the strategy frequencies of A_1, A_3, B_1, and B_3 for various values of λ. As can be seen from this figure, the QRE predicts $\{A_1$ underplayed, B_1 overplayed$\}$ for intermediate values of λ and always predicts $A_1 > B_1$. Table IV gives the maximum likelihood estimates ($\mathcal{L}^*$) of λ for the QRE. One can easily reject both the random (Rand) and Nash (NE) predictions in favor of the QRE at the 0.01 level using a likelihood ratio test.

[10] Brown and Rosenthal (1990) reexamined these data and found a number of discrepancies with the theory.

[11] Subjects chose strategies by selecting cards. Strategy 1 was a joker, and the others were Ace, Deuce, and Trey, respectively. A conjecture is that Strategy 2 was over played because of an Ace effect.

In addition to estimating the logit QRE model using the aggregate data, we also have broken the observations down into seven different experience levels. Each subject played 105 games, so each experience level corresponds to a 15-game sequence (games 1–15, 16–30, . . .). The results are displayed in Table V, and the estimated move frequencies are superimposed on the QRE graph in Fig. 4. There is no discernible trend in the estimated values of λ, in contrast to what we found in the Lieberman experiment.

Rapoport and Boebel (1992)

Rapoport and Boebel conducted experiments on a variation of O'Neill's game. The game was also two person, zero sum, and had the following payoff matrix.

	B_1	B_2	B_3	B_4	B_5
A_1	W	L	L	L	L
A_2	L	L	W	W	W
A_3	L	W	L	L	W
A_4	L	W	L	W	L
A_5	L	W	W	L	L

Rapoport and Boebel (RB) ran two versions of the game: in one W was worth (to the row player) \$10 and L was worth −\$6, while in the other W was worth \$15 and L was worth −\$1. These versions are both equivalent from the point of view of the quantal response model. The subjects were paid for a randomly chosen 3 out of 120 rounds, leading to expected payoffs (to row) of 25 cents for a win versus −15 cents for a loss in Game 1, and 37.5 cents for a win versus −2.5 cents for a loss in Game 2. In our estimates, we multiply these payoffs by 0.713 to express them in 1982 pennies.

This payoff matrix has a unique Nash equilibrium at (.375, .250, .125, .125, .125) for each player. The aggregate data for the RB experiments are given in Tables VI and VII. They found that Player 1 underplayed the strategy A_1 and overplayed A_2, whle Player 2 underplayed B_2 and overplayed B_3. Figure 5 gives the QRE as a function of λ for the RB experiments, and the maximum likelihood estimates are given in Tables VI and VII. The QRE does a fair job of predicting the behavior of Player 1, even picking up the reversal in frequency between the first two strategies for Experiment 1. It does not do as well with Player 2. The QRE does not explain Player 2's overplay of strategy B_3, although it does predict the underplay of strategy B_2. The random and Nash models are easily rejected in favor of the QRE.

RB compare the performance of alternative models to Nash equilibrium

TABLE V
DATA AND ESTIMATES FOR O'NEILL EXPERIMENTS, BROKEN DOWN BY PERIOD

Periods		A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	λ	$-\Omega^*$ QRE	Nash	Rand
1–15	Actual	0.363	0.208	0.227	0.203	0.445	0.211	0.179	0.165	1.262	995	997	1040
	Predicted	0.358	0.214	0.214	0.214	0.427	0.191	0.191	0.191				
16–30	Actual	0.349	0.187	0.229	0.234	0.421	0.221	0.181	0.176	1.120	1004	1007	1040
	Predicted	0.352	0.216	0.216	0.216	0.429	0.190	0.190	0.190				
31–45	Actual	0.376	0.205	0.216	0.203	0.400	0.213	0.200	0.187	3.313	1005	1005	1040
	Predicted	0.385	0.205	0.205	0.205	0.413	0.196	0.196	0.196				
46–60	Actual	0.331	0.237	0.216	0.216	0.424	0.216	0.187	0.173	0.798	1006	1011	1040
	Predicted	0.332	0.223	0.223	0.223	0.433	0.189	0.189	0.189				
61–75	Actual	0.347	0.227	0.211	0.216	0.432	0.227	0.165	0.176	1.034	1002	1005	1040
	Predicted	0.348	0.217	0.217	0.217	0.430	0.190	0.190	0.190				
76–90	Actual	0.379	0.248	0.208	0.165	0.435	0.219	0.163	0.184	1.823	994	996	1040
	Predicted	0.372	0.209	0.209	0.209	0.420	0.193	0.193	0.193				
91–105	Actual	0.387	0.232	0.200	0.181	0.427	0.272	0.179	0.123	2.482	995	996	1040
	Predicted	0.380	0.207	0.207	0.207	0.416	0.195	0.195	0.195				

Note. The first 15 periods were practice rounds.

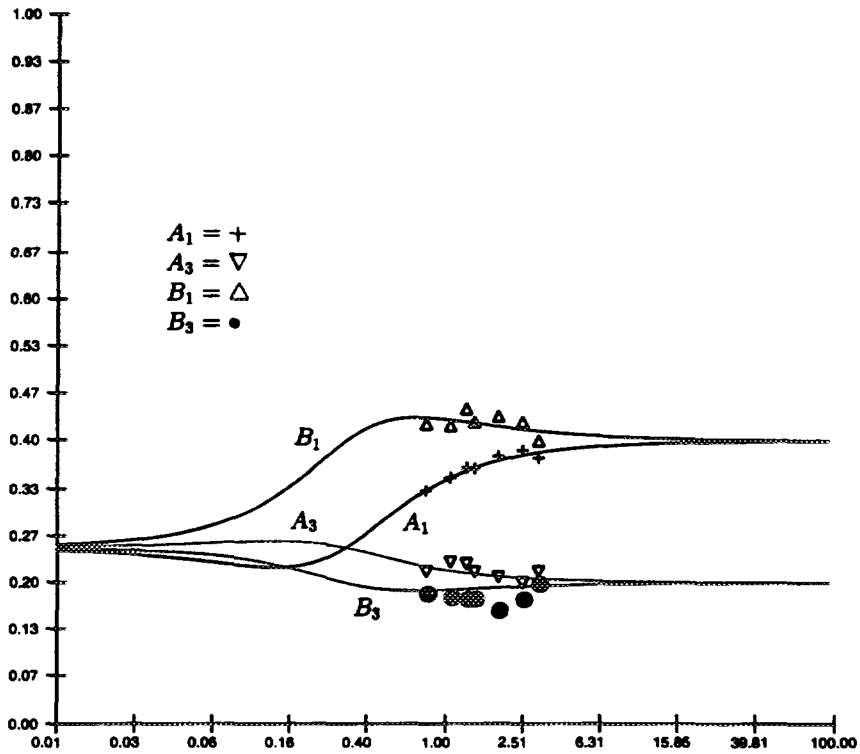

FIG. 4. QRE as a function of λ for O'Neill experiment.

including the totally random (equiprobable) choice model (corresponding to $\lambda = 0$, in our setup) and a win-weighted model, which says that players are more likely to choose strategies with more possible wins than those with less possible wins. This corresponds roughly to a nonequilibrium quantal response model where players believe opponents are choosing randomly. They find that the Nash model outperforms both of these alternative models. In contrast, we find that an *equilibrium* quantal response model significantly outperforms the Nash model in both of their experiments.

Table VIII breaks down the Rapoport–Boebel data to identify experience effects. Each subject participated sequentially in two sessions of one of the games, and each session consisted of 120 plays of the game. Each subject switched roles (row to column or column to row) between session. We break each session down into two experience levels, so that we have a total of four experience levels for each of the two games. The data and estimates are displayed on the QRE graph in Fig. 5.

TABLE VI
DATA AND ESTIMATES FOR RAPOPORT-BOEBEL, EXPERIMENT 1

	n_{ij}	f_{ij}	Rand	NE	QRE
A_1	702	0.293	0.200	0.375	0.286
A_2	732	0.305	0.200	0.250	0.302
A_3	295	0.123	0.200	0.125	0.138
A_4	287	0.120	0.200	0.125	0.138
A_5	384	0.160	0.200	0.125	0.138
B_1	845	0.352	0.200	0.375	0.412
B_2	432	0.180	0.200	0.250	0.169
B_3	523	0.218	0.200	0.125	0.140
B_4	238	0.099	0.200	0.125	0.140
B_5	362	0.151	0.200	0.125	0.140
λ			0	∞	0.2478
$-\mathfrak{L}^*$			7725	7475	7401

In Game 2 the trend from low λ to higher λ is strong and systematic, indicating monotonic convergence to the Nash equilibrium. By the second half of the second session, the data have converged to Nash play to the point where the QRE model is not (statistically) better than the Nash model. We do not find such a trend in the first game. Specifically, the estimated value of λ is quite low throughout the game. This difference between play in the two games suggests that the statistical evidence for

TABLE VII
DATA AND ESTIMATES FOR RAPOPORT-BOEBEL, EXPERIMENT 2

	n_{ij}	f_{ij}	Rand	NE	QRE
A_1	736	0.307	0.200	0.375	0.309
A_2	778	0.324	0.200	0.250	0.296
A_3	239	0.100	0.200	0.125	0.132
A_4	275	0.115	0.200	0.125	0.132
A_5	372	0.155	0.200	0.125	0.132
B_1	831	0.346	0.200	0.375	0.410
B_2	463	0.193	0.200	0.250	0.184
B_3	485	0.202	0.200	0.125	0.135
B_4	279	0.166	0.200	0.125	0.135
B_5	342	0.142	0.200	0.125	0.135
λ			0	∞	0.3274
$-\mathfrak{L}^*$			7725	7400	7345

TABLE VIII

DATA AND ESTIMATES FOR RAPOPORT-BOEBEL EXPERIMENTS, BROKEN DOWN BY PERIOD AND SESSIONS

													$-\mathcal{L}^*$		
Game	Sess.	Periods		A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	λ	QRE	Nash	Rand
		1–60	Actual	0.308	0.307	0.113	0.120	0.350	0.218	0.202	0.092	0.439	1836	1843	1931
1	1		Predicted	0.327	0.289	0.128	0.128	0.406	0.199	0.132	0.132				
		61–120	Actual	0.293	0.272	0.162	0.100	0.333	0.177	0.190	0.140	0.211	1878	1896	1931
1	1		Predicted	0.271	0.303	0.142	0.142	0.410	0.160	0.143	0.143				
		1–60	Actual	0.273	0.350	0.103	0.123	0.353	0.133	0.258	0.102	0.184	1840	1881	1931
1	2		Predicted	0.256	0.304	0.147	0.147	0.407	0.154	0.146	0.146				
		61–120	Actual	0.295	0.292	0.113	0.135	0.372	0.192	0.222	0.063	0.293	1841	1855	1931
1	2		Predicted	0.300	0.296	0.134	0.134	0.412	0.178	0.137	0.137				
		1–60	Actual	0.258	0.367	0.105	0.143	0.332	0.115	0.245	0.140	0.149	1850	1906	1931
2	1		Predicted	0.233	0.302	0.155	0.155	0.396	0.148	0.152	0.152				
		61–120	Actual	0.290	0.347	0.118	0.110	0.355	0.198	0.208	0.108	0.308	1827	1844	1931
2	1		Predicted	0.304	0.297	0.133	0.133	0.411	0.181	0.136	0.136				
		1–60	Actual	0.355	0.313	0.082	0.100	0.355	0.215	0.187	0.110	0.644	1803	1808	1931
2	2		Predicted	0.344	0.279	0.126	0.126	0.398	0.215	0.129	0.129				
		61–120	Actual	0.323	0.270	0.093	0.105	0.343	0.243	0.168	0.107	1.124	1842	1843	1931
2	2		Predicted	0.358	0.268	0.125	0.125	0.390	0.230	0.127	0.127				

Note. Data for A_5 and B_5 can be inferred from the remaining data and are omitted. The first 10 periods of each session were practice rounds and are excluded from the analysis.

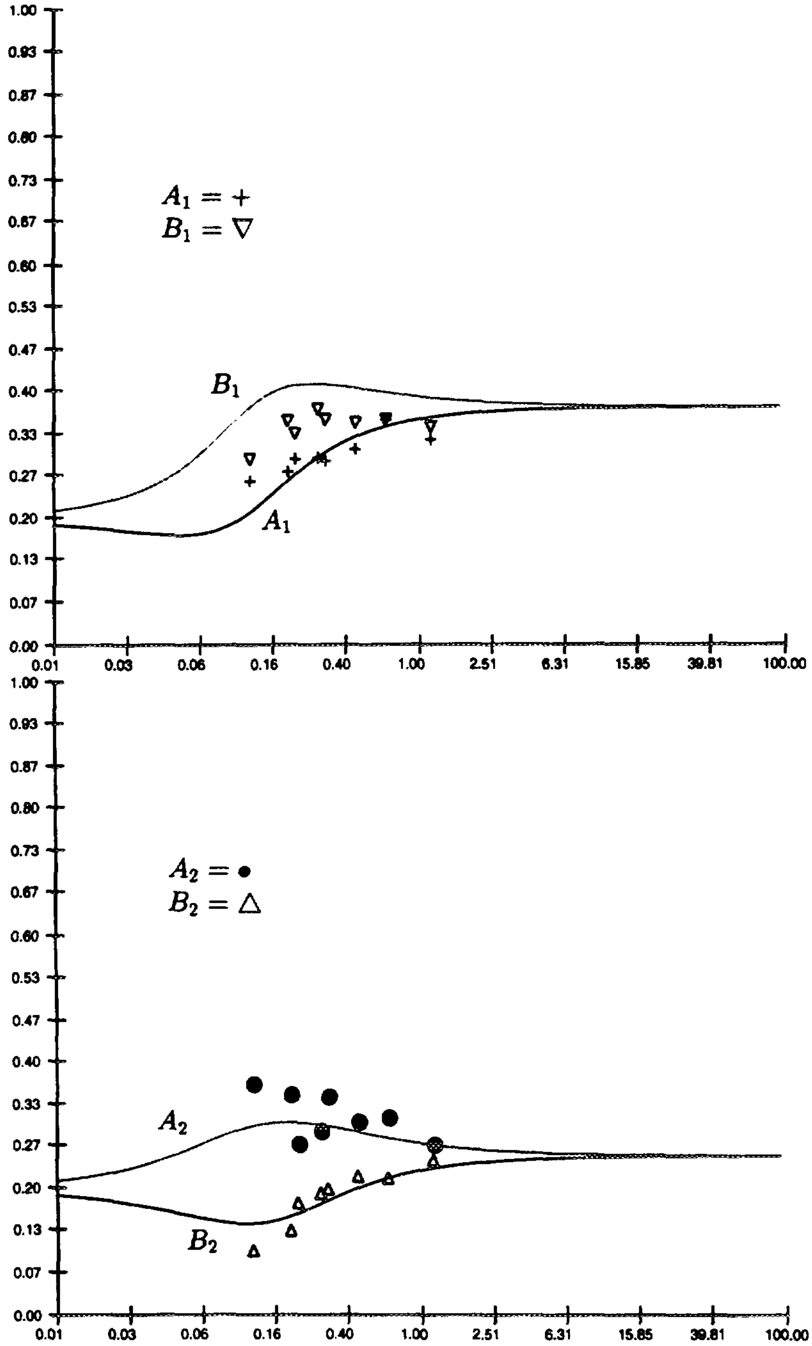

FIG. 5. QRE as a function of λ for the Rapoport–Boebel experiment.

strategic equivalence is somewhat weaker than Rapoport and Boebel's reported finding of "no evidence to reject the hypothesis of strategic equivalence" (1992, p. 279).

Ochs (1995)

Ochs recently conducted experiments on the following three two-person non-zero-sum games:

Game 1

	B_1	B_2
A_1	1, 0	0, 1
A_2	0, 1	1, 0

Game 2

	B_1	B_2
A_1	9, 0	0, 1
A_2	0, 1	1, 0

Game 3

	B_1	B_2
A_1	4, 0	0, 1
A_2	0, 1	1, 0

These experiments are designed so that the only difference between the three tables is the payoff to Player 1 in the upper left cell. In all three tables, there is a unique Nash equilibrium. Since the Nash equilibrium for Player 1 depends only on the payoffs to Player 2, this means that the Nash equilibrium probability that Player 1 chooses A_1 or A_2 is the same (.5, .5) in all three games and that the only differences are in the predicted behavior of Player 2. The Nash equilibrium specifies that Player 1 chooses A_1 with probability .5 in all three games and Player 2 chooses strategy B_1 with probability 0.5 in Game 1, 0.1 in Game 2, and 0.2 in Game 3.

In converting the above payoffs to 1982 pennies, we encounter a difficulty that did not arise in the previous, constant sum experiments. The subjects in the Ochs experiments were paid using a lottery procedure, and the probability of winning the large payoff in the lottery was determined by the total percentage of the maximum possible points that the player accumulated over the course of the experiment. Since the maximum possible number of points for each subject was different, this means that the exchange rate of points to expected payoff was different for each player. Most traditional theories of behavior in games are not affected by a positive scalar multiple of a player's payoffs. The quantal response equilibrium does change if one or both players' payoffs are multiplied by a positive scalar. We express each player's payoffs in terms of expected 1982 money payoff to that player. Since there was a $10 difference between the high and low payoffs in the lottery for each player, and there were a total of 640 games for each player, this means that the maximum payoff for a player is 1.5625 cents. Multiplying by 0.713 to express the payoff in 1982 pennies, this yields the following two games, which represent the Ochs games 2 and 3, in (expected) 1982 pennies for each player.

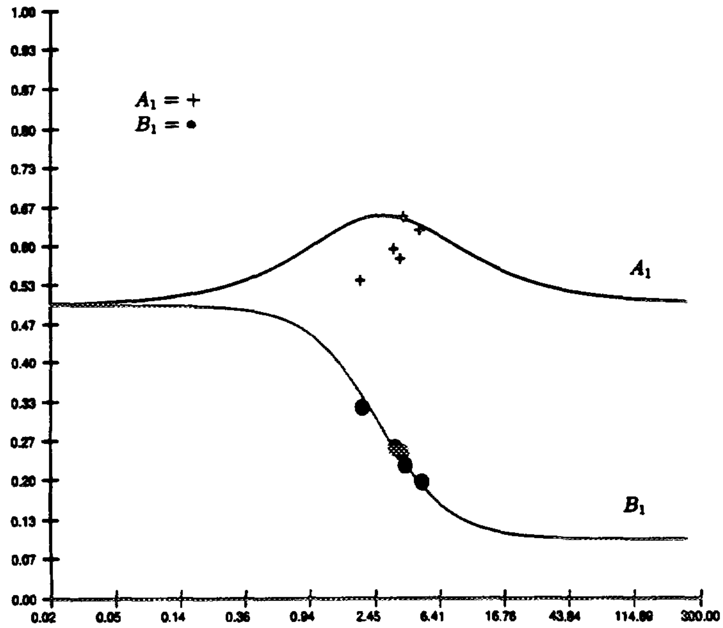

Fig. 6. QRE as a function of λ for Game 2 of the Ochs experiment.

	B_1	B_2
A_1	1.1141, 0.0000	0.0000, 1.1141
A_2	0.0000, 1.1141	0.1238, 0.0000

Game 2

	B_1	B_2
A_1	1.1141, 0.0000	0.0000, 1.1141
A_2	0.0000, 1.1141	0.2785, 0.0000

Game 3

The aggregate data from Game 1 are close to the Nash equilibrium predictions and the Logit model makes the same prediction for all values of λ. We do not analyze that data. Data from the other games are informative. Figures 6 and 7 show the QRE as a function of λ for these two games. As QRE predicts, both A_1 and B_1 are overplayed in early rounds and this overplaying declines over time, suggesting learning is taking place. This is reflected in the overall (increasing) trend in the estimated λ for each game. The Nash model is soundly rejected in both games, using the aggregate data, consistent with Ochs' own conclusions.

Tables IX and X present the data and estimates for Games 2 and 3, respectively, broken down into four experience levels (each level corresponding to 16 plays of the game). These data and estimates are also

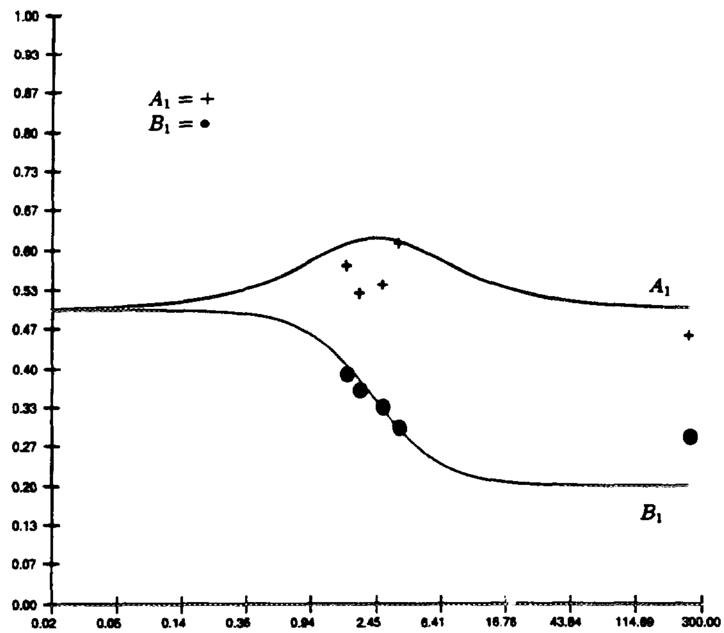

FIG. 7. QRE as a function of λ for Game 3 of the Ochs experiment.

superimposed onto Figs. 6 and 7. In both games, the QRE fitted significantly better[12] than the Nash model in early rounds and significantly better than the random model in later rounds. In Game 2, QRE fits better than both the random and Nash models in both early and later rounds. In the early rounds of Game 3, we cannot reject the random model. By the later rounds of Game 3, we cannot reject the Nash model.

6. CONCLUSION

This paper proposed a general statistical theory of equilibrium in normal form games based on the notion that better strategies are played more often than worse strategies, but best strategies are not always played. The resulting quantal response equilibrium imposes a consistency requirement that the expected payoff to a strategy in equilibrium be calculated based on the equilibrium quantal response probabilities. A parametric version of

[12] Significance is at the 1% level based on a χ^2 test with one degree of freedom.

TABLE IX
DATA AND ESTIMATES FOR OCHS, GAME 2

		Actual		Predicted			$-\mathcal{L}^*$		
Period	n	A_1	B_1	$\hat{A}_1$	$\hat{B}_1$	λ	QRE	Nash	Rand
1–16	128	0.541	0.326	0.645	0.347	1.951	1721	1938	1774
17–32	128	0.649	0.228	0.645	0.228	3.763	1517	1664	1774
33–48	128	0.578	0.250	0.648	0.241	3.475	1605	1725	1774
48–52	64	0.626	0.200	0.636	0.197	4.638	743	792	887
All	448	0.595	0.258	0.649	0.254	3.241	5612	6119	6210

this equilibrium, based on logit response functions, was analyzed and used to fit experimental data from a variety of two-person normal form games.

This model predicts systematic deviations from Nash equilibrium, in spite of the fact that the error structure of the model is unbiased, in the sense that the assumed errors are not the result of a *systematic* deviation from induced preference by the players.[13] Accordingly, this model is able to account for many of the systematic deviations from Nash equilibrium in the experiments which we reanalyze. The qualitative predictions of the direction of these systematic deviations from Nash equilibrium are borne out in most of the data. In addition, the estimated error rates (i.e., inverse of the parameter λ in the logistic quantal response functions) are generally declining with subject experience, consistent with an interpretation that learning is taking place.

TABLE X
DATA AND ESTIMATES FOR OCHS, GAME 3

		Actual		Predicted			$-\mathcal{L}^*$		
Period	n	A_1	B_1	$\hat{A}_1$	$\hat{B}_1$	λ	QRE	Nash	Rand
1–16	128	0.527	0.366	0.615	0.383	1.856	1747	1822	1774
17–32	128	0.573	0.393	0.610	0.405	1.568	1735	1870	1774
33–48	128	0.610	0.302	0.614	0.301	3.306	1640	1708	1774
48–52	128	0.455	0.285	0.500	0.200	∞	1679	1679	1774
All	512	0.542	0.336	0.619	0.331	2.656	6864	7079	7098

[13] This is in contrast to the model we used elsewhere (McKelvey and Palfrey, 1992) to explain departures from equilibrium in the Centipede Game by introducing altruistic preferences.

While the QRE model picks up some of the gross departures of the aggregate behavior from the Nash equilibrium predictions, there are also aspects of the model that remain unexplained. For example, we have expressed payoffs in a common currency (1982 pennies) in an attempt to see if there is any consistency across experiments in the estimated values of λ. Despite this normalization, we find some differences across experiments in the range of λ that is estimated. Also, while we see a tendency for λ to increase with experience, this does not occur in all cases, and in many cases, despite the length of the experiments, λ remains significantly different from the Nash equilibrium even in the later periods of the experiment.

These discrepancies suggest several research directions in which to proceed from here. In light of the above observations, one direction is to attempt to endogenize the learning. The experimental evidence suggests λ may grow over time. It may be possible to infer this from optimizing (but still error prone) behavior by individuals. Also, if one tried to incorporate into the equilibrium the decision to choose λ as part of a labor–leisure tradeoff (Smith and Walker, 1993) that might explain why λ does not approach infinity and might generate testable predictions about how the "equilibrium" values of λ would be related to the magnitude of the payoffs.

A second direction is to incorporate heterogeneity across players. There is some convincing evidence emerging that models which impose homogeneity of subject behavior are inadequate.[14] Conceptually, it is not a difficult extension of the QRE to allow for different error rates across individuals (in fact the formal definition permits this), but this complicates the estimation.

A third direction is to extend this approach to extensive form games.In McKelvey and Palfrey (1994) we extend the basic model presented here to games in extensive form and apply the analysis to several experimental multistage games, including signalling games. A specific application of the extensive form quantal response model can also be found in Fey *et al.* (in press).

APPENDIX

This appendix proves the properties of the quantal response correspondence stated in Theorem 3.

Let $N = \{1, \ldots, n\}$, and for each $i \in N$ define $m_i = J_i - 1$ and $M_i = \{1, \ldots, m_i\}$. Write $N_{-i} = N - \{i\}$ and $m = \sum_{i \in N} m_i$. Let $S = \prod_{i \in N} S_i$ and $u_i : S \to \mathbb{R}$ be the payoff function for player $i \in N$. As before, Δ_i is the set of mixed strategies on S_i. A mixed strategy is a function $p_i : S_i \to [0, 1]$,

[14] See El-Gamal and Grether (1993), Holt (1993), Stahl and Wilson (1993), and McKelvey and Palfrey (1992).

satisfying $p_i(s_{ik}) \geq 0$, for all $i \in N$ and $s_{ik} \in S_i$, and $\sum_{s_{ik} \in S_i} p_i(s_{ik}) = 1$ for all $i \in N$. Write $p_{ik} = p_i(s_{ik})$, for $s_{ik} \in S_i$, and $p_{i0} = p_{iJ_i}$. Define

$$D_i = \left\{ p_i \in \mathbb{R}^{m_i} : p_{ik} \geq 0 \text{ for all } k \in M_i, \text{ and } \sum_{k \in M_i} p_{ik} \leq 1 \right\},$$

and write D_i^0 for the interior of D_i. Using the identity $p_{iJ_i} = p_{i0} = 1 - \sum_{k \in M_i} p_{ik}$, a mixed strategy $p_i \in \Delta_i$ can be identified by the first m_i components, i.e., by a vector in D_i. Write $D = \prod_{i \in N} D_i$ and D^0 for the interior of D. A vector $p = (p_1, \ldots, p_n) \in D$ is referred to as a *mixed profile.*

Define $S_{-i} = \prod_{j \in N_{-i}} S_j$. For any $s_{-i} \in S_{-i}$, and $s_{ik}, s_{il} \in S_i$, define $u_{ikl}(s_{-i}) = u_i(s_{ik}, s_{-i}) - u_i(s_{il}, s_{-i})$. Write $u_{ik0}(s_{-i}) = u_{ikJ_i}(s_{-i})$. Define $X = D^0 \times (0, \infty)$, and $f: X \to \mathbb{R}^m$ with components given by, for any $i \in N$ and $k \in K_i$,

$$f_{ik}(p, \lambda) = \frac{1}{\lambda} \log \left\{ \frac{p_{ik}}{p_{i0}} \right\} + u_{ik0}(p),$$

where

$$u_{ikl}(p) = u_{ikl}(p_{-i}) = \sum_{s_{-i} \in S_{-i}} u_{ikl}(s_{-i}) p_{-i}(s_{-i})$$

$$p_{-i}(s_{-i}) = \prod_{j \in N_{-i}} p_j(s_j).$$

Note that $\pi^*(\lambda) = \{p : f(p, \lambda) = 0\}$. The logistic QRE graph, $\mathfrak{Q} = \{(\pi^*(\lambda), \lambda) : 0 < \lambda < \infty\}$ is given by $\mathfrak{Q} = f^{-1}(\mathbf{0})$, where $\mathbf{0}$ is the m-dimensional vector of zeros. Since f is a continuous function, it follows that $f^{-1}(\mathbf{0})$ is a closed set. This establishes property 2 of Theorem 3, that $\mathfrak{Q}$ is upper hemicontinuous.

The domain X, of f, is a manifold of dimension $m + 1$. It follows from the pre-image theorem (see e.g. Guillemin and Pollack, 1974, p. 21) that if $\mathbf{0}$ is a regular value of $f: X \to \mathbb{R}^m$ that $f^{-1}(\mathbf{0})$ is a one-dimensional manifold. Writing $\boldsymbol{\varepsilon} \in \mathbb{R}^m$, with components ε_{ik}, it follows by Sard's theorem that almost all values of $\boldsymbol{\varepsilon}$ are regular values of f (see Guillemin and Pollack, 1974, p. 39). Hence $f^{-1}(\boldsymbol{\varepsilon})$ is a one-dimensional manifold for almost all $\boldsymbol{\varepsilon} \in \mathbb{R}^m$. But

$$f_{ik}(p, \lambda) = \varepsilon_{ik} \Leftrightarrow \frac{1}{\lambda} \log \left(\frac{p_{ik}}{p_{i0}} \right) + [u_{ik0}(p) - \varepsilon_{ik}] = 0$$

$$\Leftrightarrow \frac{1}{\lambda} \log \left(\frac{p_{ik}}{p_{i0}} \right) + \tilde{u}_{ik0}(p) = 0$$

where $\tilde{u}_{ik0}$ is defined by $\tilde{u}_{ik0}(s_{-i}) - \varepsilon_{ik}$ for all $s_{-i} \in S_{-i}$. It follows that for all u, and almost all perturbations, $\tilde{u}$ of u the QRE graph, $\tilde{\mathfrak{Q}}$ of $\tilde{u}$ is a one-dimensional manifold. Hence, for almost all games, $\mathfrak{Q}$ is a one-dimensional manifold.

Note that the above argument can be extended to the case when the domain of f is bounded: For any $c = (\underline{c}, \bar{c})$ with $0 < \underline{c} < \bar{c}$, define $X_c \subseteq X$ by $X_c = D^0 \times [\underline{c}, \bar{c}]$. Then X_c is a $(m + 1)$-dimensional manifold with boundary. It follows from the pre-image theorem for manifolds with boundary (see e.g. Guillemin and Pollack, 1974, p. 60) that if $\mathbf{0}$ is a regular value of both $f: X_c \to \mathbb{R}^m$ and $\partial f: \partial X_c \to \mathbb{R}^m$, then $\mathfrak{Q}_c = f^{-1}(\mathbf{0}) \cap X_c$ is a one-dimensional manifold with boundary. Now by Sard's theorem, it follows that for almost all values of $\boldsymbol{\varepsilon}$ that ε is a regular value of both f and ∂f (see Guillemin and Pollack, p. 62). Hence $f^{-1}(\boldsymbol{\varepsilon})$ is a one-dimensional manifold with boundary for almost all $\boldsymbol{\varepsilon} \in \mathbb{R}^m$. But then by the same argument as above, it follows that for almost all games $\mathfrak{Q}_c$ is a one-dimensional manifold with boundary.

Now pick $M > 0$ so that for all $p \in \Delta$, $\sup_{ikl}|u_{ikl}(p)| \leq M$. Define $a_\lambda = e^{-\lambda M}$ and $b_\lambda = e^{\lambda M}$. Then it follows that for any $(p, \lambda) \in \mathfrak{Q}$, that

$$-\lambda \cdot M \leq \log\left(\frac{p_{ik}}{p_{i0}}\right) \leq \lambda \cdot M \Rightarrow a\lambda p_{i0} \leq p_{ik} \leq b\lambda p_{i0}.$$

But

$$p_{ik} \leq b\lambda p_{i0} \Rightarrow 1 - p_{i0} = \sum_{k \in M_i} p_{ik} \leq b\lambda m_i p_{i0}$$

$$\Rightarrow p_{i0} \geq 1/(b_\lambda m_i + 1)$$

and

$$p_{ik} \geq a_\lambda p_{i0} \Rightarrow p_{ik} \geq a_\lambda/(b_\lambda m_i + 1) = c_\lambda.$$

Since $a_\lambda < 1$, it follows that for all $0 \leq k \leq m_i$ (i.e., including $k = 0$) the above inequality holds. Define

$$W = \{(p, \lambda) \in X : p_{ik} \geq c_\lambda \text{ for all } i \in N, 0 \leq k \leq m_i\}.$$

Thus, we have shown that $\mathfrak{Q} \subseteq W \cap X$. Similarly, $\mathfrak{Q}_c \subseteq W \cap X_c$. In otherwords, the QRE graph can only "exit" X at the minimum and maximum values of λ.

We wish to show that in generic games, the QRE graph can be used to make a unique selection of a Nash equilibrium. To do this, we must establish two facts. First, we establish that for sufficiently small λ, there is a unique solution. Then we show that this branch of the correspondence converges to a unique Nash equilibrum as λ goes to ∞.

LEMMA 1. *For sufficiently small* λ, $\pi^*(\lambda)$ *is a singleton.*

Proof. To see this, define the mapping $\phi: \mathbb{R}^m \to \mathbb{R}^m$ to have components

$$\phi_{ik}(p) = \exp[-\lambda u_i(k, p_{-i})] / \left\{ \sum_{l \in S_i} \exp[-\lambda u_i(l, p_{-i})] \right\} \qquad (*)$$

Then, for any λ, $(p, \lambda) \in \mathcal{Q}$ if and only if p is a fixed point ϕ. We will show that, for λ sufficiently small, ϕ is a contraction mapping. We use three facts to prove the result, each of which follows from easy arguments.

Fact 1. For any p, $q \in \Delta^0$ (the interior of Δ), $\max_l |p_l - q_l| \leq \max_{kl} |p_l/p_k - q_l/q_k|$.

Fact 2. Since the derivative of e^x at $x = 0$ is 1, then for all $D > 1$, there is a δ such that whenever $|x_1|, |x_2| < \delta$, $|\exp[x_1] - \exp[x_2]| \leq D \cdot |x_1 - x_2|$.

Fact 3. There is an $M > 0$ such that $\max_{ikl} |u_{ikl}(p) - u_{ikl}(q)| \leq M \cdot \max_{ik} |p_{ik} - q_{ik}|$.

Pick any $D > 1$ and let δ be defined as in Fact 2 and M be defined as in Fact 3. We pick $\underline{\lambda}$ to satisfy

1. $\underline{\lambda}\, u_{ikl}(p) < \delta$ for all i, k, l, and any p.
2. $\underline{\lambda} < 1/(D \cdot M)$.

Write $\rho = \underline{\lambda} \cdot D \cdot M$. Then pick any $p, q, \in \Delta$. Then, letting $\|\cdot\|$ represent the sup norm, we let

$$\begin{aligned}
\|\phi(p) - \phi(q)\| &= \max_{il} |\phi_{il}(p) - \phi_{il}(q)| \\
&\leq \max_{ikl} |\phi_{il}(p)/\phi_{ik}(p) - \phi_{il}(q)/\phi_{ik}(q)| \\
&= \max_{ikl} |\exp[\lambda u_{ikl}(p)] - \exp[\lambda u_{ikl}(q)]| \\
&\leq \lambda \cdot D \cdot \max_{ikl} |u_{ikl}(p) - u_{ikl}(q)| \\
&\leq \underline{\lambda} \cdot D \cdot M \cdot \max_{ik} |p_{ik} - q_{ik}| = \rho \cdot \|p - q\|,
\end{aligned}$$

where $\rho < 1$. The steps follow, respectively, by the definition of $\|\cdot\|$, Fact 1, Equation (*), Fact 2, Fact 3, and the definition of $\|\cdot\|$. It follows that, for $\lambda \leq \underline{\lambda}$, ϕ is a contraction mapping. Hence it has a unique fixed point. ■

We now have shown enough to prove Property 1 of Theorem 3, that for almost all λ there are an odd number of logistic QREs. From the above argument, seting $c = (\underline{c}, \bar{c})$, we have shown that $\mathcal{Q}_c$ is a compact, one-dimensional manifold with boundary, which for small enough $\underline{c}$, has a unique intersection with $\lambda = \underline{c}$. Any such manifold has a finite number of connected, compact components, each of which must have an even number of boundary points. We have also shown that any boundary point must be at $\lambda = \underline{c}$ or $\lambda = \bar{c}$. Since there is exactly one solution at $\lambda = \underline{c}$, there must be an odd number of solutions at $\lambda = \bar{c}$.

We now show the first assertion of Property 3 of the theorem, that as $\lambda \to \infty$, the branch $\mathcal{B}$ of the manifold that passes through $\underline{\lambda}$ converges to a unique Nash equilibrium.

LEMMA 2. *Let $\underline{\lambda}$ be chosen so that $\pi^*(\underline{\lambda})$ is a singleton. Then for almost all games u, as $\lambda \to \infty$, the branch $\mathcal{B}$ of the manifold that passes through $\underline{\lambda}$ converges to a unique Nash equilibrium.*

Proof. It follows from the arguments above that for almost all games there exists an increasing sequence $\{\lambda_i\}$ with $\underline{\lambda} < \lambda_i$ for all i, such that if we define $c_i = (\underline{\lambda}, \lambda_i)$, $X_i = X_{c_i}$, and $\mathcal{Q}_i = \mathcal{Q}_{c_i} \subseteq \mathcal{Q}$,

1. $\mathcal{Q}$ is a one-dimensional manifold with a unique point, say $(\underline{p}, \underline{\lambda})$, for which $\lambda = \underline{\lambda}$ and a unique connected branch $\mathcal{B}$ that passes through $(\underline{p}, \underline{\lambda})$.
2. $\mathcal{Q}_i$ is a compact one-dimensional manifold with boundary, which has a finite number of connected components.
3. Letting $\mathcal{B}_i$ be the connected branch of $\mathcal{Q}_i$ which begins at $(\underline{p}, \underline{\lambda})$, it follows that $\mathcal{B}_i$ is a compact connected one-dimensional manifold with boundary for all i, which has a unique intersection, say (p_i, λ_i), with $\lambda = \lambda_i$.

Now for any i, define $\mathcal{A}_i = \{(p, \lambda) \in \mathcal{B} : \lambda > \lambda_i\}$ and A_i to be the closure of the projection of $\mathcal{A}_i$ onto D. Then $\{A_i\}$ is a decreasing sequence of sets. We show that for almost all games, $\cap_i A_i$ must be a unique point. First of all, since D is compact and each A_i is closed and nonempty, $\cap_i A_i$ cannot be empty. Suppose, by way of contradiction, that $\cap_i A_i$ contains two distinct points. Since generic games contain a finite number of Nash equilibria, we may assume that the game defined by u has a finite number of Nash equilibria. By Theorem 2, any point in $\cap_i A_i$ must be a Nash equilibrium. But if p^* and q^* are both in $\cap_i A_i$, then we can construct a sequence $\{(p_i, \lambda_i)\} \subseteq \mathcal{B}$ with $p_{2i-1} \to p^*$, $p_{2i} \to p^*$, $\lambda_i \to \infty$, and a homeomorphism, $\phi: \mathbb{R} \to \mathcal{B}$, satisfying $\phi(i) = (p_i^*, \lambda_i)$, and $\phi(2i) = (q_i^*, \eta_i)$. In particular, start with any δ_1 and find $(p_1^*, \lambda_1) \in \mathcal{B}$ with $\lambda_1 > 1/\delta_1$ and $\|p_1 - p^*\| < \delta_1$. Since $\mathcal{B}$ is a connected, it is path connected (Guillemin and Pollack, p. 38, Exercise 3), so one can construct $\phi[0, 1] \to \mathcal{B}$ with $\phi(0) = (\underline{p}, \underline{\lambda})$ and $\phi(1) = (p_1^*, \lambda_1)$. Since $\phi[0, 1]$ is compact, it is bounded. Pick δ_2 so that

$1/\delta_2$ exceeds the bound and find $(p_2^*, \lambda_2) \in \mathcal{B}$ with $\lambda_2 > 1/\delta_2$ and $\|p_2 - q^*\| < \delta_2$. By the same reasoning as above, one can construct $\phi[1, 2] \rightarrow \mathcal{B}$ with $\phi(1) = (p_1^*, \lambda_1)$, and $\phi(2) = (p_2^*, \lambda_2)$. Proceeding in this fashion, one can construct the sequence $\{(p_i, \lambda_i)\} \subseteq \mathcal{B}$ and a homeomorphism, $\phi: \mathbb{R} \rightarrow \mathcal{B}$ with the properties specified. Now for each i, $\phi[i, i + 1]$ is a compact one-dimensional manifold with boundary. Moreover, we can pick ϕ so that $\phi(i - 1, i) \cap \phi(i, i + 1) = \phi(i)$.

We have constructed an infinite sequence of compact manifolds with boundary, each of whose projection on D connects a point near p^* to a point near q^*. Further, for any λ_i, at most a finite number of these manifolds intersect with X_i (since a $\mathcal{B} \cap X_i$ is a compact one-dimensional manifold with boundary, which can consist of at most a finite number of components.) It follows that any separating hyperplane $H_t = \{p^* \in D: p \cdot (p^* - q^*) = t\}$ between p^* and q^* must have a nonempty intersection with $\cap_i A_i$ (by compactness of H_t.) But, since there are an infinity of such separating hyperplanes, this means that $\cap_i A_i$, is infinite and hence there are an infinite number of Nash equilibria, which is a contradiction. ■

We have established that there is a unique branch of $\mathcal{Q}$ that selects a unique Nash equilibrium as λ goes to infinity. This establishes Property 3.

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