Sampling Equilibrium, with an Application to Strategic Voting

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Abstract

We suggest an equilibrium concept for a strategic model with a large number of players in which each player observes the actions of only a small number of the other players. The equilibrium concept fits well situations where each player treats his sample as a prediction of the distribution of actions in the entire population, and responds optimally to this prediction. We apply the concept to a strategic voting model and investigate the conditions under which a centrist candidate can win the popular vote although his strength in the population is smaller than those of the right and left candidates.

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1. Introduction

A strategic situation is characterized by the dependence of each player's optimal action on his expectations of the other players' actions. Nash equilibrium is the standard tool used to analyze such a situation. This concept implicitly assumes that each player gets to know, by an unspecified process, the equilibrium behavior of all other players and responds to it optimally. In this paper, we study strategic situations in which each agent gets only partial information about the other agents' actions prior to making a decision. We have in mind situations with a large number of players where each player's interests depend on his own action and the distribution of actions in the population. A player gets to know the actions of a sample of players, treats the sample as a prediction of the distribution of actions in the entire population, and responds optimally to this prediction. In this respect our model belongs to the literature on bounded rationality and game theory.

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One class of situations that fits this framework is elections. A voter often bases his beliefs when he goes to the poll on information about the intentions of only *some* of the other voters. In our model a voter participates in an election when the information he gathers leads him to believe that his vote will make a difference, and votes strategically when this information leads him to believe that an insincere vote will influence the outcome in the direction of his favorite outcome.

Our main aim is to suggest a simple framework for discussing such situations. We wish to construct a model where players respond to the information they get about the behavior of only some of the other players, without specifying the information acquisition process. We do not want to allow a player to consider the potential effect of his decision on other players' behavior. (In contrast, such considerations are central to herding models.)

The construction of such a model is not obvious because of the difficultly, well known in game theory, of distinguishing between two types of information: "hard" information that a player perceives and "soft" information that he "deduces" from his knowledge of the game and the assumption that the players are rational. A simultaneous game, in which players must rely on "soft" information about their opponents' moves does not fit our goal because we want players to rely only on the "hard" information they gather. An extensive game does not fit our goal because it requires a specification of the exact temporal order in which the players get the hard information, something we want to avoid. Thus, we suggest a new model.

We assume that a player who gets information about the actions of a sample of players considers it to represent the true distribution of actions in the population. This assumption entails "bounded rationality" in the sense that a player does not take into account the possibility that the randomness of his sample makes his inference inaccurate. It conflicts with what seems to be a fact of life that the small sample of other voters to which a voter has access is biased towards people with views similar to his own, but our general framework could be modified to encompass such considerations.

We use an "equilibrium" approach. The sampling process is random, so a player's optimal response is random. Our equilibrium concept requires that the distribution of outcomes of this process be in a steady state, and thus be equal to the distribution of actions in the population.

Our idea is rooted in our earlier paper Osborne and Rubinstein (1998), in which we analyze an interactive situation in which each player samples each of his available actions once, and on the basis of their performance makes a choice, ignoring the fact that the realization of his sample is random. In this paper, a player observes a sample of the other players' actions, constructs a belief about the distribution of actions in the population, and responds optimally. In the earlier paper, he samples each of his actions, constructs a mapping between his actions and outcomes, and responds optimally.

After defining the solution concept, we show that it exists. Then we apply it to a voting model where agents have single peaked preferences over three candidates, a leftist, a centrist, and a rightist, and may vote strategically. This model is very simple relative to others in the literature, in that players are not required to make complicated calculations. Yet it is quite rich and allows us to address issues regarding strategic voting, like the conditions under which a centrist is elected in a polarized society.

2. The Model and Solution Concept

Our model is $\langle I, (q_i)_{i \in I}, A, k, (R_i)_{i \in I} \rangle$, with the following components.

- I is a finite set, the set of types.
- q_i is the proportion type i in the population ($q_i \geq 0$, $\sum_{i \in I} q_i = 1$).
- A is a finite set, the set of actions from which each agent can choose.
- k is a positive integer, the size of the sample that an individual takes before choosing an action.
- R_i is a function that associates an action in A with each *sample result* $(n_a)_{a \in A}$, where each n_a is a nonnegative integer (the number of players in the sample who choose a) and $\sum_{a \in A} n_a = k$. The action $R_i((n_a)_{a \in A})$ is player i's response to the observation that in his sample each action $a \in A$ is chosen by n_a players.

A candidate for our solution concept is a profile of distributions of actions, one for each type. That is, it is a profile $p = (p_i)_{i \in I}$, where $p_i \in \Pi(A)$ for each i and $\Pi(A)$ is the set of distributions over A. Any profile p of distributions induces a random sample result $s(p)$. An equilibrium is a profile p such that for all $i \in I$ and all $a \in A$ the probability that $R_i(s(p)) = a$ is exactly $p_i(a)$. We call such an equilibrium a *sampling equilibrium*. The following result follows trivially from Brouwer's fixed point theorem.

Proposition 1 *A sampling equilibrium exists.*

3. A Model of Voting

We now use our model to study strategic voting. The usefulness of the standard models of voting using the notion of Nash equilibrium is limited because knowledge of the other voters' behavior makes a single individual's vote irrelevant unless the vote totals of the top two candidates are equal or almost equal; typically all outcomes can be supported by some equilibrium.

Our approach assumes that a voter has partial information about the other voters' behavior that he acquires from a small random sample of the population. A potential voter takes the real distribution of votes in the population to be the same as the distribution of votes in his sample, and on this basis decides how to vote. He votes strategically if the sample reveals a tie between two alternatives that do not include his favorite.

We assume that the size k of each voter's sample is 2 or 3. The assumption of a larger size would make the analysis more difficult and would also strain our assumption that an agent votes strategically only if his sample indicates that he is *exactly* pivotal. Note that the fact that the sample taken by a voter is private information makes our analysis very different from the literature that analyzes elections with public polls.

In most of this section we consider a political environment with three possible positions, L, M, and R, ordered naturally along a line. The population contains three types of agents, L, M, and R, where agents of type X, which we refer to as "partisans of X", have single peaked preferences with peak at X. We assume initially that agents of type M are indifferent between L and R. We assume that the winner of the election is the candidate who obtains the largest number of votes.

Many questions may be asked in this environment. Our main aim is to demonstrate the usefulness of the model by studying a single issue: whether the middle candidate wins, due to strategic voting by partisans of the other candidates, even when he has fewer partisans than them.

Three candidates, no abstention, sample of two

First assume that each agent votes for one of the three positions—abstention is not an option. Assume also that each agent samples two others ($k = 2$) and that not all agents are of type M ($q_M < 1$). The response functions are defined as follows: an agent of type M always votes for M; an agent of type L (R) votes ("strategically") for M if one agent in his sample votes for M and one votes for R (L), and otherwise votes for his favorite candidate.

Claim 2 *The model has a unique sampling equilibrium.*

Proof. In an equilibrium, the probability $p_L(M)$ that a voter of type L votes for M equals the probability that a voter obtains a sample containing one agent who votes for R and one who votes for M. Only an agent of type R votes for R, so the sample must contain an agent of type R who votes for R and either (i) an agent of type L who votes for M, (ii) an agent of type M (who votes for M), or (iii) an agent of type R who votes for M. Symmetric considerations apply to $p_R(M)$. Thus for equilibrium we need

$$\begin{aligned} p_L(M) &= 2q_R p_R(R)\left[q_L p_L(M) + q_M + q_R p_R(M)\right] \\ p_R(M) &= 2q_L p_L(L)\left[q_R p_R(M) + q_M + q_L p_L(M)\right]. \end{aligned}$$

Using the fact that $p_R(R) = 1 - p_R(M)$ and $p_L(L) = 1 - p_L(M)$, and denoting $p_L(M) = \beta$ and $p_R(M) = \gamma$, we have

$$\beta = 2(1-\gamma)q_R x \tag{1}$$

$$\gamma = 2(1-\beta)q_L x \tag{2}$$

$$x = \gamma q_R + q_M + \beta q_L. \tag{3}$$

(The variable x is the share of the vote received by M.) Given $q_M < 1$, these equations have no solution in which $x = 1$. Given any $x < 1$, they have at most one solution for β and γ, so we need to show only that they have a unique solution for x. For $q_R = q_L = \frac{1}{2}$ the unique solution is $x = 0$, and for $q_L = q_R = 0$ the unique solution is $x = 1$. Otherwise algebraic calculations lead to $H(x) = x - q_M - 4q_L q_R(x^3 - x^2 + x) = 0$. We have $H(0) < 0$ and $H(1) > 0$ (because $q_L + q_R > 4q_L q_R$ unless q_L and q_R are either both 0 or both $1/2$). We have also $H'(x) = 1 - 4q_L q_R(3x^2 - 2x + 1)$ and $H''(x) = -4q_L q_R(6x - 2)$. Hence H is convex on $[0, \frac{1}{3}]$ and concave on $[\frac{1}{3}, 1]$. Thus if $H(\frac{1}{3}) > 0$ then there is a unique solution less than $\frac{1}{3}$, if $H(\frac{1}{3}) = 0$ then there is a unique solution equal to $\frac{1}{3}$, and otherwise there is a unique solution greater than $\frac{1}{3}$. ■

Claim 3 *In any equilibrium,*

(a) not more than 50% of the partisans of L and R vote strategically

(b) if $q_R > q_L$ then the proportion of partisans of L who vote strategically exceeds the proportion of partisans of R who do so; $q_R > q_L$ if and only if R gets more votes than does L

(c) if $q_R \geq \frac{1}{2}$ then R wins

(d) candidate M wins outright only if $q_M > \frac{1}{7}$, and if $q_L = q_R$ loses if $q_M < \frac{1}{7}$.

Proof. (a) All agents vote, so $1 = x + (1-\beta)q_L + (1-\gamma)q_R$ and thus $2x(1-x) = 2x(1-\beta)q_L + 2x(1-\gamma)q_R = \beta + \gamma$ (using (1) and (2)), which implies $\beta + \gamma \le \frac{1}{2}$.

(b) By (1) and (2) we have $\beta(1-\beta)/\gamma(1-\gamma) = q_R/q_L > 1$. Now, by (a), both β and γ are less than $\frac{1}{2}$, so that $\beta(1-\beta) > \gamma(1-\gamma)$ implies $\beta > \gamma$. Given $q_R > q_L$, we thus have $(1-\gamma)q_R > (1-\beta)q_L$.

(c) Assume that there is an equilibrium in which R does not win. Let $z = (1-\gamma)q_R$, the proportion of the population that votes for R. Then by (b), the proportion of votes for L is at most z, and candidate M wins. The probability with which the partisans of R vote strategically is $1 - z/q_R \ge 1 - 2z$. Now, the sum of the proportions of voters for L and for M is $1 - z$, so the probability that a partisan of R obtains a sample with one voter for M and one for L is at most twice the maximal value of xy where $x + y = 1 - z$ and $y \le z$, or $2z(1-2z)$. But $2z(1-2z) < 1 - 2z$ for $z < \frac{1}{2}$, a contradiction.

(d) By the proof of Claim 2, candidate M wins if and only if $H(\frac{1}{3}) > 0$. Now, $H(\frac{1}{3}) = \frac{1}{3} - q_M - \frac{28}{27}q_L q_R$ and $q_L q_R \le \frac{1}{4}(1-q_M)^2$. Thus candidate M wins only if $\frac{1}{3} - q_M - \frac{7}{27}(1-q_M)^2 > 0$, which implies $q_M > \frac{1}{7}$. If $q_L = q_R$ and $q_M > \frac{1}{7}$ then $H(\frac{1}{3}) > 0$. ■

By part (b) of the result, if $q_R > q_L$ then either R or M wins. Candidates R and M tie if $(1-\gamma)\,q_R = x$, in which case the equilibrium conditions (1), (2), and (3) imply that q_R is equal to

$$\left[2q_L^2 - 3q_L + 2 + (2-3q_L)\,(1 - 2q_L + 2q_L^2)^{1/2}\right] \Big/ \left[14q_L^2 - 16q_L + 8\right].$$

If q_R is larger, R wins, otherwise M wins. The winning candidate, as a function of q_L and q_R, is shown in Figure 1.

We have assumed so far that the partisans of M do not vote strategically. We now assume that they are split into two equal sized types, M_L, whose preference puts M on top and L next, and M_R, whose preference puts M on top and R second. An agent of type M_L who observes one vote for L and one for R votes strategically for L. If $q_L = q_R = q$ we obtain the following conditions for a symmetric equilibrium, where $\beta = p_L(M) = p_R(M)$ and 2δ is the proportion of the partisans of M who vote strategically (half for L and half for R).

$$\begin{aligned} 2\delta &= 2[q(1-\beta) + (1-2q)\delta]^2 \\ \beta &= 2[q(1-\beta) + (1-2q)\delta][2q\beta + (1-2q)(1-2\delta)] \end{aligned}$$

It is easy to see that in an equilibrium the fraction of the population that votes for M, namely $2q\beta + (1-2q)(1-2\delta)$, exceeds $\frac{1}{3}$ if and only if $q < 0.4$,

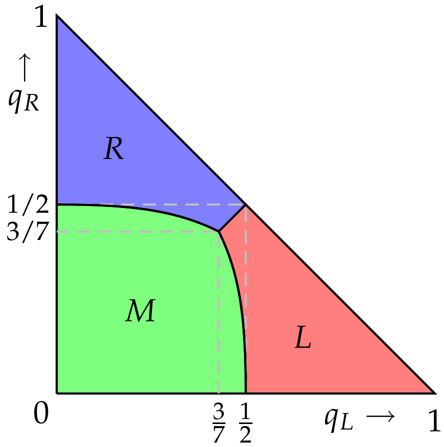

Figure 1. The winning candidate as a function of q_L and q_R, the proportion of partisans of L and R in the population. (The proportion of partisans of M is $1 - q_L - q_R$.)

or if the proportion of partisans of M is at least 20%. Thus, as expected, the possibility that M's partisans vote strategically makes it more difficult for M to win.

Two candidates, with abstention

So far, we have assumed that all agents are obliged to vote. We now assume that each agent may abstain from voting, and votes only when he concludes from his sample that his vote would make a difference, namely, whenever there is a tie in his sample result.

For simplicity, we assume that there are only two candidates, L and R, so that there is no room for strategic voting. We assume that each agent takes a sample from all agents, including those who do not vote. (One could analyze a variant of the model in which each agent's sample contains only agents who vote; this might fit the practice of poll takers to report only the results within the group of "likely voters".)

If the sample size is two ($k = 2$), an agent votes if and only if either one sampled agent votes for A and one for B, or both sampled agents abstain. Thus the fraction of partisans of A who vote is equal to the fraction of partisans of B who vote, and this fraction v satisfies

$$v = 2q_A(1 - q_A)v^2 + (1 - v)^2,$$

which yields a participation rate of

$$v = \frac{3 - \sqrt{5 - 8q_A(1 - q_A)}}{2(1 + 2q_A(1 - q_A))}.$$

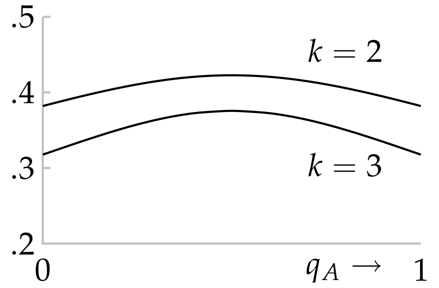

Figure 2. The participation rate as a function of q_A for $k = 2$ and $k = 3$.

This rate is shown in Figure 2 as a function of q_A. Greater polarization of the population (q_A close to $\frac{1}{2}$) increases the participation rate, but only moderately.

If $k = 3$, an agent votes if and only if his sample contains exactly two opposing voters and one who abstains, or three abstaining voters. The equilibrium condition is

$$v = 6q_A(1 - q_A)v^2(1 - v) + (1 - v)^3.$$

The solution, shown in Figure 2, has the same general shape as the solution for $k = 2$, though the participation rate is smaller.

Can a middle candidate win without any partisans?

We now return to the case of three candidates L, M and R. Claim 3(c) shows that without abstention and with a sample size 2, the middle candidate wins only if the proportion of his partisans is at least $\frac{1}{7}$. In other words, strategic voting does not generate a centrist winner when the population is very polarized. We now show that when the society is totally polarized, a centrist does not attract any votes in equilibrium.

Claim 4 *If $k = 2$ and $q_M = 0$ then there is no strategic voting in equilibrium, whether or not agents can abstain.*

Proof. For the case of no abstention, Claim 2 shows that there is a unique equilibrium. It is trivial to verify that $\beta = \gamma = 0$ is a solution of the equations characterizing this equilibrium.

For the case that agents can abstain, denote by x the fraction of votes for M. A partisan of L votes for M if he gets a sample of one vote for M and one vote for R, which occurs with probability $2xq_R p_R(R)$. Similarly, a partisan of R votes for M if he gets a sample of one vote for M and one vote for L. Thus in a sampling equilibrium we need

$$\begin{aligned} x &= q_L(2xq_R p_R(R)) + q_R(2xq_L p_L(L)) \\ &= 2xq_L q_R(p_R(R) + p_L(L)). \end{aligned}$$

Now, $q_L q_R \leq \frac{1}{4}$ and thus $2q_L q_R(p_R(R) + p_L(L)) < 1$ unless $p_R(R) = p_L(L) = 1$. In both cases $x = 0$. ■

4. Concluding Comments

We suggest a framework for analyzing strategic situations in which each player gets partial information about the behavior of the other players and responds to this knowledge systematically. We use the framework to study a simple election model, demonstrating that it can address questions that are hard to discuss using conventional tools. We believe that the framework is useful in many other contexts—but the only way to establish this assertion is to carry out such analyses.

In the election model, a question on which we focus is that of whether a compromise candidate can win in a society where agents are considering voting strategically, even when the proportion of his partisans is low. We are not aware of previous models addressing this issue. The work most closely related is that of Myatt (2002), who studies a strategic voting model in which an incumbent is disliked by the supporters of two minority parties. To defeat the incumbent, some supporters of one of the minority parties must vote strategically for the other one. Voters get noisy information about the outcome of an opinion poll, and behave rationally given this information. The imperfect information leads to the emergence of a unique equilibrium (in the spirit of the theory of global games"), in which some votes are strategic. A model less closely related is studied by McKelvey and Ordeshook (1985), who assume that agents do not possess perfect information about each others' votes, but collect information from sources like public polls, and act rationally given this information.

Many extensions of our voting model are possible. We mention one direction that we find appealing. We assume specific response functions, under which strategic voting is the outcome of a tie in the sample. An interesting alternative is to assume that an agent votes for his preferred candidate unless this candidate is the least popular according to his sample. If $k = 2$

and there is no abstention then such a response function might yield an equilibrium very different from the one we have found. Even in a totally polarized society (in which the population is evenly split in supporting the left and the right) and even for a large sample, in this case an equilibrium exists where $\frac{1}{3}$ of the agents vote for the center.

References

McKelvey, Richard and Peter C. Ordeshook "Elections with limited information: a fulfilled expectations model using contemporaneous poll and endorsement data as information sources," *Journal of Economic Theory* 36 (1985):55–85.

Myatt, David P. "A new theory of strategic voting [revised]", unpublished paper, Oxford University, 2002.

Osborne, Martin J. and Ariel Rubinstein, "Game with procedurally rational players", *American Economic Review* 88 (1998), 834–847.