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Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal–Agent Problem*

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We examine a simple repeated principal–agent model with discounting. There are a risk averse borrower with an unobservable random income and a risk neutral lender. The efficient contract is characterized. It tends to the first-best (constant consumption) contract as the discount factor tends to one and the time horizon extends to infinity. If the time horizon is infinite and the contract is legally enforceable the borrower's utility becomes arbitrarily negative with probability one. If the borrower has constant absolute risk aversion consumption is transferred between any two states at a constant interest rate which is less than the rate of time preference. *Journal of Economic Literature* Classification Numbers: 026, 315.

1. INTRODUCTION

A risk averse agent whose income fluctuates will want to stabilise consumption by borrowing and saving. If there is an infinite time horizon, no discounting of the future, and income is independently and identically distributed, consumption can be perfectly stabilised at the average value of

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income by borrowing and lending at a zero rate of interest (Yaari [14]). If, on the other hand, there is just one time period and no outsider can observe the agent's income (so there is asymmetric information) consumption cannot be stabilized at all. The purpose of this paper is to examine, when there is asymmetric informaton, how debt contracts can be used to stabilize consumption for any finite or infinite time horizon and any discount factor between zero and one.

The model we study was first presented by Townsend [13]. There is a risk averse borrower who has an i.i.d. income stream. A risk neutral lender would like to offer insurance but cannot observe the borrower's income, past or present. If the lender were to offer to stabilize the borrower's consumption, the borrower would have an incentive to claim his income is low—such a contract is not incentive compatible. On the other hand a scheme in which the amount lent is independent of the borrower's income is incentive compatible but does not share any risk. Grossman, Levhari, and Mirman [7] showed that if there were more than one period a simple loan contract with a fixed rate of interest could provide some insurance. Townsend pointed out that these simple loan schemes are not generally (constrained) efficient, there being better ways of tying future transfers to present claims. Townsend, however, only partially characterized the solution to the two-period, two-state problem. We characterize the efficient contract for any time horizon and finite state space.

Models of this kind have a number of potential applications apart from the immediate one of how best to insure an individual whose income is unobservable. One is the study of liquidity constraints. Liquidity constraints, it is often argued (for empirical support see, e.g., Flavin [4]), raise the marginal propensity to consume out of income above that predicted under perfect capital markets, and provide a greater role for stabilization policy. In our model of asymmetric information the borrower is constrained in an efficient contract by his past history: if he has borrowed heavily in the past he will be able to borrow less in the current period than otherwise, this despite the fact that his future income prospects are unaffected. Another potential application is the study of international debt contracts. Naturally no simple model can do the problem justice; nevertheless even in the model studied here, which has no capital accumulation, it is shown that the borrower will inevitably get deeper and deeper into debt. What is interesting about this result is that the "debt problem" is a consequence of an *efficient* contract.

In this paper we follow the methodology set out in Townsend [13] and view loan contracts as constrained Pareto efficient agreements between borrowers and lenders, so that no mutually beneficial gains remain unexploited. It is shown (Sections 3 and 4) that the efficient loan contract corresponds to the solution of a dynamic programming problem. This has two important consequences. First the continuation of the loan contract is efficient at every date so the borrower and lender will never mutually agree to renegotiate the contract. This is not to say that one or other party will not wish to break the contract ex post; it is however assumed (apart from Section 8) that the contract is legally enforceable and therefore cannot be broken. Second, at any time the future course of the contract is determined solely by knowledge of a state variable. In particular, this state variable can be interpreted as the "indebtedness" of the borrower, where indebtedness represents the expected future net payments due to the lender.

Section 5 presents the main and perhaps most surprising result of the paper. If the time horizon is infinite the borrower's future utility becomes arbitrarily negative with probability one. The borrower gets deeper and deeper into debt, and consumption moves down as debt increases. The contract is therefore not very good at stabilizing consumption over time; nevertheless what appears to be happening is that making future utilities low reduces the cost of inducing incentive compatibility, which is obtained by variations in future utility. So stability in consumption in the initial periods is obtained at a cost of variation in consumption over time.

While this result is of considerable interest in its own right, it also implies that if the discount factor is allowed to converge to one and the time horizon is infinite, the efficient contract cannot converge uniformly to the first-best, constant-consumption contract. Nevertheless the results of Radner [10] suggest that it should be possible to approach first-best utilities. The dynamic programming approach permits a simple and natural proof of this since as we show in Section 6, the second-best Pareto frontier converges pointwise to the first-best frontier as the discount factor tends to one and the time horizon tends to infinity.

In Section 7 we consider the special case where the borrower has an exponential utility function so that wealth effects are excluded. The efficient contract is explicitly solved for: it transfers consumption between any two states at a constant rate of interest, which is positive, but less than the common rate of time preference. Thus a kind of "soft loan" can be the best way of insuring an agent whose income is unobservable; this is very intuitive, since the first-best contract, which stabilizes consumption, involves an implicit interest rate of minus one, but is not incentive compatible, whereas a loan contract with rate of interest equal to the rate of time preference, while avoiding incentive problems, is not very good at smoothing consumption since when income is low, the agent will be discouraged from borrowing to smooth his consumption by the high interest costs.

Finally in Section 8 we drop the enforceability assumption and examine self-enforcing contracts. This is important because the results of Section 5 show that for any fixed penalty associated with reneging, the borrower will eventually want to renege with probability one. It is shown that a non-trivial self-enforcing contract will exist for discount factors near unity, and that such contracts do not use termination as an incentive device.

The dynamic programming approach to repeated asymmetric information models was introduced by Green [5] in his seminal paper on social insurance. In his model there is a continuum of risk averse agents, each with unobservable, i.i.d. zero–one income streams and exponential utility functions who would like to insure one another. Green shows that consumption is a random walk with drift added to an i.i.d. term, a result we also obtain in the exponential case. The drift term is however negative, so Green's scheme would only make sense if the community as a whole is able to borrow in the initial periods: an outside resource borrowed at the rate of time preference is necessary to finance the scheme, whereas in our model the negative drift term implies that the borrower gets deeper into debt.[1] While we consider only a two person model, our main results are valid for all utility functions satisfying non-increasing absolute risk aversion, and for general i.i.d. processes and time horizons. Moreover, we characterise the asymptotic properties of efficient contracts both as time goes to infinity and as the discount rate goes to zero.

The dynamic programming approach is also used by Spear and Srivastava [11] to study a repeated moral hazard problem in contrast to the hidden information problem studied here. Assuming the validity of the first-order approach and the existence of a differentiable solution they examine the time structure of an efficient contract. According as output exceeds or falls short of a critical level the future utility of the agent rises or falls. Similarly in our model the future utility of the borrower will rise or fall according to the size of his current income. They do not, however, examine the long-run properties of efficient contracts which are one of our main concerns here.

2. The Model

As mentioned in the introduction we use the basic model introduced by Townsend [13]. There are a borrower and a lender who both live $T+1$ periods and can trade a single non-storable consumption good which we call income. Both try to maximise lifetime utility and discount the future by the common discount factor $\alpha \in (0, 1)$. At each date $t = 0, 1, ..., T$, the borrower has a random income s_t which takes on one of N possible values, $(\theta_1, \theta_2, ..., \theta_N)$. We call this set of income values Θ and index it by $S = \{1, 2, ..., N\}$ (by convention $\theta_i > \theta_j$ for all $i > j$, $i, j \in S$). It is assumed that s_t is identically and independently distributed and that the probability of income θ_i is π_i, independent of t, where $\sum_{i \in S} \pi_i = 1$.

[1] We are grateful to an associate editor for drawing our attention to this paper.

The borrower is risk averse and has a time-separable utility function which satisfies

Assumption 1. The per period utility function of the borrower is $v: (a, \infty) \to \mathbf{R}: v$ is C^2 with $\sup v(c) < \infty$, $\inf v(c) = -\infty$, $v' > 0$, $v'' < 0$, $-v''/v'$ non-increasing and $\lim_{c \to a} v'(c) = \infty$.

There are two slightly non-standard assumptions here. First it is assumed that the borrower's absolute risk aversion is non-increasing. This is a sufficient condition for the valuation function of the dynamic programming problem solved later on to be concave. Second it is assumed that utility is unbounded below. This greatly facilitates the analysis in Section 5 and in particular fits the special case of constant absolute risk aversion which we consider in Section 7.

The lender is assumed to be risk neutral and therefore would be prepared to offer insurance to the borrower. The problem is that although the lender knows how income is distributed and knows Assumption 1 he cannot observe actual income either contemporaneously of retrospectively. Thus if he were to offer the borrower perfect insurance the borrower would always under report income. However, we know from the Revelation Principle[2] that if there is any way in which the lender can provide some insurance for the borrower there is an equivalent incentive compatible way in which the borrower reports his true income. Thus it is possible to write $b_t^{T+1}(h^t)$, (the payment at t, from the lender to the borrower if positive, a repayment if negative) as a function of history, $h^t = (s_0, s_1, ..., s_t) \in \Theta^{t+1}$, and define

DEFINITION 1. A loan contract b^{T+1} is a sequence of functions $(b_t^{T+1})_{t=0,1,...,T}$, where each $b_t^{T+1}: \Theta^{t+1} \to (a - s_t, \infty)$.

Allen [1] has shown that if the borrower can save and borrow at a rate of interest equal to the rate of time preference unobserved by the lender then there is no feasible loan contract which provides insurance. For in this case the borrower will evaluate any series of payments by its expected discounted value. Incentive compatibility then requires that all series of payments must yield the same expected discounted value, so no risk can be shared. Many contracts, however, contain explicit provisions to control outside transactions; for example bank loans usually require that other debts be disclosed and most insurance contracts are void if it is found that material facts were unreported. Therefore we adopt the assumption that the lender can monitor or control all the borrower's outside transactions.

We know from Townsend [13] or from Grossman, Levhari, and Miman [7] that there do exist incentive compatible contracts when outside

[2] The Revelation Principle holds for any time horizon and any stochastic structure.

transactions can be monitored. We let B^{T+1} be the set of these incentive compatible loan contracts. It is described by a series of inequalities similar to those typical in one period incentive contracts (e.g., Hart [8]). To be precise, for any given contract b^{T+1} and some date t let V_i be the highest expected future utility (discounted to date $t+1$) which the borrower can get if he reports his income at date t to be $r_t = \theta_i$. Since income is i.i.d. V_i depends only upon the reported history, $g^t \equiv (r_0, r_1, ..., r_t)$, and not upon the actual history, $h^t = (s_0, s_1, ..., s_t)$. Let $b_i = b_t^{T+1}(g^{t-1}, \theta_i)$ be the amount borrowed at time t if $r_t = \theta_i$ and the past reported history is g^{t-1}. Then if b^{T+1} is incentive compatible, i.e., if the borrower never has an incentive to lie,

$$v(b_i + \theta_i) + \alpha V_i \geqslant v(b_j + \theta_i) + \alpha V_j \tag{1}$$

for every g^{t-1}, $t = 1, 2, ..., T$, and all $i, j \in S$. Thus $B^{T+1} = \{b^{T+1}$ satisfying Eq. (1)$\}$.

3. The Dynamic Programming Characterisation of Efficient Contracts

An incentive compatible loan contract is efficient if it is undominated by any other incentive compatible contract. Efficient contracts can be characterized by a dynamic program. The basic idea is simple: in an efficient contract after any history h^t, with $s_t = \theta_i$, the remaining part of the contract from date $t+1$ onwards, the continuation contract, must itself be efficient. Otherwise replacing it by an efficient continuation contract which gives the borrower the same expected future utility, V_i, from date $t+1$ onwards will make the lender better off. (Note that the new continuation contract must give the borrower exactly V_i and no more; otherwise an incentive compatible constraint, Eq. (1), in some other state at date t might be violated). It only remains to check that the new contract is itself incentive compatible. If income is not i.i.d. then the future income stream of the borrower depends upon the actual history, while future loans and repayments depend upon the reported history. Therefore, although the continuation part of the contract after the history h^t is, by definition, incentive compatible, the new series of loans and repayments promised after this history might look more attractive than the old series after some other history $h^{t'}$. In this case the history h^t may be reported even when the true history is $h^{t'}$. If income is i.i.d. this cannot happen since V_i does not depend upon actual history. Thus the new contract will be incentive compatible for every possible h^t. Because the continuation contract is efficient, the future utilities of the two parties must lie on the Pareto frontier for the remaining part of the problem.[3] Moreover the utilities must be chosen on the frontier to solve a dynamic programming optimality equation. Any set of future utilities for the borrower V_i can be chosen together with the current transfers b_i so long as (1) is satisfied and the necessary overall utility for the borrower is exactly attained. So the optimality equation will say that these variables must be chosen to maximise the lender's utility subject to these constraints. It should be stressed that maximising lender's utility subject to giving the borrower a certain utility is just one way of characterising efficient contracts; doing it the other way around will lead to the same contracts.

The dynamic programming approach we adopt treats the borrower's net future utility, relative to autarky, as the state variable.[4] So the first step will be to describe how the lender's future utility depends on this state variable. First we consider contracts of a given length: a contract of length k, that is with k periods remaining, b^k, begins at $T-k+1$. Then the set of k period incentive compatible contracts that give the borrower a net expected utility of V is

$$B^k(V)=\left\{b^k \in B^k : E\sum_{\tau=0}^{k-1} \alpha^\tau(v(b_\tau^k(h^\tau)+s_\tau)-v(s_\tau))=V\right\},$$

where E is the expectations operator over all future income levels. The maximum expected discounted utility the lender can get from a k period contract when the borrower gets V can now be defined; this is just the constrained Pareto frontier for the k period problem.

DEFINITION 2. The lender's value function with k periods to go is

$$U_k(V)=\sup_{b^k \in B^k(V)} -E\sum_{\tau=0}^{k-1} \alpha^\tau b_\tau^k(h^\tau)$$

for any $V \in (-\infty, d_k)$, where $d_k=(\sup v(c)-\sum_{i\in S}\pi_i v(\theta_i))(1-\alpha^k)/(1-\alpha)$.

It represents the expected discounted value of the series of future loans and repayments, and can therefore be thought of as the level of "indebtedness" of the borrower to the lender at any particular time. While it is more convenient to work in terms of the borrower's utility V as the state variable, the level of indebtedness is directly related to it through the value function at any date.

[3] As remarked in the introduction, this argument implies that there will never be an incentive to renegotiate the contract since any renegotiation would make at least one party worse off.

[4] Since there are two agents in the problem, there should normally be two optimality equations describing the evolution of utility in terms of the state variable. Since it is possible to define the borrower's future utility as the state variable we need only consider one optimality equation as the other becomes trivial. We thank Vincent Brousseau for this remark.

For $T=\infty$ the lender's value function, which is independent of time, is denoted $U_*(V)$. The value $U_k(V)$ is not defined to the right of d_k but if we let $U_k(V)=-\infty$ for $V \geqslant d_k$ each U_k is defined on the common interval $(-\infty, d_\infty)$ and takes values in the extended reals. Thus it is possible to define the following

Definition 3. For any function $U:(-\infty, d_\infty) \to \mathbf{R} \cup \{-\infty\}$ the *one-step operator* L satisfies

$$L(U)(V)= \sup_{(b_i, V_i)_{i\in S} \in A(V)} \sum_{i \in S} \pi_i(-b_i+\alpha U(V_i)),$$

$$A(V)=\Bigg\{(b_i, V_i)_{i\in S}: b_i \in (a-\theta_i, \infty), V_i \in (-\infty, d_\infty);$$

$$\sum_{i\in S} \pi_i(v(b_i+\theta_i)-v(\theta_i)+\alpha V_i)=V,$$

$$v(b_i+\theta_i)+\alpha V_i \geqslant v(b_j+\theta_i)+\alpha V_j \text{ for all } i, j \in S\Bigg\}.$$

The fundamental optimality equation of the dynamic programming algorithm is given in Lemma 1 (all proofs, and statements of further lemmas, are relegated to the appendix).

Lemma 1. *For $k \geqslant 1$ the one-step operator L defines the value functions recursively through the optimality equation*

$$U_k(V)=L(U_{k-1})(V). \tag{2}$$

The optimality equation shows how the value functions may be computed. In the finite horizon problem U_k is found recursively by repeated application of L starting from $U_0 \equiv 0$; the optimum values of $(b_i, V_i)_{i\in S}$ which are chosen at each application of L then constitute the contract which achieves the utilities corresponding to the constrained Pareto frontier for the k period problem. In the infinite horizon problem U_* is a fixed point of L.

To study the infinite horizon problem it would be helpful for both analytical and computational reasons to know if the sequence of functions, U_k, converges to U_* as $k \to \infty$. While they do not converge in the supremum metric (since they cannot be restricted to the same metric space in this metric), pointwise convergence can nevertheless be proven. To use standard arguments it will be necessary to restrict the space of functions. There are natural bounds on $U_k(V)$. Consider first the contract which pays a constant amount y_k at all dates, where y_k satisfies $\sum_{i\in S} \pi_i(v(y_k+\theta_i)-$

$v(\theta_i))(1-\alpha^k)/(1-\alpha)=V$. It is trivially incentive compatible and gives the borrower a net utility of V. Therefore the lender's utility from this contract provides a lower bound to $U_k(V)$. On the other hand $U_k(V)$ can give the lender no more than the unconstrained first-best contract which pays $c_k - s_t$ at all dates, where c_k satisfies $\sum_{i\in S}\pi_i(v(c_k)-v(\theta_i))(1-\alpha^k)/(1-\alpha)=V$. Thus

$$-(1-\alpha^k)\,y_k/(1-\alpha)\leqslant U_k(V)\leqslant(1-\alpha^k)\sum_{i\in S}\pi_i(\theta_i-c_k)/(1-\alpha), \qquad (3)$$

and in the limit, using an obvious notation,

$$-y_\infty/(1-\alpha)\leqslant U*(V)\leqslant\sum_{i\in S}\pi_i(\theta_i-c_\infty)/(1-\alpha). \qquad (4)$$

These bounds tie down U_k quite tightly, as depicted in Fig. 1; since $\lim_{c\to a}v'(c)=\infty$, $\lim_{V\to-\infty}U_k'(V)=0$, and $\lim_{V\to d_k}U_k'(V)=-\infty$. If $a=-\infty$ then $\lim_{V\to-\infty}U_k(V)=\infty$, while it is finite if $a>-\infty$. Similarly $\lim_{V\to d_k}U_k(V)=-\infty$. Let F be the space of continuous functions on $(-\infty, d_\infty)$ lying between the bounds in (4). Since the gap between these bounds is itself bounded, F is a complete metric space in the supremum metric, and by standard arguments L is a contraction in F (Lemma 2). So U_* is the unique fixed point of L in the space F and for any $U\in F$, $\lim_{k\to\infty}L^k(U)=U_*$. To show that U_* is the limit of the finite horizon value functions is not straightforward as $U_k\notin F$. Nevertheless Lemma 3

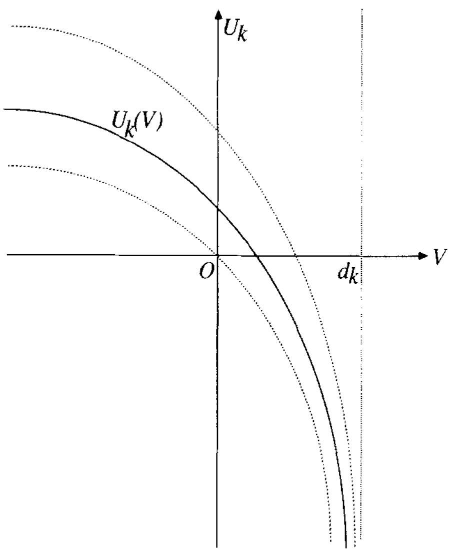

FIG. 1. The bounds on the value function.

shows that $\lim_{k\to\infty} U_k$ is a fixed point of L and since the limit belongs to F (take limits in (3)) it follows that $\lim_{k\to\infty} U_k = U^*$.

4. The Efficient Contract

To calculate the efficient contract it is necessary to solve the programming problem defined by Eq. (2). (Since L is a contraction mapping a necessary and sufficient condition for efficiency is that the supremum in Definition 3 be attained for every k. This is guaranteed by Assumption 1.) To solve this problem we need to show that the value functions are concave. Starting with $U_0 \equiv 0$, it is straightforward to see that $U_1(V)$ is strictly concave. Therefore we will invoke the induction hypothesis and assume $U_{k-1}(V)$ is strictly concave too.

To tackle the maximisation problem of Definition 3 it is first necessary to simplify the constraint set. Rewrite Eq. (1) as $C_{ij} \equiv v(b_i + \theta_i) - v(b_j + \theta_i) + \alpha(V_i - V_j) \geqslant 0$. This equation states that the borrower must be induced to report the true income θ_i rather than the false income θ_j. Since v is concave, it is easily shown (see, for example, Hart [8]) that if the local downward constraints $C_{i,i-1} \geqslant 0$ and upward constraints $C_{i,i+1} \geqslant 0$ hold for each $i \in S$ then the global constraints $C_{ij} \geqslant 0$ hold for each $i, j \in S$. It is intuitively unlikely that the borrower should wish to report a higher income than he actually has, so it is to be expected that the downward incentive compatibility constraints will bind at the optimum, and this is what Lemma 4 proves. It is shown, under the induction hypothesis that $U_{k-1}(V)$ is strictly concave, that all the local downward incentive compatibility constraints, $C_{i,i-1} \geqslant 0$, always bind and none of the local upward incentive compatibility constraints, $C_{i,i+1} \geqslant 0$, ever do. This means that the borrower is always just indifferent about reporting that his income was actually a little lower than it was, but would never want to report that his income was in fact higher.

It also follows from the concavity of v and adding $C_{i,i-1} \geqslant 0$ and $C_{i-1,i} \geqslant 0$ that $b_{i-1} \geqslant b_i$ and $V_i \geqslant V_{i-1}$. This of itself imposes considerable structure on the contract. For example, in low income states more is borrowed but at the cost of a lower future utility or a greater level of indebtedness for the borrower. (This would of course not be true of the first-best contract where what is borrowed today does not affect future indebtedness). Further, Lemma 4 shows that the optimum contract involves coinsurance so that the lender's expected future utility is also lower when the borrower's income is lower.

Having simplified the constraint set we are in a position to consider the properties of the value function $U_k(V)$. It is obviously decreasing, but it is not so obvious that it is concave because the constraint set itself is not convex. Since, however, the borrower's utility function exhibits N.I.A.R.A. (Assumption 1) it can be shown that there is a unique $(b_i, V_i)_{i \in S}$ which attains the supremum in Definition 3, so the efficient contract is unique, and strict concavity can be proved.

PROPOSITION 1. *There is a unique $(b_i, V_i)_{i \in S}$ which attains the supremum in Definition 3. $U_k(V)$ is decreasing, strictly concave, and continuously differentiable on $(-\infty, d_k)$ and $U_*(V)$ is decreasing, concave, and continuously differentiable on $(-\infty, d_\infty)$.*

The efficient contract is stationary in the sense that V incorporates all the information necessary to calculate $(b_i, V_i)_{i \in S}$. That is, V contains all the necessary information about past history. Thus the efficient contract can be determined recursively by solving the optimality equation starting with some initial value of V, which may be considered to be determined either by market forces or by some bargaining procedure if there is imperfect competition. Since there is a one-to-one correspondence between V and $U_k(V)$ the efficient contract is somewhat like a standard borrowing/lending contract in which the key variable is the borrower's indebtedness. To determine the current transfers it is only necessary to know the level of indebtedness. This analogy will be made even clearer in Section 7 which examines the constant absolute risk aversion case. These results are summarized by the following proposition:

PROPOSITION 2. *For any T there exists an efficient coinsurance contract such that after any history $b_i \leqslant b_{i-1}$, $V_i \geqslant V_{i-1}$, $i = 2, 3, ..., N$. The local upward incentive compatibility constraints never bind, the local downward incentive compatibility constraints always do.*

If we let λ, $(\mu_i)_{i \in S}$, be the multipliers associated with the constraints $\sum_{i \in S} \pi_i (v(b_i + \theta_i) - v(\theta_i) + \alpha V_i) = V$, $C_{i,i-1} \geqslant 0$, the first order necessary conditions for a solution to the optimality equation are[5]

$$\pi_i(1 - \lambda v'(b_i + \theta_i)) = \mu_i v'(b_i + \theta_i) - \mu_{i+1} v'(b_i + \theta_{i+1}), \tag{5}$$

$$\pi_i(U'_{k-1}(V_i) + \lambda) = \mu_{i+1} - \mu_i, \tag{6}$$

for $i = 2, 3, ..., N$, where $\mu_1 = \mu_{N+1} = 0$, together with the envelope condition

$$U'_k(V) = -\lambda. \tag{7}$$

[5] Ed Green has pointed out to us that a solution to these equations need not be feasible. They are however necessary at the optimum, and we only need sufficiency in calculating the solution to the example of Section 8, where the first-order conditions are indeed sufficient.

5. The Infinite Horizon Contract

We shall concentrate on the case $T = \infty$ for the rest of the paper. In this section we consider the long run properties of an efficient infinite horizon contract. Because the infinite horizon value function U_* is independent of time, the relationship between V and the current transfers does not change, so we need only look at how V varies through time. We define V^t to be the random variable representing the borrower's utility at the beginning of period t. It is possible to show that, at any date t, $V_N > V > V_1$ (Lemma 5), so that the level of indebtedness rises after the lowest income state and falls after the highest income state. A slightly stronger result can also be demonstrated: if the borrower experiences a long enough sequence of high income states he will eventually become a creditor, and if he experiences a long enough sequence of low income states he will become a debtor (Lemma 5). So V^t can rise or fall, but it would be of considerable interest to discover its long run tendency. To do this we use the fact that viewed as a stochastic process $U'_*(V^t)$ is a non-positive martingale. To see this consider increasing the borrower's utility at any date by one unit. One way of doing this is to increase every V_i by a factor of $1/\alpha$ while keeping every b_i constant. Such a change preserves incentive compatibility at a cost to the lender of $\sum_{i \in S} \pi_i U'_*(V_i)$. By the envelope theorem this is locally as good a way to increase V as any other and so is equal to $U'_*(V)$. Formally, summing (6) over $i \in S$ and using (7) yields

$$\sum_{i \in S} \pi_i U'_*(V_i) = U'_*(V). \tag{8}$$

Then using the martingale convergence theorem it is possible to prove:

Proposition 3. *If $T = \infty$, V^t converges to $-\infty$ almost surely.*

The idea behind the proof is quite simple. $U'_*(V^t)$ must converge almost surely, so it only needs to be shown that it does not converge to a non-zero limit with positive probability, since if $U'_*(V^t)$ converges to zero, V^t converges to $-\infty$. Likewise if $U'_*(V^t)$ converges to a non-zero limit, V^t converges to a finite limit. However, because the future V_i's are always spread out to aid incentive compatibility it can be shown that this only happens with zero probability.

The economic intuition behind the result seems to be that the cost of incentive compatibility is in some sense cheaper when V is low. To see this first note that the advantage of having a history dependent contract stems from using future utility, the V_i's, as inducements to truth-telling. To do this they must differ. Since U_* is concave this is costly because the lender's future utility falls as the dispersion of the V_i's increases. For example, if

$N=2$ the cost of spreading V_1 and V_2 an equal small amount z either side of their average value, V', is approximately $-(1/2)z^2U''_*(V')$. From the properties of U_* at its endpoints it follows that $\lim_{V\to-\infty} U''_*(V)=0$, and $\lim_{V\to d_\infty} U''_*(V)=-\infty$. Thus while it is not possible to assert that $U''_*(V)$ is monotonically decreasing (it *is* monotonic in the example in Section 7) it must decline on average which is all that is needed to prove the result. Thus the cost of obtaining a given spread of V_1 and V_2 is generally lower when they themselves are lower: incentive compatibility is in this sense cheaper when future utilities are lower. Therefore although it is preferable to try to keep V constant in order to smooth consumption over time there is a strong enough incentive to cause it to drift downwards. A contract then in which V declines over time can induce the borrower to tell the truth by using large variability in future utility and at the same time smooth consumption in the initial periods.

6. The Efficient Contract for Discount Factors Close to One

Although Proposition 3 shows that the efficient contract cannot converge uniformly to the first-best contract, the results of Radner [10] and Fudenberg, Holmstrom, and Milgrom [3] strongly suggest that first-best utilities can be approached as the discount factor gets close to unity. Radner uses a statistical approach based on a contract which penalizes the agent periodically if his record does not meet a specified standard. Such a rule is not usually efficient nor necessarily incentive compatible, but for discount factors close enough to one the periods of punishment become insignificant relative to the periods of first-best payments. Fudenberg, Holmstrom, and Milgrom adopt a different approach in which the agent can covertly borrow and save at a rate of interest equal to the rate of time preference. The agent's ability to self-insure in this way effectively reduces his risk aversion in any given period and in the limit he behaves as if he were risk neutral.

The dynamic programming approach affords a simple and natural proof of this convergence result. In particular it can be shown that the second-best Pareto-frontier converges pointwise to the first-best frontier[6] and that the efficient contract payments converge pointwise to the first-best levels. This second result complements Radner's result that payments are at their first-best levels in almost every period in the limit. To keep expected future utility bounded as $\alpha\to 1$ all per-period utilities are multiplied by $(1-\alpha)$ in this section. Let V_i^α be the borrower's normalised discounted future utility and $U_*^\alpha(V_i^\alpha)$ the corresponding utility of the lender. For any given V the

[6] For more details in a more general model see Lockwood and Thomas [9].

first-best contract gives the lender an expected return of $-\Sigma_{i\in S}\pi_i b_i^*$, where $v(b_i^*+\theta_i)=V$ for all $i\in S$. It is necessary to show $\lim_{\alpha\to 1}U_*^\alpha(V)=-\Sigma_{i\in S}\pi_i b_i^*$. Consider the following contract. In the first period pay b_i^*. This implies that $\Sigma_{i\in S}\pi_i V_i=V$, and it is possible to choose the V_i's to satisfy the downward incentive constraints with equality since they are all linear in V_i. Then follow the efficient contract from date two onwards. Because this almost efficient contract is incentive compatible it cannot offer more utility than the efficient contract itself, i.e., $U_*^\alpha(V)\geqslant -\Sigma_{i\in S}\pi_i b_i^* + (\alpha/(1-\alpha))\,\Sigma_{i\in S}\pi_i h_\alpha(V_i^\alpha)$, where $h_\alpha(V_i^\alpha)\equiv U_*^\alpha(V_i^\alpha)-U_*^\alpha(V)$. So it is sufficient to show that $(\alpha/(1-\alpha))\,\Sigma_{i\in S}\,\pi_i h_\alpha(V_i^\alpha)\to 0$ as $\alpha\to 1$.

PROPOSITION 4. *For given V, as $\alpha\to 1$ the utility of the lender from the efficient contract tends to the first-best level.*

By the convergence of the finite horizon value functions to the infinite horizon value functions (Lemmas 2 and 3) we have

COROLLARY. *For given V, and any $\varepsilon>0$, there are an α' and a T' such that $U_T^\alpha(V)\geqslant\Sigma_{i\in S}\pi_i b_i^*-\varepsilon$ for $T>T'$, $\alpha>\alpha'$.*

The intuition behind these two results is straightforward: for α close to unity incentive compatibility can be attained by a small divergence in the V_i^α's. The cost of this divergence is $\Sigma_{i\in S}\pi_i h_\alpha(V_i^\alpha)$ which is positive because U^α is concave. It goes to zero faster than α goes to unity because U^α is differentiable and hence locally linear.

Although the efficient contract cannot converge uniformly to the first-best contract, a weaker result can be proved.

PROPOSITION 5. *For any history h^t the efficient contract payments converge to their first-best levels as $\alpha\to 1$.*

7. CONSTANT ABSOLUTE RISK AVERSION

A special case of Assumption 1 is the constant absolute risk aversion utility function $v(c)=-\exp(-Rc)$, where $v:(-\infty,\infty)\to(-\infty,0)$. The optimality equation then is a concave programming problem since the constraints are linear in $\exp(-b_i)$ and V_i, and the maximand is concave in these variables. Then by repeated application of the operator L, and letting $A=-\Sigma_{i\in S}\pi_i\exp(-R\theta_i)/(1-\alpha)$, the discounted utility in the case of autarky, the following can be shown.

PROPOSITION 6. *If $T=\infty$ and $v(c)=-\exp(-Rc)$ then at the optimum $\exp(-Rb_i)=-a_i(V+A)$, $V_i=d_iV+(d_i-1)A$, where the a_i and d_i's are* *constants satisfying* $a_i \geqslant a_{i-1} > 0$, $d_{i-1} \geqslant d_i > 0$, $\Sigma_{i \in S} \pi_i d_i^{-1} = 1$, $\Sigma_{i \in S} \pi_i a_i^{-1} = -A$, *and* $U_*(V) = R^{-1}(1-\alpha)^{-1} \{\log(-V-A)+K\}$, *where* $K = \Sigma_{i \in S} \pi_i \log a_i + \alpha(1-\alpha)^{-1} \Sigma_{i \in S} \pi_i \log d_i$.

Let $I = U_*(V)$ denote the level of indebtedness. Then the amount borrowed in state i when indebtedness is I is $b_i = R^{-1}(K - \log a_i) - (1-\alpha) I$. It is decreasing in income since $a_i > a_{i-1}$ and decreasing in the level of indebtedness. How it varies with history depends on how indebtedness evolves. The constant absolute risk aversion utility function implies that history dependence takes a very simple form because there are no income effects.[7] Loans depend only on the number of times each state occurs and not on the order in which they occur. So if τ_j is the number of times state j occurs in history h^t and I_0 is the initial level of indebtedness, the amount lent in state i at date $t+1$ is

$$b(h^t, \theta_i) = R^{-1}\{(K - \log a_i) - \Sigma_{j \in S} \tau_j \log d_j\} - (1-\alpha) I_0. \tag{9}$$

There are a number of consequences of Eq. (9). First, the repayments from the borrower to the lender tend to minus infinity with probability one. To see this note that for t long enough τ_j can be approximated by $t\pi_j$. Then substituting in (9) proves the result since $\Sigma_{i \in S} \pi_i \log d_i > 0$ (see Proposition 7(ii) in the appendix). (This result is equivalent to Proposition 3 because of the relationship between b_i and V given in Proposition 6.) Equally expected repayments are always increasing.

Second it is possible to define an implicit rate of interest, independent of time, between any two states. Suppose that at some date state j is announced instead of state i, $j < i$, so an extra payment is received from the lender. Because the order of announcement does not matter, this can be corrected at a later date by announcing i instead of j, so that I returns to the value it would have had. In the meantime the borrower will be paying back more than he otherwise would, and so

$$r_{ij} = -\log(d_i/d_j)/\log(a_i/a_j)$$

is the implicit rate of interest between states i and j. There are $N-1$ such independent interest rates, defined between adjacent states, all others being weighted averages of these. It can be shown that $0 \leqslant r_{ij} < (1-\alpha)/\alpha$ so that each is positive and less than the rate of time preference. This is very intuitive since if it is to provide some insurance the borrower must be able to access funds relatively cheaply when necessary but must correspondingly receive a relatively low rate of return on his savings. In the limit as $\alpha \to 1$ each $r_{ij} \to 0$ so consumption risk can be eliminated (Yaari [14]).

It should be remembered that these are purely implicit rates of interest.

[7] We are grateful to Barry Nalebuff for this observation.

The borrower cannot borrow or save as much as he wants to but only what is dictated by the contract. If for example he suffers a series of low income levels his indebtedness will go up and he will only be able to borrow less and less, although his needs and future prospects have not changed. The borrower is constrained by his past history. We summarise the results of this section by

PROPOSITION 7. *If* $T=\infty$ *and* $v(c)=-\exp(-Rc)$ *then* (i) *an invariant implicit rate of interest* r_{ij} *is defined between any two states,* $0\leqslant r_{ij}<(1-\alpha)/\alpha$, *for each* $i, j\in S$; (ii) *expected payments to the lender increase over time, and* (iii) *payments to the lender tend to infinity with probability one.*

8. SELF-ENFORCING LOAN CONTRACTS

Proposition 3 showed that the borrower's future utility will become arbitrarily negative with probability one. This stretches to the limit the assumption that the contract is costlessly enforceable. Indeed for any fixed penalty associated with breach of contract, Proposition 3 shows that the benefits to reneging will in the long term almost surely exceed the costs. Certainly when loans are international the costs of reneging are likely to be low as legal sanctions are difficult to enforce across national boundaries. An efficient contract should take these possibilities into account. So in this section we examine the situation when either party can costlessly renege, under the assumption that they remain in autarky after one of them has reneged. We examine self-enforcing contracts in which neither party ever has an incentive to renege.

If the borrower has no outside opportunities and can costlessly renege he will do so whenever the net gain from the contract, looking forward, is negative, that is if $v(b_i+\theta_i)-v(\theta_i)+\alpha V_i<0$. Likewise the lender will renege if his net gain is negative. Thus the constraints

$$v(b_i+\theta_i)-v(\theta_i)+\alpha V_i\geqslant 0, \qquad i\in S \tag{10}$$

$$-b_i+\alpha U_*(V_i)\geqslant 0, \qquad i\in S \tag{11}$$

should be added to the definition of the one-step operator L to rule out the incentive to renege. Changing the definition of L in this way does not invalidate the dynamic programming approach and the efficient contract can be characterised in the same way as before. The argument is very similar to that given at the beginning of Section 3, where the words "self-enforcing" should be added to "incentive compatible." The crucial observation is that again the continuation contract should always be efficient: if it is not then replacing it by an incentive compatible, self-enforcing contract giving the borrower the same utility but the lender more means that the new contract is not only incentive compatible but also self-enforcing and Pareto superior.[8]

Without outside enforcement the infinite time horizon is strictly necessary to avoid the unravelling problem familiar from finitely repeated games with a unique stage game equilibrium. That is, (10) and (11) could only be satisfied by a zero transfer in the last period, and therefore only by a zero transfer in the penultimate period and so on, so that no non-trivial contract can exist in the finite horizon case.

It is nevertheless possible to show that for a sufficiently high discount factor a non-trivial contract exists for an infinite time horizon. The argument is as follows. Consider first a simple two-period contract which differs only slightly from autarky. In the initial period if any state other than N is announced then no transfers are made in either period. However, if in the initial period state N is announced the borrower pays an amount Δb_N to the lender and consequent upon this recieves Δb in the second period irrespective of the state. These are chosen so that the borrower is just indifferent about announcing N when it occurs. That is, $v(-\Delta b_N+\theta_N)+\alpha\Sigma_{i\in S}\pi_i v(\Delta b+\theta_i)=v(\theta_N)+\alpha\Sigma_{i\in S}\pi_i v(\theta_i)$. By the concavity of v this contract is incentive compatible and it satisfies (10). The expected profit to the lender is $\pi_N(\Delta b_N-\alpha\Delta b)$ and the borrower gets a zero net utility gain. When $\alpha=1$ a first order Taylor's approximation yields $\Delta b_N=(\Sigma_{i\in S}\pi_i v'(\theta_i)/v'(\theta_N))\,\Delta b$, where the bracketed term is greater than one. So a small enough Δb can be found that expected profit is strictly positive. Now keep Δb constant at this level and reduce α below one. The lender's profits and Δb_N vary continuously with α, so profit converges to a positive number as $\alpha\to 1$. Of course this contract violates (11) because the lender will always want to renege in the second period when he has to pay out. However, if this two-period contract is repeated every other period, thus forming an infinite horizon contract, the expected profit to the lender, $\pi_N(\Delta b_N-\alpha\Delta b)/(1-\alpha^2)$, tends to infinity as $\alpha\to 1$. The short-term gain to the lender from reneging in the second period is no more than Δb, so this will be outweighed by the future loss of not having the contract if α is near one. The long-term contract therefore satisfies (10) and (11) for α near one and we conclude that a non-trivial contract exists for such an α.

Proposition 3 implies that one of the constraints in Eq. (10) will eventually bind, so an efficient self-enforcing contract will be different from a contract which was legally enforceable. It would be interesting to know

[8] In a labour contracts model [12] we consider the pure self-enforcement problem and show how this may be tackled using dynamic programming. Broader issues relating to self-enforcing contracts are dealt with there.

whether once (10) binds the contract effectively terminates, that is no future loans or repayments are scheduled, or whether it is always worthwhile to schedule future loans and repayments. That is, can termination be used as an incentive device or not? The answer is negative: if a non-trivial contract exists at all, there must be a contract that gives the borrower a zero net gain and the lender a strictly positive net gain. This dominates a termination, where both get zero, so that by the Principle of Optimality such a termination could not be part of an efficient contract. To summarise:

PROPOSITION 8. *If both parties can renege and have no outside opportunities then there exists some α' such that a non-trivial, self-enforcing contract exists for each $\alpha \in (\alpha', 1)$. Moreover such a non-trivial efficient contract will not terminate.*

APPENDIX

LEMMA 1. $U_k(V) = L(U_{k-1})(V)$ *for* $V \in (-\infty, d_\infty)$.

Proof. (i) We first show $U_k(V) \leqslant L(U_{k-1})(V)$. Define

$$U[b^k] = -E \sum_{\tau=0}^{k-1} \alpha^\tau b(h^\tau), \qquad U[b^k : h^t] = -E\left[\sum_{\tau=t+1}^{k-1} \alpha^\tau b(h^\tau) : h^t\right].$$

$U[b^k]$ is the net gain to the lender from the contract b^k and $U[b^k : h^t]$ is the net gain after the history h^t. Define $V[b^k]$ and $V[b^k : h^t]$ analogously. So for any $V(-\infty, d_\infty)$ and any $b^k \in B^k(V)$, $U[b^k] = \Sigma_{i \in S} \pi_i(-b_i + U[b^k : s_0 = \theta_i])$. Then by the definition of U_{k-1}, $U[b^k : s_0] \leqslant U_{k-1}(V[b^k : s_0])$ and since $(b_i, V[b^k : s_0 = \theta_i])_{i \in S} \in \Lambda(V)$, $\Sigma_{i \in S} \pi_i(-b_i + U_{k-1}(V[b^k : s_0 = \theta_i]) \leqslant L(U_{k-1})(V[b^k])$. Therefore taking the supremum over all $b^k \in B^k(V)$, $U_k(V) = \sup U[b^k] \leqslant L(U_{k-1})(V)$.

(ii) We now show $U_k(V) \geqslant L(U_{k-1})(V)$. There exist some $(\beta_i, K_i)_{i \in S} \in \Lambda(V)$ and $\varepsilon > 0$ such that $\Sigma_{i \in S} \pi_i(-\beta_i + \alpha U_{k-1}(K_i)) \geqslant L(U_{k-1})(V) - \varepsilon$ for any $V \in (-\infty, d_\infty)$. Equally $U[b^k : s_0 = \theta_i] \geqslant U_{k-1}(K_i) - \varepsilon$, where $[b^k : s_0 = \theta_i] \in B^k(K_i)$. Let β^k be the contract which pays β_i in the first period and follows $[b^k : s_0 = \theta_i]$ thereafter. Since income is i.i.d. $\beta^k \in \beta^k(V[\beta^k])$. So $U[\beta^k] \geqslant \Sigma_{i \in S} \pi_i(-\beta_i + \alpha U_{k-1}(K_i)) - \alpha\varepsilon \geqslant L(U_{k-1}) - (1 + \alpha)\varepsilon$. Since ε is arbitrary, when the supremum is taken over all $b^k \in B^k(V)$, $U_k(V) \geqslant L(U_{k-1})(V)$.

LEMMA 2. *F is a complete metric space in the supremem metric and L is a contraction on F.*

Proof. To show that F is a metric space it suffices to show that the gap between the bounds in (4) is itself bounded. Since v is increasing and $\Sigma_{i\in S}\pi_i v(y_\infty+\theta_i)=v(c_\infty)$, for given V, $y_\infty+\theta_1\leqslant c_\infty$ and so $y_\infty+\theta_i+\theta_1\leqslant y_\infty+\theta_N+\theta_1\leqslant c_\infty+\theta_N$. Thus $y_\infty+\theta_i-c_\infty\leqslant\theta_N-\theta_1$ and therefore it must be the case that $y_\infty+\Sigma_{i\in S}\pi_i(\theta_i-c_\infty)/(1-\alpha)\leqslant(\theta_N-\theta_1)/(1-\alpha)$. Completeness is standard. For any $U\in F$, $L(U)$ can be no greater at any point than the upper bound in (4) since otherwise this would imply that utilities higher than the first-best could be achieved by ignoring the incentive constraints, which is impossible. Likewise, if $L(U)$ were less at any point than the lower bound in (4), utility would be less than the trivial incentive compatible contract, despite starting with a value function no smaller than that corresponding to the trivial contract, again an impossibility. So $L(U)\in F$. That L is a contraction now follows from standard arguments.

LEMMA 3. $\operatorname{Lim}_{k\to\infty} U_k=L(\lim_{k\to\infty}U_k)$.

Proof. Define $U_\infty=\lim_{k\to\infty}U_k$. It is obvious the lender can do at least as well in $k+1$ periods as he can in k periods. So for any V, $U_0\leqslant LU_0\leqslant L^2U_0\leqslant\cdots\leqslant L^kU_0\leqslant\cdots\leqslant U_\infty$. Hence $L^{k+1}U_0\leqslant LU_\infty$ and taking limits $U_\infty\leqslant LU_\infty$. Again since $L(L^kU_0)\leqslant U_\infty$, $\Sigma_{i\in S}(b_i+\alpha L^kU_0(V_i))\leqslant U_\infty(V)$ for any $(b_i, V_i)_{i\in S}\in A(V)$. So taking limits $\Sigma_{i\in S}\pi_i(-b_i+\alpha U_\infty(V_i))\leqslant U_\infty(V)$ and taking the supremum, $LU_\infty(V)\leqslant U_\infty(V)$.

LEMMA 4. *Assuming $U_{k-1}(V)$ is strictly concave, at the solution to* (2): (i) *the local downward incentive compatibility constraints always bind,* (ii) *there is coinsurance, i.e.* $-b_i+\alpha U(V_i)\geqslant-b_{i-1}+\alpha U(V_{i-1})$ *and* $v(b_i+\theta_i)+\alpha V_i>v(b_{i-1}+\theta_{i-1})+\alpha V_{i-1}$, (iii) *the local upward incentive compatibility constraints never bind.*

Proof. (i) It is first shown that $C_{i,i-1}=0$. Suppose to the contrary that $C_{i,i-1}>0$, for some $i\in S$. Then $V_i>V_{i-1}$ since $b_{i-1}\geqslant b_i$. Then consider changing $(b_i, V_i)_{i\in S}$, as follows: keep V_1 fixed and if necessary reduce V_2 until $C_{2,1}=0$. Next reduce V_3 until $C_{3,2}=0$, and so on, until $C_{i,i-1}=0$ for all $i\in S$. Add the necessary constant to each V_i to leave EV_i unchanged overall. Each (V_i-V_{i-1}) has been reduced so the lender's utility is increased. The new contract offers the borrower the same utility and is incentive compatible since $b_{i-1}\geqslant b_i$ and $C_{i,i-1}=0$ together imply $C_{i,i+1}>0$, i.e., the upward constraints hold. Hence the original contract has been improved, contrary to assumption.

(ii) The latter follows from part (i). So suppose $-b_i+\alpha U(V_i)<-b_{i-1}+\alpha U(V_{i-1})$. Then replacing b_i by b_{i-1} and V_i by V_{i-1} raises the lender's utility but leaves the borrower's utility unchanged and is also incentive compatible.

(iii) Suppose we ignore the constraint $C_{i-1,i} \geqslant 0$. If $b_{i-1} \geqslant b_i$ then by (i) the upward incentive constraint is automatically satisfied. So suppose that the solution has $b_i > b_{i-1}$. Then $V_i < V_{i-1}$ and $C_{i-1,i} < 0$. But then replacing b_{i-1} by b_i and V_{i-1} by V_i cannot decrease the lender's utility and cannot violate incentive compatibility. But $v(b_i + \theta_{i-1}) - v(b_{i-1} + \theta_{i-1}) > v(b_i + \theta_i) - v(b_{i-1} + \theta_i) = \alpha(V_{i-1} - V_i)$ since v is concave. So $v(b_i + \theta_{i-1}) + \alpha V_i > v(b_{i-1} + \theta_{i-1}) + \alpha V_{i-1}$ and the borrower's utility is improved.

Proof of Proposition 1. It is obvious that $U_k(V)$ is decreasing. Assume $U_{k-1}(V)$ is strictly concave. Consider any V and V' with the associated contracts $(b_i, V_i)_{i \in S}$, $(b'_i, V'_i)_{i \in S}$. Let $V_i^* = \delta V_i + (1-\delta) V'_i$ and define b_i^* by $v(b_i^* + \theta_i) = \delta v(b_i + \theta_i) + (1-\delta) v(b'_i + \theta_i)$, for $\delta \in (0, 1)$. So $(b_i^*, V_i^*)_{i \in S}$ gives the borrower average utility and the lender no less than average utility. Then $C^*_{i,i-1} = \delta C_{i,i-1} + (1-\delta) C'_{i,i-1} + \delta v(b_{i-1} + \theta_i) + (1-\delta) v(b'_{i-1} + \theta_i) - v(b^*_{i-1} + \theta_i)$. By Lemma 4, at the optimum, $C_{i,i-1} = 0$ and $C'_{i,i-1} = 0$ and since the risk premium is a decreasing function of income (Assumption 1) the third term is non-negative, so the downward constraints are satisfied. However, the contract $(b_i^*, V_i^*)_{i \in S}$ may violate the upward incentive contraints. Nevertheless, using a similar argument to that used in Lemma 4(i), it is possible to construct a new contract from $(b_i^*, V_i^*)_{i \in S}$ which is incentive compatible and offers both the lender and the borrower no less utility. This may be done as follows. Keep V_1 fixed and reduce V_2 until $C_{2,1} = 0$ or until $V_1 = V_2$. Then reduce V_3 in the same way and so on. Add the necessary constant to each V_i to leave EV_i unchanged overall. This will not make the lender worse off. Now if $V_2 = V_1$, which implies $b_2^* > b_1^*$, reduce b_2 until $C_{2,1} = 0$, and proceed in the same way for b_3 and so on; since $b_i + \theta_i > b_{i-1} + \theta_{i-1}$ adding a constant to each b_i to leave Eb_i constant cannot make the borrower worse off. So in this new contract $C_{i,i-1} = 0$ and $b_{i-1} \geqslant b_i$. Thus the upward incentive constraints also hold. Strict concavity then follows because it is not possible to have both $b_i = b'_i$ and $V_i = V'_i$ for all $i \in S$ and $V \neq V'$, so the contract $(b_i^*, V_i^*)_{i \in S}$ yields the lender strictly more than $\delta U_{k-1}(V) + (1-\delta) U_{k-1}(V')$. To complete the induction argument observe that $U_1(V)$ is trivially strictly concave. So each $U_k(V)$ is strictly concave, and their pointwise limit, $U_*(V)$, must be concave. Uniqueness of the optimum contract follows because it is not possible to have both $b_i = b'_i$ and $V_i \neq V'_i$ for all $i \in S$ for the same V, so non-uniqueness would imply $b_i \neq b'_i$ for some i, and the constructed contract is strictly better, true also for an infinite horizon. To prove continuous differentiability, consider a neighbourhood of values of V around any V', and construct an incentive compatible contract for each V by taking the optimum contract at V' and keeping the future utilities constant but varying the b_i's so as to maintain incentive compatibility and give V overall (there is a unique way of doing this). The lender's utility is then a concave function of V by a similar argument to that given above, and this function is differentiable and equal to U_k at V'. The result follows from applying Lemma 1 of [2].

LEMMA 5. (i) $U'_*(V_N) < U'_*(V) < U'_*(V_1)$, (ii) $V_1 < V < V_N$, (iii) *for any* $V^t \in (-\infty, d_\infty)$ *and* $\gamma < U'_*(V^t)$, *if state* N *is repeated* τ *times consecutively then* $U'_*(V^{t+\tau}) \leqslant \gamma$ *for* τ *large enough*; *likewise for* $0 > \gamma > U'_*(V^t)$ *if state* 1 *is repeated* τ *times consecutively then* $U'_*(V^{t+\tau}) \geqslant \gamma$ *for* τ *large enough.*

Proof. (i) We shall show that $U'_*(V_N) < U'_*(V)$; the argument for $U'_*(V) < U'_*(V_1)$ is symmetric. Suppose, contrary to assumption, that $U'_*(V_N) \geqslant U'_*(V)$. Since $V_N \geqslant V_i$ for all $i \in S$, (8) implies $U'_*(V_i) = U'_*(V)$ for all $i \in S$. So using the first-order conditions (5)–(7), $v'(b_i + \theta_i) = 1/\lambda$ for all $i \in S$. Hence consumption is stabilized, and U_* must be linear between V_1 and V. Consider some $V' < V$ and the associated $(b'_i, V'_i)_{i \in S}$. Let V' be the smallest value such that $U'_*(V') = U'_*(V)$. There are two cases to consider; first $U'_*(V'_N) \geqslant U'_*(V) = U'_*(V_i) = U'_*(V')$. Since $V'_N \leqslant V$ it must be true that $U'_*(V) = U'_*(V')$ by the above argument. But V' is the smallest value such that $U'_*(V) = U'_*(V')$, so $V'_i = V_i$ and $b'_i = b_i$ for all $i \in S$. From incentive compatibility, $V'_i = V'_{i-1}$ implies $b'_i = b'_{i-1}$, a contradiction. The other possibility is $U'_*(V_i) > U'_*(V'_N)$. The case $N = 2$ is dealt with for simplicity; the argument generalizes straightforwardly. From (5) and (6) $U'_*(V'_2) = -1/v'(b_2 + \theta_2)$ so $b'_2 > b_2$. From (6) $V'_1 < V_1$ and from (5) $b'_1 < b_1$. By incentive compatibility, $V'_2 - V'_1 > V_2 - V_1$ implies $v(b_2 + \theta_2) - v(b_1 + \theta_2) > v(b'_2 + \theta_2) - v(b'_1 + \theta_2)$. But since $b'_2 > b_2$ and $b'_1 < b_1$, this is a contradiction.

(ii) Follows immediately from (i) and concavity.

(iii) Consider a sequence in which state N is repeated τ times consecutively. With $V = V^{t+\tau-1}$ we have $V^{t+\tau} = V_N$, using the notation in the text where V is the current value of the borrower's and V_N is next period's if state N occurs. So $V^{t+\tau} > V^{t+\tau-1}$ since $V_N > V$, and thus $V_{t+\tau} > V^t$. Then $U'_*(V^{t+\tau}) \leqslant U'_*(V^t)$ since U_* is concave. Suppose no τ exists such that $U'(V^{t+\tau}) \leqslant \gamma$. Then $\lim_{\tau \to \infty} U'_*(V^{t+\tau}) > \gamma$ or equivalently $\lim_{\tau \to \infty} V^{t+\tau} < \phi$, where $U'_*(\phi) = \gamma$; both limits exist by the concavity of U_*. Note that the contract is continuous in V by the fact that the constraint set $\Lambda(V)$ is a continuous correspondence and the optimum contract is unique. So there is a convergent sequence of contracts as $\tau \to \infty$ with V_N tending to V. The limit contract must be optimum when $V = \lim_{\tau \to \infty} V^{t+\tau}$, and has $V_N = \lim_{\tau \to \infty} V^{t+\tau}$, so $V_N = V$, a contradiction. Thus τ as required exists. The case $0 > \gamma > U'_*(V^t)$ is proved similarly.

Proof of Proposition 3. U'_* is a non-positive martingale. Therefore by Doob's Convergence Theorem [6, p. 204] it converges almost surely to some random variable, R. Recall that $\lim_{V \to -\infty} U'_* = 0$ and $\lim_{V \to d_\infty} U'_*(V) = -\infty$; it suffices to show $R=0$ almost surely. Consider a path with the property that $\lim_{t \to \infty} U'_*(V^t) = C \neq 0$ and state N occurs infinitely often. We will show that such paths cannot exist. Take a subsequence composed of those dates when state N occurs. This sequence must have a convergent subsequence $(V^{t(\tau)})_{\tau=0,1,\ldots}$ (since it eventually belongs to $\{V : U'_*(V) \in [C-\varepsilon, C+\varepsilon]\}$ for some $\varepsilon > 0$, which is bounded), and call the limit W. Denote the general relationship between successive values of V by $V^{t+1} = f(V^t, \theta_i)$ and observe that f is continuous in V^t (see the proof of Lemma 5(iii)). So the sequence $(f(V^{t(\tau)}, \theta_N))_{\tau=0,1,\ldots}$ converges to $f(W, \theta_N)$ and since by definition $f = V^{t(\tau)}, \theta_N = V^{t(\tau)+1}$, $V^{t(\tau)+1}$ converges to $f(W, \theta_N)$ as well. But both $\lim_{\tau \to \infty} U'_*(V^{t(\tau)}) = C$ and $\lim_{\tau \to \infty} U'_*(V^{t(\tau)+1}) = C$, so by continuity of U'_* we have $U'_*(W) = U'_*(f(W, \theta_N)) = C$, which contradicts Lemma 5(i). Since paths where state N occurs finitely often have zero probability, the probability that $\lim_{t \to \infty} U'_*(V^t)$ exists and is non-zero is zero, which completes the proof.

Proof of Proposition 4. By the mean value theorem there is some K_i^α between V_i^α and V such that $h_\alpha(V_i^\alpha) = (U_*^\alpha)'(K_i^\alpha)(V_i^\alpha - V)$. From the incentive constraints $V_i^\alpha - V_{i-1}^\alpha = (1-\alpha) B_i/\alpha$, where $B_i \equiv v(b_{i-1}^* + \theta_i) - v(b_i^* + \theta_i)$ is a constant independent of α. As $\Sigma_{i \in S} \pi_i V_i^\alpha = V$, it follows that $V_i^\alpha - V = V_i^\alpha(1-\pi_i) - \Sigma_{j \neq i} \pi_j V_j^\alpha = \Sigma_{j<i} \pi_j (V_i^\alpha - V_j^\alpha) - \Sigma_{j>i} \pi_j (V_j^\alpha - V_i^\alpha)$ and for $i > j$, $V_i^\alpha - V_j^\alpha = \Sigma_{k=i}^{j+1} B_k (1-\alpha)/\alpha$, so substitution gives

$$(\alpha/(1-\alpha)) \sum_{i \in S} \pi_i h_\alpha(V_i^\alpha) = \sum_{i=2}^{N} \pi_i \sum_{j<i} \pi_j \sum_{k=i}^{j+1} B_k((U_*^\alpha)'(K_i^\alpha) - (U_*^\alpha)'(K_j^\alpha)).$$

Clearly $V_i^\alpha \to V$, since $V_i^\alpha - V_{i-1}^\alpha = B_i(1-\alpha)/\alpha$, and so $K_i^\alpha \to V$ for all $i \in S$. Let U be the pointwise limit of U_*^α. Since U_*^α is concave it converges uniformly on all compact subsets of $(-\infty, d_\infty)$ and it is differentiable almost everywhere. Let D be the set of points where $U(V)$ is differentiable. We have $(U_*^\alpha)'(K_i^\alpha) \geqslant (U_*^\alpha(V_i^\alpha + \mu) - U_*^\alpha(V_i^\alpha))/\mu$ for $\mu > 0$. So $\lim_{\alpha \to 1} (U_*^\alpha)'(K_i^\alpha) \geqslant (U(V+\mu) - U(V))/\mu$ or $\lim_{\alpha \to 1} (U_*^\alpha)'(K_i^\alpha) \geqslant U'(V)$ if $V \in D$, in the limit as $\mu \to 0$. For $\mu < 0$ the inequalities are reversed, so $(U_*^\alpha)'(K_i^\alpha) \to U'(V)$ if $V \in D$ as $\alpha \to 0$. Since b_i^* is continuous in V and $U(V)$ is differentiable almost everywhere the result is proved.

Proof of Proposition 5. For simplicity (the same argument can be generalized) assume there are just two states. By incentive compatibility in the first period $V_2^\alpha - V_1^\alpha = ((1-\alpha)/\alpha)(v(b_1^\alpha + \theta_2) - v(b_2^\alpha + \theta_2))$. The latter bracket can be shown to be bounded, so $V_i^\alpha \to V$. Since $(U^\alpha)'(V_2^\alpha) \to U'(V)$, from (5) and (6), $v'(b_i^\alpha + \theta_i) \to -1/U'(V) = v'(b_i^* + \theta_i)$. Therefore $b_i^\alpha \to b_i^*$. Second period payments in state j, b_{ij}^α depend on V_i^α, as does V_{ij}^α. So reapplying the same arguments, $V_{ij}^\alpha \to V$, $b_{ij}^\alpha \to b_j^*$ and so on.

Proof of Proposition 6. The solution is clearly feasible. It is also easy to check that U_* is a fixed point of L. The ordering of the a_i's and d_i's follows from adding the adjacent upward and downward incentive constraints. The other conditions are derived directly from (5), (6), and (7).

Proof of Proposition 7. We take each part of the proof in turn.

(i) Since $a_i \geqslant a_j$, and $d_i \leqslant d_j$ for $i > j$, $r_{ij} \geqslant 0$. As log is concave $r_{ij} = -(\log d_i - \log d_{i-1})/(\log a_i - \log a_{i-1}) < -(d_i - d_{i-1})\, a_i/(a_i - a_{i-1})\, d_i$. By Eq. (1) and Lemma 4 $-c_i(a_i - a_{i-1}) = \alpha(d_i - d_{i-1})$, where $c_i = \exp(-R\theta_i)$, so $r_{ij} < c_i a_i/\alpha d_i$. But from (5) and (6) $c_i a_i \leqslant (1-\alpha)\, d_i$. Therefore $r_{i,i-1} < (1-\alpha)/\alpha$. By definition $r_{i,i-2}$ is a convex combination of $r_{i,i-1}$ and $r_{i-1,i-2}$, so $r_{i,i-2} < (1-\alpha)/\alpha$. As it holds for $i = 2, 3, ..., N$, $r_{ij} < (1-\alpha)/\alpha$ for all $i, j \in S$.

(ii) By definition expected payments change each period by $-R^{-1}\Sigma_{i \in S}\pi_i \log d_i = R^{-1}\Sigma_{i \in S}\pi_i \log d_i^{-1} < R^{-1} \log \Sigma_{i \in S}\pi_i d_i^{-1} = 0$.

(iii) By the strong law of large numbers τ_j/t converges to π_j almost surely, so $-R^{-1}\Sigma_{i \in S}(\tau_i/t) \log d_i$ converges almost surely to $-R^{-1}\Sigma_{i \in S}\pi_i \log d_i$, which is negative by (ii). So $-tR^{-1}\, \Sigma_{i \in S}(\tau_i/t) \log d_i$ converges to $-\infty$, almost surely, which from the formula for $b(h^t, \theta_i)$ proves the result.

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