Measurable Ambiguity[†]

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Abstract

We introduce and analyze *expected uncertain utility theory (EUU)*. A prior and an *interval utility* characterize an EUU decision maker. The decision maker uses her subjective prior to transform each uncertain prospect f into an *interval-valued* prospect $\mathbf{f}$ which assigns an interval $[x, y]$ of prizes to each state. The decision maker ranks prospects according to their expected interval utilities $E(u(\mathbf{f}))$ where u is the index that specifies the utility of each interval $[x, y]$. We define risk and ambiguity aversion for EUU, use the EUU model address the Allais Paradox, the Ellsberg Paradox, the Home Bias and relate these behaviors to the individuals attitude towards risk and ambiguity.

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1. Introduction

We introduce and analyze *expected uncertain utility* theory (EUU), a model of decision making under uncertainty. The choice objects are Savage acts that associate a monetary prize to every state of nature. The goal is to provide a theory that can address three well-documented deviations from expected utility theory:

(i) Source preference. (Heath and Tversky (1991)). This evidence shows that decision makers prefer uncertain prospects if they depend on familiar rather than unfamiliar events.

(ii) Ellsberg-style evidence. (Camerer and Weber (1992)). This evidence shows decision makers may lack probabilistic sophistication;

(iii) Allais-stye evidence. (Starmer (2000). This evidence shows systematic violations of the independence axiom.

EUU decision makers are characterized by a sigma-algebra $\mathcal{E}$, a probability measure μ and a interval utility $u(x, y)$ that associates a real number with every interval of monetary prizes $[x, y]$. The sigma-algebra, like the probability and the interval utility , is subjective, that is, a parameter of the decision maker's preference. Therefore, a generic act f in the domain of preferences need note be $\mathcal{E}$-measurable.

The decision maker evaluates each act f according to its expected interval utility $U(f)$. To compute $U(f)$, we find $\mathcal{E}$-measurable acts $\mathbf{f}_1, \mathbf{f}_2$ so that $\mathbf{f}_1 \leq f$ is the largest $\mathcal{E}$-measurable lower bound and $f \leq \mathbf{f}_2$ is the smallest $\mathcal{E}$-measurable upper bound. Then, $U(f) = \int_\Omega u(\mathbf{f}_1(\omega), \mathbf{f}_2(\omega))d\mu$. If f is $\mathcal{E}$-measurable then $\mathbf{f}_1 = f = \mathbf{f}_2$ and the $U(f)$ reduces to subjective expected utility.

Consider an agent who must confront two sources of uncertainty, one is the outcome of a basketball tournament and the other is the outcome of a tennis tournament. The DM has prior beliefs about the outcomes each tournament and, therefore, acts that depend on the outcome of a single tournament give rise to well-defined lotteries over prizes $[l, m]$. Moreover, the DM ranks those acts according to their implied lotteries. That is, the DM has a (continuous and monotone[1]) *lottery preference* that describes his behavior for single source acts. However, the decision maker prefers basketball bets over tennis bets, that is,

[1] In the sense of first order stochastic dominance.

if given the choice between two bets with identical odds of winning, the decision maker prefers the one that depends on the outcome of the basketball tournament. Behavior of this type is referred to as a source-preference. Evidence for non-indifference among sources can be found in Heath and Tversky (1991) and Abdellaoui, et al. 2009. In addition, the finance literature has coined the phrase "home bias" to describe the preference of investors for domestic assets. See, for example, French and Poterba (1991) for evidence of the home bias. The home bias is puzzling because investors forgo the benefits of international diversification in favor of holding familiar assets.

We use the term *risky environment* to describe a collection of single source acts. We show that for every EUU decision maker there are many risky environments and the corresponding lottery preferences vary with the environment. Hence, our model can accommodate agents who exhibit a source preference and forgo the benefits of diversification in favor of holding only assets that depend on a preferred source.

The lottery preferences of EUU decision makers form a class of non-expected utility preferences we term *generalized quadratic utility* (GQT). GQT utility includes as a special case rank dependent utility with parameter restrictions that are commonly employed in empirical studies of lottery preferences (Starmer (2000)). Thus, EUU decision makers exhibit lottery preferences that match Allais-style experimental evidence.

When acts depend on multiple sources, the decision maker may fail probabilistic sophistication. A collection of events is *ambiguous* if they are not part of a single risky environment. When EUU decision makers must choose among bets on an ambiguous collection of events, they will violate probabilistic sophistication and exhibit preference reversals as documented in Ellsberg style experiments. We show that EUU can accommodate all Ellsberg-style urn experiments.

1.1 Related Literature

Our model is most closely related to the work of Jaffray (1989) who introduces a discrete model of expected uncertain utility. He takes the set of all discrete totally monotone capacities over prizes as a primitive and applies the von Neumann-Morgenstern axioms to preferences over such capacities to obtain a linear representation. He applies the Moebius transform to each such capacity and hence identifies it with a probability distribution over sets of prizes. Thus, he interprets linear preferences over capacities as expected utility preferences over lotteries over sets. Finally, he argues that sets that have the same best and worst elements should be indifferent and arrives at an expected uncertain utility representation.

Just as von Neumann and Morgenstern define risky lotteries as the object of choice, Jaffray defines capacities as the objects of choice for a decision maker confronting ambiguity. In contrast, we provide a Savage-style representation theorem for EUU theory. We take acts that map states into prizes as the domain of preferences and derive a the decision maker's subjective probability and interval utility from her preferences.

We defer a discussion of the relation between EUU theory, Choquet expected utility theory (Schmeidler (1989) and α-MEU theory (Ghirardato, et al. (2004)) to the final section of the paper.

2. Expected Uncertain Utility

The interval $M = [l, m]$ is the set of monetary prizes. Let Ω be the state space with the cardinality of the continuum. The decision maker has preferences over acts, that is, functions f from Ω to M. Let $\mathcal{F}$ be the set of all acts. Given any $\sigma-$algebra $\mathcal{E} \subset 2^{\Omega}$ and countably additive $\mu : \mathcal{E} \to [0, 1]$, we call $(\mathcal{E}, \mu)$ a prior if it is a complete (i.e., $A \subset E \in \mathcal{E}$ and $\mu(E) = 0$ implies $A \in \mathcal{E}$) and nonatomic (i.e., $\mu(A) > 0$ implies $0 < \mu(B) < \mu(A)$ for some $B \subset A$) probability measure.

Let $I = \{(x, y) \,|\, l \leq x \leq y \leq m\}$ be the set of all pairs of prizes. We interpret the pair (x, y) as a single (subjective) consequence. The pair (x, y) describes a situation that the decision maker interprets as getting at least x and at most y. Given a prior $(\mathcal{E}, \mu)$, a function $\mathbf{f} : \Omega \to I$ is a subjective interval act if it is measurable with respect to $\mathcal{E}$. A subjective interval act is tight if $\mu(\{\omega \in \Omega \,|\, \mathbf{f}_1(\omega) = \mathbf{f}_2(\omega)\}) = 1$. Let $\mathbf{F}_{\mathcal{E}}$ denote the set of all subjective interval acts. For $\mathbf{f} \in \mathbf{F}_{\mathcal{E}}$, let $\mathbf{f}_i$ denote the i't coordinate of $\mathbf{f}$. That is, $\mathbf{f}(\omega) = (\mathbf{f}_1(\omega), \mathbf{f}_2(\omega))$ for all $\omega \in \Omega$. Lemma 1 below reveals that given any prior $(\mathcal{E}, \mu)$, each act can be identified with a unique (up to a set of measure 0) subjective interval act.

Lemma 1: *Let $(\mathcal{E}, \mu)$ be any prior. Then, for any $f \in \mathcal{F}$, there exists an $\mathbf{f} \in \mathbf{F}_{\mathcal{E}}$ such that*

$$\mu(\{\omega \in \Omega \,|\, \mathbf{f}_1(\omega) \leq f(\omega) \leq \mathbf{f}_2(\omega)\}) = 1 \tag{22}$$

and if $\mathbf{g} \in \mathbf{F}_{\mathcal{E}}$ *also satisfies* (22)*, then*

$$\mu(\{\omega \in \Omega \,|\, \mathbf{g}_1 \leq \mathbf{f}_1(\omega) \leq \mathbf{f}_2(\omega) \leq \mathbf{g}_2(\omega)\}) = 1 \tag{23}$$

It is clear than any $\mathbf{f}$ with the property above is unique up to a set of measure 0. We call the $\mathbf{f}$ corresponding to any f its *envelope*. Note that $f \in \mathcal{F}_{\mathcal{E}}$ if and only if $\mathbf{f}_1 = f = \mathbf{f}_2$ almost $(\mathcal{E}, \mu)$-surely. That is, an act is $\mathcal{E}-$measurable if and only if its envelope is tight. Lemma 2 below is a converse of Lemma 1.

Lemma 2: *Let* $(\mathcal{E}, \mu)$ *be a prior. Then, for any* $\mathbf{f} \in \mathbf{F}_{\mathcal{E}}$*, there exists* $f \in \mathcal{F}$ *such that* $\mathbf{f}$ *is* f*'s envelope.*

Henceforth, we write $\mathbf{f}, \mathbf{g}$ and $\mathbf{h}$ to denote the envelopes of f, g and h respectively. An *interval utility* is a continuous function $u : I \to I\!R$ such that $u(x, y) > u(x', y')$ whenever $x > x'$ and $y > y'$. Let $\mathcal{U}$ be the set of all interval utility indicies. A preference $\succeq$ is a *expected uncertain utility* (EUU) if there exists a prior $(\mathcal{E}, \mu)$ and $u \in \mathcal{U}$ such that the function W defined below represents $\succeq$:

$$W(f) = \int u(\mathbf{f}) d\mu \tag{33}$$

Thus, a prior $(\mathcal{E}, \mu)$ and an interval utility u characterize an EUU decision maker. Therefore, we identify $(\mathcal{E}, \mu, u)$ with corresponding the EUU preference $\succeq$. We say that the interval utility u is *symmetric* if there exists $\alpha \in [0, 1]$ such that $u(x, y) = \alpha u(x, x) + (1 - \alpha)u(y, y)$ for all $(x, y) \in I$. We say that u is strongly symmetric if this α is $1/2$.

To illustrate the main ideas we use the following example through out the paper: let $\Omega = [0, 1) \times [0, 1)$ be the unit square. Let λ^2 be the two-dimensional Lebesque measure on the two-dimensional Borel sets $\mathcal{B}^2$ of Ω and let $\mathcal{E}$ be the sigma-algebra that contains all events of the form $[a, b] \times [0, 1]$ with $0 \leq a \leq b \leq 1$ and all zero measure sets. In this example, $\mathcal{E}$ contains all full-height rectangles as illustrated in Figure 1 below.

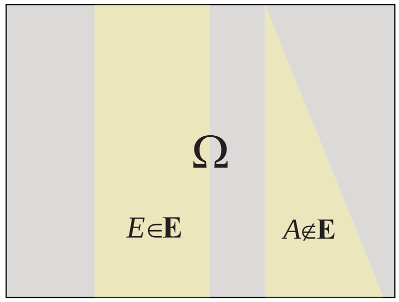

Figure 1

Consider the act f illustrated in Figure 2 below with prizes $x < y < z$. The act yields prize x on the yellow shaded region, y on the light grey shaded region and z on the dark grey region.

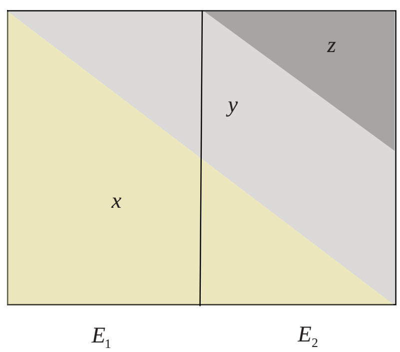

Figure 2

We write xAy for an act that yields x on A and y on A^c. The envelope $\mathbf{f}$ for the act f depicted in Figure 2 is $\mathbf{f}_1 = x, \mathbf{f}_2 = yE_1z$ and hence

$$U(f) = \mu(E_1)u(x, y) + \mu(E_2)u(x, z)$$

Theorem 1 below shows that $\succeq$ is an EUU if and only if it satisfies the following 6 axioms. Note that the axioms are analogous to their counterparts in Savage's theorem. We identify $x \in M$ with the constant act that yields x in every state. Hence, the binary relation $\succeq$ on $\mathcal{F}$ induces a binary relation on M.

Axiom 1: *The binary relation $\succeq$ is complete and transitive.*

Axiom 2: *If $f(s) > g(s)$ for all $s \in \Omega$, then $f \succ g$.*

We interpret prizes as quantities of money and Axiom 2 is a natural consequence of that interpretation. For any $f, g \in \mathcal{F}$ and $A \subset \Omega$, let fAg denote the act h such that $h(s) = f(s)$ for all $s \in A$ and $h(s) = g(s)$ for all $s \in A^c$. Hence, xAy denotes the act that yields x if A occurs and y otherwise.

Our goal is to identify a collection an ideal environment in which each decision-maker satisfies the expected utility axioms and use the decision makers preferences in this environment to calibrate his attitude towards uncertainty. Consider two acts that imply different subacts on the event E but have a common subact on E^c. If the event E is *ideal*, the ranking of acts does not depend on the common subact on E^c. Similarly, if two acts differ on E^c but have a common subact on E then the ranking of acts does not depend on the common subact.

Definition: *An event E is it ideal if $fEh \succeq gEh$ and $hEf \succeq hEf$ implies $fEh' \succeq gEh'$ and $h'Ef \succeq h'Ef$.*

An event is ideal if Savage's sure thing principle holds with respect to E and E^c. Our definition of ideal events is related to Zhang (2002), Epstein and Zhang (2001) and Sarin and Wakker (1992) notions of unambiguous events. Sarin and Wakker assume an exogenous collection of unambiguous events and require Savage's sure thing principle to hold for those events. Thus Sarin and Wakker's unambiguous events yield an environment in which the decision maker is an expected utility maximizer. Epstein and Zhang define unambiguous events to be those events for which a weakened version of the sure thing principle applies. Hence, their unambiguous events yield a single environment in which the decision maker may be an expected utility maximizer or may have some nonexpected utility functional over subjective lotteries. Our permits multiple collections of unambiguous events. The collection of ideal events yields an environment in which the stronger Sarin and Wakker's stronger requirement is satisfied. Our remaining axioms ensure the existence distinct environments; across these environments, the decision-maker reveals a rich variety of risk attitudes.

An event A is *null* if $fAh \sim gAh$ for all $f, g, h \in \mathcal{F}$. If A is not null, we call it non-null. Let $\mathcal{E}$ be the set of all ideal events and E, E', E_i etc. denote elements of $\mathcal{E}$. Let $\mathcal{E}_+ \subset \mathcal{E}$ denote the set of ideal events that are not null.

An event is *diffuse* if it and its complement intersect every non-null ideal event. Diffuse events represent outcomes in situations of complete ignorance. The decision maker cannot find any (non-null) ideal event contained in it or its complement and hence cannot bound the probability of such events. Let $\mathcal{D}$ be the set of all diffuse events and let D, D', D_i etc. denote elements of $\mathcal{D}$.

Definition: *An event D is diffuse if $E \cap D \neq \emptyset \neq E \cap D^c$ for every $E \in \mathcal{E}_+$.*

In the example above, any subset $D \in \mathcal{B}^2$ of the unit square is diffuse if and only if

$$\lambda^2(D \cap (A_1 \times [0,1))) \neq 0 \neq \lambda^2(D^c \cap (A_1 \times [0,1)))$$

whenever $\mu(A_1 \times [0,1)) > 0$; that is, whenever A_1 has strictly positive (one-dimensional) Lebesque measure.

Our maintained hypothesis (formalized in Axiom 3(ii)) is that the decision maker cannot discriminate among diffuse events. That is, the decision maker is indifferent between betting on D_1 and D_2 when both events are diffuse. This indifference reflects the decision maker's complete ignorance over diffuse outcomes. Axiom 3(i) below is Savage's comparative probability axiom (P4) applied to ideal events. Axiom 3(ii) says that diffuse events are interchangeable.

Axiom 3: *If $x > y$ and $x' > y'$, then (i) $xEy \succeq xE'y$ implies $x'Ey' \succeq x'E'y'$ and (ii) $xDy \sim xD'y$.*

Let $\mathcal{F}^o$ denote the set of simple acts, that is, acts such that $f(\Omega)$ is finite. The simple act $f \in \mathcal{F}^o$ is ideal if $f^{-1}(x) \in \mathcal{E}$ for all x. Let $\mathcal{F}^o_{\mathcal{E}}$ denote the set of ideal simple acts. A simple act f is diffuse if $f^{-1}(x) \in \mathcal{D} \cup \emptyset$. An act is constant if $f^{-1}(x) \in \Omega \cup \emptyset$. Let $\mathcal{F}^d$ be the collection of constant or diffuse simple acts. Note that constant acts are in $\mathcal{F}^d$ and in $\mathcal{F}^o_{\mathcal{E}}$.

The standard state independence assumption requires that the ranking of constant acts be the same conditional on any non-null event. Axiom 4 below requires the same for ideal events. In that sense, Axiom 4 below weakens the standard state independence assumption. However, Axiom 4 requires state independence to hold not just for constant acts but for all *diffuse acts* that is, acts that are measurable with respect to the collection of diffuse events. This strengthening of state-independence follows from our hypothesis that diffuse events are interchangeable. To see this, consider the diffuse act xDy. The event $D \cap E$ is a diffuse subset of E as is the event $D^c \cap E$. Therefore, conditional on any ideal event E, the act xDy yields x on a diffuse subset of E and y on its (diffuse) complement in E. Therefore, xDy is analogous to a constant act; it yields identical diffuse bets conditional on any ideal event. If utility is state independent, the ranking of diffuse acts must therefore be preserved when conditioning on a non-null ideal event.

Axiom 4: *If E is nonnull, then $f \succ g$ implies $fEh \succ gEh$ for all $f, g \in \mathcal{F}^d$.*

Axiom 5 is Savage's divisibility axiom for ideal. It serves the same role here as in Savage. Its statement below is a little simpler than Savage's original statement because in our setting, there is a best and a worst prize.

Axiom 5: *If $f, g \in \mathcal{F}^o_{\mathcal{E}}$ and $f \succ g$, then there exists a partition $E_1, \ldots, E_n$ of Ω such that $lE_i f \succ mE_i g$ for all i.*

Axiom 6 below is a strengthening of Savage's dominance condition adapted to our setting. We use it to extend the representation from simple acts to all acts, to establish continuity of u and to guarantee countable additivity of the prior $(\mathcal{E}, \mu)$. Notice that for ideal acts $f \in \mathcal{F}^o_{\mathcal{E}}$ Axiom 6(i) implies Arrow's (1970) monotone continuity axiom, the standard axiom used to establish countable additivity of the probability measure in SEU.

Axiom 6: *(i) If $f_n \in \mathcal{F}^o_{\mathcal{E}}$ converges pointwise to f, then $g \succeq f_n \succeq h$ for all n implies $g \succeq f \succeq h$. (ii) If $f_n \in \mathcal{F}$ converges uniformly to f, then $g \succeq f_n \succeq h$ for all n implies $g \succeq f \succeq h$.*

Theorem 1: *The binary relation $\succeq$ satisfies Axioms $1-6$ if and only if there is a prior $(\mathcal{E}, \mu)$ and an interval utility u such that $\succeq = (\mathcal{E}, \mu, u)$. Moreover, the prior is unique and the interval utility is unique up to positive an affine transformation.*

Proof: See Appendix.

Next, we provide a brief description of the proof of Theorem 1. If we restrict attention to ideal events, Axioms 1-6 yield a standard expected utility theory with a countably additive probability measure ν and a continuous utility index $v : M \to I\!R$.

A *partition act* is a simple act f with the following property. There is a partition of Ω into the ideal events $(E_1, \ldots, E_k)$ and a collection of diffuse or constant acts $(f_1, \ldots, f_k)$ such that f coincides with f_k on E_k. A key step in the proof of Theorem 1 is to show that for any simple act $\hat{f} \in \mathcal{F}^o$ we can find an equivalent partition act f. Equivalent acts differ only on null events. As part of this argument, we show that Ω can be partitioned into any finite number of diffuse sets. This step uses a Theorem by Birkhoff (1967) which in turn uses the continuum hypothesis.[2]

A *binary* partition act is a partition act where each f_k is either a constant act or takes the form xDy for some x, y and some diffuse set D. A simple monotonicity argument shows that any partition act is indifferent to a binary partition act. To see this, let D_1, D_2, D_3 be a partition of Ω into three diffuse events and consider the act xD_1yD_2z with $x < y < z$. By monotonicity

$$xD_1yD_2z \succeq xD_1 \cup D_2z$$

and

$$xD_1z \succeq xD_1yD_2z$$

and by Axiom 3,

$$xD_1 \cup D_2z \sim xD_1z$$

and therefore $xD_1 \cup D_2z \sim xD_1yD_2z \sim xD_1z$.

The diffuse act $f = xDy$ has the constant envelope $\mathbf{f} = (x, y)$. The utility $u(x, y)$ of (x, y) is the utility of this act, that is,

$$u(x, y) := U(xDy)$$

[2] Birkhoff (1967), Theorem 13 (pg. 266) shows that no nontrival (i.e., not identically equal to 0) countably additive measure such that every singleton has measure 0 can be defined on the algebra of all subsets of the continuum.

More generally, consider a binary partition act f with partition $(E_1, \ldots, E_k)$ that yields $x_i D y_i$ on E_i. The utility if this act is

$$W(\mathbf{f}) = \sum_{i=1}^{k} \mu(E_i) u(x_i, y_i)$$

The extension to all acts uses Axiom 6 and follows familiar arguments.

3. Environment and Revealed Lottery Preferences

In this section we define risky environments and characterize the lottery preferences of EUU decision makers. The lottery preferences are shown to be environment dependent and therefore EUU decision makers exhibit a source preference, that is, prefer lotteries derived from one source over the same lotteries derived from another source.

A collection $\mathcal{C}$ of subsets of Ω is a λ-*system* if (i) $\emptyset, \Omega \in \mathcal{C}$; (ii) $A \in \mathcal{C}$ implies $A^c \in \mathcal{C}$; and (iii) $A, B \in \mathcal{C}$ and $A \cap B = \emptyset$ implies $A \cup B \in \mathcal{C}$. The collection $\mathcal{C}$ is *continuous* if $A_n \in \mathcal{C}$ and $A_n \subset A_{n+1}$ for all n implies $\bigcup_n A_n \in \mathcal{C}$. For any such $\mathcal{C}$, let $\mathcal{F}_{\mathcal{C}}$ denote the set of all $\mathcal{C}$-measurable acts from Ω to M. That is, $f \in \mathcal{F}_{\mathcal{C}}$ if and only if $f^{-1}(X) \in \mathcal{C}$ for any Borel set $X \subset M$. We call such an $\mathcal{F}_{\mathcal{C}}$ an *environment*. Henceforth, when we write $\mathcal{F}_{\mathcal{C}}$ it is understood that $\mathcal{C}$ is a continuous λ-system and therefore $\mathcal{F}_{\mathcal{C}}$ is an environment.

Let $\mathcal{L}$ denote the set of all cumulative distribution functions, F, such that $F(m) = 1$ and $F(x) = 0$ for all $x < l$. Given any prior $(\mathcal{A}, \pi)$, λ-system $\mathcal{C} \subset \mathcal{A}$, and act $f \in \mathcal{F}_{\mathcal{C}}$, the cumulative distribution $G^f \in \mathcal{L}$ is defined as

$$G^f(x) = \pi(f^{-1}[l, x])$$

A preference relation $\succeq_l$ on $\mathcal{L}$ is *monotone* if $G \succ_l G'$ whenever G stochastically dominates G'. This preference is continuous if the weakly-better-than sets and the weakly-worse-than sets are closed in the topology of weak convergence. We call a monotone and continuous preference $\succeq_l$ on $\mathcal{L}$ a *lottery preference*.

We say that the EUU $\succeq$ is probabilistically sophisticated on $\mathcal{F}_{\mathcal{C}}$ if there exists a lottery preference $\succeq_l$ and a prior $(\mathcal{A}, \pi)$ such that $\mathcal{C} \subset \mathcal{A}$ and

$$f \succeq g \text{ if and only if } G^f \succeq_l G^g \tag{4}$$

for all $f, g \in \mathcal{F}_{\mathcal{C}}$.

Suppose the EUU $\succeq$ is probabilistically sophisticated on some $\mathcal{F}_{\mathcal{C}}$. Hence, there exists some prior $(\mathcal{A}, \pi)$ and $\succeq_l$ such that equation (4) is satisfied. We call $(\mathcal{A}, \pi)$ a possible prior and $\succeq_l$ a possible lottery preference (for $\succeq$ on $\mathcal{F}_{\mathcal{C}}$). If, in addition, for all $A \in \mathcal{C}$, $r \in [0, 1]$, there exists $B \in \mathcal{C}$, $B \subset A$ such that $\pi(B) = r\pi(A)$, then we say that $\mathcal{F}_{\mathcal{C}}$ is a risky environment. It is easy to check that if $\mathcal{F}_{\mathcal{C}}$ is a risky environment, then there is a unique possible lottery preference of $\succeq$ on $\mathcal{F}_{\mathcal{C}}$. Moreover, any two possible priors $(\mathcal{A}, \pi)$ and $(\mathcal{B}, \pi')$ on $\mathcal{F}_c$ for $\succeq$ will agree on $\mathcal{C} \subset \mathcal{A} \cap \mathcal{B}$. Since we are only interested in the probabilities of events in $\mathcal{C}$, we simply say that $\succeq$ reveals the prior $(\mathcal{A}, \pi)$ and the lottery preference $\succeq_l$ on $\mathcal{F}_{\mathcal{C}}$.

Next, we provide two examples of risky environments and describe the lottery preferences revealed in those environments.

Example 1 (Expected Utility): Ideal acts, that is, acts that are $\mathcal{E}$-measurable, are an example of an environment. The sigma-algebra $\mathcal{E}$ is a continuous λ-system and therefore $\mathcal{F}_{\mathcal{E}}$ is an environment. The EUU is probabilistically sophisticated on $\mathcal{F}_{\mathcal{E}}$ and therefore $\mathcal{F}_{\mathcal{E}}$ is a risky environment. The lottery preference of the EUU $(\mathcal{E}, \mu, u)$ is expected utility with utility index $v(x) = u(x, x)$.

Example 2 (Quadratic Utility): Let f, f' be two $\mathcal{E}$-measurable acts such that

(i) f and f' are uniformly distributed, i.e., $G^f(x) = G^{f'}(x) = x/(m - l)$ for $x \in [l, m]$

(ii) f and f' are independent, i.e, $\mu(\{f \leq x, f' \leq x\}) = G^f(x) \cdot G^{f'}(x)$.

Fix a diffuse set D, let $h = fDf'$, and let $\mathcal{A}$ be the smallest sigma-algebra that contains the sets $h^{-1}(B)$ for all Borel subsets of $[0, 1]$. The sigma-algebra $\mathcal{A}$ is a continuous λ-system and the collection of acts $\mathcal{F}_{\mathcal{A}}$ is a risky environment. Any act $h' \in \mathcal{F}_{\mathcal{A}}$ has the form $h' = gDg'$ for some g, g' such that g, g' are independent and identically distributed ideal acts. The acts $\mathbf{f}_1 = \min\{g, g'\}$ and $\mathbf{f}_2 = \max\{g, g'\}$ are an envelope for $h = gDg'$. The joint distribution of $\mathbf{f}_1$ and $\mathbf{f}_2$ is

$$H_2(x, y|G^g) := \begin{cases} G^g(x)^2 - (G^g(y) - G^g(x))^2 & \text{if } x \geq y \\ G^g(x) & \text{otherwise} \end{cases}$$

and therefore $U(h') = \int u(x, y) dH_2(x, y|G^g)$. Thus, the lottery preference in this environment satisfies $F \succ_l G$ if and only if $\int u(x, y) dH_2(x, y|F) \geq \int u(x, y) dH_2(x, y|G)$. This utility function is known as *quadratic utility* (Machina (1989)). Let $\phi(x, y)$ be a symmetric extension of $u(x, y)$ to all pairs $(x, y) \in M \times M$. That is,

$$\phi(x,y) = \begin{cases} u(x,y) & \text{if } x \geq y \\ u(y,x) & \text{if } y < x \end{cases}$$

Then,

$$\int u(x,y)dH_2(x,y \,|\, F) = \int\int \phi(x,y)dF(x)dF(y) \tag{2}$$

The right hand side of the above equation is the quadratic utility function as analyzed in Chew, Epstein and Segal (1991).

Examples 1 and 2 describe specific risky environments and the corresponding lottery preference of EUU decision makers. Theorem 2 below characterizes the lottery preferences of EUU decision makers in all risky environments. Let

$$Z = \left\{ a = (a_1, a_2, \ldots) \in [0,1]^\infty \mid \sum a_n = 1 \right\}$$

For $F \in \mathcal{L}$, let

$$H_n(x,y \,|\, F) = \begin{cases} (F(y))^n - (F(y) - F(x))^n & \text{if } y \geq x \\ (F(y))^n & \text{otherwise} \end{cases}$$

Note that $H_n(x, y \,|\, F)$ is the joint distribution of the 1-st and n-th order statistics of n independent draws of an F-distributed random variable.

Definition: *A function $V : \mathcal{L} \to I\!R$ is a generalized quadratic utility (GQU) if there exists $a \in Z$ and an interval utility u such that*

$$V(F) = \sum_{n=1}^{\infty} a_n \int u(x,y)dH_n(x,y \,|\, F) \tag{1}$$

We write (a, u) for a lottery preference represented by a GQU with parameters a, u. The interval utility u is *strongly symmetric* if $u(x, y) = (u(x, x) + u(y, y))/2$. We say the EUU $(\mathcal{E}, \mu, u)$ is *regular* if it is not strongly symmetric.

Theorem 2, below, shows that for any EUU $(\mathcal{E}, \mu, u)$ there is an environment where the EUU reveals the GQU (a, u). Moreover, unless the u is strongly symmetric, all lottery preferences are generalized quadratic. Strongly symmetric utility functions allow additional risky environments for which the lottery preference does not have a GQU form.

Theorem 2: *(i) For any prior $(\mathcal{E}, \mu)$ and $a \in Z$, there exists a risky environment $\mathcal{F}_{\mathcal{A}}$ such that for all $u \in \mathcal{U}$, the EUU $(\mathcal{E}, \mu, u)$ reveals (a, u) on $\mathcal{F}_{\mathcal{A}}$. (ii) If the regular EUU $(\mathcal{E}, \mu, u)$ reveals $\succeq_l$ in some risky environment then $\succeq_l$ is a GQU (a, u) for some $a \in Z$.*

The risky environments defined in Examples 1 and 2 above depend on the EUU's prior but are independent of the interval utility. As we demonstrate in Proposition 1 below, this property is general.

Proposition 1: *(i) If $\mathcal{F}_{\mathcal{A}}$ is a risky environment for the regular EUU $(\mathcal{E}, \mu, u)$ then it is a risky environment for any EUU $(\mathcal{E}, \mu, u')$ and both reveal the same prior on $\mathcal{F}_{\mathcal{A}}$. (ii) If the regular EUU $(\mathcal{E}, \mu, u)$ reveals the lottery preference $\succeq_l$ in some environment and $(\mathcal{E}', \mu')$ is any prior, then $(\mathcal{E}', \mu', u)$ reveals the same lottery preference $\succeq_l$ in some risky environment.*

Proposition 1 (i) shows that the prior rather than the interval utility defines an environment because almost all EUU preferences with a given prior have the same collection of environments. The only qualification is that a strongly symmetric interval utility yields a larger class of environments. Part (ii) of Proposition 1 is a immediate corollary of Theorem 2.

The notion of a risky environment enables us to formalize the two ways in which the EUU model achieves separation between the individual's perception of uncertainty and risk, as described by her prior, and attitude towards this uncertainty and risk, as described by u. First, as Proposition 1 shows, what constitutes a risky environment depends only on the prior and not on the interval utility. Second, every prior perceives every risky environment. That is, just as in the Savage model, a nonatomic prior forces the decision-maker to confront every risky (and uncertain) situation; two different priors may disagree on which events have probability .6 (or, in our more general model, which collection of acts constitute a risky environment), but both confront the entire range of probabilities and risky environments.

4. Measures of Uncertainty and Uncertainty Aversion

An EUU's lottery preference depends on the environment. It is risk averse in a given environment if it dislikes mean preserving spreads in that environment. It is strongly uncertainty averse if it dislikes mean preserving spreads in all environments.

Definition: *An EUU $\succeq$ is risk averse in the risky environment $\mathcal{F}_\mathcal{A}$ if for all $f, g \in \mathcal{F}_\mathcal{A}$, $f \succeq g$ whenever G^g is a mean preserving spread of G^f.*

Definition: *An EUU $\succeq$ is strongly uncertainty averse if it is risk averse in every risky environment.*

We say that u is *maximally pessimistic* if $u(x, y) = u(x, x)$ for all x, y. For any u, let $\rho_u(x) = u(x, x)$ for all x. Hence, u is maximally pessimistic, $u(x, y) = \rho_u(x)$ for all x, y. To simplify notation, we let $(\mathcal{E}, \mu, \rho)$ denote a maximally pessimistic EUU. The following proposition shows that for EUU preferences, our notion of strong uncertainty aversion is equivalent to risk aversion in the ideal environment plus maximal pessimism. Furthermore, this notion has a characterization similar to Schmeidler's notion of uncertainty aversion.

Proposition 2: *Let $(\mathcal{E}, \mu, u)$ be an EUU. Then, the following conditions are equivalent*

(ii) The EUU $(\mathcal{E}, \mu, u)$ is strongly uncertainty averse;

(ii) ρ_u is concave and $[f \in \mathcal{F}$, $\alpha \in [0, 1]$ and $\mathbf{g} = \mathbf{f}$ implies $\alpha f + (1 - \alpha) g \succeq f]$;

(iii) ρ_u is concave and u is maximally pessimistic.

Definition: *Let $\mathcal{F}_\mathcal{A}, \mathcal{F}_\mathcal{B}$ be risky environments for the prior $(\mathcal{E}, \mu)$. The EUU $(\mathcal{E}, \mu, u)$ prefers $\mathcal{F}_\mathcal{A}$ to $\mathcal{F}_\mathcal{B}$ if $f \in \mathcal{F}_\mathcal{A}, g \in \mathcal{F}_\mathcal{B}$, $G^f = G^g$ implies $f \succeq g$. The risky environment $\mathcal{F}_\mathcal{B}$ is more uncertain than $\mathcal{F}_\mathcal{A}$ if every strongly uncertainty averse $(\mathcal{E}, \mu, u)$ prefers $\mathcal{F}_\mathcal{A}$ to $\mathcal{F}_\mathcal{B}$.*

Theorem 2 shows that we can associate to each risky environment $\mathcal{F}_\mathcal{A}$ a parameter $a \in Z$ such that the EUU's lottery preference is represented by the GQT (a, u) in this environment. We refer to this $a \in Z$ as *the parameter* of the environment $\mathcal{F}_\mathcal{A}$. For $a \in Z$, let $\gamma_a(t) := \sum_{n=1}^{\infty} a_n t^n$

Proposition 3: *Let $(\mathcal{E}, \mu)$ be a prior and $\mathcal{F}_\mathcal{A}, \mathcal{F}_\mathcal{B}$ be two risky environments with parameters $a, b \in Z$. Then, $\mathcal{F}_\mathcal{B}$ more uncertain than $\mathcal{F}_\mathcal{A}$ if and only $\gamma_a(t) \geq \gamma_b(t)$ for all $t \in [0, 1]$.*

Proposition 3 provides a measure of uncertainty for environments. The ideal environment has parameter $a = (1, 0, \ldots)$ while the quadratic environment has parameter $b = (0, 1, 0, \ldots)$. Since $t \geq t^2$ for all $t \in [0, 1]$, Proposition 3 shows that the quadratic environment is more uncertain than the ideal environment. More generally, let $a^n \in Z$ be such that $a_n = 1$ (and hence $a_k = 0$ for all $k \neq n$). Then, the environment with parameter a^{n+1} is more uncertain than the environment with parameter a^n. Of course, not all environments can be ranked. For example, the environment with parameter $a' = (1/4, 0, 3/4, 0 \ldots)$ and the environment with parameter $b = (0, 1, 0, \ldots)$ cannot be ranked.

Consider a sequence of environments with parameters $a^n, n = 1, 2, \ldots$. Hence, as n goes to infinity these environments become more and more uncertain. For any lottery F let $CE_n(F)$ be the certainty equivalent of F for the GQU (a^n, u). Let $x(F)$ and $y(F)$ be the maximum and the minimum respectively of the support of F. It is straightforward to verify that

$$\lim CE_n(F) = \rho_u^{-1}(u(x(F), y(F))$$

Thus, as n goes to infinity, the agent ranks lotteries according to their support.

5. Allais Paradox

A typical Allais-style reversal occurs if a decision maker prefers the certain prize y over the lottery F but reverses this ranking if both prospects are mixed with an undesirable outcome x. We say that an interval utility is prone to Allais style reversals if we can find risky environments where such reversals occur. Recall that $y(F)$ is the minimal element in the support of F.

Definition: *The interval utility u is prone to Allais-reversals if there exist $a \in Z$, $F \in \mathcal{L}$, $x \leq y(F)$, $z \in M$ and $\alpha \in (0, 1)$ such that $V(F) < V(z)$ and $V(\alpha F + (1 - \alpha)x) > V(\alpha z + (1 - \alpha)x)$ for $V = (a, u)$.*

Definition: *The interval utility u displays risk loving under extreme uncertainty if there exists $a \in Z$ and $F \in \mathcal{L}$ such that $V(F) > V(z(F))$ for all $b \in Z$ with $\gamma_a \leq \gamma_b$ and $V = (b, u)$.*

Proposition 4: *The following conditions are equivalent*

(i) u is not maximally pessimistic;

(ii) u is prone to Allais-reversals;

(iii) u displays risk loving under extreme uncertainty.

Proposition 4 shows that to capture Allais-style reversals we must choose interval utility indices that are not maximally pessimistic. We know from Proposition 2 above that such u's cannot be risk averse in every environment. Proposition 4 shows that, more specifically, risk aversion will be violated in very uncertain environments.

An interval utility u is symmetric separable if it satisfies

$$u(x, y) = \alpha v(x) + (1 - \alpha)v(y) \tag{3}$$

We write $(\mathcal{E}, \mu, (\alpha, v))$ for an EUU with a symmetric separable interval utility. To illustrate Proposition 4, consider the example

$$u(x, y) = {}^3\!/_4 \cdot x + {}^1\!/_4 \cdot y$$

Consider a lottery F that yields $x > 0$ with probability p and 0 with probability $1 - p$. In environment a^n the utility of this lottery is

$$V(F) = x \cdot \left(\frac{3}{4} p^n + \frac{1}{4}(1 - (1 - p)^n \right)$$

If $x = 300$ and $p = .8$ and $n \geq 3$ then the decision maker prefers a certain prize of 200 over the lottery F but reverses this ranking when both prospects are combined with a 1/2 chance of 0. More generally, this decision maker is prone to Allais-style reversals if the environment is more uncertain than a^n for $n \geq 3$. To see the relation to risk loving behavior, note that for $n \geq 3$ and p sufficiently small, the decision maker prefers F to its expected value. Hence, the decision maker is risk loving for gambles that offer a small chance of winning in uncertain environments.

Next, consider the "source preference" of this decision maker. If $p = 1/2$ and therefore F offer an equal chance of winning (the prize x) and losing (the prize 0), then the decision maker prefers less uncertain environments. If $V_n(F)$ is the utility of F in environment a^n

then $V_n(F)$ is decreasing in n. However, if p is small (for example $p = .1$) and hence the lottery F offers a small chance of winning, then the ranking of environments is reversed. In that case $V_n(F)$ is increasing in n and the decision maker prefers more uncertain environments. A similar reversal of the source preference as a function of the odds of winning has been documented in experimental settings (see Curley and Yates (1989) and Camerer and Weber (1992) for a survey.)

Next, we characterize the lottery preferences of EUUs with symmetric separable interval utilities. We show below that those lottery preferences correspond to a subclass of rank dependent expected utility (RDEU) (Quiggin (1982)). RDEU is characterized by a utility index v and a continuous bijection $\nu : [0,1] \to [0,1]$ called the probability transformation function (PTF). For any cdf $F \in \mathcal{L}$ we write $\nu \circ F$ for the cdf G such that $G(x) = 1 - \nu(1 - F(x))$. The preference $\succeq_l$ is a rank-dependent expected utility preference (RDEU) if there exists a PTF ν and a continuous, strictly increasing function $v : M \to I\!R$ such that R defined by

$$R(G) = \int v(x) d\nu \circ G \tag{4}$$

represents $\succeq_l$. We write $R = (\nu, v)$ to describe a particular RDEU utility function.

Define the function $\gamma_a^*(t) = 1 - \sum_{n=1}^{\infty} a_n (1-t)^n$ and note that $\gamma_a^*(t) = 1 - \gamma_a(1-t)$.

Proposition 5: *The interval utility (α, v) reveals $\succeq_l$ in a risky environment with parameter a if and only if $\succeq_l$ is the RDEU (ν, v) for $\nu = \alpha\gamma_a + (1-\alpha)\gamma_a^*$.*

In his survey of evidence on lottery preferences, Starmer (2000) notes that PTF's with an inverted S-shape, that is, concave on $[0, t^*]$ and convex on $[t^*, 1]$ for some $t^* \in (0,1)$, provide a good fit of experimental data. PTFs of the form

$$\alpha\gamma_a + (1-\alpha)\gamma_a^*$$

are inverted S-shaped whenever the environment is sufficiently uncertain and $0 < \alpha < 1$. For example, the PTF

$$\frac{3}{4}\gamma_{a^n} + \frac{1}{4}\gamma_{a^n}^*$$

is S-shaped for all $n \geq 3$. More generally, for any $\alpha \in (0, 1)$, there exists $a \in Z$ such that if the environment is more uncertain than an environment with parameter a then the PTF is inverted S-shaped.

6. The Ellsberg Paradox

In this section, we analyze "urn experiments," that is, situations where the relevant uncertainty is described by a finite number of events corresponding to the possible draws from an urn. To describe urn experiments in our setting, let N be a finite set of events, let $\mathcal{N}$ be the set of subsets of N and let $T : \Omega \to N$ be an onto function. Hence, $\mathcal{A}_{\mathcal{M}} = \{T^{-1}(L) \,|\, L \in \mathcal{N}\}$ is the sigma-algebra of events corresponding to the urn experiment.

Some events have intuitive probabilities because the experimenter gives objective information such as the total number of balls of a particular type. For example, the urn may contain three different kinds of balls, red, blue and yellow and the experimenter may tell the subject the total number of blue or yellow balls. Then, if the total number of balls is n and there are m blue or yellow balls, then the probability of the event $L = \{B, Y\}$ is ${}^{m}\!/_{n}$.

Let $\mathcal{M}$ be the set of events with intuitive probabilities and let $\iota : \mathcal{M} \to [0, 1]$ be the function that assigns intuitive probabilities to events in $\mathcal{M}$. We require that ι satisfy the following conditions: there are probability measures p, q on $\mathcal{N}$ such that (i) $p(L) = q(L) = \iota(L)$ for all $L \in \mathcal{M}$ and (ii) $p(L) \neq q(L)$ for $L \notin \mathcal{M}$. The first property requires that the intuitive probabilities are consistent, that is, there is a way to assign probabilities to the states that matches all intuitive probabilities. The second property says that $\mathcal{M}$ contains all events that have intuitive probabilities. Any event $L \notin \mathcal{M}$ can be assigned multiple probabilities consistent with the intuitive probabilities in $\mathcal{M}$. Note that (i) and (ii) imply that $\mathcal{M}$ is a λ-system of subsets of N since disjoint unions of events with intuitive probabilities must have intuitive probabilities themselves. The triple $(N, \mathcal{M}, \iota)$ is called an *urn experiment.*

Next, we illustrate our definition of urn experiments in the context of well-known examples.

Ellsberg One-urn Experiment: The urn contains 90 balls; 30 are red and the remaining 60 balls are either black or yellow. The exact number or black balls is not known.

Hence, $N = \{R, B, Y\}$, $\mathcal{M} = \{\{R\}, \{B, Y\}\}$. The intuitive probability of drawing an R is $\iota(\{R\}) = 1/3$ and that of drawing a B or a Y is $\iota(\{B, Y\}) = 2/3$. The intuitive probabilities satisfy properties (i) and (ii) for $p(\{R\}) = p(\{B\}) = p(\{Y\}) = 1/3$ and $q(\{R\}) = 1/3; q(\{B\}) = 0, q(\{Y\}) = 2/3$.

Ellsberg Two-Urn Experiment: Next, consider the Ellsberg two-urn experiment. Each urn contains 100 balls. In urn 1, there are 50 Red balls and 50 white balls. Each ball in urn 2 is either black or yellow, but the exact number of black balls is not known. One ball is drawn from each urn. Let $N = \{(R, B), (R, Y), (W, B), (W, Y)\}$ and $\mathcal{M} = \{\{(R, B), (R, Y)\}, \{(W, B), (W, Y)\}\}$ with $\iota(\{(R, B), (R, Y)\}) = \iota\{(W, B), (W, Y)\}\} = 1/2$. The intuitive probabilities satisfy properties (i) and (ii) for $p(\{(R, B)\}) = p(\{(W, B)\}) = 0; p(\{(R, Y)\}) = p(\{(W, Y)\}) = 1/2$ and $q(\{(R, B)\}) = q(\{(W, B)\}) = 1/2; q(\{(R, Y)\}) = q(\{(W, Y)\}) = 0$.

Zhang's Four-Color Urn: The urn has 100 balls; each ball is either brown, green or red, white. The only objective information is that B+R=B+G=50. That is, the total number balls that are blue or red and the total number balls that are blue or green is both 50. From, this fact we can deduce that $G + W = R + W = 50$ and hence to collection of events to which probabilities can readily be assigned is the λ-system $\mathcal{M} = \{\emptyset, \{R, G\}, \{G, B\}, \{G, W\}, \{B, W\}, N\}$ of $N = \{B, G, R, W\}$. For each two elements subset of $\mathcal{M}$ we have $\iota(L) = 1/2$. The intuitive probabilities satisfy properties (i) and (ii) for $p(\{R\}) = p(\{G\}) = 1/2, p(\{B\}) = p(\{W\}) = 0$ and $q(\{R\}) = q(\{G\}) = 0, p(\{B\}) = p(\{W\}) = 1/2$.

Fix any prior $(\mathcal{E}, \mu)$. We say that the collection of sets $\mathcal{C}_o$ is unambiguous if there exists a risky environment $\mathcal{F}_{\mathcal{A}}$ such $\mathcal{C}_o \subset \mathcal{A}$. Otherwise, we say that $\mathcal{C}_o$ is ambiguous. If $\mathcal{C}_o$ is unambiguous but $\mathcal{C}_o \cup \{A\}$ is not, then we say that A is ambiguous with respect to $\mathcal{C}_o$.

We say that $(\mathcal{E}, \mu)$ rationalizes the urn experiment $(N, \mathcal{M}, \iota)$ if there exists an onto function $T : \Omega \to N$ such that $\mathcal{A}_{\mathcal{M}} = \{A \subset \Omega | A = T^{-1}(L), L \in \mathcal{M}\}$ is unambiguous with a possible prior $(\mathcal{A}, \pi)$ satisfying $\pi(T^{-1}(L)) = \iota(L)$ and each $A \in \mathcal{A}_{\mathcal{N}} \backslash \mathcal{A}_{\mathcal{M}}$ is ambiguous with respect to $\mathcal{A}_{\mathcal{M}}$.

Proposition 6: *Every $(\mathcal{E}, \mu)$ rationalizes every urn experiment $(N, \mathcal{M}, \iota)$.*

Proposition 6 shows that EUU theory can accommodate Ellsberg-style evidence. In Ellsberg-style experiments some events are ambiguous, that is, they do not belong to a common risky environment. Notice that ambiguity is not a property of an event but of a collection of events. The event A is ambiguous relative to the event C if there is no risky environment that contains both. At the same time, the event A may be unambiguous relative to the event B because both are contained in a single risky environment. Consider, for example, the Ellsberg two-urn experiment described above with n colors in each urn. The second urn specifies that there are 100 balls and n potential colors while the first urn provides the exact number of balls for each color. Given the symmetry of the second urn, it is plausible that the decision maker views each color to be equally likely and, more generally, ranks bets that depend only on urn 2 according to the number of winning colors. In other words, if we consider acts that depend only on urn 2, the decision maker is probabilistically sophisticated. Thus, the events "red is drawn from urn 2" and "black is drawn from urn 2" are unambiguous. At the same time, the events "black is drawn from urn 1" and "red is drawn from urn 2" are ambiguous because they do not belong to a common risky environment and therefore cannot be assigned probabilities by the decision maker. Note that this notion of ambiguity adopts the terminology of Chew and Sagi (2008).

Recently, Machina (2009) has posed the following question regarding the ability of Choquet expected utility theory (and related models) to accommodate variations of the Ellsberg paradox that appear plausible and even natural. In this subsection, we will show that within EUU theory the behavior described by Machina is synonymous with the failure of separability of u. We say that a EUU preference u is separable if there exist functions v_1 and v_2 such that $u(x, y) = v_1(x) + v_2(y)$.

Next, we describe Machina's urn experiment: Let $N = \{1, 2, 3, 4\}$. To be concrete, suppose a ball will be drawn from an urn that is known to have 20 balls. It is also known that 10 of these balls are marked 1 or 2 and the other 10 balls are marked 3 or 4. Thus, the λ-system of events with intuitive probabilities is $\mathcal{M} = \{\emptyset, \{1, 2\}, \{3, 4\}, N\}$.

We identify each $f \in F = \{g : N \to M\}$ with $(f(1), f(2), f(3), f(4)) \in M^4$. Machina (2008) observes that if $\succeq_o$ is any Choquet expected utility preference such that

$$(x_1, x_2, x_3, x_4) \sim_o (x_2, x_1, x_3, x_4) \sim_o (x_2, x_1, x_4, x_3) \sim_o (x_4, x_3, x_2, x_1) \tag{99}$$

for all $x_1, x_2, x_3, x_4 \in M$, then we must have $(x_1, x_2, x_3, x_4) \sim_o (x_1, x_3, x_2, x_4)$. He notes that this indifference may not be a desirable restriction for a flexible model. Machina arguments means that a decision maker may not be indifferent between a fifty-fifty bet over the interval $(0, 200)$ and $(50, 100)$ versus a fifty-fifty bet over the consequences $(0, 100)$ and $(50, 200)$; that is, he may prefer "packaging" 200 with 50 and 100 with 0 rather than the other way around.

Call it an M-reversal if a EUU preference is not indifferent between (x_1, x_2, x_3, x_4) and (x_1, x_3, x_2, x_4) for some $x_i \in M$, $i = 1, 2, 3, 4$, despite satisfying equation (99). Then, Machina argues that imposing no M-reversals is a possibly unwarranted restriction on a model of uncertainty. Below, we show that an EUU decision maker has no M-reversals if and only if her interval utility is not separable.

A function $\beta : M^2 \to I\!R$ is symmetric if $\beta(x, y) = \beta(y, x)$ for all $x, y \in M$. Let $\succeq_o$ be a binary relation on F. The preference $\succeq_o$ is a Machina preference if there exists a continuous, increasing function and symmetric function β such that the function V defined by

$$V(x_1, x_2, x_3, x_4) = \beta(x_1, x_2) + \beta(x_3, x_4)$$

for $f \in F$ represents $\succeq_o$.

Proposition 7: *Let β be a Machina preference and $(\mathcal{E}, \mu)$ be any prior. Then, there exists a unique $u \in \mathcal{U}$ such that $(\mathcal{E}, \mu, u)$ explains β. This u is separable if and only if β has no M-reversals.*

7. EUU, Choquet and α-MEU

In this section, we relate EUU to Choquet expected utility (Scheidler (1989)) and to expected utility theory (Ghirardato, Maccheroni and Marinacci (2004)). Specifically, we ask the following question. Suppose we observe the behavior of a decision maker in a setting with finitely many states and this decision maker has a Choquet expected utility representation or a $\alpha-$MEU representation. Under what conditions is this behavior consistent with EUU? This motivates the following definition.

Definition: *Let N be any nonempty, finite set, $F = \{f : N \to M\}$ and $\succeq_o$ a binary relation on F. We say that the EUU $\succeq$ explains $\succeq_o$ if there exists $T : \Omega \to N$ such that $f \succeq_o g$ if and only if $f \circ T \succeq g \circ T$ for all $f, g \in F$.*

Let N be a finite and nonempty set. A function $\lambda : N \to [0,1]$ is a probability if $\sum_{n \in N} \lambda(n) = 1$. The function $c : \mathcal{N} \to [0,1]$ is a *capacity* if (i) $c(L) = 1$, (ii) $c(\emptyset) = 0$; (iii) $c(K) \leq c(L)$ whenever $K \subset L \in \Omega$. The capacity c is *totally monotone* if there exists a probability $e : \mathcal{N} \to [0,1]$ such that

$$c(L) = \sum_{K \subset L} e(K) \tag{5}$$

for all $L \in \mathcal{N}$.

A binary relation $\succeq_o = (c, v)$ on F is a Choquet Expected Utility (CEU) if c is a capacity, $\rho : M \to I\!R$ is continuous and strictly increasing and the function $V : F \to I\!R$ defined by

$$V(f) = \int \rho(f) dc$$

represents $\succeq_o$, where the integral above denotes the Choquet integral.

Proposition 8: *Let c be a totally monotone capacity, $(\mathcal{E}, \mu)$ be any prior and (c, ρ) be any CEU. Then, the maximally pessimistic EUU $(\mathcal{E}, \mu, \rho)$ explains (c, ρ).*

Let Δ_o be the set of all probabilities on some finite nonempty N. Let $\Delta_L = \{\lambda \in \Delta \,|\, \sum_{n \in L} \lambda(n) = 1\}$. The set of probabilities Δ is a simple mixture if there exist a probability $e : \mathcal{N} \to [0,1]$ such that

$$\Delta = \sum_{L \in \mathcal{M}} e(L) \Delta_L$$

For $F = \{f : N \to M\}$ and $\alpha \in [0,1]$, the binary relation $\succeq$ is an α-minmax expected utility (α-MMEU) if there exists a compact set of probabilities Δ and a continuous $\rho : M \to I\!R$ such that the function V defined by

$$V(f) = \alpha \min_{\lambda \in \Delta} \sum_{n \in N} v(f(n))\lambda(n) + (1 - \alpha) \max_{\lambda \in \Delta} \sum_{n \in N} v(f(n))\lambda(n)$$

Proposition 9: *Let Δ be any simple mixture, $(\mathcal{E}, \mu)$ be any prior and (Δ, α, ρ) be any α-MMEU. Then, the symmetric separable EUU $(\mathcal{E}, \mu, (\alpha, v))$ explains (Δ, α, ρ).*

Propositions 8 and 9 show the extent to which EUU decision capture the behavior corresponding to CEU and α-MEU. As we show in Proposition 7, both CEU and α-MEU allow no M-reversals while EUU with a non-separable utility index does.

8. Appendix A: Preliminary Results

For the prior $(\mathcal{E}, \mu)$ let

$$\mu^*(A) = \inf_{\{E_i\}} \sum_i \mu(E_i)$$

where the $\{E_i\}$ ranges over all sequences such that $E_i \in \mathcal{E}$ and $A \subset \bigcup_i E_i$. Since $\mathcal{E}$ is a $\sigma-$field, this definition is equivalent to

$$\mu^*(A) = \min_{A \subset E \in \mathcal{E}} \mu(E)$$

That is, there exists $E \in \mathcal{E}$ such that $A \subset E$ and $\mu(E) = \mu^*(A)$. Call such an E a *sheath* of A. Clearly, the symmetric difference between any two sheaths of a given set A has measure 0.

Lemma A1: *For any set $A \subset \Omega$, there exists a partition $E_1, E_2, E_3 \in \mathcal{E}$ of Ω such that (i) $E_1 \subset A \subset E_1 \cup E_2$ and (ii) $\mu^*(E_2 \cap A) = \mu^*(E_2 \cap A^c) = \mu(E_2)$. (iii) If $\hat{E}_1, \hat{E}_2, \hat{E}_3$ also satisfy (i) and (ii), then $\mu([E_i \cap \hat{E}_i^c] \cup [\hat{E}_i \cap E_i^c]) = 0$ for all $i = 1, 2, 3$*

Proof: Choose sheaths $\hat{E}$ E for A and A^c respectively. Then, let $E_1 = E^c$, $E_2 = E \cap \hat{E}$ and $E_3 = \Omega \backslash (E_1 \cup E_2)$. Clearly, $E_1 \subset A$. Note that $\hat{E}^c \subset A^c \subset E$. Then, $x \notin E_1 \cup E_2$ implies $x \in E_1^c \cap E_2^c = E \cap [E \cap \hat{E}^c]^c = E \cap \hat{E}^c = \hat{E}^c$ and hence $x \notin A$. Thus, $A \subset E_1 \cup E_2$.

Finally, note that $\mu^*(A) = \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) \geq \mu^*(E_1) + \mu^*(E_2 \cap A)$. Since μ^* is subadditive, we have $\mu^*(E_1) + \mu^*(E_2 \cap A) \geq \mu^*(E_1 \cup [E_2 \cap A]) = \mu^*(A)$. Thus, $\mu^*(E_2) = \mu(E_2) = \mu^*(E_2 \cap A)$. A symmetric argument yields $\mu(E_2) = \mu^*(E_2 \cap A^c)$ establishing (i) and (ii).

To prove the uniqueness claim, note that the argument above showed that if $\hat{E}$ is a sheath for A and E is a sheath for A^c, then $E_1 = E^c$, $E_2 = E \cap \hat{E}$ and $E_3 = \Omega \backslash (E_1 \cup E_2)$

have the desired properties. It is easy to see that the converse is true as well: if E_1, E_2, E_3 have the desired properties, then $E_1 \cup E_2$ is a sheath for A and $E_2 \cup E_3$ is a sheath for A^c*. This establishes the uniqueness assertion. $\square$

For any $E \in \mathcal{E}$, let $[E]$ denote the equivalence class of sets in $\mathcal{E}$ that differ from E by a set of measure 0. Then, define $[E] \wedge [E'] = [E^*]$ for $E^* \in [E \cap E']$, $[E] \vee [E'] = [E^*]$ for $E^* \in [E \cup E']$ and $\neg E = [E^*]$ for $E^* \in [E^c]$. Let $S(A) = [E^*]$ for some sheath E^* of $A \subset \Omega$. By Lemma A2, $S(A)$ is well-defined for all $A \subset \Omega$. Let $[\mathcal{E}] = \{[E] \,|\, E \in \mathcal{E}\}$. It is easy to verify that $[\mathcal{E}]$ is a Boolean $\sigma-$algebra partially ordered by the binary relation $\leq$, where $[E] \leq [E']$ if and only if $[E] \wedge [E'] = [E]$. When there is no risk of confusion, we omit the brackets and write $E \vee E', \neg E$ etc.

We say that $\{E_1, \dots E_n\} \in [\mathcal{E}]$ is a partition if (i) $[E_i] \wedge [E_j] \neq [\emptyset]$ if and only if $i = j$ and (ii) $\Omega = E_1 \vee E_2, \dots, \vee E_n$. A partition act $\bar{f}$ is defined as a one-to-one map from some partition $\mathcal{P}$ of $([\mathcal{E}])$ to the set of nonempty finite subsets of M. We let $\mathcal{P}_{\bar{f}}$ denote the partition that is the domain of $\bar{f}$. The partition act $\bar{f}$ is equivalent to the act f if for all $E \in \mathcal{P}_{\bar{f}}$

$$\begin{aligned} &\text{(i)} \qquad \bar{f}(E) = \bigcap_{\hat{E} \in [E]} f(\hat{E}) \\ &\text{(ii)} \qquad \bar{f}(E) \subset f(E') \text{ for all } E, E' \text{ such that } \mu(E) > 0 \text{ and } E' \subset E \end{aligned} \tag{$*$}$$

Lemma A2: *For every simple act $f \in \mathcal{F}^o$, there exists a unique partition act $\bar{f}$ that is equivalent to f.*

Proof: Define

$$[\mathcal{E}_f^0] = \{S(f^{-1}(Z)) \in [\mathcal{E}] \,|\, Z \subset M\}$$

and let $[\mathcal{E}_f]$ be the smallest sub $\sigma-$algebra of $[\mathcal{E}]$ that contains $[\mathcal{E}_f^0]$. Note that $[\mathcal{E}_f^0]$ and therefore $[\mathcal{E}_f]$ are both finite. Let

$$\mathcal{P}_f = \{[E] \in [\mathcal{E}_f] \backslash [\emptyset] \,|\, E' \in [\mathcal{E}_f] \text{ implies } E' \wedge E \in \{[E], [\emptyset]\}$$

Hence, $\mathcal{P}_f$ is the set of minimal elements in $[\mathcal{E}_f] \backslash [\emptyset]$. We claim that $\mathcal{P}_f$ is a partition. Clearly, $\mathcal{P}_f$ is nonempty and condition (i) is satisfied by construction. So, we need only show that $\bigvee_{\hat{E}\in\mathcal{P}_f}\hat{E}=\Omega$. Let $\mathcal{T}=\{E\in[\mathcal{E}_f]\,|\,[\emptyset]<E\leq\bigwedge_{\hat{E}\in\mathcal{P}_f}\neg\hat{E}\}$. If $\mathcal{T}$ is nonempty, then it must contain a minimal element E and therefore, $E\in\mathcal{P}_f$. Hence, $[\emptyset]<E\leq\bigwedge_{\hat{E}\in\mathcal{P}_f}\neg\hat{E}\leq\neg E$ and therefore $E=[\emptyset]$, a contradiction. This proves that $\mathcal{T}$ is empty and therefore $\bigwedge_{\hat{E}\in\mathcal{P}_f}\neg\hat{E}=[\emptyset]$ which implies $\bigvee_{\hat{E}\in\mathcal{P}_f}\hat{E}=\Omega$ as desired.

Define $\bar{f}(E)=\bigcap_{\hat{E}\in[E]}f(\hat{E})$ for all $E\in\mathcal{P}_f$. Choose $E'\subset E$ such that $\mu(E')>0$. Let $Z=f(E')$ and let $E^*=S^{-1}(Z)\in[\mathcal{E}_f^0]$. Note that $f(E')\cap\bar{f}(E)\neq\emptyset$. Since $E^*\in[\mathcal{E}_f^0]$ and $[E]\in\mathcal{P}_f$, we conclude $[E]\leq E^*$ and therefore $\bar{f}(E)\subset f(E^*)=f(E')$ proving $(*)$.

Next, we will show that $\bar{f}$ is a partition act with the domain $\mathcal{P}_{\bar{f}}=\mathcal{P}_f$. If $x\notin\bar{f}(\hat{E})$, then $\mu(f^{-1}(x)\cap\hat{E})=0$. Since $\mu(E)>0$ for $E\in\mathcal{P}_f$ it follows that $\bar{f}(E)$ cannot be empty for $E\in\mathcal{P}_f$. We have established above that $\mathcal{P}_f$ is a partition. So to prove that $\bar{f}$ is a partition act with $\mathcal{P}_{\bar{f}}=\mathcal{P}_f$, we need only show that $\bar{f}$ is one-to-one. Assume that $\bar{f}(E)=\bar{f}(\hat{E})=Z$, for $E,\hat{E}\in\mathcal{P}_f$. Then, for all $E_0\in[\mathcal{E}_f^0]$, $E\subset E_0$ if and only if $\hat{E}\subset E_0$. Therefore, the same holds for all $E_0\in[\mathcal{E}_f]$. Since $E,\hat{E}\in[\mathcal{E}_f]$, we have $E=\hat{E}$ as desired.

Finally, let $\bar{g}$ be some other partition act that is equivalent to f. Choose $E_1\in\mathcal{P}_{\bar{f}}$ and $E_2\in\mathcal{P}_{\bar{g}}$ such that $E_1\wedge E_2\neq 0$. Then, $(*)$ implies

$$\bar{f}(E_1)=\bigcap_{\hat{E}\subset E_1\wedge E_2}f(\hat{E})=\bar{g}(E_2)$$

Then, the one-to-oneness of $\bar{f}$ ensures that $E_1\wedge E_2\neq 0\neq E_1'\wedge E_2$ implies $E_1=E_1'$ for all $E_1,E_1'\in\mathcal{P}_{\bar{f}}$ and $E_2\in\mathcal{P}_{\bar{g}}$. Hence, $\mathcal{P}_{\bar{f}}=\mathcal{P}_{\bar{g}}$ and since both $\bar{f},\bar{g}$ are equivalent to f, we have $\bar{f}=\bar{g}$ as desired. $\square$

Henceforth we write $\bar{f}$ to denote the partition act that is equivalent to f.

Lemma 1, below, was stated in the text in section 2.

Lemma 1: *Let $(\mathcal{E},\mu)$ be any prior. Then, for any $f\in\mathcal{F}$, there exists an $\mathbf{f}\in\mathbf{F}_{\mathcal{E}}$ such that*

$$\mu(\{\omega\in\Omega\,|\,\mathbf{f}_1(\omega)\leq f(\omega)\leq\mathbf{f}_2(\omega)\})=1 \tag{22}$$

and if $\mathbf{g}\in\mathbf{F}_{\mathcal{E}}$ also satisfies (22), then

$$\mu(\{\omega\in\Omega\,|\,\mathbf{g}_1\leq\mathbf{f}_1(\omega)\leq\mathbf{f}_2(\omega)\leq\mathbf{g}_2(\omega)\})=1 \tag{23}$$

Proof: We first proof the result for simple acts. Let $\bar{f}$ be the unique partition act equivalent to $f \in \mathcal{F}^0$. Choose a partition $\mathcal{P}$ of Ω such that $\{[E]|E \in \mathcal{P}\} = \mathcal{P}_{\bar{f}}$ and let E_ω be the unique element of $\mathcal{P}$ that contains ω. Define

$$\mathbf{f}_1(\omega) := \min \bar{f}([E_\omega])\}$$

$$\mathbf{f}_2(\omega) := \max \bar{f}([E_\omega])$$

From the construction of interval acts it follows that $\mu(\{\omega : \mathbf{f}_1(\omega) \leq f(\omega) \leq \mathbf{f}_2(\omega)\}) = 1$ and $\mathbf{f} \in \mathbf{F}_{\mathcal{E}}$. By the definition of interval acts $\bar{f}([E_\omega]) = \cap_{\hat{E} \in [E_\omega]} f(\hat{E})$ and $f(E) \supset \bar{f}([E_\omega])$ for any $E \subset E_\omega$ with $\mu(E) > 0$. Therefore any $\mathbf{g} \in \mathbf{F}_{\mathcal{E}}$ such that

$$\mu(\{\omega \in \Omega \,|\, \mathbf{g}_1(\omega) \leq f(\omega) \leq \mathbf{g}_2(\omega)\}) = 1$$

must satisfy $\mu(\{\omega \in \Omega \,|\, \mathbf{g}_1(\omega) \leq \min \bar{f}([E_\omega])\}) = 1$ and $\mu(\{\omega \in \Omega \,|\, \mathbf{g}_2(\omega) \geq \max \bar{f}([E_\omega])\})$ which in turn implies (23).

Next, consider a general act f. Let $w = m - l$ and $z_i^n = l + wi2^{-n}$ for all $i = 0, 1, \ldots, 2^n$. For any $x, y \in M$, let $i(n, x) = \max\{i \,|\, z_i^n \leq x\}$ and $j(n, y) = \min\{j \,|\, z_j^n \geq y\}$. The function i is increasing in both arguments while j is decreasing in the first argument and increasing in the second argument. Let $g^n(\omega) = i(n, f(\omega))$ and $h^n(\omega) = j(i, f(\omega))$. Since g^n and h^n are simple functions the first part of the proof implies that there are interval acts $\mathbf{g}^n$ and $\mathbf{h}^n$ corresponding to g^n and h^n. Note that $\mathbf{h}^n(\omega)$ is a decreasing sequence and therefore has a limit $\mathbf{f}(\omega)$. Similarly, $\mathbf{g}^n(\omega)$ is an increasing sequence and therefore has a limit $\mathbf{g}^\infty$. Since $\mathbf{h}_i^n(\omega) - \mathbf{g}_i^n(\omega) \leq w2^{-n}$ for $i = 1, 2$ it follows that $\lim \mathbf{g}^\infty = \mathbf{f}$. We claim that $\mathbf{h}$ is an interval act for f. To see this, first note that for all n

$$\mu(\{\omega \in \Omega \,|\, \mathbf{g}_1^n(\omega) \leq f(\omega) \leq \mathbf{h}_2^n(\omega)\}) = 1$$

since $h^n \geq f \geq g^n$. Therefore,

$$\mu(\{\omega \in \Omega \,|\, \mathbf{f}_1(\omega) \leq f(\omega) \leq \mathbf{f}_2(\omega)\}) = 1$$

Next, observe that if $\mathbf{g}$ satisfies

$$\mu(\{\omega \in \Omega \,|\, \mathbf{g}_1(\omega) \leq f(\omega) \leq \mathbf{g}_2(\omega)\}) = 1$$

then for all n

$$\mu(\{\omega \in \Omega \,|\, \mathbf{g}_1(\omega) \leq \mathbf{g}_1^n(\omega), \mathbf{h}_2^n(\omega) \geq \mathbf{g}_2(\omega)\}) = 1$$

and therefore

$$\mu(\{\omega \in \Omega \,|\, \mathbf{g}_1(\omega) \leq \mathbf{f}_1(\omega) \leq \mathbf{f}_2(\omega) \leq \mathbf{g}_2(\omega)\}) = 1$$

This completes the proof of Lemma 1. □

A set D is diffuse if $\mu^*(D) = \mu^*(D^c) = 1$.

Lemma A3: *Assume the continuum hypothesis holds. If $(\mathcal{E}, \mu)$ is a nonatomic probability then (i) there exists a diffuse set $D \subset \Omega$. (ii) For any natural number n, there exists a partition $(D_1, \ldots, D_n)$ of Ω with $D_i \in \mathcal{D}$ for $i = 1, \ldots, n$.*

Proof: Birkhoff (1967) page 266, Theorem 13 proves the following: no nontrival (i.e., not identically equal to 0) measure such that every singleton has measure 0 can be defined on the algebra of all subsets of the continuum.

For each $A \subset \Omega$, let $E_1 \subset A$ be such that $\mu(E_1) = \mu(A)$ and let $E_3 \subset A^c$ be such that $\mu(E_3) = \mu(A^c)$. Define $N(A) = (E_1 \cup E_3)^c$. Call $N(A) \cap A$ the completely nonmeasurable part of A. Let $\alpha = \sup\{\mu(N(A)) \,|\, A \subset \Omega\}$. We note that this α is attained. To see this, let A_i be a sequence such that $\lim \mu(N(A_i)) = \alpha$. Define B_i as follows: $B_1 = A_1 \cap N(A_1)$ and $B_{i+1} = A_i \cap N(A_{i+1}) \cap (\bigcup_{j \leq i} N(A_i)^c)$. Note that $N(B_1 \cup \ldots \cup B_i) = N(A_1) \cup \ldots \cup N(A_i)$ and $\bigcup_{i=1}^{\infty} B_i$ is completely nonmeasurable in $\bigcup_{i=1}^{\infty} N(A_i)$. Since $\lim \mu(N(A_i)) = \alpha$, we have $\mu(\bigcup_{i=1}^{\infty} N(A_i) \geq \alpha$ showing that α is attained.

If $\alpha < 1$, then we would find A such that $\mu(N(A)) = \alpha$ and use Birkhoff result to find $B \subset N(A)^c$ with $\mu(N(B)) > 0$ to get $C = B \cup (A \cap N(A))$ with $\mu(N(C)) > \mu(N(A))$ contradicting the maximality of α. Hence, $\alpha = 1$. Then, choose D such that $\mu(N(D)) = 1$ and note that D is a diffuse set. This proves part (i).

Next, we will show any diffuse set can be partitioned into two diffuse sets. Then, a simple inductive argument yields part (ii). Let D be any diffuse set and define $\Sigma_1 =$

$\{E \cap D \,|\, E \in \mathcal{E}\}$, $\mu_1(E \cap D) = \mu(E)$. Note that since D is diffuse, $\mu(E \cap D) = \mu(E' \cap D)$ implies that E, E' differ by a set of measure 0. Hence, (D, Σ_1, μ_1) is a probability space and $\mu_1(\{s\}) = 0$ for $s \in D$. Since $\inf_{E \supset D} \mu(D) = 1$, D cannot be countable. Then, by the Continuum Hypothesis, the cardinality of D must be the continuum. Repeated the argument in part (i) above yields a diffuse subset of D_1 of D. Then, for any E such that $\mu(E) > 0$, we have $\mu_1(E \cap D) > 0$ and therefore $E \cap D_1 \neq \emptyset$. A symmetric argument yields $E \cap [D \backslash D_1] \neq \emptyset$. Hence, $D_1, D \backslash D_1$ are diffuse in Ω. $\square$

Lemma A3 is used to establish Lemma 2 (stated in the text). For completeness, we restate Lemma 2 below.

Lemma 2: *Let $(\mathcal{E}, \mu)$ be a prior. Then, for any $\mathbf{f} \in \mathbf{F}_{\mathcal{E}}$, there exists $f \in \mathcal{F}$ such that $\mathbf{f}$ is f's envelope.*

Proof: Since $(\mathcal{E}, \mu)$ is a prior, there exists a diffuse set D. Let $f = \mathbf{f}_1 D \mathbf{f}_2$. We claim that $\mathbf{f}$ is an interval act for f. First, note that $\mathbf{f}_1(\omega) \leq f(\omega) \leq \mathbf{f}_2(\omega)$ for all ω. Second, note that since D is diffuse it follows that $\mu^*(D \cap E) = \mu(E)$ and $\mu^*(D^c \cap E) = \mu(E)$ for all $E \in \mathcal{E}$. It follows that for any $\mathbf{g} \in \mathbf{F}_{\mathcal{E}}$ with $\mu(\{\mathbf{g}_1(\omega) \geq f(\omega) \geq \mathbf{g}_2(\omega)\}) = 1$ we must have $\inf \mathbf{g}_1(E) \geq \sup \mathbf{f}_1(E)$ for all $E \in \mathcal{E}$ with $\mu(E) > 0$. It is straightforward to show that this implies $\mu(\mathbf{g}_1(\omega) \geq \mathbf{f}_1(\omega) = 1$. An analogous argument shows that $\mu(\mathbf{f}_2(\omega) \geq \mathbf{g}_2(\omega)) = 1$. $\square$

9. Appendix B: Proof of Theorem 1

In this section we prove Theorem 1. The proof is divided into a series of Lemmas. It is understood that Axioms 1-6 hold throughout.

Lemma B1: *(i) $f(s) \geq g(s)$ for all $s \in \Omega$ implies $f \succeq g$. (ii) $f \succ g$ implies $f \succ z \succ g$ for some $z \in M$. (iii) $f_n, g_n \in \mathcal{F}$, f_n converges uniformly to f, g_n converges uniformly to g, $g \succ f$ implies $g_n \succ f_n$ for some n. (iv) $f_n, g_n \in \mathcal{F}^o_{\mathcal{E}}$, f_n converges pointwise to f, g_n pointwise to g, $g \succ f$ implies $g_n \succ f_n$ for some n.*

Proof: To prove (i), let $f_n = \frac{1}{n} l + (\frac{n-1}{n}) f$ and $g_n = \frac{1}{n} l + (\frac{n-1}{n}) g$. Then, f_n converges to f uniformly and g_n converges to g uniformly. By Axiom 2, $f_n \succ g_n$. Then, by Axiom 6, $f \succeq g_n$ and applying Axiom 6 again yields $f \succeq g$ as desired.

To prove (ii), assume $f \succ g$ and let $y = \inf\{z \in M \,|\, z \succeq f\}$ and let $x = \sup\{z \in M \,|\, g \succeq z\}$. By (i) above, x and y are well-defined. Axiom 6 ensures that $y \sim f$ and $z \sim g$ and therefore $y \succ x$. Then, for $z = \frac{x+y}{2}$, we have $f \succ z \succ g$.

To prove (iii), let $g \succ f$ and apply (ii) three times to get z, y, x such that $g \succ z \succ y \succ x \succ f$. Axiom 6 ensures that $g_n \succ y$ and $y \succeq f_n$ for all n large enough. Therefore, $g_n \succ f_n$ for all such n. Analogous argument proves (iv). $\square$

Lemma B2: *The collection $\mathcal{E}$ is a $\sigma-$field.*

Proof: First, we note that $\mathcal{E}$ is a field. That $E \in \mathcal{E}$ implies $E^c \in e$ is obvious as is the fact that $\emptyset \in \mathcal{E}$. Hence, to show that $\mathcal{E}$ is a field, we need to establish that $E, E \in \mathcal{E}$ implies $E \cap E \in \mathcal{E}$.

Suppose $fE \cap E'h \succeq gE \cap E'h$. We must show that $fE \cap E'h' \succeq gE \cap E'h'$. Note that $fE \cap E'h = (fEh)E'h$. Since $E' \in \mathcal{E}$ we have $(fEh)E'h' \succeq (gEh)E'h'$. Next, observe that $(fEh)E'h' = (fE'h')E(hE'h')$. Since $E \in \mathcal{E}$ we have $fE \cap E'h' = (fE'h')E(h'E'h') \succeq (gE'h')E(h'E'h') = gE \cap E'h'$ as required. A symmetric argument yields $h'E \cap E'f \succeq h'E \cap E'g$ if $hE \cap E'f \succeq hE \cap E'g$ and therefore $\mathcal{E}$ is a field.

To prove that the field $\mathcal{E}$ is a $\sigma-$field, it is enough to show that if $E_i \in \mathcal{E}$ and $E_i \subset E_{i+1}$, then $\bigcup E_i \in \mathcal{E}$. Let $E_i \subset E_{i+1}$ for all i. Note that $\hat{f}E_i\hat{g}$ converges pointwise to $\hat{f} \bigcup E_i \hat{g}$ for all $\hat{f}, \hat{g} \in \mathcal{F}$. Hence, if $g \bigcup E_i h' \succ f \bigcup E_i h'$ or $h' \bigcup E_i g \succ h' \bigcup E_i f$ for some $f, g, h, h' \in \mathcal{F}^o_{\mathcal{E}}$, by (iv) above, we have $gE_nh' \succ fE_nh'$ or $h'E_ng \succ h'E_nf$ for some n, proving that $E_i \in \mathcal{E}$ for all n implies $\bigcup_i E_i \in \mathcal{E}$. $\square$

Lemma B3: *There exists a finitely additive, convex-ranged probability measure μ on $\mathcal{E}$ and a function $v : \Omega \to I\!R$ such that the function $V : \mathcal{F}^o_{\mathcal{E}} \to I\!R$ define by*

$$V(f) = \sum_{x \in M} v(x)\mu(f^{-1}(x))$$

represents the restriction of $\succeq$ to $\mathcal{F}^o_{\mathcal{E}}$.

Proof: Note that Axiom 1 implies Savages P1, Axiom 2 implies P2. By definition P3 is satisfied for acts in $\mathcal{F}^o_{\mathcal{E}}$, Axiom 3 yields P4, Axiom 4 yields P5, and finally, Axiom 5 yields P6. Then applying the proof of Savage's Theorem to all acts in $\mathcal{F}^o_{\mathcal{E}}$ yields the desired conclusion. This is true despite the fact that Savage's theorem assumes that the underlying $\sigma-$field is the set of all subsets of Ω; the arguments work for any $\sigma-$field. $\sigma-$field. Hence, the result follows from Savage's theorem restricted to simple acts (i.e., $\mathcal{F}^o$). $\square$.

Lemma B4: *The probability measure μ on $\mathcal{E}$ is countably additive and complete.*

Proof: To show that μ is countably additive, we need to prove that given any sequence E_i such that $E_{i+1} \subset E_i$ for all i and $E^* := \bigcap_i E_i = \emptyset$, $\lim \mu(E_i) = 0$. Suppose $\lim \mu(E_i) > 0$. Then, by Axiom 5, there exists E such that $\lim \mu(E_i) > \mu(E) > 0$. Hence, $\mu(E_i) > \mu(E)$ for all i; that is $mE_il \succ mEl$ for all i. But $mE_il \in \mathcal{F}^o_{\mathcal{E}}$ and converges pointwise to mE^*l. Hence, $mE^*l \succeq mEl \succ l$. Therefore, $\mu(E^*) > 0$ as desired.

To see that μ is complete, let $fEg \sim g$ for all f, g. Since $E \in \mathcal{E}$ it follows that for $A \subset E$, $(fAg)Eg \sim g$ for all f, g and therefore $fAg \sim g$ for all f, g. This implies that $A \in \mathcal{E}$ and therefore μ is complete. $\square$

Lemma B5: *The function v is strictly increasing and continuous.*

Proof: That v is strictly increasing follows from $y \succ x$ whenever $y > x$. To prove continuity, let $E_r\mathcal{E}$ be any event such that $\mu(E_r) = r$. Suppose $r' = \lim v(x_n) < v(x)$ for some sequence x_n in X. Then, choose $r \in (r', v(x)$ and note that $x \succ hE_rl \succeq x_n$ for n large. Therefore, $x \succ hE_rl \succeq \lim x_n = x$, a contradiction. Hence, $r' \geq v(x)$. A symmetric argument proves $r' = v(x)$ and yields the continuity of v. $\square$

Lemma B6: *For any y, x and diffuse act D, there exists a unique $z \in X$ such that $yDx \sim z$.*

Proof: Let $z = \sup\{w \in X \,|\, yDx \succeq w\}$. Since, $yDx \succeq l$ by Axiom 2, z is well-defined. Then, we can construct two sequences $y_n \geq z$ and $z \geq x_n$ such that both sequences converge to z and $y_n \succeq yDx$, $yDx \succeq x_n$. Hence, by Axiom 6, $z \succeq yDx \succeq z$ as desired. $\square$

Lemma B7: *Let $D_1, \ldots, D_n \in \mathcal{D}$ be a partition of Ω and $y_{i+1} \geq y_i$ for $i = 1, \ldots, n-1$ and define $f : \Omega \to X$ as follows: $f(s) = y_i$ whenever $s \in D_i$. Then, $f \sim y_nDy_1$ for all $D \in \mathcal{D}$.*

Proof: By monotonicity, $y_n[D_2 \cup \ldots \cup D_n]y_1 \succeq f y_n D_n y_1$. By Axiom 5, $y_n[D_2 \cup \ldots \cup D_n]y_1 \sim y_n D_n y_1 \sim y_n D y_1$. $\square$

Lemma B8: *(i) For any partition act g, there exists some simple act f such that $g = \bar{f}$. (ii) For any partition act $\bar{g}$ and $E \in \mathcal{P}_{\bar{g}}$, there exist $h \in \mathcal{F}^o$ and $f \in \mathcal{F}^d$ such that $hEf = f$ and $\bar{h} = \bar{g}$.*

Proof: Let $\bar{f}$ be the partition act and n be the maximum of the cardinality of $\bar{f}(E)$ for $E \in \mathcal{P}_{\bar{f}}$. Define an onto function $t_E : \{D_1, \ldots D_n\} \to \bar{f}(E)$ for each $E \in \mathcal{P}_{\bar{f}}$. Let $D_1, \ldots, D_n \mathcal{D}$ be a partition of $\mathcal{D}$. Choose a partition $\mathcal{P} \subset \mathcal{E}$ of Ω such that $\{[E] \parallel, E \in \mathcal{P}\} = \mathcal{P}_{\bar{f}}$. Define the act f as follows: for all $s \in E \cap D_n$ $f(s) = t_E(D_n)$. Then, $\bar{f}$ is equivalent to f. This proves (i).

Let $\bar{g}(E) = \{y_1, \ldots, y_n\}$ and $D_1, \ldots, D_n$ be a diffuse partition, the existence of which is guaranteed in Lemma A3 (ii). By part (ii), we can choose $h' \in \mathcal{F}^o$ so that $\bar{h}' = \bar{g}$. Define $f(s) = y_i$ if and only if $s \in D_i$. Let $h = fEh'$. Hence, $\bar{h} = \bar{g}$ as well. Note that h, f have the desired properties. $\square$

For any partition act $\bar{f}$ and $E \in \mathcal{P}_{\bar{f}}$, we define

$$x(E, \bar{f}) := \min \bar{f}(E)$$
$$y(E, \bar{f}) := \max \bar{f}(E)$$

Let $\mathcal{E}_{xy}(\bar{f}) = \{E \in \mathcal{P}_{\bar{f}} \,|\, x = x(E, \bar{f}), y = y(E, \bar{f})\}$ and define

$$E_{xy}(\bar{f}) = \bigvee_{E \in \mathcal{E}_{xy}(\bar{f})} E$$

We define

$$\mathcal{P}(\bar{f}) = \{E_{xy}(\bar{f}) \ x, y \in M\}$$

Note that $\mathcal{P}(\bar{f})$ is a partition that is coarser than $\mathcal{P}_{\bar{f}}$; that is, for $E \in \mathcal{P}_{\bar{f}}$ there exists a unique $\hat{E} \in \mathcal{P}(\bar{f})$ such that $E = E \wedge \hat{E}$. Finally, we define the partition act f^* on $\mathcal{P}(\bar{f})$ as

$$f^*(E_{xy}) = \{x\} \cup \{y\}$$

We call f^* the binary partition act of f.

Lemma B9: *(i) $\bar{f} = \bar{g}$, then $f \sim g$. (ii) If $f^* = \bar{g}$, then $f \sim g$.*

Proof: For all f, g such that $\bar{f} = \bar{g}$ and $E \in \mathcal{P}_{\bar{f}}$, let $T(E) = 0$ if there exists $E' \in [E]$ such that $f(s) = g(s)$ for all $s \in E'$ and $T(E) = 1$ otherwise. We will prove the result by induction on the cardinality of the set of $E \in \mathcal{P}_{\bar{f}}$ such that $T(E) = 1$.If this set is empty, then f, g differ on a set $E \in \mathcal{E}$ such that $\mu(E) = 0$. Hence, $gE^c m \succeq f \succeq gE^c l$ by Lemma B1(i). Similarly, we have $fE^c m \succeq f \succeq fE^c l$. Since E is null, we have $fEm = gEm \sim gEl = fEl$ and therefore $f \sim g$. Next, assume the assertion holds whenever the cardinality of $E \in \mathcal{P}_{\bar{f}}$ such that $T(E) = 1$ is k and consider $E \in \mathcal{P}_{\bar{f}}$ for some f, g for which this cardinality is $k+1$. Choose $E' \in [E]$ such that $T(E) = 1$ and let n be the cardinality of $\bar{f}(E)$. Since $T(E) = 1$, $n > 1$. Hence, $\bar{f}(E) = \{y_1, \ldots, y_n\}$ for $y_i < y_{i+1}$. Choose a partition $D_1, \ldots D_n \in \mathcal{D}$ and let h be the act yields y_i on D_i for all i. Let $D_i^* = [D_i \cap E^c] \cup [f^{-1}(y_i) \cap E]$ for all i. It is easy to verify that D_i^* is diffuse for all i. Consider the act h' that yields y_i on each D_i^*. By Lemma B9(i) and Axiom 4(ii), $h' \sim h$. That is, $fEh \sim h$. A similar argument yields $fEh \sim h$, therefore $fEh \sim gEh$ and finally, $fEg \sim g$. But notice that for f and fEg, the cardinality of the set of $[E'] \in \mathcal{P}_{\bar{f}}$ such that $T(E') = 1$ is k and hence by the inductive hypothesis $f \sim fEg$ and therefore $f \sim g$ proving part (i).

The proof is by induction on the cardinality of the set

$$\{E \in \mathcal{P}_{\bar{f}} \,|\, \bar{f}(E) \neq \bar{g}(\hat{E}) \text{ for } \hat{E} \text{ such that } E \leq \hat{E}\}$$

When that cardinality is 0, the one-to-oneness of partition functions ensures that $\bar{f} = \bar{g} = f^*$ and then part (i) yields $f \sim g$. Suppose the cardinality of that set is $k+1$ and pick any element E of that set. Let $\hat{E}$ be the element of $\mathcal{P}_{\bar{g}}$ such that $E \cap \hat{E} = E$. Choose $f' \in \mathcal{F}^d$ and $h \in \mathcal{F}^o$ such $hEf' = f'$ and $\bar{h} = \bar{f}$. Similarly, choose $g' \in \mathcal{F}^d$ and $h_* \in \mathcal{F}^o$ such $h_* \hat{E} g' = g'$ and $\bar{h}_* = \bar{g}$. Since $\bar{f}(E) \neq \bar{g}(\hat{E})$, we know that both the cardinality of $\bar{f}(E)$ and that of $\bar{g}(\hat{E})$ must be greater than 1. Hence, by Lemma B7, $h \sim h_*$. Set $f' = h^* E f$. It follows from Axiom 3 that $f' \sim hEf = f$. Note that $\mathcal{P}_{f'} = \mathcal{P}_f$ and the cardinality of the set $\{E \in \mathcal{P}_{\bar{f}'} \,|\, \bar{f}'(E) \neq \bar{g}(\hat{E}) \text{ for } \hat{E} \text{ such that } E \leq \hat{E}\}$ is one smaller than that of $\{E \in \mathcal{P}_{\bar{f}} \,|\, \bar{f}(E) \neq \bar{g}(\hat{E}) \text{ for } \hat{E} \text{ such that } E \leq \hat{E}\}$. Hence, by the inductive hypothesis, $f' \sim g$ yielding $f \sim g$. $\square$

Definition: *Let $u : I \to I\!R$ be defined as $u(x, y) = v(z)$ for z such that $yDx \sim z$.*

Lemma B10: *The function u is increasing and continuous.*

Proof: Suppose $yDx \sim z$ and $\hat{y}D\hat{x} \sim \hat{z}$. If $\hat{y} > y$ and $\hat{x} > x$, then Axiom 2 implies $\hat{z} \succ z$ and applying Axiom 2 again yields $\hat{z} > z$ as desired. If $\hat{y} \geq y$ and $\hat{x} \geq x$, then by Lemma B1(i), $\hat{z} \succeq z$. Then, applying Axiom 2 again yields $\hat{z} \geq z$.

To prove continuity, assume $y_iDx_i \sim z_i$ for $i = 1, \ldots$ and $\lim(x_i, y_i) = (x, y)$. Since z_is are in a compact set in proving continuity, we can assume this sequence converges to some z. Suppose $yDx \succ z$ and note that since y_iDx_i converges uniformly to y_iDx_i and the act z_i converges uniformly to z, we have by Lemma B1(iii), $y_iDx_i \sim z_i$ for some i, a contradiction. A symmetric argument yield $y_iDx_i \sim z_i$ and establishes continuity. □

Define

$$W(f) = \int u(\mathbf{f})d\mu$$

Lemma B11: *(i) For all $f \in \mathcal{F}^o$,*

$$U(f) = \sum_{[E]\in\mathcal{P}_{f^*}} \mu([E])u(x([E], f^*), y([E], f^*))$$

(ii) If $u(x([E], f^*), y([E], f^*)) = u(z, z)$ for $[E] \in \mathcal{P}_{f^*}$, then $U(zEf) = U(f)$ for $E \in [E]$.

Proof: For part (i) let f be a simple act and let $[E] \in \mathcal{P}_{f^*}$ and $E \in [E]$. Then,

$$\mu(\{\omega \in \Omega \,|\, \mathbf{f}(\omega) = (x([E], f^*), y([E], f^*)\} = \mu(E) = \mu([E])$$

and therefore part (i) follows.

If $\mathcal{P}_{f^*} = \mathcal{P}_{(zEf)^*}$, part (ii) follows immediately from part (i). If not, then there exists $E' \in \mathcal{P}_{f^*}$ such that $f^*(E') = \{z\}$. Then, part (i) together with the fact that $\mu(E \cup E') = \mu(E) + \mu(E')$ yield the desired conclusion. □

Let $d(f)$ be the cardinality of the set $\{E \in \mathcal{P}_{f^*} \,|\, f^*(E) \text{ is not a singleton}\}$. Hence, if $d(f) = 0$, then $f \in \mathcal{F}^o_{\mathcal{E}}$.

Lemma B12: *The function U represents the restriction of $\succeq$ to $\mathcal{F}^o$.*

Proof: Let $\mathcal{F}^n = \{f \in \mathcal{F}^o \,|\, d(f) \leq n\}$. The proof is by induction on $\mathcal{F}^n$. Note that for $f \in \mathcal{F}^0$

$$\sum_{x \in M} v(x)\mu(f^{-1}(x)) = U(f)$$

Hence, by Lemma B3, the restriction of U to $\mathcal{F}^0$ represents $\succeq$. Suppose U represents the restriction of $\succeq$ to $\mathcal{F}^n$ and choose $f, g \in \mathcal{F}^{n+1}$. Define h_f as follows: if $f \in \mathcal{F}^n$, then $h_f = f$. Otherwise, choose $E \in \mathcal{P}_{f^*}$ such that the cardinality of $f^*(E)$ is not 1. Hence, $y(E, f^*) > x(E, f^*)$. Lemma B6 ensures that there exists a unique z such that $u(z, z) = u(x(E, f^*), y(E, f^*))$. By construction, $y(E, f^*)Dx(E, f^*) \sim z$. Hence, by Axiom 3, $zEf \sim f$. By Lemma B9(ii), $U(zEf) = U(f)$. Hence, set $h_f = zEf$. Construct an h_g in the same fashion. Then, $f \succeq g$ if and only if $h_f \succeq h_g$. By the inductive hypothesis, $h_f \succeq h_g$ if and only if $U(h_f) \geq U(h_g)$. Since $U(h_f) = U(f)$ and $U(h_g) = U(g)$, the desired result follows. □

Lemma B13 shows that U as defined above represents the preference for all acts. Lemma B13 completes the proof of Theorem 2.

Lemma B13: *The function U represents $\succeq$.*

Proof: Note that for all f, there exists x_f such that $U(x_f) = u(x_f, x_f) = U(f)$. This follows from that fact that u is increasing in both arguments and continuous which implies $u(m, m) \geq U(f) \geq u(l, l)$ and by the intermediate value theorem $u(x_f, x_f) = U(f)$ for some $x_f \in [l, m]$. The monotonicity of u ensures that this x_f is unique. Next, we show that $f \sim x_f$.

Without loss of generality, assume $l = 0$ (if not let $l^* = 0$ and $m^* = m - l$ and identify each f with $f^* = f - l$ and apply all previous results to acts $\mathcal{F}^* = \{f - l \,|\, f \in \mathcal{F}\}$.) Define for any $x \geq 0$ and $\epsilon > 0$, $z^*(x, \epsilon) = \min\{n\epsilon \,|\, n = 0, 1, \ldots \text{ such that } n\epsilon \geq x\}$. Similarly, let $z_*(x, \epsilon) = \max\{n\epsilon \,|\, n = 0, 1, \ldots \text{ such that } n\epsilon \leq x\}$. Clearly,

$$0 \leq z^*(x, \epsilon) - x \leq z^*(x, \epsilon) - z_*(x, \epsilon) < \epsilon \tag{4}$$

and the first two inequalities above are equalities if and only if x is a multiple of ϵ.

Set $f^n(\omega) = z^*(f(\omega), m2^{-n})$ and $f_n(\omega) = z_*(f(\omega), m2^{-n})$ for all $n = 0, 1, \ldots$. Equation (4) above ensures that $f^n \geq f \geq f_n$ and f^n, f_n converge uniformly to f. Note also that $f^n, f_n \in \mathcal{F}^o$ with $f^n \downarrow f$. This implies that (for a measure 1 subset) $\mathbf{f}^n \downarrow \mathbf{f}$ and therefore $\int u(\mathbf{f}^n) d\mu \to \int u(\mathbf{f}) d\mu$.

Since $f^n \geq f$, we have $U(f^n) \geq U(f) = U(x_f)$ for all n. Since U represents the restriction of $\succeq$ to $\mathcal{F}^o$, we conclude that $f^n \succeq x$ for all n. Then, Axiom 6 implies $f \succeq x$. A symmetric argument with f_n replacing f^n yields $x_f \succeq f$ and therefore $x_f \sim f$ as desired.

To conclude the proof of the Lemma, suppose $f \succeq g$, then $U(x_f) = U(f)$ and $U(x_g) = U(g)$ and $x_f \sim f \succeq g \sim x_g$. Since U represents the restriction of $\succeq$ to $\mathcal{F}^o$, we conclude that $U(x_f) \geq U(x_g)$ and hence $U(f) \geq U(g)$. Similarly, if $U(f) \geq U(g)$ we conclude $f \sim x_f \succeq x_g \sim g$ and therefore $f \succeq g$. □

Uniqueness follows from standard arguments and is therefore omitted. □

10. Appendix C: Proof of Theorem 2 and Proposition 1

Recall that for any prior $(\mathcal{E}, \mu)$ and $A \subset \Omega$,

$$\mu_*(A) = \max_{E \subset A, E \in \mathcal{E}} \mu(E)$$

Hence, μ_* is the inner extension of μ to 2^Ω.

Lemma C1: *Suppose $\mathcal{C} \subset \mathcal{A}$ is a continuous λ-system, $(\mathcal{A}, \pi)$ is a prior and there exists $a \in Z$ such that $\mu_*(A) = \gamma_a(\pi(A))$ for all $A \in \mathcal{C}$. Then, for any interval utility u, $\mathcal{F}_\mathcal{C}$ is a risky environment for $(\mathcal{E}, \mu, u)$, $(\mathcal{A}, \pi)$ is the revealed prior and the GQU (a, u) represents the lottery preference.*

Proof: It follows from Theorem 1 that

$$U(f) = \int_\Omega u(\mathbf{f}_1(\omega), \mathbf{f}_2(\omega)) d\mu(\omega) = \int_I u(x, y) dH^f(x, y)$$

where H^f is the two dimensional distribution of $\mathbf{f}$, the envelope of f. Note that

$$H(x, y) = \mu_*(\{f(\omega) \leq y\}) - \mu_*(\{f(\omega) \in (x, y]\})$$

Let G^f be the cdf of f for the prior $(\mathcal{A}, \pi)$. Since $\mu_*(A) = \gamma_a(\pi(A))$ for all $A \in \mathcal{A}$ it follows that

$$H(x, y) = \gamma_a(G^f(y)) - \gamma_a(G^f(y) - G^f(x))$$

This demonstrates that (i) for any two acts $f, g \in \mathcal{F}_{\mathcal{A}}$, $G^f = G^g$ implies $f \sim g$; (ii) the lottery preference is the GQU (a, u). It is straightforward to show that the GQU is continuous and increasing in the sense of first order stochastic dominance. Hence, $\mathcal{F}_c$ is a risky environment with a revealed prior $(\mathcal{A}, \pi)$. $\square$

Lemma C2: *Suppose $\mathcal{F}_{\mathcal{C}}$ is a risky environment with a possible prior (A, π) for some EUU $(\mathcal{E}, \mu, u)$ such that u is not strongly symmetric. Then, there exists $a \in Z$ such that $\mu_*(A) = \gamma_a(\pi(A))$ for all $A \in \mathcal{C}$.*

Proof: Let $\mathcal{F}_{\mathcal{C}}$ be a risky environment, $(\mathcal{A}, \pi)$ be the prior and $\succeq_l$ be the lottery preference that $\succeq$ reveals on $\mathcal{F}_{\mathcal{C}}$. Fix a partition $\mathcal{P}^k = \{A_1, \ldots, A_k\}$ of Ω such that $A_i \in \mathcal{A}$ and $\pi(A_i) = 1/k$ for all i. Let $Z = \{z_1, \ldots, z_k\} \subset M$ be a k-element set and let f be the simple act that yields z_i on A_i for $i = 1, \ldots, k$. Finally, let $\bar{f}$ be the partition act corresponding to f. Recall that $\bar{f}$ is a one to one map from $\mathcal{P}^k$ to the non-empty subsets of Z. To simplify the notation below, we define $\bar{f}^{-1} : 2^Z \to \mathcal{P}^k \cup \emptyset$ as the inverse of $\bar{f}$ extended to all subsets of prizes. If a set of prizes X is not attained by any element of $\mathcal{P}_f$, then $\bar{f}^{-1}(X) = \emptyset$. Let $|X|$ denote the cardinality of the set X.

Step 1: $\mu(\bar{f}^{-1}(X)) = \mu(\bar{f}^{-1}(X'))$ if $|X \cap Z| = |X' \cap Z|$.

Take any $y, x \in M$ such that $u(x, y) \neq [u(y, y) + u(x, x)]/2$. Since u is not strongly symmetric such a x, y exit. Then, without loss of generality, (if necessary, by taking an positive affine transformation of u) assume $u(y, y) = 1$, $u(x, x) = 0$ and $u(x, y) = u^* \neq 1/2$. To prove step 1, let $\alpha_i = \mu(\bar{f}^{-1}(z_i))$, $\alpha_j = \mu(\bar{f}^{-1}(z_j))$, $\beta_i := \sum_{X : z_i \notin X} \mu(\bar{f}^{-1}(X))$ and $\beta_j = \sum_{X : z_j \notin X} \mu(\bar{f}^{-1}(X))$, for some i, j. Let $g = y f^{-1}(z_i) x, g' = y f^{-1}(z_j) x$, $h = x f^{-1}(z_i) y, g' = x f^{-1}(z_j) y$. Since $g, g', h, h' \in \mathcal{F}_c$, g, g' yield the same lotteries and h, h' yield the same lotteries, we have $U(g) = U(g')$ and $u(h) = U(h')$ and hence,

$$\alpha_i + (1 - \alpha_i - \beta_i) u^* = \alpha_j + (1 - \alpha_j - \beta_j) u^*$$

$$\beta_i + (1 - \alpha_i - \beta_i) u^* = \beta_j + (1 - \alpha_i - \beta_i) u^*$$

Some simple manipulations reveal that since $u^* \neq 1 - u^*$ the two equations above can only be satisfied if $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$.

Next, assume Step 1 is true for all sets of prizes with cardinality less than l. Let $X, X' \subset Z$ be two sets of cardinality $l + 1$ and let $\alpha = \sum_{Y \subset X} \mu(\bar{f}^{-1}(Y))$ and $\alpha' =$

$\sum_{Y' \subset X'} \mu(\bar{f}^{-1}(Y'))$, $\beta = \sum_{Y \subset Z \setminus X} \mu(\bar{f}^{-1}(Y))$ and $\beta' = \sum_{Y' \subset Z \setminus X'} \mu(\bar{f}^{-1}(Y'))$. Let $A = \bigcup_{z \in X} f^{-1}(z), A' = \bigcup_{z \in X'} f^{-1}(z), g = yAx, g' = yA'x$, $h = xA_i y, g' = xA'_i y$ and arguing as above, we note that g and g' yield the same lotteries and h and h' yield the same lotteries. Therefore, $U(g) = U(g')$ and $U(h) = U(h')$ and the two equations above again yield

$$\alpha = \sum_{Y \subset X} \mu(\bar{f}^{-1}(Y)) = \sum_{Y \subset X'} \mu(\bar{f}^{-1}(Y)) = \alpha'$$

The inductive hypothesis ensures that $\mu(\bar{f}^{-1}(Y)) = \mu(\bar{f}^{-1}(Y'))$ whenever $|Y| = |Y'|$, $Y \subset X$ and $Y' \subset X'$. This together with the equation above ensures that $\mu(\bar{f}^{-1}(X)) = \mu(\bar{f}^{-1}(X'))$ and completes the proof of Step 1.

Using step 1, we can define $a(t,k) = \binom{k}{t}\mu(\bar{f}^{-1}(X))$ for $|X| = t$. Note that

$$\sum_{t=1}^{k} a(t,k) = 1$$

Define $\gamma_k : \{1/k, 2/k, \ldots, 1\} \to [0,1]$ as

$$\gamma_k(t/k) := \sum_{j=1}^{t} \binom{t}{j} \binom{k}{j}^{-1} a(j,k)$$

Step 2: $\mu_*(f^{-1}(X)) = \gamma_k(|X|/k)$ for all $X \subset Z$.

To prove Step 2, note that $\mu_*(f^{-1}(X)) = \sum_{X' \subset X} \mu(\bar{f}^{-1}(X'))$ by the definition of a partition act. Therefore,

$$\begin{aligned}
\mu_*(f^{-1}(X)) &= \sum_{j=1}^{|X|} \sum_{\{X' \subset X : |X'| = j\}} \binom{k}{j}^{-1} a(j,k) \\
&= \sum_{j=1}^{|X|} \binom{|X|}{j} \binom{k}{j}^{-1} a(j,k) \\
&= \gamma_k(|X|/k)
\end{aligned}$$

This proves step 2.

Step 3: For every m, there is $N < \infty$ such that for all $k \geq N$,

$$a(m,m) \geq \sum_{i=N}^{k} a(i,k)/2$$

First, assume that k is a multiple of m that is, $k = mr$ for some r. Consider $\mathcal{P}^k$ and let $Z = \{z_1, \ldots, z_k\}$ and f be defined as above. Define g as follows:

$$g(A_{jr+l}) = f(A_{jr+1})$$

for all $j = 0, 1, \ldots, m-1$ and $l = 1, \ldots, r$. Hence, on each block of r consecutive events A_t, as t ranges between two multiples of r, say $t = jr+1, \ldots, (j+1)r$, g yields $f(A_{jr+1})$, the outcome of f on the first of these events. For $j = 0, \ldots, m$, let $A^j = A_{jr+1} \cup \ldots \cup A_{(j+1)r}$ be the union of each such sequence of events. Let

$$\begin{aligned}
\Psi^+ &= \{X \subset Z \,|\, \forall j = 0, \ldots, m \; \exists l = 1, \ldots, r \text{ s.t. } f(A_{jr+l}) \in X\} \\
\Psi^- &= \{X \subset Z \,|\, X \notin \Psi^+\} \\
\Psi_i^- &= \{X \subset Z \,|\, X \in \Psi_i^- \text{ and } |X| = i\}
\end{aligned}$$

Note that

$$\begin{aligned}
a(m, m) &= \mu(\bar{g}^{-1}(z_1, z_{r+1}, z_{2r+1}, \ldots, z_k)) \\
&= \sum_{X \in \Psi^+} \mu(\bar{f}^{-1}(X))
\end{aligned}$$

By definition, $\bigcup_{i=1}^m \Psi_i^- = \Psi^-$ and $\Psi_i^- \cap \Psi_j^- = \emptyset$ whenever $i \neq j$. Hence, for all N,

$$\begin{aligned}
\sum_{X \in \Psi^-} \mu(\bar{f}^{-1}(X)) &= \sum_{i=1}^{k} \sum_{X \in \Psi_i^-} \mu(\bar{f}^{-1}(X)) \\
&\leq \sum_{i=1}^{N} a(i, k) + \sum_{i=N+1}^{k} \sum_{X \in \Psi_i^-} \mu(\bar{f}^{-1}(X)) \\
&= \sum_{i=1}^{N} a(i, k) + \sum_{i=N+1}^{k} \frac{|\Psi_i^-|}{\binom{k}{i}} a(i, k)
\end{aligned}$$

Some manipulation reveals that $\frac{|\Psi_i^-|}{\binom{k}{i}} \leq m \left(\frac{m-1}{m}\right)^i$. Hence,

$$\sum_{X \in \Psi^+} \mu(\bar{f}^{-1}(X)) \leq \sum_{i=1}^{N} a(i, k) + \sum_{i=N+1}^{k} m \left(\frac{m-1}{m}\right)^i a(i, k)$$

and since

$$\sum_{\{X\in\Psi^-\}}\mu(\bar{f}^{-1}(X))+\sum_{\{X\in\Psi^+\}}\mu(\bar{f}^{-1}(X))=1=\sum_{i=1}^{k}a(i,k)$$

we have

$$\sum_{\{X\in\Psi^+\}}\mu(\bar{f}^{-1}(X))\geq\sum_{i=N+1}^{k}\left(1-m\left(\frac{m-1}{m}\right)^{i}\right)a(i,k)$$

Choose N so that $m\left(\frac{m-1}{m}\right)^{N}<1/2$. Then,

$$a(m,m)=\sum_{\{X\in\Psi^+\}}\mu(\bar{f}^{-1}(X))>\sum_{i=N}^{k}a(i,k)/2$$

as desired.

Step 4: For every $\epsilon>0$ there is $N<\infty$ such that $\sum_{j=N}^{rk}a(t,rk)\leq\epsilon$ for all $rk\geq N$.

Since $\succeq_l$ is a continuous preference on $\mathcal{F}_{\mathcal{A}}$ it follows that for every $\epsilon>0$ there is k such that $\mu_*(f^{-1}(X))>1-\epsilon/2$ for $|X|=k-1$ and therefore $1-\gamma_k((k-1)/k)=a(k,k)<\epsilon/2$. Now the result follows from Step 3.

Let $b(j), j=1,2,\ldots$ be the pointwise limit of a convergent subsequence of $\mu(j,kr), j=1,2\ldots,kr$ as $r\to\infty$. From step 4 it follows that $\sum_{j=1}^{\infty}b(j)=1$. Let $\gamma(t)=\sum_{j=1}^{\infty}b(j)t^{j}$.

Step 5: $\gamma_k(j/k)=\gamma(j/k)$.

Step 2 implies that $\gamma_k(j/k)=\gamma_{rk}(j/k)$ for all $r=1,2,\ldots$.

$$\gamma_{rk}(t/k):=\sum_{j=1}^{rt}\binom{rt}{j}\binom{rk}{j}^{-1}a(j,rk)$$

Fix ϵ and choose N so that $\sum_{j=N}^{rk}a(t,rk)\leq\epsilon$ for all $rk\geq N$. Then,

$$\sum_{j=1}^{N}\binom{rt}{j}\binom{rk}{j}^{-1}a(j,rk)+\epsilon\geq\gamma_{rk}(t/k)\geq\sum_{j=1}^{N}\binom{rt}{j}\binom{rk}{j}^{-1}a(j,rk)$$

for all r. Note that

$$\lim_{r\to\infty}\binom{rt}{j}\binom{rk}{j}^{-1}=(t/k)^{j}$$

for j fixed and therefore

$$\sum_{j=1}^{N}\left(\frac{t}{k}\right)^{j} b(j)+\epsilon \geq \gamma_k(t/k) \geq \sum_{j=1}^{N}\left(\frac{t}{k}\right)^{j} a(j,rk)$$

Since $\sum_{j=N}^{\infty} b(j) \leq \epsilon$ it follows that

$$\sum_{j=1}^{\infty}\left(\frac{t}{k}\right)^{j} b(j)+\epsilon \geq \gamma_k(t/k) \geq \sum_{j=1}^{\infty}\left(\frac{t}{k}\right)^{j} b(j)-\epsilon$$

Since ϵ was arbitrary this proves Step 4.

Step 5 proves the Lemma for all $A \in \mathcal{A}$ with $\pi(A)$ rational. Since $\succeq_l$ is continuous it is straightforward to extend the argument to all $A \in \mathcal{A}$. $\square$

Lemma C3: *For every $a \in Z$ there exists a prior $(\mathcal{A}, \pi)$ such that $\mu_*(A) = \gamma_a(A)$.*

Proof: First, we construct a risky environment corresponding to a^n, i.e., $\gamma_a(t) = t^n$. Let f_j, $j = 1, \ldots, n$ be a collection of acts such that

(i) $f_j \in \mathcal{F}_\mathcal{E}$ for all j;

(ii) f_j is uniformly distributed for all j; That is, if λ is Lebesgue measure and $\mathcal{B}$ is the Borel σ-algebra on $[l,m]$ then $\mu(f_j^{-1}(B)) = \lambda(B)/(m-l)$ for $B \in \mathcal{B}$.

(iii) f_i and f_j are independent for all $i, j \in \{1, \ldots, n\}$; That is, $\mu(f_j^{-1}(B) \cap f_i^{-1}(B')) = \lambda(B)\lambda(B')$ for all $B, B' \in \mathcal{B}$ and all i, j.

Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a partition of Ω into n diffuse sets and let

$$f = f_1 D_1 f_2 D_2 \ldots f_n D_n$$

Define $\mathcal{A}$ be the collection of sets $\{f^{-1}(B) \,|\, B \in \mathcal{B}\}$ and let $\pi : \mathcal{A} \to [0,1]$ be such that $\pi(f^{-1}(B)) = \lambda(B)$ for all $B \in \mathcal{B}$. Note that $(\mathcal{A}, \pi)$ is a prior.

Next, we show that $\mu_*(A) = \pi(A)^n$ for $A \in \mathcal{A}$. Let $B \in \mathcal{B}$ and note that $\mu^*(A \cap D) = \mu(A)$ if D is diffuse. Therefore,

$$\mu^*(A^c) = \mu^*\left(\bigcup_{j=1}^{k}\left(f_j^{-1}(B^c) \cap D_j\right)\right) = \mu\left(\bigcup_{j=1}^{k}\left(f_j^{-1}(B^c)\right)\right)$$

Since the f_j's are independent

$$\mu\left(\bigcup_{j=1}^{k}\left(f_j^{-1}(B^c)\right)\right) = 1 - \lambda(B)^n$$

Since $\mu_*(A) = 1 - \mu^*(A^c)$ this implies that $\mu_*(A) = \pi(B)^n$ as desired.

Next, consider arbitrary $a \in Z$. Let $\{E_1, E_2, \ldots\}$ be a partition of Ω such that $\mu(E_n) = a_n$. Let $f_1, f_2 \ldots$ be a countable set of independent acts such that each f_i is uniformly distributed on every E_j. That is, if λ is Lebesgue measure and $\mathcal{B}$ is the Borel σ-algebra on $[l, m]$ then $\mu(f_j^{-1}(B) \cap E_j)/\mu(E_j) = \lambda(B)/(m-l)$ for $B \in \mathcal{B}$. Let $\{D_1, D_2, \ldots\}$ be a partition of Ω into countably many diffuse sets. Let f be such that $f = f_k$ on $D_i \cap E_j$ for $k = \min\{i, j\}$. Hence, on E_n the act f yields $f_1, \ldots, f_n$ on n disjoint diffuse sets. Define $\mathcal{A}$ be the collection of sets $\{f^{-1}(B) \mid B \in \mathcal{B}\}$ and let $\pi : \mathcal{A} \to [0, 1]$ be such that $\pi(f^{-1}(B)) = \lambda(B)$ for all $B \in \mathcal{B}$. Note that $(\mathcal{A}, \pi)$ is a prior. The argument above applied to each E_n shows that $\mu_*(A) = \gamma_a(\pi(A))$. □

Proof of Theorem 2: Lemma C1 and Lemma C3 prove part (i) of the Theorem. Lemma C1 and C2 prove part (ii). □

Proof of Proposition 1: Lemma C1 proves part (i) of Proposition 1. Part (ii) is a corollary of Theorem 2. □

11. Appendix D: Proof of Propositions 2 -9

Proof of Proposition 2: First, we show that (i) implies (iii). Let $u(x, y) > u(x, x)$ and let F be the lottery that yields y with probability ϵ and x with probability $1 - \epsilon$. Choose ϵ so that $u(x, y) > (1 - \epsilon)u(x, x) + \epsilon u(y, y)$. Consider an environment with parameter a^n. Then,

$$V(F) = (1 - \epsilon)^n u(x, x) + \epsilon^n u(y, y) + (1 - (1 - \epsilon)^n - \epsilon^n)u(x, y)$$

As $n \to \infty$ this converges to $u(x, y)$ and hence the decision maker is not risk averse for n sufficiently large. From a standard argument it follows that $v(x) := u(x, x)$ must be concave.

Next, we show that (iii) implies (i). Assume that $u(x, y) = v(x)$ for some concave v. Then, the risk preference in the environment with parameter $a \in Z$ is

$$V(G) = \int_l^m v(x) d[1 - \gamma_a(1 - G(x))]$$

Since $[1 - \gamma_a(1 - G^f(x))]$ is concave it we can apply Theorem 1 in Chew, Karni and Safra (1987) to conclude V is risk averse.

Next, we show that (iii) if and only if (ii). Let f, g be such that $\mathbf{f}$ is an interval act for both f and g. Let $h = \alpha f + (1 - \alpha)g$ and note that $\mathbf{f}_1 \leq \mathbf{h}_1 \leq \mathbf{h}_2 \leq \mathbf{f}_2$. It follows that $U(h) \geq U(f)$. To prove the reverse direction, let D_1, D_2, D_3 be a partition of Ω into three diffuse sets. Let $f = x(D_1 \cup D_2)y$ and $g = x(D_1 \cup D_3)y$ with $y > x$. Clearly f and g share the same interval act $\mathbf{f}_1 = x, \mathbf{f}_2 = y$. Let $h = \alpha f + (1 - \alpha)g$ and note that $\mathbf{h}_1 = x, \mathbf{h}_2 = \alpha x + (1 - \alpha)y$ is an interval act for h. Therefore, we have $U(f) = U(g) = u(x, y)$ and $U(h) = u(x, \alpha x + (1 - \alpha)y)$. Uncertainty aversion requires that $U(h) \geq U(f)$ and therefore $u(x, \alpha x + (1 - \alpha)y) \geq u(x, y)$. Since α was arbitrary and u is continuous it follows that $u(x, y) = u(x, x)$ as desired. $\square$

Proof of Proposition 3: Let F be the lottery that yields m with probability t and l with probability $1 - t$. Then, for the GQU (a, u) with maximally pessimistic u,

$$V(F) = \gamma_a(t)u(m, m) + (1 - \gamma_a(t))u(l, l)$$

It follows that $\gamma_a(t) \geq \gamma_b(t)$ for all t if $\mathcal{F}_{\mathcal{B}}$ is more uncertain than $\mathcal{F}_{\mathcal{A}}$.

For the converse, let $(\mathcal{E}, \mu, u)$ with $u(x, y) = v(x)$ be an EUU. Let $\mathbf{f}$ be an interval act for f. Let H be the cumulative corresponding to $\mathbf{f}_1$. Note that

$$\begin{aligned} H(x) &= \mu(\{\omega \in \Omega : \mathbf{f}_1(\omega) \leq x\}) \\ &= 1 - \mu_*(\{\omega \in \Omega : f(\omega) > x\}) \end{aligned}$$

If $f \in \mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{A}}$ is a risky environment with PTF γ, then

$$U(f) = \int_l^m v(x) dH_1(x) = \int_l^m v(x) d[1 - \gamma(1 - G^f(x))]$$

Let $(\mathcal{A},\pi),(\mathcal{B},\rho)$ be two issues with PTF γ and γ' respectively. Let $f\in\mathcal{F}_{\mathcal{A}}, g\in\mathcal{F}_{\mathcal{B}}$ and $G^f - G^g = G$ and assume $\gamma'\leq\gamma$. Then, $[1-\gamma(1-G(x))\leq[1-\gamma'(1-G(x))]$ and therefore

$$U(f)=\int_l^m v(x)d[1-\gamma(1-G^f(x))]\geq\int_l^m v(x)d[1-\gamma'(1-G^f(x))]=U(g)$$

□

Proof of Proposition 4: First, we show that (i) implies (ii). Since u is continuous and not maximally pessimistic, there exists x,y such that $u(x,y)>u(x,x)$ and $u(x,z)<u(x,y)$ for $z<y$. Let $w=\rho_u^{-1}(u(x,y)$. Let F be the lottery that yields x and y with equal probabilities. If $\gamma_a(3/4)$ is sufficiently close to zero, then the GQU (a,u) prefers the sure prospect $(w+y)/2$ over F, but prefers the lottery $G={}^1\!/_2F+{}^1\!/_2x$ over the lottery $H={}^1\!/_2\frac{w+y}{2}+{}^1\!/_2x$. To see this, note that the certainty equivalent of F and of G both converge to w as $\gamma_a(3/4)\to 0$. The certainty equivalent of H converges to $\rho^{-1}(u(\frac{w+y}{2},x))$ and therefore the assertion follows from the fact that $u(x,y)$ is strictly decreasing in its second argument at (x,y).

Next, we show that (ii) implies (i). Fix an environment $a\in Z$ and assume u is maximally pessimistic and let $v(x)=u(x,y)$. Then,

$$V(F)=\int v(x)d[1-(1-\gamma_a(F(x))]$$

Let F,x,z be such that $V(z)>V(F)$ and $x\leq y(F)$. Thus, $v(y)>V(F)$ and therefore

$$\gamma_a(\alpha)v(y)+(1-\gamma_a(\alpha))v(x)>\gamma_a(\alpha)V(F)+(1-\gamma_a(\alpha))v(x)\qquad(\dagger)$$

for all $\alpha\in(0,1)$. Let H be the cdf such that

$$H(z)=1-\gamma_a(\alpha)+\gamma_a(\alpha)(1-\gamma_a(1-F(z))$$

for all $z\in[x,m]$ and let G be the cdf such that

$$G(z)=1-\gamma_a(1-\alpha F(x))$$

for all $z \in [x, m]$. Then,

$$\gamma_a(\alpha)V(F) + (1 - \gamma_a(\alpha))v(x) = \int v dH$$

and

$$V(\alpha F + (1 - \alpha)x) = \int v dG$$

Below, we show that H first order stochastically dominates G. Together with inequality (†) this proves the assertion. Note that

$$\begin{aligned} G(z) &= 1 - \gamma_a(1 - \alpha + \alpha(1 - F(z)) \\ &\geq 1 - \gamma_a(1 - \alpha) - \gamma(\alpha(1 - F(z)) \\ &\geq 1 - \gamma_a(1 - \alpha) - \gamma_a(\alpha)\gamma_a(1 - F(z)) \\ &= H(z) \end{aligned}$$

The first inequality follows because γ_a is superadditive for all a. To see the second inequality, note that

$$\gamma_a(\alpha\beta) = \sum_{n=1}^{\infty}(\alpha\beta)^n \leq \left(\sum_{n=1}^{\infty}\alpha^n\right)\left(\sum_{n=1}^{\infty}\beta^n\right) = \gamma_a(\alpha)\gamma_a(\beta)$$

Next, we show that (i) implies (iii). Let $u(x, y) > u(x, x)$, $z = \rho_u^{-1}(u(x, y))$ and let $\alpha = \frac{y-z}{2(y-x)}$. Let F be the lottery that yields x with probability α and $y > x$ with probability $1 - \alpha$ and assume . Then, for $\gamma_a(1 - \alpha)$ sufficiently close to zero, $V_a(F)$ is close to $u(x, y)$ and therefore the certainty equivalent of F is close to z. Since the expected value of F is less than z this shows that the decision maker is risk loving in environments that are sufficiently uncertain.

To show that (iii) implies (i) let u be maximally pessimistic. Let F be a simple lottery, let x be the lowest prize in its support and let α be the probability of x. Then, $V(F) \geq \gamma_a(1 - \alpha)u(m, m) + (1 - \gamma_a(1 - \alpha))u(x, x)$. Hence, if $\gamma_a(1 - \alpha)$ is close to zero then $V(F)$ is close to $u(x, x)$ and therefore the DM prefers the expected value of F to F under extreme uncertainty. By continuity, this argument can be extended to all lotteries and hence the decision maker is not risk loving under extreme uncertainty. □

Proof of Proposition 5: Let (α, v) be the interval utility and consider an environment with parameter a^n. Then,

$$
\begin{aligned}
V(F) = &\int(\alpha v(x) + (1-\alpha)v(y))dH(x,y|F) \\
= &\alpha \int v(x)d[1-(1-F(x))^n] + (1-\alpha)\int v(x)d[F(x)^n] \\
= &\alpha \int v(x)d[\gamma_{a^n} \circ F] + (1-\alpha)\int v(x)d[\gamma^*_{a^n} \circ F] \\
= &\int v(x)d[\alpha\gamma_{a^n} \circ F + (1-\alpha)\gamma^*_{a^n} \circ F]
\end{aligned}
$$

and hence the proposition holds for all a^n. Since all $a \in Z$ are convex combinations of a^n's the proposition follows. □

Proof of Proposition 6: To simplify the notation, we assume that $M = [0,1]$, that is, the prizes are between 0 and 1. As in the proof of Lemma C3, we construct a risky environment corresponding to a^2, i.e., $\gamma_a(t) = t^2$. Let f_j, $j = 1, 2$ be a collection of acts such that

(i) $f_j \in \mathcal{F}_\mathcal{E}$ for all j;

(ii) f_j is uniformly distributed for all j; That is, if λ is Lebesgue measure and $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$ then $\mu(f_j^{-1}(B)) = \lambda(B)$ for $B \in \mathcal{B}$.

(iii) f_1 and f_2 are independent; that is, $\mu(f_1^{-1}(B) \cap f_2^{-1}(B')) = \lambda(B)\lambda(B')$ for all $B, B' \in \mathcal{B}$.

Let D be a diffuse subset of Ω and define $\mathcal{A}$ be the collection of sets $\{f^{-1}(B) \,|\, B \in \mathcal{B}\}$ and let $\pi : \mathcal{A} \to [0,1]$ be such that $\pi(f^{-1}(B)) = \lambda(B)$ for all $B \in \mathcal{B}$. As we show in the proof of Lemma C3, $(\mathcal{A}, \pi)$ is a prior and $\mathcal{F}_\mathcal{A}$ is a risky environment.

Let p and q be two possible probabilities (as in the definition of an urn experiment). Recall that $\iota(L) = p(L) = q(L)$ for all $L \in \mathcal{M}$ and $p(L) \neq q(L)$ for $L \notin \mathcal{M}$. Define the acts $g_1 : \Omega \to N, g_2 : \Omega \to N$ as follows.

$$
\begin{aligned}
g_1(\omega) = j \text{ if } \sum_{i=1}^{j-1} p_i \leq f_1(\omega) \leq \sum_{i=1}^{j} p_i \\
g_2(\omega) = j \text{ if } \in \sum_{i=1}^{j-1} q_i \leq f_2(\omega) \leq \sum_{i=1}^{j} q_i
\end{aligned}
$$

Let $T : \Omega \to N$ be defined as follows:

$$T(\omega) = \begin{cases} g_1(\omega) & \text{if } \omega \in D \\ g_2(\omega) & \text{if } \omega \in D^c \end{cases}$$

Clearly, $T^{-1}(L) \in \mathcal{F}_{\mathcal{A}}$ for all $L \in \mathcal{M}$. Next, we show that there is no risky environment that contains $\mathcal{F}_{\mathcal{A}}$ and $T^{-1}(L)$ for $L \notin M, L \subset N$. Recall that for $A \in \mathcal{F}_{\mathcal{A}}$, the revealed prior is $\pi(A) = \sqrt{\mu_*(A)}$. Let $E_i = g_i^{-1}(L)$ and define $B := T^{-1}(L) = E_1 D E_2$. Consider the act $h = xBy$

$$\begin{aligned} U(h) =& \mu_*(B)u(x,x) + \mu_*(B^c)u(y,y) + (1 - \mu_*(B) - \mu_*(B^c))u(x,y) \\ =& p(L)q(L)u(x,x) + (1 - q(L))(1 - p(L))u(y,y) \\ &+ (p(L)(1 - q(L)) + q(L)(1 - p(L)))\, u(x,y) \end{aligned}$$

Then, by construction, $\mu_*(B) = p(L) \cdot q(L)$. By Lemma C2, if $B \in \mathcal{F}_A$ then $p := \pi(B) = \sqrt{p(L) \cdot q(L)}$ and therefore h must correspond to a lottery F that yields x with probability p and y with probability $1 - p$. The utility of such a lottery in environment $\mathcal{F}_{\mathcal{A}}$ is

$$V(F) = p^2 u(x,x) + (1-p)^2 u(y,y) + 2p(1-p)u(x,y)$$

Since $2p(1-p) > p(L)(1-q(L)) + q(L)(1-p(L))$ this implies that $V(F) > U(h)$ for all u such that $u(x,y) < u(y,y)$ and therefore $B \notin \mathcal{F}_A$. This completes the proof of Proposition 6. $\square$

Proof of Proposition 7: Let ρ be a Machina preference and define $u(x,y) = 2\rho(x,y)$ for all $(x,y) \in I$. Choose a diffuse set D and an ideal set $E \in \mathcal{E}$ for the probability $(\mathcal{E}, \mu)$ such that $\mu(E) = 1/2$. Let $A_1 = D \cap E$, $A_2 = D^c \cap E$, $A_3 = D \cap E^c$ and $A_4 = D^\cap E^c$. For $S = \{1,2,3,4\}$ and $\phi \in \mathcal{F}^S$, let $f(\omega) = \phi(s)$. Then, let $y_1 = \max\{\phi(1), \phi(2)\}$, $y_2 = \max\{\phi(3), \phi(4)\}$, $x_1 = \min\{\phi(1), \phi(2)\}$ and $x_2 = \min\{\phi(1), \phi(2)\}$ and

$$\int u(\mathbf{f}) d\mu = .5[u(x_1,y_1) + u(x_2,y_2)] = \rho(x_1,y_1) + \rho(x_2,y_2) = \rho(\phi(1),\phi(2)) + \rho(\phi(3),\phi(4))$$

as desired.

That separability precludes M-reversals is obvious. To conclude the proof, we will show that if there are no M-reversals, then the u that satisfies the above equation must be separable. No M-reversals implies

$$u(x_1, y_1) + u(x_2, y_2) = u(x_1, y_2) + u(x_2, y_1) \tag{55}$$

whenever $(x_1, y_2), u(x_2, y_1) \in I$. Define $v_2(y) = u(l, y)$ and $v_1(x) = u(x, m) - u(l, m)$. Then, $v_1(x) + v_2(y) = u(x, m) - u(l, m) + u(l, y)$ and equation (55) ensures that $u(x, m) - u(l, m) = u(x, y) - u(l, y)$. Therefore, $v_1(x) + v_2(y) = u(x, y)$ for all x, y, proving the separability of u. $\square$

Proof of Proposition 8: Theorem 1 implies that for the maximally pessimistic EUU, and any simple act h with prizes in the finite set $Z \subset [l, m]$, $U(h) = \sum_{x \in Z} u(x)\mu_*(h^{-1}(z' \geq z))$. Therefore, to proof Proposition 7 it suffices to show that there is $T : \Omega \to N$ such that $\mu_*(T^{-1}(M)) = \rho(M)$ for all $M \subset N$. Since ρ is totally monotone, there is a probability $e : \mathcal{N} \to [0, 1]$ such that $\rho(M) = \sum_{M' \subset M} e(M)$. Partition Ω into $2^n - 1$ ideal events $\{E_M, M \in \mathcal{M}\}$ so that $\mu(E_M) = e(M)$. For $|M| = 1$ let $T(E_M) = i$ such that $\{i\} = M$. For $|M| > 1$ partition E_M into $|M|$ diffuse sets and assign each of those sets one elements of M, so that $T^{-1}(E_M) = M$. Then,

$$\mu_*(T^{-1}(M)) = \sum_{M' \subset M} \mu(E_{M'}) = \sum_{M' \subset M} e(M) = \rho(M)$$

as desired. This proves Proposition 7. $\square$

Proof of Proposition 9: Let $(\mathcal{E}, \mu, (\alpha, v))$ be the EUU. Partition Ω into $2^n - 1$ ideal events $\{E_M, M \in \mathcal{M}\}$ so that $\mu(E_M) = e(M)$. For $|M| = 1$ let $T(E_M) = i$ such that $\{i\} = M$. For $|M| > 1$ partition E_M into $|M|$ diffuse sets and assign each of those sets one elements of M, so that $T^{-1}(E_M) = M$. Let $f : N \to [l, m]$ and let $h : \Omega \to [l, m]$ such that $h(T^{-1}(j)) = f(j)$. Then,

$$\begin{aligned} U(h) = &\sum_{M \in \mathcal{M}} e(M) \left(\alpha \min_{j \in M} v(f(j)) + (1 - \alpha) \max_{j \in M} v(f(j) \right) \\ =& V(f) \end{aligned}$$

where $V(f)$ is the $\alpha-$MEU utility of act f. $\square$

References

Mohammed Abdellaoui, Aurlien Baillon, Ltitia Placido, and Peter P. Wakker, "The Rich Domain of Uncertainty," mimeo 2009.

Camerer, C. and M. Weber, (1992), "Recent developments in modeling preferences: Uncertainty and ambiguity ," *Journal of Risk and Uncertainty*, 5, pp. 325-70.

Chew, S. H., L. Epstein, and U. Segal, (1991), "Mixture Symmetry and Quadratic Utility," *Econometrica*, Vol. 59, No. 1 (Jan., 1991), pp. 139-163.

Chew Soo Hong, Edi Karni and Zvi Safra, (1987)"Risk aversion in the theory of expected utility with rank dependent probabilities," *Journal of Economic Theory*, Volume 42, Issue 2, pp. 370-381.

Chew S. and J. Sagi, (2008) "Small worlds: Modeling attitudes toward sources of uncertainty," *Journal of Economic Theory*, 139, Issue 1, pp. 1-24.

Curley, Shawn E and Frank J. Yates. (1989). "An Empirical Evaluation of Descriptive Models of Ambiguity Reactions in Choice Situations," *Journal of Mathematical Psychology* 33, pp. 397-427.

Ellsberg, D. (1961): "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics*, 75, 643–669.

Epstein, L. G., (1999): "A Definition of Ambiguity Aversion," *Review of Economic Studies*, 66, 579–608.

Epstein, L. G., and J. Zhang (2001): "Subjective Probabilities on Subjectively Unambiguous Events," *Econometrica*, 69, 265–306.

French, Kenneth R., and James M. Poterba, (1991): "Investor diversification and international equity markets", *American Economic Review* 81, 222-226.

Ghirardato, P. and M. Marinacci (2001): "Ambiguity Made Precise: A Comparative Foundation," forthcoming *Journal of Economic Theory*.

Ghirardato, P., F. Maccheroni and M. Marinacci (2004) "Differentiating Ambiguity and Ambiguity Attitude, *Journal of Economic Theory*, 118, pp. 133-173.

Gilboa, I, and D. Schmeidler (1994): "Additive Representations of Non-additive Measures and the Choquet Integral," *Annals of Operations Research,* 52, 43–65.

Heath, C. and A. Tversky, (1991): "Preference and Belief: Ambiguity and Competence in Choice under Uncertainty," *Journal of Risk and Uncertainty*, 4, 5-28.

Jaffray, J.Y. (1989): "Linear Utility Theory for Belief Functions," *Operations Research Letters,* 8, 107–112.

Jaffray, J.-Y., and Wakker, P. P. (1993): “Decision Making with Belief Functions: Compatibility and Incompatibility with the Sure-Thing Principle,” *Journal of Risk and Uncertainty,* 7, issue 3, 255–71.

Kopylov, I. (2007): “Subjective Probabilities on Small Domains,” *Journal of Economic Theory,* 133, Issue 1, March 2007, 236–265.

Lehrer, E. (2007): “Partially Specified Probabilities: Decisions and Games,” mimeo.

Maccheroni, F., M. Marinacci and A. Rustichini (2006): “Ambiguity Aversion, Robustness, and the Variational Representation of Preferences,” *Econometrica,* 74, 1447–1498.

Machina, M. J. (2009): “Risk, Ambiguity, and the Rank-Dependence Axioms,” *American Economic Review*, 99:1, pp. 385392.

Machina, M. J. and D. Schmeidler (1992): “A More Robust Definition of Subjective Probability,”*Econometrica*, 60, 745–780.

Savage, L. J. (1954): The Foundations of Statistics, Wiley, New York.

Siniscalchi, M., (2007): “Vector Expected Utility and Attitudes towards Variation,” mimeo.

Zhang, J., (2002): “Subjective Ambiguity, Expected Utility and Choquet Expected Utility,” *Economic Theory*, 20, 159–181.