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TOWARD A THEORY OF DISCOUNTED REPEATED GAMES WITH IMPERFECT MONITORING

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This paper investigates pure strategy sequential equilibria of repeated games with imperfect monitoring. The approach emphasizes the equilibrium value set and the static optimization problems embedded in extremal equilibria. A succession of propositions, central among which is "self-generation," allow properties of constrained efficient supergame equilibria to be deduced from the solutions of the static problems. We show that the latter include solutions having a "bang-bang" property; this affords a significant simplification of the equilibria that need be considered. These results apply to a broad class of asymmetric games, thereby generalizing our earlier work on optimal cartel equilibria. The bang-bang theorem is strengthened to a necessity result: under certain conditions, efficient sequential equilibria have the property that after every history, the value to players of the remainder of the equilibrium *must* be an extreme point of the equilibrium value set. General implications of the self-generation and bang-bang propositions include a proof of the monotonicity of the equilibrium average value set in the discount factor, and an iterative procedure for computing the value set.

KEYWORDS: Asymmetric repeated games, extremal equilibria, self-generation, bang-bang reward functions, algorithm.

1. INTRODUCTION

A RECENT PAPER of ours (Abreu, Pearce, and Stacchetti (1986), hereafter APS) demonstrates the existence of equilibria of the Green-Porter model (Green and Porter (1984), Porter (1983)) that are optimal in terms of the degree of implicit collusion they sustain, and yet have an unexpectedly simple intertemporal structure. Here we exploit the same analytic approach to develop a theory for a broad class of asymmetric discounted repeated games with imperfect monitoring. The results characterize efficient sequential equilibria, facilitate their computation, and establish a strong relationship between the equilibrium value set and the discount factor. More generally, they demonstrate the advantages of a perspective which views these repeated games in terms of a particular intertemporal decomposition.

Our analysis is in the spirit of dynamic programming, whose impact on game theory has, of course, been substantial (see, for example, Shapley (1953), Abreu (1988), and Radner, Myerson, and Maskin (1986)). It proceeds via a succession of propositions, central among which is "self-generation" (see Section 3), which reduce the study of the equilibria in question to the solution of a class of static

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problems. The remainder of this section is devoted to an informal exposition of our results. This overview abstracts from the measure-theoretic issues that are dealt with in the analysis.

The supergames studied here involve the indefinite repetition of a simultaneous, N-person stage game: see Section 2 for formal definitions. In each period t, players independently select actions from their respective pure strategy sets in the stage game. The vector of strategies selected determines the probability distribution of a payoff-relevant random variable P, whose realization[2] is publicly observed at the end of period t. At no time does any player i observe the actions chosen earlier by other players. Nor can i infer this information from the signal realizations: the *support* of P is independent of the action profile. A player's expected payoff at the end of t depends on his own action directly, and on the profile of actions insofar as the latter affects the distribution of the signal. Payoffs are discounted (to the beginning of period 1) according to the common discount factor $\delta \in (0,1)$.

We study sequential equilibria (adapted from Kreps and Wilson (1982)) in pure strategies (hereafter abbreviated as S.E.). With each S.E. of the supergame is associated a profile of discounted payoffs, one for each player; the set of S.E. payoff vectors is denoted V. Without loss of generality (as we show in Section 3), we restrict attention to S.E.'s in which each player makes his actions depend only upon past signal realizations (not on his own previous actions). After any first period history, an S.E. induces a "continuation profile" on the remaining subtree. Because the first period signal realization is publicly observed, this profile is common knowledge, and is itself a sequential equilibrium. The value of the continuation profile is therefore always in V.

In order to obtain some powerful characterizations of the equilibrium value set, it is useful to regard an S.E. as specifying a profile of actions q for players in the first period, and a continuation reward function that "promises" some expected payoff $u(p) \in V$ for the remainder of the game (the value of the "continuation equilibrium"), depending on the value p of the first period signal. The value of the S.E. can also be viewed as being the value of the pair (q, u). Equilibrium requires that certain incentive constraints be satisfied: for each player i, the choice q_i must maximize the sum of his first period payoff and the expected value of the reward function. (Note that the choice of action affects the distribution of the signal and hence the expectation of the continuation value.) Think now of an arbitrary pair (q, u), with q an action profile of the stage game, and u a measurable function from the signal space into R^N, but not necessarily associated with any equilibrium. We say that (q, u) is *admissible with respect to* V if it satisfies the incentive constraints explained above, and if for all $p, u(p) \in V$. Let $B(V) \subseteq R^N$ be the set of all values of pairs admissible with respect to V. For any S.E., it is clear that the associated pair (the period 1 action profile and continuation reward function) is admissible with respect to V,

[2] We follow the standard practice of using upper and lower cases, respectively, to distinguish between a random variable and a particular realization.

therefore $V \subseteq B(V)$. Conversely, from any pair (q, u) admissible with respect to V, construct an S.E. as follows: the first period action profile specified by the S.E. is q, and the continuation equilibrium induced after any first period signal p is some S.E. $\sigma(p)$ having value $u(p)$ (by the definition of V, $u(p) \in V$ implies the existence of an S.E. $\sigma(p)$ with value $u(p)$). It is straightforward to check that the profile constructed is indeed an S.E., therefore $B(V) \subseteq V$. We conclude that $V = B(V)$, which is the content of Theorem 2 of Section 3. This result is referred to as "factorization" because it follows from the factorization or decomposition (in dynamic programming fashion) of an equilibrium into an admissible pair.

Although factorization proves to have a number of applications, much more can be said about V by studying admissible pairs in a broader context. For an arbitrary set $W \subseteq R^N$, a pair (q, u) is *admissible with respect to* W if it satisfies the relevant incentive constraints as above, and for all signal values $p, u(p) \in W$. The set of all values of pairs admissible with respect to W is called $B(W)$. Notice that the elements of $B(W)$ are "generated" by reward functions that draw values from W, just as elements of V are generated by continuation reward functions that draw values from V itself. Any set W such that $W \subseteq B(W)$ is called *self-generating* (as all elements of W can be generated using rewards from W). Theorem 1 of Section 3 establishes that if W is a bounded, self-generating set, then $B(W) \subseteq V$. This result, which implies that all the values in any bounded self-generating set are sequential equilibrium values, is called *self-generation*. An informal treatment of the argument is deferred until Section 3.

Factorization and self-generation generalize theorems established in APS for symmetric equilibria of symmetric games. Some of their theoretical applications are illustrated in what follows; for instance, self-generation permits a general proof that the value set, expressed in average terms, is increasing in the discount factor (see Section 6). But self-generation is also of practical use in studying specific examples. It is sometimes relatively simple to choose a set of points in R^N and show that they constitute a self-generating set for the supergame under investigation. Then one has established that each of the points is the payoff of some sequential equilibrium of the infinite horizon game.

Theorem 3 asserts that for any pair $(q, \hat{u})$ admissible with respect to a compact set W, there exists $\bar{u}$ such that $(q, \bar{u})$ has the same value as the original pair, and is admissible with respect to ext W (the extreme points of the convex hull of W). As Section 4 explains, this allows a major simplification of supergame equilibria: without loss of generality, one can restrict attention to S.E.'s whose continuation values, after any history, are extreme points of V. If V is a rectangle, for example, after the first period at most four continuation values and four associated action profiles arise in equilibrium. A corresponding "bang-bang" result for *symmetric* equilibria is found in APS. If one generalized the argument of APS to asymmetric equilibria, the result would be weaker. Namely, continuation equilibria may be taken to be boundary points of V, but not necessarily extreme points of V. The distinction is at times critical: in the 2-person rectangular example just mentioned, V has a continuum of boundary points, but only four extreme points. Our proof of the stronger result is a straightforward application of a technical theorem of Aumann (1965).

The result that it is *sufficient* to consider reward functions of the bang-bang form is open to objections concerning the appropriateness of the restriction. If the "natural" solution were a smooth function, which could be replaced by one with the bang-bang property at the cost of creating a complex pattern of rapid alternations among extremal values, one kind of simplicity would be traded against another. Reassurance is provided by a much stronger characterization, new to this paper, in Section 7. Under certain conditions, the reward functions faced by players in Pareto-efficient equilibria *must* be bang-bang: efficiency demands that nonextremal points of the payoff set are never used.

The dynamic programming technique of value iteration (Howard (1960)) has an analogue in repeated games which is discussed in Section 5. It is an iterative procedure for computing the set of equilibrium values. The novelty here is the presence of sequential incentive constraints and the fact that the map that is iterated is set-valued. Apart from its importance for the numerical computation of equilibria of specific supergames, the algorithm is an alternative characterization of the equilibrium value set, and as such will have a variety of theoretical applications. Suppose that for any $W \subseteq R^N$, one is able to compute $B(W)$ (this may be a substantial task). The algorithm works as follows. Begin with a compact set W_0 sufficiently large that it is known a priori to satisfy $V \subseteq B(W_0) \subseteq W_0$. Apply the operator B repeatedly to obtain the decreasing sequence of sets $\{W_n\}_{n=0}^{\infty}$, where for each n, $W_{n+1} = B(W_n)$. Theorem 5 shows that this sequence converges to the supergame value set V. The relationship to some earlier results by Fudenberg and Levine (1983) linking infinite and finite horizon games is explained briefly in Section 5.

The ways in which this paper furthers the research reported in APS may be summarized as follows. First, it relaxes the restriction of symmetry, showing the theory capable of embracing both asymmetric equilibria of symmetric games and arbitrary asymmetric games. Secondly, the sufficiency of using bang-bang reward functions in efficiently collusive equilibria is strengthened to a *necessity* theorem. Finally, we provide an algorithm useful in computing the sequential equilibrium value set.

Except for Section 6, this paper takes the discount factor δ to be fixed, and studies the value set and the nature of constrained efficient equilibria for that degree of patience. In this way it complements the literature initiated by Radner (1985) and Radner, Myerson, and Maskin (1986)[3] which focuses on the limiting behavior of the value set of supergames with imperfect monitoring as δ approaches 1. A dynamic programming approach again proves useful for folk theorems with discounting: this is powerfully demonstrated by Fudenberg and Maskin (1986) and Fudenberg, Levine, and Maskin (1988). There is also a growing body of work on the related topic of repeated agency theory: see, for

[3] This line of work is ultimately inspired by the early papers on folk theorems without discounting, especially Aumann and Shapley (1976), Rubinstein (1979a), and Rubinstein (1979b).

example, Fudenberg, Holmström, and Milgrom (1988), Rogerson (1985), and Spear and Srivastava (1987).

2. THE MODEL

The model outlined below features unobservable actions, stochastic outcomes, and a publicly observable random variable correlated with players' private choices. It lends itself naturally to the study of a number of economic questions. Important examples are oligopoly (Green and Porter (1984), Porter (1983)) and partnership problems (Radner (1986)) of various kinds.

The Stage Game

The N-person stage game is denoted G. Each player i has a finite strategy set S_i and a payoff function $\Pi_i : S \to R$, where $S := S_1 \times \cdots \times S_N$. For $q \in S, \Pi_i(q)$ is an expected value. Payoffs actually received $\pi_i(p, q_i)$, are stochastic and depend on realizations of a random variable P which takes values in $\Omega \subseteq R^a$. The distribution of P is parameterized by the vector of actions $q \in S$, and is denoted $\Psi(\cdot; q)$. Realized payoffs π_i depend on $q_{-i} := (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N)$ only through the effect of the latter on the distribution of P. Finally, $\Pi_i(q) = \int_\Omega \pi_i(p, q_i) \Psi(dp; q)$.

The Repeated Game

We denote by $G^\infty(\delta)$ the infinitely repeated game with component game G and discount factor $\delta \in (0,1)$. Players can observe (and therefore condition upon) only their *own* past actions and past realizations of the random variable P. Hence, a strategy σ_i for player i in $G^\infty(\delta)$ is a sequence of Lebesgue measurable functions $\{\sigma_i(t)\}_{t=1}^\infty$, where $\sigma_i(1) \in S_i$, and for $t > 1$, $\sigma_i(t) : \Omega^{t-1} \times S_i^{t-1} \to S_i$. Let $p^t = (p(1), \ldots, p(t))$ and $q^t = (q(1), \ldots, q(t))$ denote t-period signal and action histories, respectively. As is standard $\sigma|_{p^t, q^t}$ denotes the strategy profile induced by σ after the t-period history (p^t, q^t). In each period, p is drawn independently according to the distribution $\Psi(\cdot; q)$. Associated with any strategy profile σ of $G^\infty(\delta)$ is a stochastic stream of payoff vectors. The expected present discounted value of this stream is denoted $v(\sigma) = (v_1(\sigma), \ldots, v_N(\sigma))$. Note for later use that period t payoffs are received at the end of period t and discounted to the beginning of period 1. We assume that:

(A1) *S_i is finite, $i = 1, \ldots, N$.*

(A2) *For each $q \in S, \Psi(\cdot; q)$ is absolutely continuous. Let $g(\cdot; q)$ be the corresponding probability density.*

(A3) *$\{p \in \Omega \mid g(p; q) > 0\}$ is independent of $q \in S$.*

(A4) *$\pi_i(p, q_i)$ is continuous in p.*

(A5) *G has a Nash equilibrium in pure strategies.*

Without loss of generality we take Ω to equal $\{p \mid g(p; q) > 0\}$.

Assumptions (A1) and (A4) guarantee that $v(\sigma)$ is well defined. Theorems 3 and 7 depend upon (A2). The solution concept used is the natural generalization of sequential equilibrium[4] (see Kreps and Wilson (1982)) to the repeated games under consideration. Hereafter, we use S.E. to denote a sequential equilibrium in pure strategies, and denote by $V := \{v(\sigma) \mid \sigma \text{ is an S.E.}\}$ the set of S.E. payoffs. Assumption (A5) implies that V is nonempty; the strategy profile specifying that in every period independently of the history each player uses his one-period Nash equilibrium action, is an S.E. Further discussion of the assumptions is deferred until Section 3.

3. FACTORIZATION AND SELF-GENERATION

Consider the maximization problem faced by a player in the first period of an equilibrium σ. Recall that his choice of action q_i has two consequences: it affects payoffs in period 1, and also influences the distribution of the first-period signal $p(1)$. The player is in effect maximizing the sum of current payoffs and the expectation of the future reward (a function of $p(1)$) implicitly "promised" by σ. The reward function must be drawn from V: an S.E. can offer only S.E. rewards. Furthermore, $\sigma_i(1)$ must yield at least as high a value of the sum as any other action available to i. The same remarks apply to player i's choice after any t-period history.

We proceed rather abstractly by studying structures suggested by the above observations but no longer in the context of any particular equilibrium.

Let $L^\infty(\Omega; R^N)$ denote the set of equivalence classes of essentially bounded Lebesgue measurable functions u from Ω into R^N. For any pair $(q,u) \in S \times L^\infty(\Omega; R^N)$, $E(q;u) := \delta\{\Pi(q) + \int_\Omega u(p)g(p;q)\,dp\}$. Clearly $E(q;u)$ is continuous in u when $L^\infty(\Omega; R^N)$ is endowed with the weak-* topology. For any set $W \subseteq R^N$, $L^\infty(\Omega; W)$ will denote the set of functions $u \in L^\infty(\Omega; R^N)$ such that $u(p) \in W$ a.e. $p \in \Omega$.

DEFINITION: For any set $W \subseteq R^N$, a pair $(q,u) \in S \times L^\infty(\Omega; R^N)$ is called *admissible with respect to* W if

(i) $\quad u(p) \in W$ a.e. $p \in \Omega$, and

(ii) $\quad E_i(q;u) \geqslant E_i(\gamma_i, q_{-i}; u)$ for all $\gamma_i \in S_i$ and $i = 1, \ldots, N$.

A profile $q \in S$ is *supportable* by W if there exists $u \in L^\infty(\Omega; R^N)$ such that (q,u) is admissible with respect to W.

[4] The original definition applies to finite extensive games. In our usage, a sequential equilibrium will be a pure strategy profile σ such that for each player i, each t-period history q_i^t of actions by i, every supergame strategy $\tilde{\sigma}_i$ for i, almost every t-period signal history p^t, and the t-period history of action profiles q_{-i}^t for players other than i induced by σ and p^t,

$$v_i(\sigma|_{q^t, p^t}) \geqslant v_i(\tilde{\sigma}_i, \sigma_{-i}|_{q_{-i}^t, p^t}).$$

Note that the issue of consistency does not arise, because the constant support assumption (A3) makes the calculation of conditional beliefs at any information set unambiguous. See also the remark following Theorem 1.

These conditions mimic the two requirements noted above on pairs of the form (recommended action, reward function) arising in an S.E.

DEFINITION: For each set $W \subseteq R^N$, $B(W) := \{E(q;u) | (q,u)$ is admissible w.r.t. $W\}$.

By the definition of $B(W)$, for each $w \in B(W)$ there exists a pair admissible with respect to W with value w. Hence (by the Axiom of Choice) there exist functions Q: $B(W) \to S$ and $U: B(W) \to L^\infty(\Omega, W)$ such that $(Q(w), U(w))$ is admissible with respect to W and $E(Q(w); U(w)) = w$ for all $w \in B(W)$. Furthermore, for any bounded Borel set W, the domain of Q and U can be extended to R^N, and the functions Q and $(p,w) \to U(w)(p)$ from $\Omega \times R^N$ into R^N can be taken to be universally measurable. The last remark needs some justification; we provide a proof in Lemma C of the Appendix. The functions Q and U are used extensively below. The extension of Q and U to R^N, the requirement that W is Borel, and various measure-theoretic qualifications are unnecessary if Ω is countable (or finite).

That admissibility successfully captures the information essential for studying V is evidenced in Theorems 1 (self-generation) and 2 (factorization). These combine to say that V is the largest bounded fixed point of the set-valued map B. This is a powerful result insofar as the definition of B is quite simple and makes no reference to the complex strategic structure of an infinite horizon game.

The proofs of the theorems below are very similar to those presented for the symmetric case in APS. We have included them to provide a self-contained treatment.

DEFINITION: $W \subseteq R^N$ is said to be self-generating if $W \subseteq B(W)$.

THEOREM 1 (Self-Generation): *For any bounded Borel set $W \subseteq R^N$, if W is self-generating, then $B(W) \subseteq V$.*

Before giving a proof, we provide an intuitive discussion of self-generation; for simplicity, qualifications such as "almost everywhere" are ignored. If a bounded set W is self-generating, any value in W is also in its image $B(W)$. This permits us to choose any element of $B(W)$ and "transform" it period-by-period into an S.E., say σ, having the same value. Begin by choosing a pair (q,u) admissible with respect to W, with value w. Set $\sigma(1) = q$. For any $p \in \Omega$, we would like to ensure a continuation value of $u(p)$ in equilibrium. As a first step, select a pair (q', u') admissible with respect to W, and having value $u(p)$ (this is possible because $u(p) \in W \subseteq B(W)$). Set the action profile in period 2 (given that p arose in the first period) equal to q', and for each $p' \in \Omega$ choose a new admissible pair (q'', u'') with value $u'(p')$. In this way strategies for the first t

periods (for any t) are generated. A recursive step allows this process to determine a complete supergame profile σ. The profile has the desired value w, because each time an admissible pair was substituted for a continuation value, the value was preserved. Moreover the action and reward function after each history comprise an admissible pair by construction, so no player has a profitable "one-shot" deviation. Backward induction then implies that no deviations at a finite number of information sets can benefit a player. Finally, the fact that period t payoffs are bounded and are discounted heavily if t is large, implies that cheating infinitely often is unprofitable (otherwise, some deviation at a finite number of information sets would also be profitable).

PROOF: The proof is constructive. For all $w \in B(W)$ we specify sequential equilibria $\hat{\sigma}(w)$ such that $v(\hat{\sigma}(w)) = w$. For each $w \in B(W)$, consider the function $U(w) \in L^\infty(\Omega, W)$ as defined earlier. Recursively define the functions $U^t(w): \Omega^t \to R^N$ as follows: $U^1(w) = U(w), U^t(w)(p^t) = U(U^{t-1}(w)(p^{t-1}))(p_t)$, $t = 2,3,\ldots$. Since $U(w)(p) \in W$ a.e. $p \in \Omega$, for each $t = 1,2,\ldots, U^t(w)(p^t) \in W$ a.e. $p^t \in \Omega^t$, so that $U^t(w) \in L^\infty(\Omega^t, W)$. The required strategy profiles $\hat{\sigma}(w)$ are $\hat{\sigma}(w)(1) = Q(w), \hat{\sigma}(w)(t+1)(p^t, q^t) = Q(U^t(w)(p^t))$, $t = 1,2,\ldots$. Observe that the $\hat{\sigma}(w)$'s are independent of past actions. We will write $\hat{\sigma}(w)|_{p^t}$ for $\hat{\sigma}(w)|_{p^t,q^t}$. Note also that the $\hat{\sigma}(w)(t)$'s are Lebesgue measurable functions, being the composition of universally measurable functions (see the Appendix).

It may be checked that $v(\hat{\sigma}(w)) = w$ for all $w \in B(W)$. By construction, given $\hat{\sigma}_{-i}(w)$, the strategy $\hat{\sigma}_i(w)$ is "unimprovable" (after almost all histories, no one-shot deviation improves a player's payoff), and hence optimal for player i (see, for instance, Whittle (1983), Theorem 2.1, Chapter 24, and, for the genesis of this idea, Howard (1960)). Thus, $\hat{\sigma}_i(w)$ is a best response to $\hat{\sigma}_{-i}(w)$ for all i, and $\hat{\sigma}(w)$ is a Nash equilibrium.

It now remains only to show that $\hat{\sigma}(w)$ is an S.E. Consider any history (p^t, q^t). Since players' strategies do not depend on past actions, and expected payoffs in any period depend only on actions in that period, beliefs about past actions are irrelevant, and we need only check that $\hat{\sigma}(w)|_{p^t}$ is a Nash equilibrium for almost all $p^t \in \Omega^t$, $t = 1,2,\ldots$. But a.e. $p^t \in \Omega^t, \hat{\sigma}(w)|_{p^t} = \hat{\sigma}(x)$ for some $x \in W$ and we have just shown that $\hat{\sigma}(x)$ is a Nash equilibrium for all $x \in B(W) \supseteq W$. *Q.E.D.*

REMARK: The assumption of constant support (A3) implies that all possible price histories occur in equilibrium. As a consequence, there is no material difference between Nash and sequential equilibria. For each Nash equilibrium there is a payoff-equivalent sequential equilibrium which (modulo events of measure zero) differs from the former only after histories corresponding to a player's own deviations. Both strategy profiles hence generate the same equilibrium behavior.

The next result can be viewed as a strategic, set-valued expression of Bellman's equation.

THEOREM 2 (Factorization): $V = B(V)$.

PROOF: By Theorem 1, it suffices to establish that V is a bounded Borel set and that $V \subseteq B(V)$. We first show that V is self-generating. Consider $w \in V$ and an S.E. σ such that $v(\sigma) = w$. Let (q, u) be a pair such that $q := \sigma(1)$ and $u(p) := v(\sigma|_{p,\sigma(1)})$ for all $p \in \Omega$. We must prove that (q, u) is admissible with respect to V, and that $E(q; u) = w$. Observe that u is Lebesgue measurable, since it may be written as the discounted sum of Lebesgue measurable functions.[5] Clearly,

$$w = \delta\left[\Pi(\sigma(1)) + \int_{\Omega} v(\sigma|_{p,\sigma(1)}) g(p; \sigma(1))\, dp\right] = E(q; u).$$

By (A3), the information sets $(p, \sigma_i(1))$ are reached in equilibrium for all $p \in \Omega$. Hence player i can use Bayes' rule to predict what player j's future behavior will be. Since players are using pure strategies, player i's conditional beliefs about player j's behavior are concentrated at $\sigma_j|_{p,\sigma_j(1)}$, which is a strategy in $G^\infty(\delta)$. Hence $\sigma|_{p,\sigma(1)}$ is an S.E. of $G^\infty(\delta)$. In other words, under our assumptions, the repeated game has a recursive structure. It follows from the preceding discussion that $u(p) \in V$, for almost all $p \in \Omega$.

For any i and $\gamma_i \in S_i$, consider $\tilde{\sigma}_i$ such that $\tilde{\sigma}_i(1) = \gamma_i$ and $\tilde{\sigma}_i|_{p,\gamma_i} = \sigma_i|_{p,q_i}$ for all $p \in \Omega$. Then $(\tilde{\sigma}_i, \sigma_{-i})|_{p,(\gamma_i, q_{-i})} = \sigma|_{p,q}$. Since σ is an S.E., $v_i(\sigma) \geqslant v_i(\tilde{\sigma}_i, \sigma_{-i})$, which implies $E_i(q; u) \geqslant E_i(\gamma_i, q_{-i}; u)$. This establishes the admissibility of (q, u).

Note that $V \subseteq [\delta/(1-\delta)]\text{co}\{\Pi(q) | q \in S\}$ (co := convex hull). Since S is finite, this implies that V is bounded. We defer proving that V is a Borel set; this will be an immediate corollary of Theorem 4. *Q.E.D.*

Take the self-generating set W in the statement of Theorem 1 to be V. It is worth noting that for any $w \in V$, the profile $\hat{\sigma}(w)$ constructed in the proof of Theorem 1 is a sequential equilibrium in which no player conditions his choice of action in any period on actions he has previously taken.

In establishing that $V \subseteq B(V)$, the proof of Theorem 2 constructs pairs admissible with respect to V that mimic the first-period incentive structure of an S.E. That this is possible depends on the fact that equilibrium continuation values after the first period are elements of V. This accounts for some limitations in the scope of our inquiry. First, mixed strategies are excluded from consideration. If players randomize in the first period, player 1 cannot infer (from the signal and the equilibrium hypothesis) what other players' continuation strategies are: player 2's continuation strategy may depend on his first-period action, which is unobservable to 1, and is no longer specified deterministically by the S.E. The same problem arises in models in which players observe private

[5] That is, $u(p) = \sum_{t \geqslant 2} \delta^{t-1} E_t^\sigma(\pi(p(t), q(t)) | p(1) = p)$, where $E_t^\sigma(\pi(p(t), q(t)) | p(1) = p)$ is the conditional expectation of players' payoffs in period t, when they follow the strategy profile σ, given that the first period signal realization is $p(1) = p$.

signals: because they are conditioning their actions on information that is not publicly observed, players cannot compute one another's continuation strategies. Continuation profiles need not be equilibria; the link between sequential equilibria and admissibility with respect to V is broken. (By assuming that player i's payoff is determined by his own action and the publicly observed signal, we ensure that payoffs do not serve as privately observed signals.) Finally, we assume in Section 2 that the support of the signal is independent of the action profile. Suppose, instead, that there are three players who, in the first period of an S.E. σ, are supposed to play the profile q, and that some value $\bar{p}$ is outside the support of the signal, given q. If player 3 cheats and $\bar{p}$ arises, player 1 concludes that either player 2 or player 3 deviated; suppose that his posterior gives equal weight to both alternatives. Similarly, suppose player 2 gives equal weight to the possibilities that 1 or 3 deviated. The continuation profile need not be an equilibrium; moreover, the continuation value for player 3 might now be worse than his least-preferred S.E. value. Consequently a pair admissible with respect to V is unable to match the severity with which σ punishes the deviation by player 3. In this case $V \not\subseteq B(V)$.

4. BANG-BANG REWARD FUNCTIONS AND THE STRUCTURE OF EQUILIBRIA

This section proves that any reward function can be replaced by one yielding each player the same expected value (without affecting incentive compatibility) and taking on values only in the set of extreme points of V. Apart from the obvious practical advantages this offers in working with particular games, it has theoretical applications: examples are provided in the proofs of Theorems 5 and 6 and Lemma 1.

For $W \subseteq R^N$, let $\operatorname{co} W$ denote the convex hull of W and $\operatorname{ext} W$ the set of extreme points of $\operatorname{co} W$.

DEFINITION: $u \in L^\infty(\Omega; W)$ has the *bang-bang* property if $u(p) \in \operatorname{ext} W$ a.e. $p \in \Omega$.

Theorem 3 below implies that the function U of Section 3 can be chosen so that for each $w, U(w)$ has the bang-bang property. Now consider the nature of an equilibrium with value w, and summarized by (Q, U) with U chosen as above. For almost all signals $p(1)$ arising in the first period, an extremal reward $U(w)(p(1))$ is "delivered" by the pair $(Q(U(w)(p(1))), U(U(w)(p(1))))$. When $p(2)$ is observed, a new reward function comes into effect, and so on. Since after any t-period history, players' future payoffs are in $\operatorname{ext} V$, a play of the game can be viewed as an alternation among extreme points of V, where the particular pattern of extreme points is determined by the sequence of realized outcomes of the random signal. For the special case in which V is one-dimensional (as it is, for example, when attention is restricted to symmetric equilibria of symmetric games), this means that only two extreme points, and hence two action profiles, ever arise after the first period of the game.

THEOREM 3: *Let $W \subseteq R^N$ be compact and $(q, \hat{u})$ be an admissible pair with respect to* co W. *Then there exists a function $\bar{u} \in L^\infty(\Omega; \text{ext}\, W)$ such that $(q, \bar{u})$ is admissible with respect to W and $E(q; \bar{u}) = E(q; \hat{u})$.*

PROOF: Let

$$F := \{u \in L^\infty(\Omega, \text{co}\, W) \mid (q, u) \text{ is admissible w.r.t. co}\, W \text{ and } E(q; u) = E(q; \hat{u})\}.$$

By assumption F is nonempty ($\hat{u} \in F$), and it may easily be checked that F is convex. By Alaoglu's theorem, F is compact when $L^\infty(\Omega; \text{co}\, W)$ is endowed with the weak-* topology. Hence, by the Krein-Milman theorem, F has an extreme point.

By (A1), the set of integral constraints defining F is finite and Proposition 6.2 of Aumann (1965) applies directly. It implies that any extreme point $\bar{u}$ of F satisfies $\bar{u}(p) \in \text{ext}\, W$ a.e. p. Since $\text{ext}\, W \subseteq W, (q, \bar{u})$ is also admissible with respect to W, and the proof is complete. *Q.E.D.*

COROLLARY: *Let $W \subseteq R^N$ be compact. Then $B(W) = B(\text{co}\, W)$.*

The proofs of Lemma 1 and Theorem 4 below are analogous to those of Proposition 4 and Corollary 2, respectively, of APS. It is often useful to know that the operator B preserves compactness; for example, this guarantees that when applying the algorithm of Section 5, each element of the sequence of sets generated is compact (so that the bang-bang result can be invoked to simplify the calculations at each stage). Similarly, it is critical to much of the analysis to follow that the bang-bang result be applicable to pairs admissible with respect to V; this depends on the compactness of V.

LEMMA 1: *The operator B is monotone and preserves compactness*: (i) *if $W \subseteq W' \subseteq R^N, B(W) \subseteq B(W')$*; (ii) *if $W \subseteq R^N$ is compact, $B(W)$ is compact.*

PROOF: Part (i) follows immediately from the definition of admissibility.

By the Corollary to Theorem 3, $B(W) = B(\text{co}\, W)$. For each $q \in S$, let

$$l(q) := \{u \in L^\infty(\Omega; \text{co}\, W) \mid (q; u) \text{ is admissible w.r.t. co}\, W\}.$$

Alaoglu's theorem implies that $l(q)$ is weak-* compact. Since $E(q; u)$ is continuous in u, and S is finite,

$$B(\text{co}\, W) = \bigcup_{q \in S} E(\{q\} \times l(q))$$

is compact as a finite union of compact sets. *Q.E.D.*

THEOREM 4: *V is compact.*

PROOF: Recall from the proof of Theorem 2 that V is bounded and self-generating. Let cl(V) denote the closure of V. Since V is bounded, cl(V) is compact. By monotonicity, $V = B(V) \subseteq B(\mathrm{cl}(V))$, and by Lemma 1, $B(\mathrm{cl}(V))$ is compact. Hence, $\mathrm{cl}(V) \subseteq B(\mathrm{cl}(V))$, and self-generation implies $\mathrm{cl}(V) \subseteq V$. Thus, V is closed and compact. *Q.E.D.*

Since V is compact, V is a Borel set, as claimed in Theorem 2.

5. COMPUTATION

For many purposes it is important to have an algorithm capable of finding the set V in particular supergames. To do so, it is necessary to find the largest bounded fixed point of the set-valued map B. It turns out that V may be computed by a procedure analogous to Howard's "value-iteration" (Howard (1960)) for dynamic programs. The algorithm starts with a set $W_0 \subseteq R^N$ such that $V \subseteq B(W_0) \subseteq W_0$. It then proceeds by computing the monotonically decreasing sequence of sets $W_n := B(W_{n-1})$, $n = 1, 2, \dots$. The limit of this process is $V = \lim_{n\to\infty} W_n := \cap_{n=1}^{\infty} W_n$.

The next two lemmas follow directly from factorization and the monotonicity of B; their proofs are left to the reader. For Lemma 2, recall that $V \subseteq [\delta/(1-\delta)]\mathrm{co}\, \Pi(S) =: W$. Furthermore, if (q, u) is admissible with respect to W, $E(q; u) = \delta\{\Pi(q) + [\delta/(1-\delta)]x\} = [\delta/(1-\delta)]\{(1-\delta)\Pi(q) + \delta x\} =: w$ for some $x \in \mathrm{co}\, \Pi(S)$, which implies $w \in W$.

LEMMA 2: *Let* $W := [\delta/(1-\delta)]\mathrm{co}\{\Pi(q) | q \in S\}$. *Then* $V \subseteq B(W) \subseteq W$.

LEMMA 3: *If* $W \subseteq R^N$ *satisfies* $V \subseteq B(W) \subseteq W$, *then* $V \subseteq B(B(W)) \subseteq B(W)$.

LEMMA 4: *Let* $\{W_n\}$ *be a decreasing sequence of compact sets in* R^N. *Then* $\mathrm{co} \cap W_n = \cap\, \mathrm{co}\, W_n$.

PROOF: See Appendix.

THEOREM 5 (Algorithm): *Let* $W \subseteq R^N$ *be compact and satisfy* $V \subseteq B(W) \subseteq W$. *Define* $W_0 := W$ *and for* $n = 1, 2, \dots$ *let* $W_n := B(W_{n-1})$. *Then* $\{W_n\}$ *is a decreasing sequence and* $V = \lim_{n\to\infty} W_n$.

PROOF: By Lemmas 1 and 3, $\{W_n\}$ is a decreasing sequence of compact sets, so $W_\infty := \lim_{n\to\infty} W_n = \cap W_n$ and W_∞ is compact. Again by Lemma 3, $V \subseteq W_\infty$. To complete the proof we need to show that $W_\infty \subseteq V$. By self-generation and the corollary to Theorem 3, it is sufficient to show that $W_\infty \subseteq B(\mathrm{co}\, W_\infty)$. Consider any $w \in W_\infty$. By definition, for each $n = 1, 2, \dots$, there exists (q^n, u^n) admissible with respect to W_n such that $E(q^n, u^n) = w$. Since $q^n \in S$, where S is finite, and $L^\infty(\Omega; W_n) \subseteq L^\infty(\Omega; \mathrm{co}\, W)$, where $L^\infty(\Omega; \mathrm{co}\, W)$ is a weak-* compact set, we may without loss of generality assume $q^n = q$ and $u^n \xrightarrow{*} u$ for some $q \in S$ and $u \in L^\infty(\Omega; \mathrm{co}\, W)$. We argue that (q, u) is an admissible pair with respect to $\mathrm{co}\, W_\infty$, and $w = E(q; u)$. Since for all $n = 1, 2, \dots, u^m(\Omega) \subseteq \mathrm{co}\, W_m \subseteq \mathrm{co}\, W_n$ for all

$m \geqslant n$ (modulo sets of measure 0), we have $u(\Omega) \subseteq \mathrm{co}\, W_n$ for all n. Hence, by Lemma 4, $u(\Omega) \subseteq \cap \mathrm{co}\, W_n = \mathrm{co} \cap W_n = \mathrm{co}\, W_\infty$. Since $E(q;\cdot): L^\infty(\Omega; \mathrm{co}\, W) \to R^N$ is continuous when $L^\infty(\Omega; \mathrm{co}\, W)$ is endowed with the weak-* topology,

$$E(q,u) = \lim_{n\to\infty} E(q;u^n) = w.$$

Finally $E_i(q;u^n) \geqslant E_i(\gamma_i, q_{-i}; u^n)$ for each $n = 1,2,\ldots$ imply $E_i(q;u) \geqslant E_i(\gamma_i, q_{-i}; u)$ for all $\gamma_i \in S_i$ and each $i = 1,\ldots,N$. Hence (q,u) is admissible with respect to co W_∞, as required. *Q.E.D.*

Fudenberg and Levine (1983) showed that for a substantial class of dynamic games including discounted repeated games, supergame perfect equilibria are limits of ε-perfect equilibria of T-period truncations of the supergame, as $\varepsilon \to 0$ and $T \to \infty$. Although their result is not presented in terms of value iteration, our algorithm is closely related to their limit theorem. Instead of increasing the equilibria of the T-period game $G^T(\delta)$ by computing ε-equilibria, augment the equilibrium value set by supplementing period T payoffs with reward functions drawn from any set W_0 of the kind specified in Theorem 5. If $T=1$, for example, the resulting value set is $B(W_0)$. Hence, for $T=2$, the set of supplemented values is $W_2 = B(B(W_0))$, and for arbitrary T one has the supplemented value set W_T. Thus, as $T \to \infty$, this procedure approximates the value set V of $G^\infty(\delta)$, since $\lim_{T\to\infty} W_T = V$.

6. COMPARATIVE STATICS: MONOTONICITY IN δ

Intuition suggests that the equilibrium set should in some sense increase with the discount factor. Plausibly "cooperation" becomes easier as players become more patient and thereby increasingly willing to forego immediate gains for a possible future reward. One is led to conjecture a monotonic relationship between equilibrium outcomes and the number δ, where outcomes are thought of as average discounted payoffs. Despite the complexity and generality of the model, this conjecture can be proved correct without invoking any assumptions beyond those of Theorem 3. When the discount factor increases from δ_1 to δ_2, and payoffs are appropriately normalized, the original set of equilibrium values is contained in the new set of values associated with δ_2. The proof is short and simple and illustrates the power of self-generation as an analytical tool.

We now write $V(\delta), B(W|\delta)$, and $E(q;u|\delta)$ to make explicit the dependence on the particular value of the discount factor.

THEOREM 6 (Monotonicity in Discount Factor): *Let δ_1 and δ_2 be two discount factors such that* $0 < \delta_1 < \delta_2 < 1$. *Then* $[(1-\delta_1)/\delta_1]V(\delta_1) \subseteq [(1-\delta_2)/\delta_2]V(\delta_2)$.

PROOF: As may be easily checked, we need to show $(1+k)V(\delta_1) \subseteq V(\delta_2)$, where $k := (\delta_2 - \delta_1)/(\delta_1(1-\delta_2))$. For any $w \in V(\delta_1)$ let (q,u) be an admissible pair with respect to $V(\delta_1)$ such that $w = E(q;u|\delta_1)$. Define the function u^+ on Ω by $u^+(p) = u(p) + kw$. Then it may be verified that (q,u^+) is an admissible pair with respect to $\{kw\} + V(\delta_1)$, and $E(q; u^+ | \delta_2) = (1+k)w$. Hence, $(1+k)w \in B(\{kw\} + V(\delta_1) | \delta_2)$ for all $w \in V(\delta_1)$. Let $\lambda := 1/(1+k)$; clearly $\lambda \in (0,1)$. Since for any $z \in R^N, z + kw = \lambda(1+k)z + (1-\lambda)(1+k)w$, we have $\{kw\} + V(\delta_1) \subseteq \text{co}(1+k)V(\delta_1)$. Therefore, $(1+k)V(\delta_1) \subseteq B(\text{co}(1+k)V(\delta_1)|\delta_2)$. Finally, by the corollary of Theorem 3 and self-generation, $(1+k)V(\delta_1) \subseteq V(\delta_2)$. *Q.E.D.*

7. OPTIMIZATION AND THE NECESSITY OF BANG-BANG REWARD FUNCTIONS

This section explores the idea that efficient incentive schemes must necessarily have a bang-bang structure. Consider $W \subseteq R^N$ compact and some $q \in S$ which is the first element of an admissible pair yielding an extremal payoff in the set $B(W)$. An implication of Theorem 3 is that among the reward functions which support q and maximize a linear function of player payoffs, at least one has the bang-bang property. We show here that under certain conditions *all* optimal solutions must be bang-bang. The proof takes a dual approach to the optimization problem which highlights the way in which considerations of efficiency lead to the use of rewards that are extreme points of V (or, more generally, of the compact set W from which rewards are to be drawn).

Establishing the necessity of bang-bang solutions requires several conditions not needed for the sufficiency result. Precise statements of the conditions involve the following definitions. The four conditions invoked in the statement of Theorem 7 are discussed immediately following the proof.

For any $W \subseteq R^N$, let $J(W)$ denote the set of all player indices i for which the projection of W onto the ith coordinate space is not a singleton. That is,

$$J(W) := \{i | w_i \neq w_i' \text{ for some } w, w' \in W\}.$$

A pair (q, u) is admissible w.r.t. W only if for each $i \notin J(W)$, $\Pi_i(q) \geqslant \Pi_i(\gamma, q_{-i})$ for all $\gamma \in S_i$, since $E_i(q; u) - E_i(\gamma, q_{-i}; u) = \Pi_i(q) - \Pi_i(\gamma, q_i)$ for all $\gamma \in S_i$.

DEFINITION: $q \in S$ satisfies the *Slater constraint qualification* with respect to W if there exists u such that (q, u) is admissible w.r.t. co W and

$$E_i(q; u) > E_i(\gamma, q_{-i}; u) \quad \text{for all } \gamma \in S_i \backslash \{q_i\} \text{ and } i \in J(W).$$

Let $A \subseteq R^N$. Denote "$\langle \alpha, x - y \rangle = 0$ for all $x, y \in A$" by $\alpha \perp A$, and "not $\alpha \perp A$" by $\alpha \not\perp A$. We refer to $\alpha \perp A$ as "α is perpendicular to A."

DEFINITION: For all $\beta \in R^N, \beta \neq 0$, and $W \subset R^N$ compact, let $F(\beta, W) := \arg\min\{\langle \beta, w \rangle | w \in W\}$ and $F(W) := \{F(\beta, W) | \beta \in R^N, \beta \neq 0 \text{ and } F(\beta, W) \not\subset \text{ext } W\}$.

If W were convex, $F(\beta, W)$ would be a face of W; $F(W)$ is comprised of all those "pseudo-faces" of W that contain nonextreme points of W. Notice that every pseudo-face is a subset of the boundary of W.

A rough intuitive explanation of Theorem 7 is as follows. Variations in continuation payoffs are needed to create incentives for players to choose the implicitly agreed upon actions. Some regions of the signal space Ω are particularly useful for the provision of these incentives. In a Green-Porter model, very low prices might be much less likely to occur if all players conform than if one deviates; this favorable likelihood ratio identifies a good region in which to "throw away surplus." In a less symmetric model, some subset of Ω might have higher probability if player 1 were to cheat than if 2 deviated. Here, perhaps one should transfer surplus from 1 to 2. Moreover, because of the linearity of the relevant optimization problems, there are no "decreasing returns" in using such areas intensively (making large transfers, or large movements in a particular direction). Only the size of the continuation value set W limits the intensity of the exploitation of very informative regions of Ω. Optimization consequently tends to push the continuation rewards to the extreme points of W. One of the cases in which it need not do so is discussed before the statement of Lemma 5.

THEOREM 7: *Let $W \subseteq R^N$ be compact, and consider*

$$(\bar{q}, \bar{u}) \in \arg\min\{\langle \alpha, E(q;u)\rangle \mid (q,u) \text{ is admissible w.r.t. } W\}$$

for some $\alpha \in R^N$, $\alpha \neq 0$. Suppose that (i) $g(p;\bar{q})$ is analytic in p, (ii) $\bar{q}$ satisfies the Slater constraint qualification with respect to W, (iii) $F(W)$ is a countable collection of sets,[6] *and (iv) $\alpha \not\perp F$ for all $F \in F(W)$. Then $\bar{u}$ satisfies the bang-bang property.*

PROOF: Let $\alpha, \bar{q}$, and $\bar{u}$ be as above. Then, $\bar{u}$ is a solution to:

$$\text{(P1)} \qquad \min\left\langle \alpha, \int_\Omega u(p) g(p;\bar{q})\, dp \right\rangle \qquad \text{subject to } u \in L^\infty(\Omega; \operatorname{co} W),$$

$$E_i(\gamma, \bar{q}_{-i}; u) \leqslant E_i(\bar{q}; u) \text{ for each } \gamma \in S_i \text{ and } i \in J(W),$$

where we have used Theorem 3 to replace W by $\operatorname{co} W$. The remark following the definition of $J(W)$ makes it clear that we may ignore incentive constraints for $i \notin J(W)$. We show that any solution to (P1) that has range W must have the bang-bang property. The Lagrangean associated with (P1) is

$$L(u,\lambda) = \begin{cases} +\infty & \text{if } u \notin L^\infty(\Omega; \operatorname{co} W), \\ \int_\Omega \langle u(p), \xi(p,\lambda)\rangle\, dp + b(\lambda) & \text{if } u \in L^\infty(\Omega; \operatorname{co} W) \\ & \quad \text{and } \lambda \geqslant 0, \\ -\infty & \text{if } u \in L^\infty(\Omega; \operatorname{co} W) \\ & \quad \text{and } \lambda \ngeqslant 0, \end{cases}$$

[6] Not all sets have this property. Consider a cylinder in R^3. There are an uncountable number of one-dimensional faces parallel to the axis of the cylinder. However, *every* convex set in R^2 has at most a countable number of faces of positive dimension. Otherwise, either its upper or lower boundary would include uncountably many faces. Projecting these onto the horizontal axis would give an uncountable number of disjoint, nondegenerate intervals, a contradiction.

where λ is the vector of Lagrange multipliers $\{\lambda_{i\gamma} | \gamma \in S_i,\ i \in J(W)\}$,

$$\xi_i(p,\lambda) := \left(\alpha_i - \sum_{\gamma \in S_i} \lambda_{i\gamma}\right) g(p;\bar{q}) + \sum_{\gamma \in S_i} \lambda_{i\gamma} g(p;\gamma,\bar{q}_{-i})$$

and

$$b(\lambda) := \sum_{i \in J(W)} \sum_{\gamma \in S_i} \lambda_{i\gamma}\left[\Pi_i(\gamma,\bar{q}_{-i}) - \Pi_i(\bar{q})\right].$$

Note for later use that the index function $\xi_i(p,\lambda)$ is analytic in p, and

$$\int_{\Omega} \xi(p,\lambda)\, dp = \alpha.$$

Also, by (ii), optimal Lagrange multipliers $\bar{\lambda} \geqslant 0$ exist (see Rockafellar (1974)) and any solution to (P1) also solves

(P2) $\quad \min L(u,\bar{\lambda})$ subject to $u \in L^{\infty}(\Omega; \operatorname{co} W)$.

It is clear that any optimal solution u of (P2) which has range W must be such that $u(p) \in \operatorname{arg\,min}\{\langle \xi(p,\bar{\lambda}), w\rangle \mid w \in W\} = F(\xi(p,\bar{\lambda}),W)$ a.e. $p \in \Omega$. That is, a.e. $p, u(p)$ lies in a "pseudo-face" of W and to complete the proof it suffices to show that there does not exist a subset of signals of positive measure for which $u(p)$ lies in pseudo faces which are not composed entirely of extreme points of W. Suppose there does exist $\Omega_0' \subset \Omega$ such that $\mu(\Omega_0') > 0$ (μ denotes the Lebesgue measure) and $F(\xi(p,\bar{\lambda}),W) \in F(W)$ for all $p \in \Omega_0'$. By assumption (iii), $F(W)$ is a countable collection. Hence, there exist $\Omega_0 \subseteq \Omega_0'$ and $\eta \in R^N$ such that $\mu(\Omega_0) > 0$ and $F(\xi(p,\bar{\lambda}),W) = F(\eta,W) \in F(W)$ for all $p \in \Omega_0$. By assumption, $\langle \xi(p,\bar{\lambda}), x-y\rangle = 0$ for all $p \in \Omega_0$ and any $x, y \in F(\eta,W)$. Since $\langle \xi(p,\bar{\lambda}), x-y\rangle$ is analytic in p and $\mu(\Omega_0) > 0, \langle \xi(p,\bar{\lambda}), x-y\rangle = 0$ for all $p \in \Omega$. Therefore, for each $x, y \in F(\eta,W)$, $0 = \int_{\Omega} \langle \xi(p,\bar{\lambda}), x-y\rangle\, dp = \langle \int_{\Omega} \xi(p,\bar{\lambda})\, dp, x-y\rangle = \langle \alpha, x-y\rangle$, where the last equality was established above. Hence, $\alpha \perp F(\eta,W)$, contradicting (iv). *Q.E.D.*

Condition (i) of Theorem 7 is a technical assumption that facilitates the dual line of proof we pursue. It is used to guarantee that the function $\langle \xi(p,\bar{\lambda}), x-y\rangle$ appearing near the end of the proof is analytic. This immediately implies that the function either has isolated zeroes, or is zero everywhere. Conditions (ii), (iii), and (iv) are stated in terms of restrictions on endogenous entities. Although ideally one would like to make assumptions on primitives (see the Corollary following Lemma 6), we think the theorem is broadly applicable because the conditions will arguably "often" be satisfied. When the set W is taken to be the equilibrium value set V, the Slater constraint qualification (condition (ii)) and condition (iv) hold generically in senses made precise by Lemmas 5 and 6. We do not have an analogous result for condition (iii), but as footnote 5 explains, (iii) is satisfied in *all* two-person games, or for any value set that is of no more than two dimensions. We remark that Theorem 7 does not apply to games with finite signal spaces.

Condition (iv) serves to exclude exceptional cases such as the following. Suppose that for signals in some set Ω_0 of positive measure, a reward function u supporting q optimally with respect to α takes on values in some face F of W. (For simplicity, the reader might think of the extreme case where W consists of a single face.) Consider another function u' supporting q, which on Ω_0 also takes on values in F, and coincides with u elsewhere. Such a u' will typically exist, as there will be many ways to satisfy the incentive constraints. If α happens to be perpendicular to F, u' yields the same value of the objective function as does u, and hence is a distinct solution to the optimization problem. Any nondegenerate convex combination $\hat{u}$ of u and u' is also an optimal solution. Of course, $\hat{u}$ fails to have the bang-bang property.

To understand the precise sense in which the Slater constraint qualification is satisfied generically, consider the following notation. For each $q \in S$, let $\underline{\delta}(q) := \inf\{\delta | q \text{ is supportable by } V(\delta)\}$ (if q cannot be supported by $V(\delta)$ for any $\delta \in (0,1)$, let $\underline{\delta}(q) := 1$). The independence condition used in Lemma 5 is analogous to those introduced by Fudenberg, Levine, and Maskin (1988).

LEMMA 5: *Let $q \in S$. Assume that the collection $\{g(\cdot\,; q)\} \cup \{g(\cdot\,; \gamma, q_{-i}) | \gamma \in S_i \setminus \{q_i\}, i = 1, \ldots, N\}$ is linearly independent. Then q satisfies the Slater constraint qualification with respect to $V(\delta)$ for any $\delta \in (\underline{\delta}(q), 1)$.*

PROOF: See the Appendix.

Condition (iv) is unrestrictive in the following sense: the set of directions for which it is violated has measure 0 in R^N.

LEMMA 6: *Let $W \subseteq R^N$ be compact, and assume $F(W)$ is a countable collection. Then the set $\Lambda := \{\beta \in R^N | \beta \perp F \text{ for some } F \in F(W)\}$ is of first category and has measure 0.*

PROOF: See the Appendix.

Suppose that players are cooperating as efficiently as possible, given the incentive constraints they face. Then for some welfare weights $\alpha \in R^N$, they are playing an S.E. σ that minimizes $\langle \alpha, v(\gamma) \rangle$ over all S.E.'s γ. Correspondingly, the pair (q, u), where $q = \sigma(1)$ and $u(p) = v(\sigma|_p)$ for all $p \in \Omega$, solves the minimization problem appearing in the statement of Theorem 7 (with $V = W$). Lemma 6 indicates that, assuming that V satisfies (iii), α will generically satisfy (iv) of Theorem 7; then by Theorem 7, $v(\sigma|_p)$ is an extreme point of V for almost all $p \in \Omega$. In order to extend this result on the necessity of bang-bang continuation reward functions beyond the first period, one needs to know that the continuation value $v(\sigma|_p)$ at the end of period 1 is an extreme point that can be supported by some welfare weights $\beta \in R^N$ satisfying (iv), relative to V. But for almost all $p, \xi(p, \lambda)$ is such a vector of weights (see the proof of Theorem 7). Thus, with probability 1 the continuation values of σ at the end of periods 1 and 2 (and, by induction, any period t) must be extreme points of V.

One sees in the proof of Theorem 7 that the reward function that supports a given $q \in S$ optimally for some welfare weights is essentially unique. Another implication of the necessity result is that in the analysis of strongly symmetric equilibria of symmetric repeated games, the maximum of the value set will be strictly lowered if punishment severity is reduced. In a finite action Green-Porter model, for example, losses will result from restricting attention to punishments no worse than "Cournot-Nash reversion." Even if this restriction *is* imposed (perhaps for considerations of simplicity), it is best to use punishments involving permanent reversion, rather than temporary reversion followed by resumed coöperation (take the set W in Theorem 7 to be the correspondingly restricted equilibrium value set). If in a particular example one requires the criterion for punishment to be a "tail test" when this is in fact inappropriate, a moderate punishment value may be constrained optimal; this explains the interior solutions for reversion time reported for certain cases in Porter (1983). Notice that in models having value sets of higher dimension, payoffs can be extremal without being in any sense severe.

Finally, observe that an immediate implication of Theorem 7, footnote 5, and Lemmas 5 and 6 is the following result expressed in terms of primitives.

COROLLARY: *In a two player game, let $q \in S$ and suppose $\delta \neq \underline{\delta}(q)$, $g(\cdot; q)$ is analytic, and that $\{g(\cdot; q)\} \cup \{g(\cdot; \gamma, q_{-i}) | \gamma \in S_i \setminus \{q_i\}, i = 1, 2\}$ is linearly independent. Then for almost all $\alpha \in R^N$, if $\underline{u} \in \arg\min\{\langle \alpha, E(q; u)\rangle | (q, u)$ is admissible w.r.t. $V(\delta)\}$, then $\underline{u}$ has the bang-bang property.*

8. CONCLUSION

Our purpose in this paper has been to contribute to the foundations of a systematic theory of repeated discounted games with imperfect monitoring. The results suggest that ultimately a fairly tractable and satisfying theory will emerge. Already available for a broad class of these games are powerful characterizations of the equilibrium value set, a variety of results on the nature of implicit reward functions generated by extremal equilibria, and comparative static and computational theorems. In addition, the limiting case as δ approaches 1 is particularly well understood as a result of the folk theorem literature mentioned in the Introduction. While some of our theorems, notably the bang-bang principle, specifically address the problems caused by imperfect monitoring, those in Section 3 (and, with appropriate qualifications, Sections 5 and 6) apply also to games with perfect monitoring. Not yet covered are hybrid cases falling between models with perfect monitoring and those having a publicly observed random signal with constant support. Also awaiting study are mixed strategy equilibria of repeated discounted games. These problems deserve much attention.

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APPENDIX

Measurability of the Strategies $\hat{\sigma}$

Lemma C below states that the functions Q and U can be chosen to satisfy certain measurability properties. These imply that the strategies $\hat{\sigma}$, constructed in the proof of the self-generation theorem, are indeed measurable functions. It is not sufficient to demonstrate the Lebesgue measurability of Q and U, because $\hat{\sigma}$ is comprised of compositions of these functions, and Lebesgue measurability is not preserved under composition. We work instead with universal measurability, which has the required composition property.

Let $\mathscr{B}$ denote the Borel σ-algebra of R^K, and for each Borel probability measure μ, let $\mathscr{B}(\mu)$ be the completion of $\mathscr{B}$ with respect to μ. $\mathscr{B}(\mu)$ is a σ-algebra which contains $\mathscr{B}$. The *universal σ-algebra* $\mathscr{U}$ is $\cap \mathscr{B}(\mu)$, where the intersection is over all Borel probability measures on R^K. We remark that $\mathscr{B}$ is contained in $\mathscr{U}$, which in turn is contained in the σ-algebra of Lebesgue measurable sets.

Function f: $R^K \to R^L$ is *universally measurable* if for every Borel set $Y \subseteq R^L$, $f^{-1}(Y) \in \mathscr{U}$. Every universally measurable function is Lebesgue measurable. The reader may find it helpful to consult the excellent treatment of this material in Bertsekas and Shreve (1978).

Define h: $\Omega \to R$ by $h(p) := \max_{q \in S} g(p;q)$ for each $p \in \Omega$, and for any Lebesgue measurable function u: $\Omega \to R^N$, let

$$\|u\|_h := \int_{\Omega} \|u(p)\| h(p)\, dp.$$

We denote by $L^1(\Omega; R^N; h)$ (L^1 for short) the set of equivalence classes of Lebesgue measurable functions u: $\Omega \to R^N$ such that $\|u\|_h < +\infty$, and endow it with the norm topology induced by $\| \ \|_h$. L^1 is a separable Banach space, and for any bounded set $W \subseteq R^N$, $L^\infty(\Omega; W)$ can be viewed as a subset of L^1.

For each $u \in L^1$, $p \in \Omega$, and $n \in N$, let

$$R(p,n) := \Omega \cap \bigtimes_{k=1}^{a} [p_k - 1/n, p_k + 1/n] \qquad (\text{recall } \Omega \subset R^a),$$

$$u^n(p) := (2n)^a \int_{R(p,n)} u(t)\, dt,$$

and $u^*(p) := \lim u^n(p)$ if the limit exists, and $u^*(p) = 0$ otherwise. If $v(E) := \int_{\Omega \cap E} u(p)\, dp$, u^* is almost everywhere the derivative of v with respect to the Lebesgue measure and coincides with u almost everywhere.

For a given bounded Borel measurable set $W \subseteq R^N$, and for any $q \in S$, let

$$\hat{W}_q := \{E(q;u) | (q,u) \text{ is admissible w.r.t. } W\},$$

and for each $p \in \Omega$ and $\hat{w} \in \hat{W}_q$, let

$$\Phi_q(p,\hat{w}) := \{u^*(p) | (q,u) \text{ is admissible w.r.t. } W \text{ and } E(q;u) = \hat{w}\}.$$

Lemmas A and B below prove that $\hat{W}_q$ and graph(Φ_q) are analytic sets. The set of continuous real functions on R^N, vanishing at infinity, is denoted by $C_0(R^N)$. In the proof of Lemma A we use the following result: the smallest class of real functions on R^N containing $C_0(R^N)$ and closed under bounded pointwise limits[7] is the class of bounded Borel functions.

LEMMA A: *For any Borel set $W \subseteq R^N$, $F := \{u \in L^1 | u(p) \in W$ a.e. $p \in \Omega\}$ is a Borel set of $L^1(\Omega; R^N; h)$.*

PROOF: The function $z \to \langle z, h\rangle := \int_\Omega z(p)h(p)\,dp$ from $L^1(\Omega; R; h)$ into R is continuous. Below we show that for each bounded Borel function f: $R^N \to R$, the function $u \to f \circ u$ from $L^1(\Omega; R^N; h)$ into $L^1(\Omega; R; h)$ is Borel measurable. Therefore, the function θ: $L^1(\Omega; R^N; h) \to R$ defined by $\theta(u) := \langle f \circ u, h\rangle$ is Borel measurable for any bounded Borel function f. In particular, let $f = 1_W$; then

$$F = \theta^{-1}\left(\left\{\int_\Omega h(p)\,dp\right\}\right),$$

and F is a Borel set as the preimage of a closed set.

Clearly, if $f \in C_0(R^N)$, the function $u \to f \circ u$ from $L^1(\Omega; R^N; h)$ into $L^1(\Omega; R; h)$ is continuous (and hence Borel measurable). Suppose that $\{f_n\}$ is a sequence of bounded Borel functions such that $u \to f_n \circ u$ is Borel measurable, $\sup_{n,x}|f_n(x)| < +\infty$, and $f(x) = \lim_n f_n(x)$ exists for each x. Then $f \circ u = \lim_n(f_n \circ u)$, and $u \to f \circ u$ is Borel as the limit of Borel measurable functions. Therefore, $u \to f \circ u$ is Borel measurable for any bounded Borel function f. *Q.E.D.*

LEMMA B: *For any bounded Borel set $W \subseteq R^N$, and for any $q \in S$, $\hat{W}_q$ is an analytic set in R^N and graph(Φ_q) is an analytic set in $\Omega \times R^N \times R^N$.*

PROOF: Let

$$G := \{(p,\hat{w},w,u) \in \Omega \times R^N \times R^N \times L^1 | (q,u) \text{ is admissible w.r.t. } W,\ \hat{w} = E(q;u), \text{ and } w = u^*(p)\}.$$

We will show that G is a Borel set. Since $\hat{W}_q$ and graph(Φ_q) are the projections of G into R^N and $\Omega \times R^N \times R^N$, respectively, they are analytic sets.

To show that G is a Borel set, we note that $G = G_1 \cap G_2 \cap G_3$, where

$$G_1 := \{(p,\hat{w},w,u) | \hat{w} = E(q;u) \text{ and for each } i \text{ and } \gamma_i \in S_i,\ \hat{w}_i \geqslant E_i(\gamma_i, q_{-i}; u)\},$$

$$G_2 := \{(p,\hat{w},w,u) | u(p) \in W \text{ almost everywhere } p \in \Omega\},$$

$$G_3 := \{(p,\hat{w},w,u) | w = u^*(p)\}.$$

Since for each $\gamma \in S$, $E(\gamma; \cdot): L^1 \to R^N$ is continuous, G_1 is a closed set. Lemma A implies that G_2 is a Borel set. Finally, let $G_4 := \{(p,\hat{w},w,u) | \lim u^n(p) \text{ exists}\}$. Then

$$[(p,\hat{w},w,u) \in G_3 \cap G_4] \text{ iff } [\forall m \in N\ \exists r \in N\ \forall n \geqslant r,\ |w - u^n(p)| < 1/m].$$

Since the set of $(p,\hat{w},w,u)$ satisfying $|w - u^n(p)| < 1/m$ is open, $G_3 \cap G_4$ is a Borel set, being formed by countable unions and intersections of Borel sets. Also, $(p,\hat{w},w,u) \in G_4$ iff $\{u^n(p)\}$ is a Cauchy sequence, and a similar argument shows that G_4 (and consequently G_4^c) is Borel; therefore, G_3 is a Borel set. *Q.E.D.*

Since graph(Φ_q) is an analytic set, the von Neumann selection theorem implies that the multifunction Φ_q admits a universally measurable selection ϕ_q: $\Omega \times \hat{W}_q \to R^N$ (see Bertsekas and

[7] By this we mean that if $\{f_n\}$ is a sequence in the class such that $\sup_{n,x}|f_n(x)| < +\infty$ and $f(x) = \lim f_n(x)$ exists for each x, then f is also in the class.

Shreve (1978)). Suppose $S = \{q(1), \ldots, q(r)\}$, and let $W_{q(1)} := \hat{W}_{q(1)}$ and

$$W_{q(k)} := \hat{W}_{q(k)} \setminus \bigcup_{l=1}^{k-1} \hat{W}_{q(l)}, \quad k = 2, \ldots, r.$$

Each W_q is a universally measurable set and $\{W_q\}$ is a partition of $\mathscr{B}(W)$. Fix $q^* \in S$. Define Q: $R^N \to S$ and ϕ: $\Omega \times R^N \to R^N$ by

$$Q(\hat{w}) := \begin{cases} q & \text{for each } \hat{w} \in W_q \text{ and } q \in S, \\ q^* & \text{for each } \hat{w} \notin B(W), \end{cases}$$

$$\phi(p, \hat{w}) := \begin{cases} \phi_q(p, \hat{w}) & \text{for each } p \in \Omega, \hat{w} \in W_q \text{ and } q \in S, \\ 0 & \text{for each } p \in \Omega, \text{ if } \hat{w} \notin B(W). \end{cases}$$

Finally, for each $p \in \Omega$ and $\hat{w} \in R^N$ let $U(\hat{w})(p) := \phi(p, \hat{w})$; for every $\hat{w} \in R^N$, $U(\hat{w}) \in L^1$. These observations are summarized in the next Lemma.

LEMMA C: *For any bounded Borel set $W \subseteq R^N$, $B(W)$ is a bounded universally measurable set of R^N, and there exists a pair of functions Q: $R^N \to S$ and U: $R^N \to L^1$ such that* (i) *for each $\hat{w} \in B(w)$, $(Q(\hat{w}), U(\hat{w}))$ is admissible w.r.t. W and $E(Q(\hat{w}); U(\hat{w})) = \hat{w}$; and* (ii) *the functions Q and ϕ, where ϕ: $\Omega \times R^N \to R^N$ is defined by $\phi(p, \hat{w}) := U(\hat{w})(p)$, are universally measurable.*

The composition of universally measurable functions is universally measurable. Therefore, for example, for each $\hat{w} \in \mathscr{B}(W)$, the function $\hat{\sigma}(\hat{w})(3)(p_1, p_2) = Q(U(U(\hat{w})(p_1))(p_2)) = Q(\phi(p_2, \phi(p_1, \hat{w})))$ is universally measurable in (p_1, p_2). Similarly, each $\hat{\sigma}(\hat{w})(t), t \geqslant 2$, is universally measurable and (a fortiori) Lebesgue measurable. Consequently, each $\hat{\sigma}(\hat{w})$ defined in the proof of self-generation is indeed a strategy profile.

Proof of Lemma 4

LEMMA 4: *Let $\{W_n\}$ be a decreasing sequence of compact sets in R^N. Then $\mathrm{co} \cap W_n = \cap \mathrm{co}\, W_n$.*

PROOF: Clearly $\cap W_n \subseteq \cap \mathrm{co}\, W_n$, and since $\cap \mathrm{co}\, W_n$ is convex, $\mathrm{co} \cap W_n \subseteq \cap \mathrm{co}\, W_n$. Conversely, we argue that $\mathrm{co} \cap W_n \supseteq \cap \mathrm{co}\, W_n$. Let $x \in \cap \mathrm{co}\, W_n$. By Carathéodory's Theorem, for each n there exist $\lambda^n \in R^{N+1}$ and $(w_1^n, \ldots, w_{N+1}^n) \in W_n^{N+1}$ such that $\lambda^n \geqslant 0$, $\Sigma_{i=1}^{N+1} \lambda_i^n = 1$, and $x = \Sigma_{i=1}^{N+1} \lambda_i^n w_i^n$. Since $\{\lambda^n\}$ is bounded, $\{(w_1^n, \ldots, w_{N+1}^n)\} \subseteq W_1^{N+1}$, and W_1 is compact, we can assume without loss of generality that $\lambda^n \to \lambda$ and $(w_1^n, \ldots, w_{N+1}^n) \to (w_1, \ldots, w_{N+1})$, where $\lambda \geqslant 0$ and $\Sigma_{i=1}^{N+1} \lambda_i = 1$. Since $(w_1^m, \ldots, w_{N+1}^m) \in W_n^{N+1}$ for each $m \geqslant n$, and W_n is compact, $(w_1, \ldots, w_{N+1}) \in W_n^{N+1}$ for all n. Thus $(w_1, \ldots, w_{N+1}) \in [\cap W_n]^{N+1}$, and by continuity $\Sigma_{i=1}^{N+1} \lambda_i w_i = x$. Therefore $x \in \mathrm{co} \cap W_n$. *Q.E.D.*

Genericity Results for Section 7

We show that under the following assumption on the density functions g, each $q \in S$ satisfies the Slater constraint qualification with respect to $V(\delta)$ for all $\delta > \underline{\delta}(q)$.

(LI) *The collection $\{g(\cdot; q)\} \cup \{g(\cdot; \gamma, q_{-i}) \mid \gamma \in S_i \setminus \{q_i\}, i = 1, \ldots, N\}$ is linearly independent.*[8]
For each $q \in S, \delta \in (0,1)$, and partition $\{I_k\}_{k=1}^K$ of Ω, let $S_i^* := S_i \setminus \{q_i\}$, and

$$g_k^0 := \int_{I_k} g(p; q)\, dp, \qquad k = 1, \ldots, K;$$

$$g_{\gamma k}^i := \int_{I_k} g(p; \gamma, q_{-i})\, dp, \quad k = 1, \ldots, K; \gamma \in S_i^*; i \in J(V(\delta));$$

$$a_{\gamma k}^i := g_k^0 - g_{\gamma k}^i, \qquad k = 1, \ldots, K; \gamma \in S_i^*; i \in J(V(\delta)).$$

[8] This assumption will not be satisfied in a symmetric model at a symmetric $q \in S$. The results below may be easily modified to deal with this case.

Then define the matrices $A^i := [a^i_{\gamma k}]_{\gamma, k}$ for $i \in J(V(\delta))$, and

$$A := \begin{bmatrix} A^i \\ \vdots \\ A^j \end{bmatrix}, \quad \text{where} \quad J(V(\delta)) = \{i, \ldots, j\}.$$

LEMMA D: *For each $q \in S$ and $\delta \in (0,1)$, there exists a partition $\{I_k\}_{k=1}^K$ of Ω such that I_k is measurable for each $k = 1, \ldots, K$, and the corresponding matrix A (as defined above) is full (column) rank.*

This technical result follows directly from (LI), and we omit its proof.

LEMMA 5: *Each $q \in S$ satisfies the Slater constraint qualification with respect to $V(\delta)$ for any $\delta \in (\underline{\delta}(q), 1)$.*

PROOF: Let $q \in S$ be such that $\underline{\delta}(q) < 1$, and let $\delta \in (\underline{\delta}(q), 1)$. Choose $e^0, e^1, \ldots, e^d \in V(\delta)$ such that $\min_l e^l_i \neq \max_l e^l_i$ for all $i \in J(V(\delta))$, and define $x^l := e^l - e^0, l = 1, \ldots, d$. Let $\delta' \in (\underline{\delta}(q), \delta)$. By Theorem 6 (Monotonicity), q is supportable by $V(\delta')$, and

$$\frac{(1-\delta')\delta}{\delta'(1-\delta)} V(\delta') \subseteq \operatorname{co} V(\delta).$$

Let Z be the subspace spanned by $\{x^l\}_{l=1}^d$ and let B_Z denote the unit ball in Z. It follows that there exist $t \in R^N$ and $\varepsilon > 0$ such that

$$(*) \qquad \{t\} + V(\delta') + \varepsilon B_Z \subseteq \operatorname{co} V(\delta).$$

Let $\{I_k\}_{k=1}^K$ and $\{A^i\}_{i \in J(V(\delta))}$ be as defined in Lemma D. Suppose that $J(V(\delta)) = \{i, \ldots, j\}$, and consider the system of linear equations

$$\begin{bmatrix} x^1_i A^i & x^2_i A^i & \ldots & x^d_i A^i \\ \vdots & & & \\ x^1_j A^j & x^2_j A^j & \ldots & x^d_j A^j \end{bmatrix} \begin{bmatrix} \lambda^1 \\ \vdots \\ \lambda^d \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

with unknowns $\lambda^l \in R^K$, $l = 1, \ldots, d$. Since by construction, for each $i \in J(V(\delta))$ there is l such that $x^l_i \neq 0$, the previous lemma implies that the matrix above is full rank, and so the system has a solution $\bar{\lambda}$. Let u' be such that (q, u') is admissible with respect to $V(\delta')$. Define

$$u(p) := t + u'(p) + \eta \sum_{l=1}^d \bar{\lambda}^l_k x^l \qquad \text{when} \quad p \in I_k, k = 1, \ldots, K.$$

By (*), for small enough $\eta > 0$, $u(p) \in \operatorname{co} V(\delta)$ for all $p \in \Omega$. Furthermore, since for each $\gamma \in S_i^*$ and $i \in J(V(\delta))$,

$$\begin{aligned} E_i(q; u|\delta) - E_i(\gamma, q_{-i}; u|\delta) &= \frac{\delta}{\delta'}\left[E_i(q; u'|\delta') - E_i(\gamma, q_{-i}; u'|\delta')\right] \\ &\quad + \delta\eta \sum_{k=1}^K a^i_{\gamma k}\left(\sum_{l=1}^d \bar{\lambda}^l_k x^l_i\right) \\ &\geqslant \delta\eta > 0, \end{aligned}$$

and for each $\gamma \in S_i^*$ and $i \notin J(V(\delta))$,

$$E_i(q; u|\delta) - E_i(\gamma, q_{-i}; u|\delta) = \frac{\delta}{\delta'}\left[E_i(q; u'|\delta') - E_i(\gamma, q_{-i}; u'|\delta')\right] \geqslant 0,$$

we are done. *Q.E.D.*

LEMMA 6: *Let $W \subseteq R^N$ be compact, and assume $F(W)$ is a countable collection. Then the set $\Lambda := \{\beta \in R^N \mid \beta \perp F \text{ for some } F \in F(W)\}$ is of first category and has measure 0.*

PROOF: For each $F \in F(W)$, define $F^{\perp} := \{\beta \in R^N | \beta \perp F\}$. Since $F^{\perp}$ is a subspace of R^N of dimension at most $N-1, F^{\perp}$ is nowhere dense and has measure 0. Since $F(W)$ is countable, it follows that $\Lambda := \cup_{F \in F(W)} F^{\perp}$ is of first category and has measure 0. *Q.E.D.*

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