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Fictitious Play, Shapley Polygons, and the Replicator Equation*

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For many normal form games, the limiting behavior of fictitious play and the time-averaged replicator dynamics coincide. In particular, we show this for three examples, where this limit is not a Nash equilibrium, but a Shapley polygon. *Journal of Economic Literature* Classification Numbers: C72, C73. © 1995 Academic Press, Inc.

1. INTRODUCTION

In this paper we study the two most important dynamic approaches to noncooperative game theory and point out some striking similarities, even coincidences, of their asymptotic behavior. The first is the classical *Brown–Robinson procedure,* or *fictitious play* (FP), the oldest and still most fundamental (deterministic) learning process. For simplicity we restrict our attention to the continuous time version (see Brown, 1951; Rosenmüller, 1971, which is given by

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$$\frac{d\mathbf{x}(t)}{dt} = \frac{1}{\alpha + t}(BR(\mathbf{x}(t)) - \mathbf{x}(t)), \tag{CFP}$$

where $BR(\mathbf{x})$ is a (or the set of all pure and mixed) best response(s) to the strategy profile $\mathbf{x}$ and $\alpha > 0$.

Without changing the paths we can omit the factor $1/(\alpha + t)$ (which only means a slowing down of time scale). The resulting autonomous differential equation[1] reads

$$\dot{\mathbf{x}} = BR(\mathbf{x}) - \mathbf{x}. \tag{BR}$$

This is the *best response dynamics* (see Matsui, 1992). The usual interpretation of this dynamic is that—in an infinite population of players—in each small time interval, a small fraction of players revises their strategies and changes to the present best choice. It is the prototype of modeling rational behavior.

The second dynamics we consider is the prototype of evolutionary dynamics, the *replicator equation* (RE); see, e.g., Hofbauer and Sigmund (1988) for 2-person games and Ritzberger and Vogelsberger (1990) for general n-person games. The interpretation is that—in an infinite population of replicating players—the growth rate of the frequencies of pure strategies is linearly related to their payoffs. This dynamics is smooth, in contrast to BR, and hence amenable to classical and modern tools of smooth dynamical systems.

In the following we compare these two dynamics for three particular examples: the rock–scissors–paper game (a symmetric 3×3 game, played within one population), whose state space is the two dimensional simplex; then a 3-person ($2 \times 2 \times 2$) matching pennies type game due to Jordan (1993), where the dynamics takes place on the three-dimensional cube; and finally the 3×3 bimatrix game of Shapley (1964), a four-dimensional example. To make life easy (for the reader and for us), we discuss the first example, where the dynamics is easy to visualize, in great detail, while the last example is only sketched.

[1] Since the right-hand side is discontinuous, this cannot really be viewed as a differential equation. A rigorous mathematical treatment is most naturally provided by the framework of differential inclusions. Since the best response correspondence $x \mapsto BR(x)$ is upper-semicontinuous, closed, and convex, the existence of at least one solution through each initial value, defined for all positive times, is guaranteed. These more basic questions are addressed in detail in Hofbauer (1994). In the three examples which we treat below, due to their cyclic character, solutions (which are continuous and piecewise linear) are easily seen to be unique, and we have in fact a continuous semiflow.

2. The Rock–Scissors–Paper Game

The payoff matrix of the rock–scissors–paper game is given by

$$A = \begin{pmatrix} 0 & -\mu_2 & \lambda_3 \\ \lambda_1 & 0 & -\mu_3 \\ -\mu_1 & \lambda_2 & 0 \end{pmatrix} \qquad (\lambda_i, \mu_i > 0). \tag{2.1}$$

Hence strategy 2 (rock) beats strategy 1 (scissors), 3 (paper) beats 2, and 1 beats 3. We assume neither cyclic symmetry for the payoffs nor that the game is zero sum. Let x_i denote the relative frequency of strategy i in a large population of players and $S_3 = \{x \in \mathbb{R}^3 : x_i \geq 0, \sum x_i = 1\}$ be the two-dimensional simplex, which is the set of all possible states of the population. The replicator dynamics (RE)

$$\dot{x}_i = x_i[(Ax)_i - x \cdot Ax] \qquad \text{on } S_3 \tag{2.2}$$

for this game was completely analyzed by Zeeman (1980). The pure strategies form a cycle of best responses. For (2.2) this implies that the boundary ∂S_3 forms a *heteroclinic cycle* (see Fig. 1a). This cycle may be either attracting or repelling, depending on the payoffs. The local dynamics near this cycle complements the local dynamics near the unique, interior Nash equilibrium (NE) E, which is given by

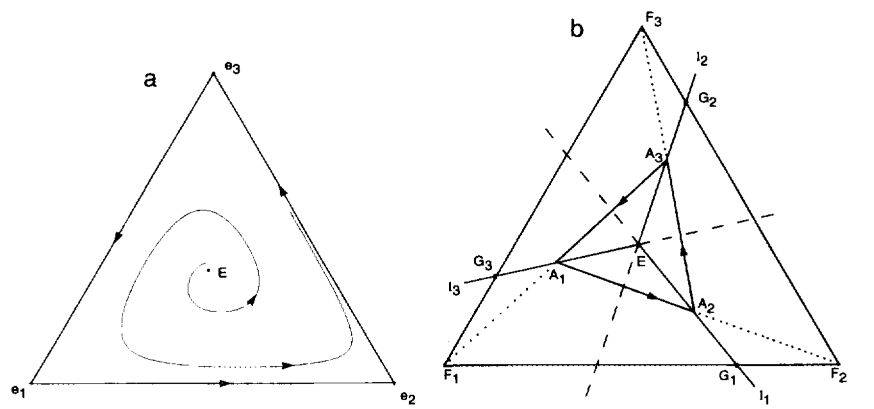

Fig. 1. Dynamics for the rock–scissors–paper game in case (c) of Theorem 1. (a) Divergence to the boundary in the replicator dynamics and (b) the Shapley triangle as attractor for BR.

$$E = \frac{1}{\Sigma}(\lambda_2\lambda_3 + \lambda_2\mu_3 + \mu_2\mu_3, \lambda_3\lambda_1 + \lambda_3\mu_1 + \mu_3\mu_1,$$
$$\lambda_1\lambda_2 + \lambda_1\mu_2 + \mu_1\mu_2) \tag{2.3}$$

with Σ the appropriate normalizing constant. The mean payoff, $\pi(x) = x \cdot Ax$ takes the equilibrium value

$$\pi(E) = \frac{\lambda_1\lambda_2\lambda_3 - \mu_1\mu_2\mu_3}{\Sigma}. \tag{2.4}$$

Zeeman (1980, Theorem 6) proved the following result on the dynamics of (2.2):

THEOREM 1. (a) *If $\lambda_1\lambda_2\lambda_3 > \mu_1\mu_2\mu_3$, or $\pi(E) > 0$, then the heteroclinic cycle is repelling and the equilibrium E is globally attracting for* (2.2).

(b) *If $\lambda_1\lambda_2\lambda_3 = \mu_1\mu_2\mu_3$, or $\pi(E) = 0$, then all interior orbits are closed. The time average over one full period equals E.*

(c) *If $\lambda_1\lambda_2\lambda_3 < \mu_1\mu_2\mu_3$, or $\pi(E) < 0$, then E is repelling and the heteroclinic cycle is the global attractor for* (2.2).

The situation is particularly simple when there is cyclic symmetry: then the condition simplifies to $\lambda \gtreqless \mu$. In case (a), the NE is then even an ESS.[2]
We now compare this with the limiting behavior of fictitious play and best response paths.

THEOREM 2. *Fictitious play and best response dynamics converge in cases* (a) *and* (b) *to the equilibrium E.*[3] *In case* (c) *the limit of* BR *paths is a triangle $A_1A_2A_3$ given by* (3.6) *below; moreover, the time averages*[4]

$$z(T) = \frac{1}{T}\int_0^T x(t)\,dt \tag{AR}$$

of the solutions x(t) of the replicator equation (2.2) *"converge" to the same triangle.*

We suggest calling such polygons, which are attractive or at least invariant

[2] In general, the NE is an ESS iff $\lambda_i > \mu_{i+1}$ and the three numbers $\sqrt{\lambda_i - \mu_{i+1}}$ form the sides of a triangle; see Hofbauer and Sigmund (1988, p. 131).

[3] Case (b) behaves like a zero-sum game: FP converges like $1/t$ and BR like e^{-t}. In case (a), both processes reach the equilibrium in *finite* time! AR goes like t^{-1} in both (a) and (b).

[4] We suggest calling this the AR (averaged replicator) process. Its paths satisfy the differential equation $\dot{z}(t) = (1/t)(x(t) - z(t))$, which is similar to CFP.

under the BR dynamics, *Shapley polygons*, as the first such example was given by Shapley (1964).

The creation of this Shapley triangle during transition from case (a) to (c) is reminiscent of a (supercritical) Hopf bifurcation: In the critical case (b), E is still attractive; increasing the μ_i in case (c), a small triangle (a stable limit cycle for the BR dynamics) grows out of E. The only difference is that the size of the triangle grows linearly with the bifurcation parameter (see subsection 3.5), whereas in the typical Hopf bifurcation of smooth dynamics the radius of the periodic orbits grows like the square root of the parameter, which is much faster. The replicator equation also undergoes a Hopf bifurcation during the transition from case (a) to (c), but it is degenerate (like for linear differential equations): All periodic orbits occur simultaneously in the critical case (b).

For the replicator equation (and more general ODEs), this peculiar divergent behavior of the time averages in case (c) was shown to (one of) us by Christopher Zeeman, when we raised this question during the Animal Conflicts Meeting in Sheffield in 1980. According to Takens (1994, and personal communication), Rufus Bowen (who died in 1978) was also aware of this nonconvergence of time averages near an attracting heteroclinic cycle. A detailed proof of this phenomenon was first published by Gaunersdorfer (1992) for general heteroclinic cycles in the plane and even for some three-dimensional examples. For Lotka–Volterra equations and the replicator equation, the special structure allows simpler proofs; see Hofbauer and Sigmund (1988, pp. 69, 70), Akin (1993), and Section 3 below.

That this limiting triangle for AR coincides with the Shapley triangle (i.e., the attractor for the BR process) has also been known to us for some time. We discovered this soon after Eric van Damme made us aware of Shapley's (1964) famous example, after a game theory meeting in 1984 in Bielefeld. We are indebted to I. Bomze for the initial collaboration on this problem back in 1985.

An example of a rock–scissors–paper game with a Shapley triangle as attractor for BR was recently given by Gilboa and Matsui (1991), as a simple and instructive example for their notion of a "cyclically stable set" (CSS).[5]

Remark. We agree with Gilboa and Matsui (1991) and Matsui (1992) that in case (c) the Shapley polygon is a much better candidate for a

[5] In dynamical systems jargon, a CSS is essentially an attractor. More precisely, it is a basic set (a component of the chain recurrent set) that is Lyapunov stable or, equivalently, an invariant set where one (or each) complete Lyapunov function attains a (local) minimum. See Akin (1993) for an excellent modern survey of topological dynamics. The definitions of CSS used in the subsequent paper by Matsui (1992) are different and much weaker: they lack Lyapunov stability. The equilibrium E would be "cyclically stable" in the latter sense even in case (c)! For a more detailed discussion see Hofbauer (1994).

"solution" of the game than the unstable Nash equilibrium. The condition for case (c) is equivalent to

$$\pi(E) < 0 \qquad \text{or} \qquad \pi(E) < \sum_i \bar{x}_i \pi(e_i) \tag{2.5}$$

if the payoff matrix (2.1) is not normalized to have zeroes in the diagonal. Here $E = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is the equilibrium (2.3), e_i is the ith unit vector (corresponding to pure strategy i), and $\pi(e_i) = a_{ii}$. So (2.5) means that the population at equilibrium E would gain from separating itself into three pure subpopulations. Hence in case (c) the Shapley polygon results in a higher payoff or "value" compared to the Nash equilibrium. As we will show below (see subsections 3.3 and in particular 3.6), each of the three strategies enjoys the payoff 0 (which is larger than $\pi(E)$) along one of the sides of the Shapley triangle. The Shapley polygon is in this sense more Pareto efficient than the Nash equilibrium.

3. Proofs of Theorem 2

In the following, we present several proofs of this result. We indicate at the beginning of each subsection which of the dynamics (RE or BR) is treated. Some of these proofs can be used also in higher dimensions. Some of them are simple adaptations of standard techniques from smooth dynamical systems to the piecewise linear best response dynamics, like Lyapunov functions, Poincaré sections, and the Bendixson test for the uniqueness of a limit cycle. Although this may be tiresome for the reader, we think that some of these techniques may be useful for related problems. The most appealing proof for a game theorist is probably the (very) last proof in subsection 3.6.

3.1 (RE). The *first proof* consists in computing the sojourn times near pure strategies and determining their rate of exponential increase. For RE this requires a computation of Poincaré maps. (CFP will be considered in subsection 3.2.) The merits of this method lie in its generality: It applies to more complicated dynamics (see Section 4) and would apply also to nonlinear payoff functions.

Now to the details. Let $\dot{x} = f(x)$ be an ODE defined in a subset of $\mathbb{R}^2$ and let $F_1, \ldots, F_n$ be saddle points, which together with n connecting orbits from F_i to F_{i+1} (i counted cyclically modulo n) form a heteroclinic cycle Γ. Each saddle point F_i of Γ has a positive eigenvalue $\lambda_i > 0$ and a negative eigenvalue $-\mu_i < 0$. Then Γ is attracting if

$$\rho := \prod_{i=1}^{n} \frac{\mu_i}{\lambda_i} > 1 \tag{3.1}$$

(i.e., the product of the "outgoing velocities" is smaller than the product of the "incoming velocities"), and it is repelling if $\rho < 1$; see Hofbauer and Sigmund (1988, p. 305). In the former case, consider a solution $x(t)$ that tends to Γ as $t \to \infty$. Denote by t_i the time it spends near the fixed point F_i. Then these sojourn times increase geometrically. More precisely, see Gaunersdorfer (1992, Lemma 1),[6]

$$\frac{t_{i+1}}{t_i} \to \rho_{i+1} := \frac{\mu_i}{\lambda_{i+1}} \tag{3.2}$$

as the orbit approaches Γ. As a consequence, for consecutive sojourn times near F_i, we have

$$\frac{t_{i+n}}{t_i} \to \rho := \prod_{i=1}^{n} \rho_i = \prod_{i=1}^{n} \frac{\mu_i}{\lambda_i}. \tag{3.3}$$

The proof of this requires the approximate computation of the Poincaré map near Γ, assuming that the vector field is linear near the saddle points. This can be achieved due to a theorem of Hartman on smooth linearization in two dimensions.

Knowing the rate of increase of the sojourn times, one can compute the accumulation points of the time averages using the following result in Gaunersdorfer (1992).

LEMMA 1. *Let $x(t)$ be a curve in $\mathbb{R}^n$ that cycles between n points $F_1, \ldots, F_n$, such that the sojourn times t_i near F_i increase geometrically as in (3.2), i.e., $t_{i+1}/t_i \to \rho_{i+1}$ with $\rho = \prod_{i=1}^{n} \rho_i > 1$, and the intermediate times are bounded. Then the accumulation points, as $T \to \infty$, of the time average $1/T \int_0^T x(t)\, dt$ form the boundary of the polygon $A_1 \ldots A_n$, where*

$$A_i = \frac{F_{i+1} + \rho_{i+2}F_{i+2} + \cdots + \rho_{i+2} \cdots \rho_{i+n}F_i}{1 + \rho_{i+2} + \cdots + \rho_{i+2} \cdots \rho_{i+n}}. \tag{3.4}$$

The points A_i, A_{i+1}, and F_{i+1} are collinear.

Thus, asymptotically, the time averages move on a line from A_i to A_{i+1} in the direction to F_{i+1}. In higher dimensions, the computation of ρ_i,

[6] Here the index i of t is a natural number, while for ρ, λ, μ it is reduced modulo n.

the asymptotic ratio of the sojourn times near F_i and F_{i-1}, is more involved than the simple formula (3.2) valid in $\mathbb{R}^2$; see Sections 5 and 6 below.

Now, the rock–scissors–paper game has the heteroclinic cycle

$$\Gamma: F_1 = (1, 0, 0) \to F_2 = (0, 1, 0) \to F_3 = (0, 0, 1) \to F_1.$$

It is attracting (compare (3.1)) if and only if

$$\mu_1\mu_2\mu_3 > \lambda_1\lambda_2\lambda_3 \tag{3.5}$$

(note that λ_i and $-\mu_i$ are the eigenvalues at F_i). By Lemma 1 and Eq. (3.2), the time averages approach the triangle $A_1A_2A_3$ (see Fig. 1), with

$$\begin{aligned} A_1 &= \frac{1}{\lambda_1\lambda_3 + \lambda_1\mu_2 + \mu_2\mu_3}(\mu_2\mu_3, \lambda_3\lambda_1, \lambda_1\mu_2) \\ A_2 &= \frac{1}{\lambda_1\lambda_2 + \lambda_2\mu_3 + \mu_1\mu_3}(\lambda_2\mu_3, \mu_3\mu_1, \lambda_1\lambda_2) \\ A_3 &= \frac{1}{\lambda_2\lambda_3 + \lambda_3\mu_1 + \mu_1\mu_2}(\lambda_2\lambda_3, \lambda_3\mu_1, \mu_1\mu_2). \end{aligned} \tag{3.6}$$

It is easy to check that in the limiting case (b) $\lambda_1\lambda_2\lambda_3 = \mu_1\mu_2\mu_3$, the three points A_i coincide with the equilibrium E. On the other hand, if $\lambda_i \to 0$, then A_i converges to the corner F_i.

3.2 (CFP). To obtain the corresponding result for the FP process, we follow Shapley (1964) to compute the time periods. The following geometric picture is helpful (see Fig. 1b).

We consider the three lines through the equilibrium E, where two of the pure strategies have the same payoff $\{x \in S_3 : (Ax)_i = (Ax)_j\}$ and the rays where the third strategy has a smaller payoff $l_k = \{x \in S_3 : (Ax)_i = (Ax)_j > (Ax)_k\}$ (with $\{i, j, k\} = \{1, 2, 3\}$.) This ray l_k is the borderline between the two regions where i and j, respectively, are the best replies.

If the process starts with x, say on l_2, then the solution of CFP is given by

$$x(t) = \frac{\alpha x + te_1}{\alpha + t} \tag{3.7}$$

for $0 \le t \le t_1$, as long as 1 is the best reply to $x(t)$. Hence we have the two equations (w.l.o.g. we can scale $\alpha = 1$)

$$-\mu_2 x_2 + \lambda_3 x_3 = -\mu_1 x_1 + \lambda_2 x_2 \qquad (x \in l_2) \tag{3.8a}$$

$$-\mu_2 x_2 + \lambda_3 x_3 = \lambda_1(x_1 + t_1) - \mu_3 x_3 \qquad (x(t_1) \in l_3). \tag{3.8b}$$

After another period of time t_2, during which 2 is the best reply for $x(t)$, the CFP path reaches l_1:

$$\lambda_1(x_1 + t_1) - \mu_3 x_3 = -\mu_1(x_1 + t_1) + \lambda_2(x_2 + t_2) \qquad (x(t_1 + t_2) \in l_1). \tag{3.8c}$$

Comparing the right-hand sides of (3.8a) and (3.8c), we obtain $t_2/t_1 = \mu_1/\lambda_2 = \rho_2$ for the times t_i during which strategy i is played. Repeating this argument, comparing with (3.2), and applying Lemma 1, the "convergence" of any CFP path (different from E) to the Shapley triangle (3.6) follows in case (c). In cases (a) and (b), the times t_i between changes of direction along the CFP path go to zero or remain constant. Since the speed of a CFP path goes to zero, this implies convergence to the equilibrium, reaching it already after a finite amount of time in case (a).

3.3 (AR) *Another proof of Theorem* 2. (The AR result in case (c)) uses the following method, also employed by Hofbauer and Sigmund (1988, pp. 69–70) and by Akin (1993, p. 183).

Let z be an accumulation point of the time averages $z(T) = 1/T \int_0^T x(t)\,dt$, i.e., $z = \lim_{k\to\infty} z(T_k)$ for some sequence $T_k \to \infty$. By refining this sequence, we can assume that also the limit $\bar{x} := \lim_{k\to\infty} x(T_k)$ exists.

Dividing (2.2) by x_i and integrating from 0 to T_k, we obtain

$$\lim_{k\to\infty} \frac{\log x_i(T_k) - \log x_i(0)}{T_k} = \lim_{k\to\infty} \frac{1}{T_k} \sum_j a_{ij} \int_0^{T_k} x_j(t)\,dt - \lim_{k\to\infty} \frac{1}{T_k} \int_0^T x \cdot Ax\,dt.$$

The second limit on the right-hand side is zero: The orbit spends most of the time near the corners of S_3, where the mean payoff $x \cdot Ax = 0$, by our assumption $a_{ii} = 0$. Thus,

$$\lim_{k\to\infty} \frac{\log x_i(T_k) - \log x_i(0)}{T_k} = \sum_j a_{ij} z_j. \tag{3.9}$$

For $\bar{x}$ there are two possibilities:

(i) $\bar{x}$ lies on one of the saddle connections $F_j \to F_{j+1}$, w.l.o.g. we assume $\bar{x}_1 = 0, \bar{x}_2, \bar{x}_3 \neq 0$. Then (3.9) yields

$$\sum_j a_{1j} z_j \leq 0 \qquad \text{and} \qquad \sum_j a_{ij} z_j = 0 \qquad \text{for } i = 2, 3. \tag{3.10}$$

From the two equations, together with $z_1 + z_2 + z_3 = 1$, one can compute $z = A_2$ which gives (3.6).

(ii) $\bar{x}$ is one of the corners, say F_1. Then the equality and inequality signs in (3.10) are interchanged and (3.10) determines the line segment A_3A_1. ■

Note that along these line segments, the largest payoff, $\max_i (Ax)_i$, is 0. Note further that *the interior of the triangle $A_1A_2A_3$ is the region where all payoffs $(Ax)_i$ are negative.* Compare this with the concluding remark in Section 2.

3.4 (AR, BR) *Next proof.* This proof has a more geometric flavor. We first show that, in case (c), AR behaves asymptotically like BR paths. Then we discuss the asymptotic behavior of the BR dynamics.

As we have seen in subsection 3.2, the rays l_k are the turning points for the CFP and BR paths: Before the path reaches l_3, it points toward F_1, afterwards toward F_2. In contrast, the vector field of the replicator equation (and any monotone selection dynamics in the sense of Nachbar, 1990) points straight away from F_3, as a kind of compromise, on the ray l_3. Since it changes smoothly, it does not adapt to the new orderings as abruptly as BR. RE points exactly toward F_1 only at the instant when strategies 2 and 3 are equally worse than 1. As soon as 2 fares slightly better than 3 (while both are worse than 1), the RE anticipates the next leader and lets 2 grow relative to 3. In contrast, after the RE path crosses l_3, RE is more conservative than BR in reducing the proportion of 1: In fact, x_1 continues to increase for some time. However, on the average, the effect is the same.

Consider now, for RE in case (c), the change of the ratios

$$\left(\frac{x_i}{x_j}\right)^{\cdot} = \left(\frac{x_i}{x_j}\right)[(Ax)_i - (Ax)_j]. \tag{3.11}$$

Integration from 0 to T gives

$$\frac{1}{T}\left(\log \frac{x_i}{x_j}(T) - \log \frac{x_i}{x_j}(0)\right) = (Az(T))_i - (Az(T))_j. \tag{3.12}$$

Suppose the sequence $T_k \to \infty$ is such that $(x_i/x_j)(T_k) \to c > 0$, a finite constant, and $z(T_k) \to \bar{z}$. Then $(A\bar{z})_i = (A\bar{z})_j$. So while orbit $x(T)$ moves from F_1 to F_2, the time average sits somewhere on (or rather close to) the ray l_3. During the long period of time when the orbit stays near F_2, the time average $z(T)$ will move (nearly) along a line toward F_2. When the orbit $x(T)$ starts moving away from F_2, toward F_3, the time average $z(T)$ has just reached l_1. This shows that, in the long run, the time averages $z(T)$ behave like a BR path. So they will finally cycle along a Shapley triangle. The argument will be complete if we can show that there is a unique Shapley triangle.

LEMMA 2. *For the rock–scissors–paper game* (2.1), *there is at most one Shapley triangle.*

First proof.[7] Suppose there are two Shapley triangles $A_1A_2A_3$ and $B_1B_2B_3$, such that $A_i, B_i \in l_{i-1}$. Then these two triangles are perspective from the equilibrium E. Since their (extended) sides A_1A_2 and B_1B_2 intersect in the corner F_2, etc., *Desargues' theorem* implies that the three intersection points F_1, F_2, F_3 are collinear. Obviously, this is not the case. ∎

Unfortunately, this geometric argument does not seem helpful for the question of existence of a Shapley triangle, nor does it give its position. Its main drawback is that it does not extend to higher dimensions. The following modification of this argument from projective geometry does.

Second proof. We introduce a distance on each of the rays l_k, which strictly decreases from one turn to the next. So the result follows from a contraction argument. The usual Euclidean distance would not do that job, but a projective version does. For A, B on l_k define $d_k(A, B) = |\log (AE/BE : AG_k/BG_k)|$, with G_k being the intersection point of l_k with the boundary of the simplex S_3; see Fig. 1b. It is easy to see (and well known) that this defines a metric. Now the double ratio of four collinear points is invariant under projective maps. But since G_k is mapped to an interior point of l_{k+1}, we get $d_{k+1}(A', B') < d_k(A, B)$ for the images A', B' on l_{k+1} of A, B. Hence there can be at most one Shapley triangle. ∎

Third proof. In the interior of each of the three regions which are bounded by the rays l_1, l_2, l_3 and where the best response is a unique pure strategy, the flow of (BR) moves straight toward a corner of the triangle. Hence this (linear) flow contracts area (at the exponential rate -2). At the equilibrium this need not be the case, because of the discontinuity of the vector field. Also on the border lines l_k the same reservation holds, but there it does not matter, since BR paths spend only one time instant on

[7] We owe this beautiful proof to our colleague, G. Kowol.

some l_k (while these time instants can accumulate near E). Hence, after excision of an (arbitrarily small) neighborhood of E, we obtain an area contracting flow on the remaining annular region. By a variant of Bendixson's test[8] this flow has at most one periodic orbit. ■

The last two proofs actually show that if there is a Shapley triangle, then it attracts all BR paths that start off the equilibrium E. On the other hand, if there is no Shapley triangle, then all BR paths converge to the equilibrium (because they move in from the boundary).

Hence, we have obtained another proof of Theorem 2, with the exception that this geometric approach gives neither the explicit condition (3.5) for the existence of the Shapley triangle nor its precise position (3.6).

We conclude our lengthy discussion of the rock–scissors–paper game with the following, probably most straightforward and obvious proof of Theorem 2 (the BR part).

3.5 (BR) *Last proof.*[9] Let s_i denote the (usual) distance of a certain point X_i on l_i from E. If X_i, X_{i+1}, and F_{i+1} are collinear, then an easy calculation (or a geometric argument) shows that $s_{i+1} = a_i s_i/(1 + b_i s_i)$. Hence the full return map τ: $l_1 \to l_1 : X_1 \mapsto X_4$ is of the same form. More precisely, a computation shows that $\tau(s) = \rho s/(1 + bs)$ with ρ from (3.1) and some positive constant b. Obviously iteration of this map leads to $\tau^n(s) \to 0$ if $\rho \leq 1$ (cases (a) and (b)) and, in case (c), to convergence to a positive fixed point, $\tau^n(s) \to (\rho - 1)/b$. This fixed point of the return map determines the Shapley triangle. Note the linear dependence on the parameter ρ which explains a comment in Section 2. ■

The rate of convergence (see footnote 3) of CFP and BR in case (b) and the fact that paths reach the equilibrium E even in finite time (after infinitely many revolutions around it) in case (a) are easily established from either this or the following proof. Similarly, in case (c), BR and CFP orbits inside the Shapley polygon have left E a finite amount of time ago. This behavior of BR is in contrast to that of smooth dynamical systems.

3.6 (BR) *Very last proof.* This proof uses a Lyapunov function for the BR dynamics:[10] Consider $V(x) = \max_i (Ax)_i$. In the region where this maximum is attained by strategy k, we have $\dot{V} = (A\dot{x})_k = a_{kk} - (Ax)_k =$

[8] The better known version says that an area contracting flow in a simply connected region in the plane has no closed orbit. The proof is a simple application of the Gauss–Green formula; see, e.g., Hofbauer and Sigmund (1988, p. 149). The same argument shows that in an annular region such a flow can have at most one periodic orbit.

[9] In dynamical systems jargon this runs under computation of the Poincaré map. For smooth dynamical systems this can rarely be done explicitly. For the piecewise linear BR dynamics it is easy.

[10] See also Hofbauer (1994).

$-V(x)$. Hence $V(x(t))$ decreases as long as it is positive and increases when it is negative. The minimum value of $V(x)$ is attained at the equilibrium E (due to our normalization (2.1) with zeroes in the diagonal), where it coincides with the mean payoff $\pi(E)$. Hence in cases (a) and (b), V decreases monotonically to $\pi(E)$ (the value of the game). In case (c), the absolute value, $|V|$, decreases monotonically. Its minimum 0 is attained at the Shapley triangle. ■

This argument probably justifies our concluding remark in Section 2.

4. Other Dynamics

One might think that—since both processes CFP and AR are obtained by taking time averages—this result should extend to (averages of) more general dynamics, e.g., the *monotone selection dynamics* of Nachbar (1990). However, this is not the case. Take for example a dynamics of the form

$$\dot{x}_i = x_i \left[f((Ax)_i) - \sum_j x_j f((Ax)_j) \right], \tag{4.1}$$

with f a strictly increasing function $\mathbb{R} \to \mathbb{R}$ with $f(0) = 0$. If $f'(0) > 0$ then condition (3.5) still decides the *local* (in)stability of E, as can be easily seen from linearization around E. However, the behavior near the boundary may be different. Indeed, the eigenvalues at the pure strategy e_j are given by $(\dot{x}_i/x_i)|_{e_j} = f(a_{ij}) - f(a_{jj})$. For the payoff matrix (2.1) we obtain the eigenvalues $f(\lambda_i)$ and $f(-\mu_i)$ at e_i. From (3.1), the condition for stability of the boundary cycle becomes $\prod f(\lambda_i) < \prod |f(-\mu_i)|$. Hence (even in the case of cyclic symmetry or when E is an ESS), the local dynamics of (4.1) near the heteroclinic cycle may be different from that of the RE, and the triangle of time averages (which is determined by the expressions in (3.6), but with λ_i, $-\mu_i$ replaced by the new eigenvalues)—if it is a triangle at all and not a single point—is in general different from the Shapley triangle (3.6).

An interesting phenomenon happens if we add small *mutation terms* to the replicator equation,[11] as in Hofbauer and Sigmund (1988, Chap. 25): In case (c), this results in a globally stable limit cycle inside (but close to the boundary of) the simplex S_3. The time average of this periodic orbit (taken over one period) is a certain point in S_3, which may depend on the

[11] This idea can be used to give a simple proof for the existence of Nash equilibria and the odd number theorem (see Hofbauer and Sigmund, 1988, pp. 166–168, 274; Ritzberger, 1995). It also leads to new, evolutionary equilibrium refinements; see Boylan (1994).

details of the mutation rates. The question arises, which points can be obtained that way, using arbitrarily small mutations? The answer is: *Any point inside or on the Shapley triangle, but no point outside it!* (The proof of this is a bit technical and will not be given here.) Maybe this is another "evolutionary" justification for the Shapley polygon as the "solution" of the game.

Another point of departure might be to try *discrete time dynamics.* The standard discrete time version of the replicator equation, used already by Maynard Smith, is given by

$$x_i' = x_i \frac{C + (Ax)_i}{C + x \cdot Ax}, \tag{4.2}$$

with C, the "background fitness," a sufficiently large positive constant to compensate for negative entries in the payoff matrix. The rock–scissors–paper game was analyzed for (4.2) by Hofbauer (1984, Theorem 4) (see also Hofbauer and Sigmund, 1988, p. 134) and by Weissing (1991). In the case of cyclic symmetry, the dynamical behavior is completely analogous to Theorem 1 above, but in the general case the situation can be more complicated, e.g., with multiple "limit cycles"; see Section 7 in Weissing (1991). Anyway, the local behavior near the boundary can be analyzed with the same method as above. The result resembles the situation for (4.1): It follows from Gaunersdorfer (1992, p. 1488) that the stability condition of the boundary cycle is again different from (3.5), and hence also the limiting triangle differs in general from the Shapley triangle. Only in the limit $C \to \infty$ do the results become the same.

However, there is a different, exponential, version of discrete time dynamics, which has been studied in collaboration with I. Bomze and K. Sigmund (1985, unpublished) and by Cabrales and Sobel (1992):

$$x_i' = x_i \frac{e^{(Ax)_i}}{\sum_j x_j e^{(Ax)_j}}. \tag{4.3}$$

This dynamics shows more complicated behavior than (RE) or even (4.2), but it has similar averaging properties like (RE). In particular, for the rock–scissors–paper game (2.1), the stability conditions for the boundary cycle are again exactly determined by the inequality (3.5), and the time averages of nearby orbits behave as in Theorem 2: Orbits approaching the boundary cycle have time averages "converging" to the Shapley triangle (3.6), while orbits staying away from the boundary have time averages converging to E. This follows again from Gaunersdorfer (1988, p. 1488) and Cabrales and Sobel (1992, Prop. 2), respectively. The only difference is that in case (c), the heteroclinic cycle is not necessarily globally attractive, and hence both types of orbits may coexist.

We conclude this section with a remark on 2×2 bimatrix games: Here even for the considerably more restricted class of *aggregate monotone dynamics* of Samuelson and Zhang (1992) the asymptotic behavior of time averages may be completely different. Take the "matching pennies" game, with the following payoff matrix and dynamics:

1, −1	−1, 1
−1, 1	1, −1

$$\begin{aligned} \dot{x} &= x(1 - x)(2y - 1)a(x, y) \\ \dot{y} &= y(1 - y)(1 - 2x)b(x, y) \end{aligned} \tag{4.4}$$

For $a = b \equiv 1$ we have the replicator equation. It is well known that its solutions are periodic and the time averages over one period coincide with the unique equilibrium E. Also fictitious play and best response dynamics converge for this (zero-sum) game to the Nash equilibrium. However, for (4.4), with suitable positive functions a, b, a calculation of the eigenvalues at the four pure strategy profiles, inserted into the stability criterion (3.1), shows that the boundary may be an attractive heteroclinic cycle, and the time averages of nearby orbits "converge" to a quadrangle. For other choices of the functions a, b the boundary is repelling, and there may be limit cycles. But again, there is no reason for the time average over such a periodic solution to coincide with the equilibrium E. Similar remarks apply to the examples in the following two sections: changing from RE to an aggregate monotone dynamics will move the limiting polygon of the time averages away from the Shapley polygon.

So these examples show a remarkable coincidence of the asymptotic behavior of BR (CFP) and AR which is not shared by (the time averages of) other evolutionary dynamics.[12]

5. Jordan's Example: A $2 \times 2 \times 2$ Game

The original example of Jordan (1993) is the following version of a *matching pennies game.* Three persons throw a coin. Person 1 wins if he matches person 2, 2 wins if he matches 3, 3 wins if he *does not* match 1. We

[12] We like to interpret this as a mutual support of the two classical dynamics BR (FP) and RE, against the present overwhelming invasion of more general adjustment and learning processes.

consider the following modification (which is actually only a renumbering of the strategies).

"*Wer imitiert, verliert.*" Three persons on a round table (cyclic symmetry) each throw a coin. Anybody who matches his right neighbor loses. In particular, everybody loses if all three coins match. Hence everybody tries to behave differently from his right neighbor. This leads to a (best reply) cycle of six pure strategies (see Fig. 2)

$$100 \rightarrow 110 \rightarrow 010 \rightarrow 011 \rightarrow 001 \rightarrow 101 \rightarrow 100. \quad (5.1)$$

If gain has value $+1$, loss has value -1, and $0 \leq x_i \leq 1$ denotes the frequency of strategy 1 (heads), used by person i, the replicator dynamics on the cube $[0, 1]^3$ reads

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1)(1 - 2x_2) \\ \dot{x}_2 &= x_2(1 - x_2)(1 - 2x_3) \\ \dot{x}_3 &= x_3(1 - x_3)(1 - 2x_1). \end{aligned} \quad (5.2)$$

This system is volume preserving and competitive. This is best seen after a change of variables $u_i = \log(x_i/(1 - x_i))$, which transforms (5.2) into

$$\dot{u}_i = \frac{1 - e^{u_{i+1}}}{1 + e^{u_{i+1}}}, \quad (5.3)$$

which is a divergence-free and competitive vector field ($\sum (\partial \dot{u}_i/\partial u_i) = 0$

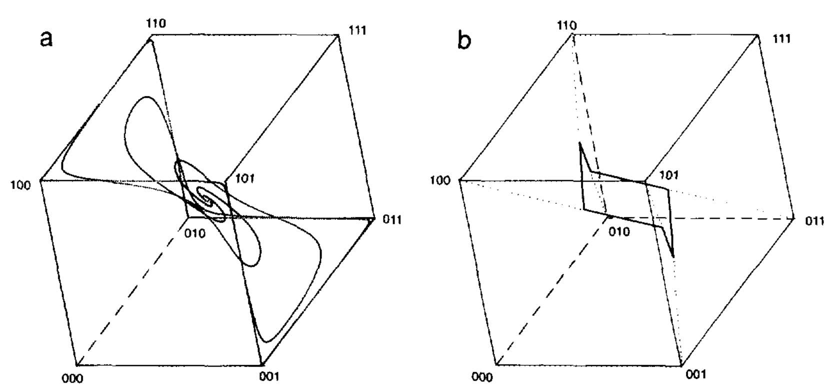

FIG. 2. Jordan's game. (a) The attracting invariant manifold for RE and (b) the Shapley hexagon, the attractor for AR and FP.

and $\partial \dot{u}_i / \partial u_j \leq 0$ for $i \neq j$). There is a unique interior equilibrium $E = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, corresponding to $u = 0$, which is attracting on the invariant line of symmetry $x_1 = x_2 = x_3$. By a theorem of Hirsch (1988) there is a unique smooth manifold (which extends the local unstable manifold of E up to the boundary of the cube), which attracts all orbits in the interior of the cube; see Fig. 2a. Since the 3d flow is volume preserving, its restriction to this 2d attracting manifold must be area expanding. By the Bendixson–Dulac test (see Hofbauer and Sigmund, 1988, p. 149), there is no periodic orbit on this manifold. Hence all interior orbits (except those on the invariant line of symmetry) converge to the boundary of the cube. Since the two corners 000 and 111 are sources, the only possible ω-limit set on the boundary is the *heteroclinic cycle* which is formed by the above six points in the best reply cycle (5.1) and their connecting orbits (edges of the cube).

We will now show that the time averages of all these interior orbits "converge" to a hexagon in the interior of the cube. Again, this coincides with the Shapley hexagon found by Jordan (1993) as the limit set for fictitious play and more general learning processes.

Following Lemma 1, we determine the sojourn times of a solution of (5.2) in the vicinity of the six corners F_i of the heteroclinic cycle. The linearization at each of these saddle points has two negative eigenvalues $-\mu$ (along the heteroclinic cycle) and $-\sigma$ ("transverse" to it, coming from one of the sources) and one positive eigenvalue λ. For (5.2) one has $\lambda = \mu = \sigma = 1$. As in subsection 3.1 one can then compute the Poincaré maps and the sojourn times. The technical details have been carried out for a similar case on the 3d simplex S_4 instead of the cube in Hofbauer and Sigmund (1988, 29.3) and Gaunersdorfer (1992). Because of the cyclic symmetry, the rate of increase of the sojourn times, ρ_i, is independent of i and turns out to be the leading eigenvalue of the matrix

$$P = \begin{pmatrix} \sigma/\lambda & 1 \\ \mu/\lambda & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

which is the golden mean $g := (1 + \sqrt{5})/2 = \rho_i \approx 1.618 > 1$. Thus, by Lemma 1, the vertices A_i of the limit hexagon for AR paths are given by

$$A_i = \frac{g-1}{g^6-1}(F_{i+1} + g F_{i+2} + g^2 F_{i+3} + g^3 F_{i+4} + g^4 F_{i+5} + g^5 F_i). \quad (5.4)$$

(the indices are counted modulo 6). This is precisely the Shapley hexagon found by Jordan (1993); see Fig. 2b. Without cyclic symmetry, the resulting formulae would be rather awful (see Gaunersdorfer, 1992), so we do not treat this case here. Instead, we give an alternative geometric argument, which shows that the hexagons for AR and BR must coincide for payoffs more general than those considered above.

First we show that AR paths $z(t) = 1/t \int_0^t x(s)ds$ ultimately behave like BR paths. We proceed as in 3.4. (5.2) can be rewritten as

$$\left(\log \frac{x_i}{1 - x_i}\right)^{\cdot} = \frac{\dot{x}_i}{x_i(1 - x_i)} = 1 - 2x_{i+1} \tag{5.5}$$

and integrated to

$$\frac{1}{T}\left(\log \frac{x_i(T)}{1 - x_i(T)} - \log \frac{x_i(0)}{1 - x_i(0)}\right) = 1 - 2z_{i+1}(T). \tag{5.6}$$

We know already that almost all solutions $x(t)$ "converge" to the heteroclinic cycle given by (5.1). When $x_i(T)$ is close to 1, (5.6) is positive; when $x_i(T)$ is close to 0, (5.6) is negative. And for any sequence $T_n \to \infty$, for which $x_i(T_n)$ converges to some constant $c \in (0, 1)$, the time averages $z_{i+1}(T_n) \to \frac{1}{2}$. Since $x(t)$ spends most of the time near one of the corners (5.1), $z(t)$ will ultimately move almost straight toward one of the corners and make turns exactly when hitting the planes of equal payoff $x_i = \frac{1}{2}$.[13]

Next, we show that there is only one Shapley polygon, using the idea in 3.4 (second proof of Lemma 2). The three planes of equal payoff $\{x : x_i = \frac{1}{2}\}$ divide the cube into eight octants, which we can number by their adjacent corner. These octants are separated by squares of the form $L_{0.01} = \{x \in \mathbb{R}^3 : x_1 = \frac{1}{2}, x_2 < \frac{1}{2}, x_3 > \frac{1}{2}\}$, etc.; see Fig. 3. The two octants 000 and 111 are negatively invariant for both RE and BR: Every orbit starting there will leave it and enter one of the remaining six octants and cycle between them according to (5.1). For BR, every path in, say octant 001, is attracted toward corner 101, until it reaches the square $L_{.01}$ through which it enters octant 101 and where it turns toward corner 100. We will show that the transition map τ: $L_{.01} \to L_{10.}$ is a contraction for a suitable projective metric and so will be the complete return map. The distance between two points X, Y in one of the squares is given by

$$d(X, Y) = \left|\log\left(\frac{XG}{XH} : \frac{YG}{YH}\right)\right|,$$

with G, H being the intersection points of the line through X, Y with the

[13] In this argument the *linear* dependence of the right-hand side of (5.5) on the frequencies x_i is essential. This is true for 2-person games, but not for general $n \geq 3$-person games. The argument works for *games with linear incentives* in the terminology of Selten (1995).

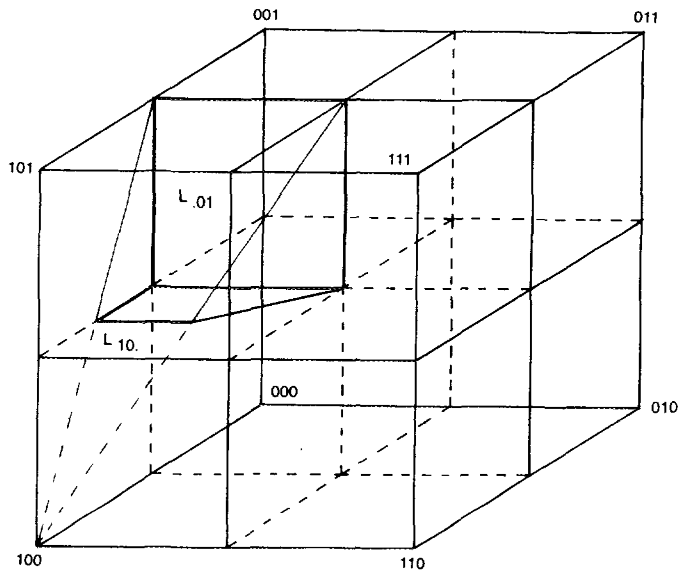

FIG. 3. The transition map τ: $L_{.01} \to L_{10.}$ mapping the square onto a trapezoid.

boundary of the square. It is well known that this defines a metric, even for any convex subset C of $\mathbb{R}^n$ instead of a square, and enlarging C makes this projective distance between two points smaller. Furthermore, along a central projection, the double ratio of four collinear points remains constant. Since $L_{.01}$ is mapped by τ *into* $L_{10.}$ (the image is a trapezoid; see Fig. 3), the projective distance (with respect to the squares) decreases: $d(\tau(X), \tau(Y)) < d(X, Y)$.

Repeating this argument along the cycle (5.1), we obtain a contraction map, which can have at most one fixed point (corresponding to the Shapley polygon). However, we cannot use the Banach fixed point theorem to conclude its existence since τ is not a uniform contraction. Near the equilibrium E, which sits on the boundary of these squares, at *infinite* projective distance from interior points, the rate of contraction goes to 1. Hence we still have to show the instability of E under BR.

We now compute the transition maps τ explicitly, as in 3.5. A simple computation shows that τ: $L_{.01} \to L_{10.}$ is given by

$$\left(\frac{1}{2}, x_2, x_3\right) \mapsto \left(1 - \frac{1}{4x_3}, \frac{x_2}{2x_3}, \frac{1}{2}\right). \tag{5.7}$$

It is convenient to make the following change of variables, which puts E into the origin, and parametrizes all six squares by $0 < u, v < \frac{1}{2}$:

$$u = \frac{1}{2} - x_2, v = x_3 - \frac{1}{2} \qquad \text{on } L_{01},$$
$$u = x_1 - \frac{1}{2}, v = \frac{1}{2} - x_2 \qquad \text{on } L_{10}. \tag{5.8}$$

Then τ: $(u, v) \mapsto (u', v')$ is given by

$$u' = \frac{v}{1 + 2v}, \qquad v' = \frac{u + v}{1 + 2v},$$

or in vector notation

$$\begin{pmatrix} v \\ u \end{pmatrix}' = \frac{1}{1 + 2v} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}. \tag{5.9}$$

In these coordinates, because of cyclic symmetry, the other five transition maps take exactly the same form. So we need only consider (5.9). We see immediately that the equilibrium E ($u = v = 0$) is unstable under this map. There is a unique fixed point, which is the eigenvector $\binom{v}{u}$ of the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with $1 + 2v$ being the positive eigenvalue of this matrix, which is the golden mean g as above. So the fixed point, which corresponds to the corners of the Shapley polygon via (5.8), is given by

$$v = \frac{g - 1}{2} = \frac{\sqrt{5} - 1}{4} \approx 0.309, \qquad u = 1 - \frac{g}{2} \approx 0.191.$$

This confirms (5.4) above and Jordan's result. The global stability of this fixed point under the transition map (resp. the Shapley polygon under BR) follows either from direct calculation, using the simple form (5.9), or from the contraction property of the projective distance shown above. The simple form (5.9) of the return map allows a complete understanding of the BR dynamics in this example. One could even write down the solutions explicitly.

6. Shapley's Example

Shapley (1964) considered 3×3 bimatrix games with a certain circular structure. In particular, he explicitly computed the limiting behavior of FP for the following payoff matrices, in the case $\varepsilon = 0$:

$$A = \begin{pmatrix} 1 & -\varepsilon & 0 \\ 0 & 1 & -\varepsilon \\ -\varepsilon & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -\varepsilon & 1 \\ 1 & 0 & -\varepsilon \\ -\varepsilon & 1 & 0 \end{pmatrix}. \tag{6.1}$$

The replicator dynamics for such bimatrix games is given by

$$\begin{aligned} \dot{x}_i &= x_i[(Ay)_i - x \cdot Ay], \qquad i = 1, 2, 3, \\ \dot{y}_j &= y_j[(B^t x)_j - x \cdot By], \qquad j = 1, 2, 3, \end{aligned} \tag{6.2}$$

on $S_3 \times S_3$ (see Hofbauer and Sigmund, 1988, Chap. 27, where a different notation is used: B is replaced by B^t).

The system exhibits a cycle of best replies

$$11 \rightarrow 13 \rightarrow 33 \rightarrow 32 \rightarrow 22 \rightarrow 21 \rightarrow 11 \tag{6.3}$$

(see Fig. 4). The eigenvalues at these six fixed points of (6.2) are 1 and -1 (along the cycle) and $-1 - \varepsilon$ and $-\varepsilon$ in the "transverse" directions. In the case $\varepsilon = 0$ studied by Shapley, one of the transverse eigenvalues is 0, so

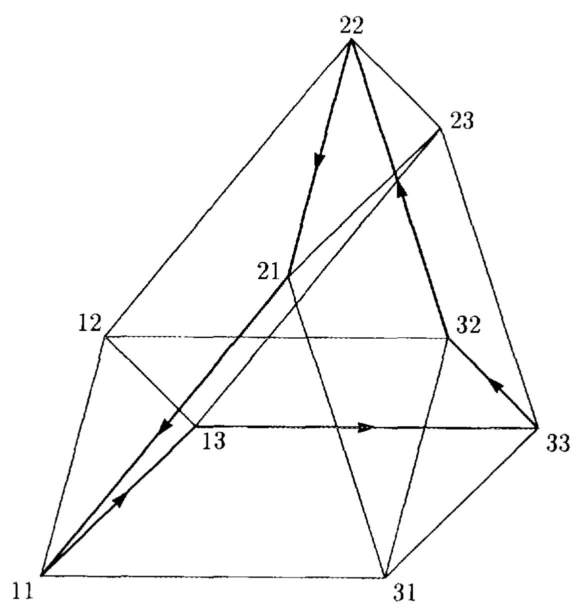

FIG. 4. The heteroclinic cycle for RE in Shapley's example.

there are fixed point edges for (6.2) which make the analysis more difficult for the RE. Hence we prefer to analyze (6.2) for $\varepsilon > 0$ and then take the limit $\varepsilon \to 0$.

For $\varepsilon > 0$, the heteroclinic cycle defined by (6.3) is asymptotically stable for RE. Hence interior orbits close to the heteroclinic cycle converge to it and spend longer and longer times near the six corners (6.3) of the 4d polytope $S_3 \times S_3$. A similar calculation as in Section 5 shows that consecutive sojourn times increase exponentially at rate ρ_i (independent of i, because of cyclic symmetry), which is determined by the leading eigenvalue of the nonnegative matrix

$$P = \begin{pmatrix} 1+\varepsilon & 1 & 0 \\ \varepsilon & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.4}$$

This eigenvalue is the largest root θ of the equation $\theta^3 - (1 + \varepsilon)\theta^2 - \varepsilon\theta - 1 = 0$. Obviously $\theta > 1$, so that Lemma 1 applies. Hence the time averages of (6.2) converge to a hexagon $A_1 \ldots A_6$, where the A_i are given as in (5.4), but with g replaced by θ and F_i running through the cycle (6.3). In the limit case $\varepsilon = 0$, this polygon is obviously identical to the hexagon found by Shapley (1964) to be the attractor for FP.

7. Conclusion

The results of this paper suggest the following *principle: The time averages of the solutions of the replicator equation seem to have the same asymptotic behavior as the best response paths.* There are many more instances for this "*AR–BR principle.*" Most of them concern the case when both processes converge to Nash equilibrium.

Two-person zero-sum games. The convergence of FP, CFP, and BR to the set of equilibria follows from the classical work of Brown (1951) and Robinson (1951); see also Hofbauer (1994) for a simple proof. RE has a constant of motion; it is even a Hamiltonian system. Its orbits are periodic, quasiperiodic, or more complicated, but their time averages converge to the set of equilibria; see Hofbauer and Sigmund (1988, pp. 277, 278).

Partnership games, games with strongly identical interests, and weighted potential games. FP, BR, RE converge to the set of Nash equilibrium; see Monderer and Shapley (1993a, 1993b); Hofbauer and Sigmund (1988, Chap. 27.2). The maxima of the potential function (which are strict NE in general) are the stable equilibria. However, their basins of attraction will depend on the dynamics.

Evolutionarily stable strategies. Evolutionarily stable strategies (if completely mixed) are (globally) asymptomatically stable for both RE (see Hofbauer and Sigmund, 1988), and BR (see Hofbauer, 1994).

Supermodular games. Almost all orbits (for a large class of dynamics, including FP, BR, RE) converge to Nash equilibrium, due to the theory of Hirsch on monotone dynamical systems; see Milgrom and Roberts (1990); Krishna (1992); Brock and Samuelson (1992).

Dominance solvable games. Convergence to Nash equilibrium was established for BR and RE by Milgrom and Roberts (1991) and Samuelson and Zhang (1992), respectively.

On the other hand, as our three examples show, cycles of pure best (or better) responses, which result in heteroclinic cycles for the replicator equation, are often attractors for the dynamics. In such situations, *Shapley polygons* arise as limiting behaviors for FP, BR, as well as the time averages of the replicator dynamics, and those might be considered the more reasonable solution of the game, instead of the (unstable) Nash equilibrium.

We conclude with two problems:

(1) For which games does this AR–BR principle hold?

It is definitely not true for all games, as simple examples of 2×2 games show. However, these counter examples are "degenerate" in the sense that there are continua of NE: In these examples the rationally minded BR dynamics selects the perfect NE, whereas the evolutionary dynamics like RE do not have such preferences.

Serious counterexamples can be constructed for $n \geq 3$-person games (with nonlinear incentives), as already indicated in Section 5. There are regular $2 \times 2 \times 2$ games with no interior equilibrium, but with an interior periodic orbit for RE (this is not possible for 2-person games or $n \geq 3$-person games with linear incentives). Hence the time-averaged replicator dynamics AR converges for some initial values, whereas BR does not. These examples form an open set in parameter space; see Hofbauer and Plank (1995).

It is an instructive exercise to analyze the BR dynamics for symmetric 3×3 games and compare this with Zeeman's (1980) classification of the RE. Doing this, one finds that for one (and only one!) class of Zeeman there is a certain discrepancy between AR and BR. See Hofbauer (1994) for this example. But for the majority (more than 99%) of randomly chosen 3×3 payoff matrices the AR–BR principle is definitely true.

(2) Are there more difficult possibilities for CSS (attractors for BR) other than NE and Shapley polygons? Are there examples of quasiperiodic or chaotic attractors for BR?

Note added in proof. An example of a 3 × 3 bimatrix game with chaotic BR dynamics has been constructed by Stuart Cowan in his Ph.D. thesis, "Dynamical Systems Arising From Game Theory," at the University of California, Berkeley, under the supervision of M. Hirsch.

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