Economics Department of the University of Pennsylvania
Institute of Social and Economic Research -- Osaka University

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Author(s): Mark Feldman
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AN EXAMPLE OF CONVERGENCE TO RATIONAL EXPECTATIONS WITH HETEROGENEOUS BELIEFS*

By MARK FELDMAN[1]

1. INTRODUCTION

The purpose of this paper is to investigate the stability of rational expectations equilibria (REE) in a model with Bayesian agents who initially possess a correct specification of the underlying structure of the economy but are uncertain of the values of some parameters. This can be an extraordinarily complicated problem because of an infinite regress in expectations. In making their optimizing decisions, agents must consider not only their own beliefs regarding parameter values, but also the beliefs of other agents, the beliefs of other agents regarding other agents' beliefs, etc. The contribution of this paper is to demonstrate that in a specific partial equilibrium setting adapted from Townsend [1978] there is convergence to the rational expectations equilibrium in spite of the initial heterogeneity of beliefs.

The topic of convergence to rational expectations has recently been the focus of considerable attention (for a survey of the literature see Blume, Bray, and Easley [1982]). The papers in the literature can be characterized according to whether the learning mechanism of agents is Bayesian or "boundedly rational".[2] In a Bayesian model, agents are Bayesian decision makers whose prior beliefs are consistent with the underlying structure of the world they inhabit. The Bayesian paradigm implicitly assumes that agents are able to discern the (possibly stochastic) functional relationship between parameter values and equilibrium outcomes. If in the original model agents don't "know" the rational expectations equilibrium it is because they are uncertain of the prameter values. Inevitably, the Bayesian "solution" entails augmenting the probability space by embedding within it all conceivable parameter values, so a state of the world includes a specification of the realization of all parameter values, and imposing the rational expectations equilibrium concept for the augmented model.

In the boundedly rational models of learning, agents are typically portrayed as

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[1] I am indebted to Christian Gilles, Jack Marshall, Jon Sonstelie and especially two anonymous referees for their helpful suggestions. I also wish to acknowledge beneficial conversations David Easley, James Jordan, and Chris Sims. It should not be inferred that the above individuals share the opinions expressed in this paper and of course only I am responsible for any errors.

[2] Recent papers that adopt a Bayesian approach are Blume-Easley [1984] and Feldman [1986], while Jordan [1985, 1986], Marcet-Sargent [1986] and Woodford [1986] are boundedly rational models of learning.

classical or Bayesian statisticians who erroneously perceive themselves as being in a stationary environment. But because of the feedback effects induced by changing beliefs, the sequence of equilibrium outcomes in learning models is a non-stationary stochastic process. Possible interpretations of boundedly rational behavior are: (1) each agent naively assumes that no other agents are engaged in learning estimation, or (2) agents do not recognize (or consider to be negligible) the dynamic impact of aggregate learning.

The equivalence of the modern neoclassical theory of von Neumann-Morgenstern expected utility maximization with the Bayesian paradigm (for a discussion, see Arrow [1970]) provides strong justification for analyzing intertemporal learning in an explicitly Bayesian framework. The alternative approach of assuming bounded rationality has been justified on grounds of the plausibility or reasonableness of the specific forecasting scheme. (For instance, see Bray [1982], Blume-Easley [1982], DeCanio [1979], and Marcet-Sargent [1986].) It has been emphasized (especially by Bray-Kreps [1981]) that in contrast the sophistication and computational skill required of agents in correctly specified Bayesian models is beyond human capability.

While this argument is not without merit, many of the boundedly rational models provide an incomplete framework to address the asymptotic issue of whether or not there is convergence to a stationary rational expectations equilibrium. Typically these models yield convergence to a rational expectations equilibrium for a set of parameter values, the "stable set", but there is also a non-negligible complement in the parameter space for which with positive probability there is either no limit stochastic process or else the limit is not a rational expectations equilibrium. When convergence to a rational expectations equilibrium does not occur, the limiting behavior of agents is implausible. They continue to abide by their forecasting scheme despite overwhelming statistical evidence of model misspecification. The robustness of rational expectations as a long-run equilibrium concept cannot be challenged by such a scenario. In practice, agents would ultimately revise their model rather than persisting with an estimation procedure evidently flawed.

Jordan [1985] has a general equilibrium model with non-Bayesian learning which is exempt from the above criticism, in that there is a.s. convergence to a REE for all parameter values. But to guarantee the existence of a temporary equilibrium, Jordan assumes that within each period there is learning in "virtual" time. Agents acquire information despite the absence of any genuine economic activity. The within period learning can be interpreted as an informational tatonnemont process. Since individuals may acquire more information from this process than could be inferred from the equilibrium price, this modeling strategy may exaggerate the information transmitted via the market mechanism.

In much recent work, especially in macroeconomics, the economy is modeled as a stationary REE. A distinct issue from how best to model how agents "actually learn" is whether the stationary REE can be embedded in an internally consistent theory of decision making when we allow individuals to be initially uninformed regarding parameter values that they know in the REE. A frequent defense of the REE concept is the assertion that rational agents will make optimal use of all available information, i.e., they will act as Bayesian decision makers. Often, there is an auxiliary, albeit tacit, assumption that a consequence of such behavior is convergence to a stationary REE. But what if this conjecture of convergence is false? This would place those who advocate the stationary REE as the appropriate long-run equilibrium concept in the peculiar position of having to reject Bayesian theories of learning while simultaneously endorsing models in which agents rigorously adhere to the tenets of Bayesian decision theory. The usual normative arguments advanced to support the stationary REE as an equilibrium hypothesis are not compelling without demonstration that such an equilibrium will be attained as the limit of a Bayesian learning scheme.

An important but unresolved issue in analyzing learning is whether the estimation scheme in Jordan [1985], or in other models with bounded rationality, has a superior claim to plausibility or reasonableness than the Bayesian methodology. Until now, all of the positive results in the boundedly rational learning literature rely upon an extreme degree of coordination in agents' forecasting strategies. Jordan [1985], recognizing this remarked, "One especially troublesome feature of the scheme we have constructed is that the convergence of a trader's estimated expectations depends on the use of the same estimation procedure by the other traders. This raises the possibility that convergence could be impaired if traders seek to somehow tailor their estimation procedures more closely to their own characteristics." Radner, also an advocate of the boundedly rational approach, made a similar comment (Radner [1982, p. 992]) regarding the Bray [1982] model.

This implicit (or chance) coordination of estimation schemes is akin to the common knowledge assumption made in this paper. And, while the degree of sophistication of Bayesian agents with a correct specification of the structure of the economy may seem beyond human capability, the *a priori* likelihood that agents will adopt any particular *ad hoc* rule is surely nil. So even if the family of boundedly rational rules collectively offer a plausible description of learning, this is insufficient to conclude that any single boundedly rational scheme is plausible. Hence, without demonstration that the qualitative results are robust under mild behavioral deviations, neither the Bayesian or any boundedly rational learning rule can provide a fully satisfactory positive theory of learning. Unfortunately, this research program of verification of robustness appears difficult. To start with, it is not obvious what is an appropriate topology to place on the space of sequential decision rules. (Kadane-Chuang [1978] have investigated this issue in a non-sequential setting and have some mildly positive results.) But the conclusions in Diaconis-Freedman [1986] on the sensitivity of consistency of Bayes procedures with respect to the prior probability, starkly limit the scope of theorems one could hope to obtain regarding Bayesian robustness.

The model used in this paper is essentially that in Townsend [1978]. Townsend [1978] is a model with a continuum of producers with quadratic cost functions facing a linear stochastic demand function with a parameter θ which is not only unknown, but also the beliefs of others firms regarding θ are unknown. Townsend (section V) succeeds in deriving closed form solutions for the infinite order beliefs of agents for each time period. He conjectures, but is unable to prove, that beliefs of all orders converge to some limit that induces convergence to a REE. The principal result of this paper is that Townsend's conjecture is correct; beliefs of all orders do converge and in the limit rational expectations prevails.

The uncertainty agents have regarding θ is treated in this paper in accordance with the approach to games of incomplete information advocated by Harsanyi [1967, 1968]. Bray and Kreps [1981] and Blume and Easley [1984] have previously adopted this framework for modeling the behavior of agents with heterogeneous information. Similarly, the asymptotic convergence results in these two papers and my paper are in large measure consequences of the Martingale Convergence Theorem. In contrast to the model in this paper, the Blume-Easley model has the merit of being embedded in a general equilibrium framework. But to achieve this generality they are forced to assume: (1) that the parameter space Θ of possible probability laws governing the economy is finite, and (2) the period t behavior of an initially uninformed trader depends solely upon current characteristics (endowment and preferences) and their beginning-of-period t probability measure on Θ.[3] No such assumptions are needed in this paper.

The remainder of the paper is organized as follows. Section 2 consists of a formal description of the model. Existence of an equilibrium and convergence in L_1 to the REE are proven in Section 3. A sufficient condition to prove a.s. convergence and characterize the equilibrium as a function of beliefs regarding θ is provided in Section 4. Section 5 contains conclusions and remarks regarding possible generalizations.

2. DESCRIPTION OF THE MODEL

There is a measure space of firms $(I, \mathscr{B}, \lambda)$ where $I=(0, 1]$, $\mathscr{B}$ is the Borel σ-field and λ is Lebesgue measure. To avoid extraneous measure-theoretic technicalities the assumption is made that there are a finite number of types of firms. Firms of the same type are identical in all respects. The set of types is denoted by $L=\{1, 2,\ldots, l\}$. Firms of type s consist of the set $A_s \in \mathscr{B}$ with $\lambda(A_s)=\lambda_s$, $\bigcap_{s\neq v} A_s \cap A_v=\phi$, and $\bigcup_{s=1}^{l} A_s=I$. The type function $\tau: I\to L$ is defined by $\tau(i)=s$ for $i \in A_s$.

The probability space on which all random variables are defined is $(\Omega, \mathscr{F}, P)$. The prior probability P is shared by all firms. As in the seminal paper of Harsanyi

[3] In choosing their consumption, the traders in Blume-Easley [1984] maximize with respect to their beginning-of-period marginal probabilty distribution of the state of the world s_t, ignoring any information in the current price realization. But, in updating their beliefs they use a correctly specified model.

[1967, 1968], the viewpoint here is that any divergence in beliefs among firms arises from them receiving different information. The information available to a type s firm prior to period t is represented by the sub-σ-field $\mathscr{F}_{s,t-1}$ where $t \in T = \{1, 2, \ldots\}$. The collective information available prior to period t is $\mathscr{F}^L_{t-1} = \bigvee_{s=1}^{l} \mathscr{F}_{s,t-1}$.

The firms produce a homogeneous product for which the period t per firm demand function is $D_t(p_t) = -bp_t + \theta + \varepsilon_t$. The price coefficient b is a parameter, in the sense that its value does not depend upon the realization of $\omega \in \Omega$. θ is the exogenous random variable in the model for which firms may have asymmetric initial information. It is the possiblity that $P(\theta \in A \| \mathscr{F}_{s,0}) \neq P(\theta \in A \| \mathscr{F}_{v,0})$ for an arbitrary Borel set A, that induces the intricacy of the learning process. Initially the only assumption made with regard to θ is that θ is strictly positive and integrable. This is sufficient to prove that outputs of firms converge in L_1 to the REE outputs. Further restrictions made in Section 4 yield a proof of a.s. convergence and allow a simple characterization of output as a function of beliefs.

The sequence of random variables $\{\varepsilon_t\}_{t=1}^{\infty}$ is i.i.d. with mean zero. Also, the the sequence $\{\varepsilon_t\}$ is independent of θ and the sub-σ-field F^L_0.

The output of firm i in period t is $\tilde{q}_{it}$ which is a random variable since it is chosen in accordance with firm i's expectation at time $t-1$ of $\tilde{P}_t$, the period t price which is also a random variable. The realization of $\tilde{q}_{it}$ is denoted by q_{it} which is produced at cost $\frac{1}{2a} q_{it}^2$ where $a > 0$ and $q_{it} \geq 0$.[4] Firms are assumed to be risk-neutral so in an equilibrium $\tilde{q}_{it} = \mathrm{Max}\{0, aE[P_t \| \mathscr{F}_{\tau(i),t-1}]\}$. The per capita output is $\tilde{Q}_t$ defined by $\tilde{Q}_t(\omega) = \int \tilde{q}_{it}(\omega)\lambda(di)$ for $\omega \in \Omega$.

To guarantee the existence of an equilibrium, restrictions must be imposed upon a and b. Define c_s by $c_s = \sum_{u \neq s} \frac{a\lambda_u}{a\lambda_u + b}$ and c_{Max} by $c_{\mathrm{Max}} = \mathop{Max}_{s \in L} c_s$. We shall assume that $c_{\mathrm{Max}} < 1$. Either $a < b$ or $l \leq 2$ are sufficient to imply that $c_{\mathrm{Max}} < 1$.

Initially all firms of type s form conditional expectations of all relevant random variables by conditioning with respect to the sub-σ-field $\mathscr{F}_{s,0}$ (and all type s firms use the same version of conditional expectation). In a manner which is explicitly described in Section 3, this determines $\tilde{q}_{i1}$ for $i \in \tau^{-1}(s)$. The equilibrium price function $\tilde{P}_1(\cdot)$ is the solution to the equation $\tilde{Q}_1(\omega) = \int \tilde{q}_{i1}(\omega)\lambda(di) = -b\tilde{P}_1(\omega) + \theta(\omega) + \varepsilon_1(\omega)$.

Upon observing the price p_1, the realization of $\tilde{P}_1$, firms revise their beliefs. That is, firms of type s condition with respect to $\mathscr{F}_{s,1} = \mathscr{F}_{s,0} \vee \sigma(\tilde{P}_1)$. This enables them to choose period 2 output, etc.

It is easy to calculate that if θ has a degenerate distribution that the rational expectations equilibrium is $\tilde{P}_t = \frac{\theta}{a+b} + \frac{\varepsilon_t}{q}$ with $\tilde{q}_{it} = \frac{a\theta}{a+b}$ for all $i \in I$. The main result of the paper is that even when θ is a non-degenerate random variable,

[4] In this context there is no sensible interpretation of negative outputs. But the equilibrium price can be negative if ε_t is sufficiently negative.

asymptotically all firms can infer the realization of θ and that this inference is common knowledge. This implies that $\tilde{q}_{it} \xrightarrow{L_1} \frac{a\theta}{a+b}$ and that $\tilde{P}_t - \left[\frac{\theta}{a+b} + \frac{\varepsilon_t}{b}\right] \xrightarrow{L_1} 0$. In other words, the sequence of temporary equilibria converges to the rational expectations equilibrium.

3. FORMATION OF PRICE EXPECTATIONS

The first task in determining the evolution of prices and outputs over time is to define a *temporary rational expectations equilibrium* (TREE), the equilibrium concept that is assumed to describe the behavior of firms at a given point in time.

3.1. *Definition.* A period t TREE is a family of output functions $(\tilde{q}_{it}) i \in I$, a price function $\tilde{P}_t$, and a vector of sub-σ-fields $\mathscr{F}_{t-1} = (\mathscr{F}_{1,t-1}, \mathscr{F}_{2,t-1}, \ldots, \mathscr{F}_{l,t-1})$ such that:

i) $q_{it}(\cdot)$ is jointly measurable as a function of i and ω,

ii) $\tilde{P}_t\colon \Omega \to R$ satisfies $\int \tilde{q}_{it}(\omega)\lambda(di) = -b\tilde{P}_t(\omega) + \theta(\omega) + \varepsilon_t(\omega)$,

iii) $\tilde{q}_{it}(\omega) = \text{Max}\{0, aE[\tilde{P} \| \mathscr{F}_{\tau(i),t-1}](\omega)\}$ for all $i \in I$ for almost all ω.

The assumption of a continuum of firms implies that the output choice of any single firm does not affect the equilibrium price. Hence, in this context competitive behavior is equivalent to Nash behavior.

3.2. *Definition.* A *period t Bayesian-Nash* equilibrium is a family of output functions $(\tilde{q}_{it}) i \in t$, and a vector of sub-σ-fields $\mathscr{F}_{t-1} = (\mathscr{F}_{1,t-1}, \mathscr{F}_{2,t-1}, \ldots, \mathscr{F}_{l,t-1})$ such that

i) $\tilde{q}_{it}(\cdot)$ is jointly measurable as a function of i and ω,

ii) $\tilde{q}_{it}(\omega) = \text{Max}\left\{0, \frac{a}{b}[E(\theta \| \mathscr{F}_{\tau(i),t-1})(\omega) - E(\tilde{Q}_t \| \mathscr{F}_{\tau(i),t-1})(\omega)]\right\}$ for all $i \in I$ for almost all ω, where $\tilde{Q}_t(\omega) = \int \tilde{q}_{jt}(\omega)\lambda(dj)$.

3.3. PROPOSITION. *For a given vector $\mathscr{F}_t$ of sub-σ-fields, $(\tilde{q}_{it})$ are equilibrium output functions in a TREE iff $(\tilde{q}_{it})$ are a period t Bayesian Nash equilibrium.*

PROOF. *Trivial.*

Because of the structure of the model certain useful refinements can without loss of generality be imposed upon the definition of an equilibrium. Since all firms are risk-neutral and the cost function is strictly convex, in an equilibrium all firms of the same type must have identical output functions. Let $\tilde{q}_t^s$ denote the period t equilibrium output for type s firms. To characterize $\tilde{q}_t^s$ suppose that $(\tilde{q}_t^1, \tilde{q}_t^2, \ldots, \tilde{q}_t^{s-1}, \tilde{q}_t^{s+1}, \ldots, \tilde{q}_t^{l-1}, \tilde{q}_t^l)$ is the vector of equilibrium output functions for the types other than s. If $[E(\theta \| \mathscr{F}_{s,t-1})(\omega) - \sum_{u \neq s} \lambda_u E(\tilde{q}_t^u \| \mathscr{F}_{s,t-1})(\omega)] \leq 0$ then $\tilde{q}_t^s(\omega) = 0$ since $E[\tilde{P}_t \| \mathscr{F}_{s,t-1}](\omega) \leq 0$. If $[E(\theta \| \mathscr{F}_{s,t-1})(\omega) - \sum_{u \neq s} \lambda_u E(\tilde{q}_t^u \| \mathscr{F}_{s,t-1})(\omega)] > 0$, then for $\tilde{q}_{it} = \tilde{q}_t^s$ to constitute a Nash equilibrium in the sub-game among the set of firms A_s, it is necessary that $q_t^s(\omega)=\frac{a}{b}[E(\theta\|\mathscr{F}_{s,t-1})(\omega)-\sum_{u\neq s}\lambda(q_t^u\|\mathscr{F}_{s,t-1})(\omega)-\lambda_u q_t^s(\omega)]$. Upon rearranging we have $q_t^s(\omega)=\frac{a}{a\lambda_s+b}[E(\theta\|\mathscr{F}_{s,t-1})(\omega)-\sum_{u\neq s}\lambda_s E[q_t^u\|\mathscr{F}_{s,t-1})]$. This motivates the following definition of an equilibrium that will apply for the remainder of the paper.

3.4. *Definition. A Bayesian-Nash equilibrium* is a sequence of output functions $\{\tilde{q}_t\}_{t=1}^{\infty}$ where $\tilde{q}_t=(\tilde{q}_t^1, \tilde{q}_t^2,\ldots, \tilde{q}_t^l)$, a sequence of price functions $\{\tilde{P}_t\}_{t=1}^{\infty}$ and a sequence of sub-σ-fields $\{\mathscr{F}_{s,t}\}_{s=1}^{l}{}_{t=1}^{\infty}$ such that:

i) $\tilde{q}_t^s$: $\Omega\to R$ is $\mathscr{F}_{s,t-1}$ measurable,

ii) $\tilde{P}_t$: $\Omega\to R$ is defined by $\sum_s \lambda_s\tilde{q}_t^s=-b\tilde{P}_t+\theta+\varepsilon_t$,

iii) $\mathscr{F}_{s,t}=\mathscr{F}_{s,t-1}\vee\sigma(\tilde{P}_t)$,

iv) for all $u\in L$, for almost all ω,

$$\begin{aligned}\tilde{q}_t^u(\omega) &= \operatorname{Max}\{0,\ aE[\tilde{P}_t\|\mathscr{F}_{u,t-1}](\omega)\}\\ &= \operatorname{Max}\left\{0,\ \frac{a}{b}[E(\theta\|\mathscr{F}_{u,t-1})(\omega)-E(\sum_s \lambda_s\tilde{q}_t^s\|\mathscr{F}_{u,t-1})](\omega)\right\}.\end{aligned}$$

Firms observe the realization of $\tilde{P}_t$ but no other aggregate market data. In particular, $\tilde{Q}_t(\omega)$ is not revealed to the firms. The formal statement of the revision of beliefs that occurs upon observing $\tilde{P}_t(\omega)=p_t$ is that $\mathscr{F}_{s,t}=\mathscr{F}_{s,t-1}\vee\sigma(\tilde{P}_t)$ for all $s\in L$.[5] Upon determining the posterior beliefs, firms can choose period $t+1$ outputs, etc.

Before plunging into the technical details of proving existence of an equilibrium and limit theorems, as an aid to the reader the basic conceptual ideas will first be sketched. The first step to define a space $\mathscr{F}^*$ of information structures exploiting mathematical results introduced into the economics literature by Allen [1983] and subsequently Cotter [1986]. A space $\mathscr{L}$ of output functions is defined along with a function Γ: $\mathscr{F}^*\times\mathscr{L}\to\mathscr{L}$ that is continuous. For $\mathscr{S}\in\mathscr{F}^*$, $\Gamma(\mathscr{S},\cdot)$ is a contraction mapping with the fixed point being the equilibrium output function for information structure $\mathscr{S}$. Since the modulus of the contraction mapping is uniform over $\mathscr{F}^*$, the fixed point mapping E: $\mathscr{F}^*\to\mathscr{L}$ is continuous and so convergence of a sequence of information structures implies convergence of the corresponding sequence of equilibrium output strategies.

For $s\in L$, define $\mathscr{F}^{s,**}$ to be the family of all sub-σ-fields of $\mathscr{F}$. Define an equivalence relation $\sim$ on $\mathscr{F}^{s,**}$ by $\mathscr{S}\sim\mathscr{S}'$ if for every $G\in\mathscr{S}$, there exists $G'\in\mathscr{S}'$ such that $P(G\Delta G')=0$ (and vice versa). Define $\mathscr{F}^{s,*}$ to be the family of equivalence classes of $\mathscr{F}^{s,**}$. Before endowing $\mathscr{F}^{s,*}$ with a topology we need

[5] All the results of this paper extend to the case where firms receive information in addition to that generated by prices. As long as $\sigma(\mathscr{F}_{s,t-1}\vee\sigma(\tilde{P}_t))\subseteq\mathscr{F}_{s,t}$ for all s and t, none of the proofs require modification.

two technical results. The first result proved by Boylan [1971] is that the random variable generated by conditioning with respect to a sub-σ-field $\mathscr{S}$ depnds upon $\mathscr{S}$ only through its equivalence class.

The other result is that if a sub-σ-field $\mathscr{S}$ is generated by the union of two other sub-σ-fields (this is the technical representation of updating of beliefs), then the equivalence class that $\mathscr{S}$ is a member of depends only upon the equivalence classes of the other two sub-σ-fields.

3.5. LEMMA. *Let $\mathscr{S}^* \in \mathscr{F}^{s,*}$ and let $\mathscr{S}_1, \mathscr{S}_2 \in \mathscr{S}^*$ then $E[X \| \mathscr{S}_1] = E[X \| \mathscr{S}_2]$ a.s. for every $X \in L_1(\Omega, \mathscr{F}, P)$.*

PROOF. (Boylan [1971, Theorem 2]).

3.6. LEMMA. *If $\mathscr{S}^*, \mathscr{H}^* \in \mathscr{F}^{s,*}$, $\mathscr{S}_1, \mathscr{S}_2 \in \mathscr{S}^*$ and $\mathscr{H}_1, \mathscr{H}_2 \in \mathscr{H}^*$, then $\mathscr{S}_1 \vee \mathscr{H}_1 \sim \mathscr{S}_2 \vee \mathscr{H}_2$.*

PROOF. *See Appendix.*

Collectively, the two above results allow us to dispense with the formalism of distinguishing between a sub-σ-field and the equivalence class of which it is a member.

If Y and Z are two arbitrary Banach spaces let $BL(Y, Z)$ denote the space of bounded linear operators from Y to Z. $\mathscr{F}^{s,*}$ can be identified with a subset of $BL(L_1, L_1)$, where $L_1 = L_1(\Omega, \mathscr{F}, P)$, by associating with $\mathscr{S}^* \in \mathscr{F}^{s,*}$ the map $X \to E[X \| \mathscr{S}]$ for $X \in L_1$ and $\mathscr{S} \in \mathscr{S}^*$. Because of Lemma 3.5 this map is independent of the choice of $\mathscr{S} \in \mathscr{S}^*$.

Adopting a suggestion of Coffer (1986), we endow $F^{s,*}$ with the topology T of pointwise convergence. Since for $X \in L_1$ and $G \in F^{s,**}$, $\|E(X \| G)\| \leq \|X\|$, $\mathscr{F}^{s,*}$ is an equicontinuous family.

3.7. LEMMA. *The function J: $\mathscr{F}^{s,*} \times L_1 \to L_1$ defined by $J(\mathscr{S}, X) = E(X \| \mathscr{S})$ is continuous.*

PROOF. (Kelley [1955, Theorem 7.15]).

For $s \in L$, let $L_1^s = L_1$ and define the Banach space $\mathscr{L} = \mathop{X}\limits_{s=1}^{l} L_1^s$ with $\|q\| = \sum_s \lambda_s \|q^s\|$ for $q = (q^1, q^2, \ldots, q^l) \in \mathscr{L}$. Define $\mathscr{F}^* = \mathop{X}\limits_{s=1}^{l} \mathscr{F}^{s,*}$ and endow $\mathscr{F}^*$ with the product topology. For $\mathscr{S} = (\mathscr{S}^1, \mathscr{S}^2, \ldots, \mathscr{S}^l) \in \mathscr{F}^*$ define ψ^u: $\mathscr{F}^* \times \mathscr{L} \to L_1^u$ by $\psi^u({}^1\mathscr{S}, \ldots, \mathscr{S}^l, q^1, \ldots, q^l) = \frac{a}{a\lambda_u + b}[E(\theta \| \mathscr{S}^u) - \sum_{s \neq u} \lambda_s E(q^s \| \mathscr{S}^u)]$. Ignoring nonnegativity constraints, ψ^u is the response function of type u firms based on their

[6] I am grateful to a referee for suggesting that a contraction mapping argument could be invoked to prove uniqueness and generalize the existence theorem in a previous version of this paper.

information of the output strategies of the other types. Define $\psi\colon \mathscr{F}^* \times \mathscr{L} \to \mathscr{L}$ by $\psi(\mathscr{S}, q) = (\psi^1(\mathscr{S}, q), \psi^2(\mathscr{S}, q), \ldots, \psi^l(\mathscr{S}, q))$, where $\mathscr{F}^*$ is defined as in the proof of Lemma 3.7.

We now proceed to prove that ψ is continuous for all $\mathscr{S} \in \mathscr{F}^*$, and $\psi(\mathscr{S}, \cdot)\colon \mathscr{L} \to \mathscr{L}$ is a contraction mapping with a uniform (in $\mathscr{S}$) modulus.[6]

3.8. LEMMA. *Let X be a Banach space with $u \in X$ and f a linear operator from X into X such that $\|f\| < 1$. Then $\phi\colon X \to X$ defined by $\phi(x) = u + f(x)$ is a contraction mapping.*

PROOF.
$$\|\phi(x) - \phi(y)\| = \|u + f(x) - u - f(y)\| = \|f(x) - f(y)\| \le \|f\| \cdot \|x - y\| < \|x - y\|.$$

3.9. PROPOSITION. *ψ is continuous and for all $\mathscr{S} \in \mathscr{F}^*$ the function $\psi(\mathscr{S}, \cdot)\colon \mathscr{L} \to \mathscr{L}$ is a contraction with modulus less than or equal to c_{Max} (recall $c_{\mathrm{Max}} = \operatorname{Max}_{u \in L} \left\{ \sum_{s \neq u} \frac{a\lambda_s}{a\lambda_s + b} \right\}$ is by assumption less than one).*

PROOF. Continuity is a direct consequence of Lemma 3.7. For $\mathscr{S} = (\mathscr{S}^1, \ldots, \mathscr{S}^l) \in \mathscr{F}^*$, define $f_{\mathscr{S}}\colon \mathscr{L} \to \mathscr{L}$ by $f_{\mathscr{S}}(q^1, \ldots, q^l) = \Big(-\frac{a}{a\lambda_1 + b} \sum_{s \neq 1} \lambda_s F(q^s \| \mathscr{S}^1),$ $-\frac{a}{a\lambda_2 + b} \sum_{s \neq 2} \lambda_s E(q^s \| \mathscr{S}^2), \ldots, -\frac{a}{a\lambda_l + b} \sum_{s \neq l} \lambda_s E(q^s \| \mathscr{S}^l) \Big)$.
Since $\psi(\mathscr{S}, q) = \left[\frac{a}{a\lambda_1 + b} E(\theta \| \mathscr{S}^1), \frac{a}{a\lambda_2 + b} E(\theta \| \mathscr{S}^2), \ldots, \frac{a}{a\lambda_l + b} E(\theta \| \mathscr{S}^l) \right] +$ $f_{\mathscr{S}^*}(q)$, it suffices to demonstrate that $\|f_{\mathscr{S}}\| \le c_{\mathrm{Max}}$. Because the conditional expectation operator is a linear map with norm equal to one (Neveu [1965, p. 123]), $\| \sum_{s \neq u} \frac{a\lambda_s}{a\lambda_s + b} E(q^u \| \mathscr{S}^s) \| \le \sum_{s \neq u} \frac{a\lambda_s}{a\lambda_s + b} \|q^u\|$. So $\|f_{\mathscr{S}}(q)\| \le \sum_u \Big[\lambda_u \sum_{s \neq u} \frac{a\lambda_s}{a\lambda_s + b} \|q^u\| \Big] \le c_{\mathrm{Max}} \sum_u \lambda_u \|q^u\| = c_{\mathrm{Max}} \|q\|$ and hence $\|f_{\mathscr{S}}\| \le c_{\mathrm{Max}} < 1$.

We can now define the best response function for type u, $\Gamma^u\colon \mathscr{F}^* \times \mathscr{L} \to L_1$ by $\Gamma^u(\mathscr{S}, q)(\omega) = \operatorname{Max}\{0, \psi^u(\mathscr{S}, q)(\omega)\}$. The collective best response function is $\Gamma\colon \mathscr{F}^* \times \mathscr{L} \to \mathscr{L}$ defined by $\Gamma(\mathscr{S}, q) = (\Gamma^1(\mathscr{S}, q), \Gamma^2(\mathscr{S}, q), \ldots, \Gamma^l(\mathscr{S}, q))$.

3.10. PROPOSITION. *Γ is continuous and for all $\mathscr{S} \in \mathscr{F}^*$, $\Gamma(\mathscr{S}, \cdot)\colon \mathscr{L} \to \mathscr{L}$ is a contraction with modulus less than or equal to c_{Max}.*

PROOF. To prove continuity it is sufficient to prove that Γ^u is continuous. But $\Gamma^u(\mathscr{S}, q) = M(\psi^u(\mathscr{S}, q))$ where $M\colon L_1 \to L_1$ is defined by $M(X)(\omega) = \operatorname{Max}\{0, X(\omega)\}$, and since M is continuous, Γ^u is continuous.

From Proposition 3.9 $\sum_s \lambda_s \|\psi^s(\tilde{q}) - \psi^s(\hat{q})\| \le c_{\mathrm{Max}} \|\tilde{q} - \hat{q}\|$ for $\tilde{q}, \hat{q} \in L$. So the conclusion is implied by proving that $\sum_s \lambda_s \|\Gamma^s(\hat{q}) - \Gamma^s(\tilde{q})\| \le \sum_s \lambda_s \|\psi^s(\tilde{q}) - \psi^s(\hat{q})\|$. But $\|\Gamma^s(\tilde{q}) - \Gamma^s(\hat{q})\| = \int_\Omega |\operatorname{Max}\{0, \psi^s(\tilde{q})(\omega)\} - \operatorname{Max}\{0, \psi^s(\hat{q})(\omega)\}| P(d\omega)$ $\le \int_\Omega |\psi^s(\tilde{q})(\omega) - \psi^s(\hat{q})(\omega)| P(d\omega) = \|\psi^s(\hat{q}) - \psi^s(\hat{q})\|$.

Define the one-period equilibrium output function E: $\mathscr{F}^* \to \mathscr{L}$ by $E(\mathscr{S}) = \Gamma(\mathscr{S}, E(\mathscr{S}))$. Because of Proposition 3.10 and the Banach Fixed Point Theorem this definition is without ambiguity. The vector of period one equilibrium output functions is $\tilde{q}_1 = E(\mathscr{F}_0)$, $\mathscr{F}_1$ is defined by $\mathscr{F}_1 = (\mathscr{F}_{0,1}, \vee \tilde{P}_1, \mathscr{F}_{0,2} \vee \tilde{P}_1, \ldots, \mathscr{F}_{0,l} \vee \tilde{P}_1)$, $\tilde{q}_2 = E(F_1)$, etc.

The above results are summarized with the following theorem.

3.11. THEOREM. *There is an (essentially) unique Bayesian-Nash equibrium. The equilibrium satisfies the recursive equations*:

i) $\tilde{q}_{t+1} = E(\mathscr{F}_t)$

ii) $\mathscr{F}_t = (\mathscr{F}_{1,t-1} \vee \sigma(\tilde{P}_t), F_{2,t-1} \vee \sigma(\tilde{P}_t), \ldots, \mathscr{F}_{l,t-1} \vee \sigma(\tilde{P}_t))$.

Defining $F_{s,\infty} = \bigvee_{t=0}^{\infty} F_{s,t}$ and $\mathscr{F}_\infty = (\mathscr{F}_{1,\infty}, \mathscr{F}_{2,\infty}, \ldots, \mathscr{F}_{l,\infty})$ a consequence of a corollary to the Martingale Convergence Theorem is that $\mathscr{F}_{s,t} \to \mathscr{F}_{s,\infty}$ and so $\mathscr{F}_t \to \mathscr{F}_\infty$. So proving that E is continuous implies that $\tilde{q}_t \to \tilde{q}_\infty$ where $\tilde{q}_\infty$ is defined by $\tilde{q}_\infty = E(F_\infty)$. Note that this is convergence in $\mathscr{L}$ but not a.s. convergence of $\{\tilde{q}_t\}$ to $\tilde{q}_\infty$.

3.12. LEMMA. $\mathscr{F}_{s,t} \to \mathscr{F}_{s,\infty}$ *and* $\mathscr{F}_t \to \mathscr{F}_\infty$

PROOF. (Billingsley [1979, Theorem 35.5]) implies $\mathscr{F}_{s,t} \to \mathscr{F}_{s,\infty}$. Since $\mathscr{F}^*$ has the product topology, $\mathscr{F}_t \to \mathscr{F}_\infty$.

3.13. PROPOSITION. *E is continuous and* $\tilde{q} \xrightarrow{L_1} \tilde{q}_\infty$..

PROOF. To prove continuity let $(\mathscr{F}_\alpha)$ be a net (Kelley [1955, Chapter 2] is a standard reference on nets) converging to $\mathscr{F}_{\alpha_0}$, and define $q_\alpha = E(\mathscr{F}_\alpha)$. Let O_ε be an open sphere centered at q_{α_0} with radius ε. It suffices to prove that $(E(\mathscr{F}_\alpha))$ is eventually in O_ε. Let V be an open sphere centered at q_{α_0} with radius less than $\varepsilon \cdot (1 - c_{\text{Max}})$. Γ is continuous so $\Gamma(\mathscr{F}_\alpha, q_{\alpha_0}) \to q_{\alpha_0}$ and is eventually in V. Applying a standard successive approximation result (see e.g., Smart [1974, Remark 1.2.3 (iii)]) $\Gamma(\mathscr{F}_\alpha, q_{\alpha_0}) \in V$ implies $E(q_\alpha) \in O_\varepsilon$, so $E(q_\alpha)$ is eventually in O_ε. E continuous and $\mathscr{F}_t \to \mathscr{F}_\infty$ imply that $\tilde{q}_t \xrightarrow{L_1} \tilde{q}_\infty$.

The final step is to demonstrate that $\tilde{q}^s_\infty = \left(\frac{a\theta}{a+b}\right)$ for all $s \in L$. Define Z_t by $Z_t = b^{-1}[\theta - \sum_s \lambda_s \tilde{q}^s_t]$, so $Z_t = \tilde{P}_t - \frac{\varepsilon_t}{b}$. Define Z_∞ by $Z_\infty = b^{-1}[\theta - \sum_s \lambda_s \tilde{q}^s_\infty]$ and observe that $Z_t \xrightarrow{L_1} Z_\infty$.

3.14. LEMMA. *For all* $s \in L$, $E[Z_t \| \mathscr{F}_{s,t-1}] \xrightarrow{L_1} E[Z_\infty \| \mathscr{F}_{s,\infty}]$ *and* $E[Z_\infty \| \mathscr{F}_{s,\infty}] \overset{a.s.}{=} Z_\infty$.

PROOF. Since $Z_t \xrightarrow{L_1} Z_\infty$, by direct application of an extension of the Martingale Convergence Theorem (Blackwell-Dubins, [1962, Theorem 2]), $E(Z_t \| \mathscr{F}_{s,t-1}) \to E[Z_\infty \| \mathscr{F}_{s,\infty}]$.

We now demonstrate that $E[Z_\infty \| \mathscr{F}_{s,\infty}] = Z_\infty$ a.s.. Choose a subsequence $\{Z_{t_k}\}$ such that $Z_{t_k} \xrightarrow{a.s.} Z_\infty$. Define Z'_∞ by $Z'_\infty = \overline{\lim} \frac{1}{n} \sum_{k=1}^{n} \tilde{P}_{t_k} = \overline{\lim} \frac{1}{n} \sum_{k=1}^{n} (Z_{t_k} + \varepsilon_{t_k})$. Z'_∞ is $F_{s,\infty}$ measurable and by the strong law of large numbers $Z'_\infty = Z_\infty$. So $Z_\infty = Z'_\infty \overset{a.s.}{=} E[Z'_\infty \| \mathscr{F}_{s,\infty}] = E[Z_\infty \| \mathscr{F}_{s,\infty}]$.

3.15. LEMMA. *$Z_\infty > 0$ P a.s.*.

PROOF. Let $A = \{\omega | Z_\infty(\omega) \leq 0\}$ and since $\tilde{q}^s_\infty = \operatorname{Max}\{0, aE[Z_\infty \| \mathscr{F}_{s,\infty}]\} = \operatorname{Max}\{0, aZ_\infty\}$, $q^s_\infty(\omega) = 0$ for almost all $\omega \in A$. Define the exceptional subset $E \subset A$ by $E = \{\omega \in A \| s \in L \text{ s.t. } \tilde{q}^s_\infty(\omega) \neq 0\}$, $P(E) = 0$. For $\omega \in A - E$, $\tilde{Q}_\infty(\omega) = 0$ so $Z_\infty(\omega) = b^{-1}(\theta(\omega) - Q_\infty(\omega)) = b^{-1}\theta(\omega) > 0$, contradicting $\omega \in A$. Hence, $A - E = \emptyset$ and $P(A) = 0$.

3.16. THEOREM. *$Z_\infty = \dfrac{\theta}{a+b}$ and $\tilde{q}^s_\infty = \dfrac{a\theta}{a+b}$ P a.s. for all $s \in L$*

PROOF. Observe that $\tilde{q}^s_\infty \overset{a.s.}{=} \operatorname{Max}\{0, aE[Z_\infty \| \mathscr{F}_{s,\infty}] = \operatorname{Max}\{0, aZ_\infty\} = aZ_\infty$. The first equality is by definition, the second by Lemma 3.14, and the third by Lemma 3.15. $\tilde{q}^s_\infty = aZ_\infty$ for all $s \in L$ implies $\tilde{Q}_\infty = aZ_\infty$ and so $\tilde{Q}_t \xrightarrow{L_1} aZ_\infty$. But $\tilde{Q}_t = \theta - bZ_t$ and since $Z_t \xrightarrow{L_1} Z_\infty$, $Q_t \xrightarrow{L_1} \theta - bZ_\infty$ So $aZ_\infty = \theta - bZ_\infty$ or $Z_\infty = \dfrac{\theta}{a+b}$ and $\tilde{q}^s_\infty = \dfrac{a\theta}{a+b}$.

Since $Z_t \xrightarrow{L_1} Z_\infty$, $\tilde{q}^s_t \xrightarrow{L_1} \tilde{q}^s_\infty$, and $\tilde{P}_t = Z_t + \frac{\varepsilon_t}{b}$, Theorem 3.16 implies convergence (in L_1) to the rational expectations equilibrium.

4. EQUILIBRIUM AND BELIEFS REGARDING θ

In this section, we impose a strong restriction on the support of θ. This restriction yields a uniform (in s and t) bound on $\tilde{q}^s_t$ that enables us to represent q^s_t as a function of type s expectation of θ, type s expectations of average expectations of θ, type s expectations of average expectations of average expectations, etc. Applying an extension of the Martingale Convergence Theorem we prove that there is a.s. convergence of expectations of all orders and this in turn implies that $\tilde{q}^s_t \xrightarrow{a.s.} \dfrac{a\theta}{a+b}$.

The restriction on a, b, and the support of θ that is assumed throughout this section is:

ASSUMPTION 4.1. *The support of θ is contained in an interval supp $\theta \subseteq [\theta_{\text{Min}}, \theta_{\text{Max}}]$ with $1 > \dfrac{\theta_{\text{Min}}}{\theta_{\text{Max}}} > \dfrac{a}{b}$.*

4.2. PROPOSITION. *P a.s. for all $s \in L$ and for all t, $t = 1, 2, \ldots$, $\dfrac{a}{b}(\theta_{\text{Min}} - \dfrac{a}{b}\theta_{\text{Max}}) < \tilde{q}^s_t < \dfrac{a}{b}\theta_{\text{Max}}$.*

PROOF. For all $u \in L$ and all t, $\tilde{q}_t^u \geq 0$ so $E[\tilde{q}_t^u \| \mathscr{F}_{s,t-1}] \geq 0$ P a.s. and similarly $E[\theta \| \mathscr{F}_{u,t-1}] \leq \theta_{\text{Max}}$ P a.s.. So $\tilde{q}_t^s = \text{Max}\left\{0, \frac{a}{a\lambda_s + b}[E(\theta \| \mathscr{F}_{s,t-1}) - \sum_{u \neq s} \lambda_u E(\tilde{q}_t^s \| \mathscr{F}_{s,t-1})]\right\} \overset{as.}{\leq} \text{Max}\left\{0, \frac{a}{a\lambda_s + b}\theta_{\text{Max}}\right\} < \frac{a}{b}\theta_{\text{Max}}$.

To prove that $\tilde{q}_t^s > \frac{a}{b}\left(\theta_{\text{Min}} - \frac{a}{b}\theta_{\text{Max}}\right)$, we observe that since $\tilde{q}_t^u < \frac{a}{b}\theta_{\text{Max}}$, $[E(\theta \| \mathscr{F}_{s,t-1}) - \sum_{u \neq s} \lambda_u E(\tilde{q}_t^u \| \mathscr{F}_{s,t-1})] \geq 0$ and so $\tilde{q}_t^s = \frac{a}{a\lambda_s + b}[E(\theta \| \mathscr{F}_{s,t-1}) - \sum_{u \neq s} \lambda_u E(\tilde{q}_t^u \| \mathscr{F}_{s,t-1})]$. The last equality and $\tilde{q}_t^u < \frac{a}{b}\theta_{\text{Max}}$ imply that $\tilde{q}_{it} = \frac{a}{b}[E(\theta \| \mathscr{F}_{\tau(i),t-1}) - E(\tilde{Q}_t \| \mathscr{F}_{\tau(i),t-1})] > \frac{a}{b}\left[\theta_{\text{Min}} - \frac{a}{b}\theta_{\text{Max}}\right]$.

With slight modification, Proposition 2 of Townsend [1978] provides an explicit characterization of the equilibrium as a function of beliefs, beliefs about beliefs, etc. To facilitate comparison with the work of Townsend [1978] his notational conventions are adopted.

The expection of θ for firm i prior to time $t+1$ when ω occurs is denoted by $m_{0,t}(i, \omega)$ where by definition $m_{0,t}(i, \omega) = E[\theta \| \mathscr{F}_{\tau(i),t}](\omega)$. The average expectation of θ across firms is denoted by the random variable $\theta_{1,t}$ defined by $\theta_{1,t}(\omega) = \int_I m_{0,t}(i, \omega)\lambda(di)$. The expectation of $\theta_{1,t}$ for firm i is the random variable $m_{1,t}(i, \cdot)$ defined by $m_{1,t}(i, \omega) = E[\theta_{1,t} \| \mathscr{F}_{\tau(i),t}](\omega)$. Continuing in a recursive manner, $\theta_{k,t}(\cdot) = \int_I m_{k-1,t}(i, \cdot)\lambda(di)$ and $m_{k,t}(i, \cdot) = E[\theta_{k,t} \| \mathscr{F}_{\tau(i),t}](\cdot)$. Sometimes I will write $m_{k,t}(i)$ as a shorthand for the random variable $m_{k,t}(i, \cdot)$.

I define α_n for $n = 0, 1, 2, \ldots$ by $\alpha_n = \left(\frac{a}{b}\right)^{n+1}(-1)^n$.

4.3. PROPOSITION. *$(\tilde{q}_{it})$ is a Bayesian-Nash equilibrium in output strategies iff* $\tilde{q}_{it} = \sum_{n=0}^{\infty} \alpha_n m_{n,t-1}(i)$ *for all* $i \in I$.

PROOF. In Appendix.

Define $m_{0,\infty}(i)$ *by* $m_{0,\infty}(i) = E[\theta \| \mathscr{F}_{\tau(i),\infty}]$

4.4. PROPOSITION. $m_{0,t}(i) \to m_{o,\infty}(i)$ for all $i \in I$ P a.s..

PROOF. This follows from the assumption that all agents of the same type use the same version of conditional expectation and that for arbitrary $i \in I$, $m_{0,t}(i) \xrightarrow{a.s.} m_{0,\infty}(i)$ (Billingsley, [1979, Theorem 35.5]).

4.5. COROLLARY. *With* $\theta_{1,\infty}$ *defined by* $\theta_{1,\infty}(\omega) = \int m_{0,\infty}(i, \omega)\lambda(di)$, *$P$ a.s.* $\theta_{1,t} \to \theta_{1,\infty}$.

PROOF. This is a direct consequence of Proposition 4.4, the boundedness of θ, and the Dominated Convergence Theorem (Billingsley [1979, Theorem 16.4]).

I define $m_{k-1,\infty}(i)$ by $m_{k-1,\infty}(i) = E[\theta_{k-1,\infty} \| \mathscr{F}_{\tau(i),\infty}]$ and define $\theta_{k,\infty}$ by $\theta_{k,\infty} = \int m_{k-1,\infty}(i)\lambda(di)$ for $k = 2, 3, \ldots$. Since $m_{0,\infty}(i)$ and $\theta_{1,\infty}$ have been defined above, $m_{k,\infty}(i)$ and $\theta_{k,\infty}$ are well-defined for all k. An induction argument is now invoked to prove that for all k, $\theta_{k,t} \xrightarrow{a.s.} \theta_{k,\infty}$ and $m_{k,t}(i) \xrightarrow{a.s.} m_{k,\infty}(i)$ for all i and k.

4.6. PROPOSITION. *Let* $k=1, 2, 3, \ldots$. *Suppose* $m_{k-1,t}(i) \to m_{k-1,\infty}(i)$ *for all* $i \in I$ *P a.s. and* $\theta_{k,t} \xrightarrow{a.s.} \theta_{k,\infty}$. *Then* $m_{k,t}(i) \to m_{k,\infty}(i)$ *for all* $i \in I$ *P a.s. and* $\theta_{k+1,t} \xrightarrow{a.s.} \theta_{k+1,\infty}$.

PROOF. By a generalization of the Martingale Convergence Theorem (Chung [1974, Theorem 9.4.8]), $\theta_{k,t} \xrightarrow{a.s.} \theta_{k,\infty}$ implies $E[\theta_{k,t} \| \mathscr{F}_{s,t}] \xrightarrow{a.s.} E[\theta_{k,\infty} \| \mathscr{F}_{s,\infty}]$. So $m_{k,t}(i) \to m_{k,\infty}(i)$ for all $i \in A_s$, P a.s.. Since there are a finite number of types, $m_{k,t}(i) \to m_{k,\infty}(i)$ for all $i \in I$, P a.s..

Since θ is bounded, by the Dominated Convergence Theorem $m_{k,t}(i) \to m_{k,\infty}(i)$ for all $i \in I$ P a.s. implies $\int m_{k,t}(i)\lambda(di) \xrightarrow{a.s.} \int m_{k,\infty}(i)\lambda(di)$ or equivalently $\theta_{k+1,t} \xrightarrow{a.s.} \theta_{k+1,\infty}$.

4.7. COROLLARY. *For all* $k=0, 1, 2, \ldots$, $m_{k,t}(i) \to m_{k,\infty}(i)$ *for all* $i \in I$ *P a.s. and* $\theta_{k+1,t} \xrightarrow{a.s.} \theta_{k+1,\infty}$.

PROOF. Follows from Propositions 4.4 and 4.6, the Corollary 4.5 and a standard induction argument.

4.8. PROPOSITION. $\tilde{q}_{i,\infty} = \sum_{n=0}^{\infty} \alpha_n m_{n,\infty}(i)$, $\tilde{q}_{i,t} \xrightarrow{a.s.} \tilde{q}_{i,\infty}$, *and* $\tilde{Q}_t \xrightarrow{a.s.} \tilde{Q}_\infty$.

PROOF. Because θ is bounded, $a<b$, and $m_{n,t}(i) \to m_{n,\infty}(i)$, the Weierstrass M-test for series (Billingsley [1979, p. 180]) implies that $\tilde{q}_{i,t} = \sum_{n=0}^{\infty} \alpha_n m_{n,t}(i) \cdot \xrightarrow{a.s.} \sum_{n=0}^{\infty} \alpha_n m_{\infty,t}(i)$. Since $\tilde{q}_{i,t}$ is uniformly bounded, $\tilde{q}_{i,t} \xrightarrow{L_1} \sum_{n=0}^{\infty} \alpha_n m_{n,\infty}(i)$. So by Proposition 3.13, $\tilde{q}_{i,\infty} = \sum_{n=0}^{\infty} \alpha_n \cdot m_{n,\infty}(i)$. Since $\tilde{q}_{i,t} \to \tilde{q}_{i,\infty}$ for all i, P a.s.,

$$\int \tilde{q}_{i,t}(\omega)\lambda(di) \xrightarrow{a.s.} \int \tilde{q}_{i,\infty}(\omega)\lambda(di) \quad \text{or} \quad \tilde{Q}_t \xrightarrow{a.s.} \tilde{Q}_\infty .$$

Summarizing the above results, we have almost sure convergence to the rational expectations equilibrium.

4.9. PROPOSITION. $\tilde{q}_{i,t} \to \dfrac{a\theta}{a+b}$ and $\tilde{Q}_t \to \dfrac{a\theta}{a+b}$, *P a.s.*.

PROOF. By Theorem 3.16 $\tilde{q}_{i,\infty} = \dfrac{a\theta}{a+b}$ and $\tilde{Q}_\infty = \dfrac{a\theta}{a+b}$, P a.s.. The conclusion follows from Proposition 4.8.

5. SCOPE FOR GENERALIZATION AND CONCLUDING COMMENTS

The results of this paper validate the logical possibility of convergence to a rational expectations equilibrium in a world inhabited by Bayesian agents for whom there is not only parameter uncertainty, but also uncertainty regarding the beliefs of other agents.[7] The assumptions in the body of the paper are so stringent, though, that no general conclusions can be inferred. So we are compelled to ask the extent to which the convergence result is roubust to a more general specification.

Within a partial equilibrium setting with risk neutral firms, generalization is not difficult. For example one could model b as well as θ as a random variable, making sufficient restrictions on the joint support to insure existence of a temporary REE. While this can result in an identification problem for b and θ individually, $\frac{\theta}{a+b}$ will remain identifiable. The assumptions of linearity of the demand function and quadratic cost functions can also relaxed.

A different sort of generalization would be to allow multiple shocks each period or to allow the shocks to be governed by a more complex stochastic process. For instance if the demand function was $D_t(p_t) = -bp_t + \theta X_t + \varepsilon_t$ where the distribution of X_t is common knowledge, independent of θ, and $\{X_t\}$ is i.i.d. then the techniques used in this paper to prove convergence to the REE are still valid.[8] But if $\{X_t\}$ or $\{\varepsilon_t\}$ is a Markov process then the analytical issues are much more difficult. It is relatively easy to bound the asymptotic deviation from the REE, but it is not apparent how to prove convergence to the REE when the shocks are Markov.

It would also be of interest to further investigate the implications of Bayesian learning in general equilibrium models. To pursue this topic one has to construct a model with an associated notion of equilibrium such that the temporary REE are non-revealing.[9] Suppose now that the parameter space is Θ, a separable metric space, and that the existence problem is resolved. Then as in Blume-Easley [1984] and Bray-Kreps [1981] the Martingale Convergence Theorem guarantees that beliefs regarding Θ converge to some (possibly random) limit beliefs. The technique used in Section 4 can be extended to assure that beliefs of all orders converge. But as stressed by Bray-Kreps [1981] convergence of beliefs will not in general imply convergence of the sequence of temporary REE. The smoothness (and other) assumptions needed so that the temporary equilibria are continuous function of beliefs may be quite severe.

So whether the static REE studied in microeconomic theory can be viewed as the limit of a Bayesian learning process remains an open question.

[7] It is asserted by Frydman [1982] in a similar framework that asymmetric initial information precludes firms from learning over time the knowledge that is necessary to sustain a rational expectations equilibrium. The techniques of this paper can be adapted to demonstrate that Frydman's claims are incorrect if the firms in his model act as Bayesians with a common prior along the lines of Harsanyi [1967, 1968].

[8] A referee encouraged me to consider this case.

[9] It is not yet resolved how best to model the economy to assure the existence of a non-revealing REE. Allen ([1985a] (circulated as Allen [1982a])), Allen ([1985b] (circulated as Allen [1982b])), and Anderson-Sonnenschein [1982] have made some progress in this direction.

University of Illinois at Urbana-Champaign U.S.A.

APPENDIX

PROOF OF LEMMA 3.6. Let $\mathscr{D}=\{D_1 \in \mathscr{S}_1 \vee \mathscr{H}_1 : \exists D_2 \in \mathscr{S}_2 \vee \mathscr{H}_2 \text{ s.t. } D_1 \sim D_2\}$. It suffices to prove that $\mathscr{D}=\mathscr{S}_1 \vee \mathscr{H}_1$. D has the following properties: (1)$\Omega \in \mathscr{D}$, (2) $A \in \mathscr{D}$, $B \in \mathscr{D}$, and $A \subseteq B$ imply $B-A \in \mathscr{D}$, and (3) $A_n \in \mathscr{D}$, $A_n \uparrow A$, imply $A \in \mathscr{D}$. So by definition $\mathscr{D}$ is a λ-system [Billingsley (p. 33)]. Define the family $\mathscr{W}$ of subsets of $\mathscr{F}$ by $\mathscr{W}=\{F \in \mathscr{F} | F=G_1 \cap H_1 \text{ s.t. } \mathscr{D}_1 \in \mathscr{S}_1 \text{ and } \mathscr{H}_1 \in \mathscr{H}_1\}$. $\mathscr{W}$ is closed under intersection and hence is a π-system. $\mathscr{W} \subset \mathscr{D}$ and so by the $\pi-\lambda$ Theorem (Billingsley [1979, Theorem 3.2]) $\sigma(\mathscr{W}) \subset \mathscr{D}$. But $\sigma(\mathscr{W})=\mathscr{S}_1 \vee \mathscr{H}_1$ so $\mathscr{S}_1 \vee \mathscr{H}_1 = \mathscr{D}$.

PROOF OF ROPOSITION 4.3. This requires only minor modifications of Proposition 2 in Townsend [1978]. Suppose that $\tilde{q}_{jt}=\sum_{n=0}^{\infty} \alpha_n m_{n,t-1}(j)$. By Assumption 4.1 $\|m_{n,t}(j)\|$ is uniformly bounded in n, t, and j. By the series version of the Dominated Convergence Theorem (Billingsley [1979, Theorem 16.7]), $\int_{-1} \tilde{q}_{jt}\lambda(d_j) = \sum_{n=0}^{\infty} \int \alpha_n m_{n,t-1}(j)\lambda(d_j)= \sum_{n=0}^{\infty} \alpha_n \theta_{n+1,t-1}$. So $\tilde{P}_t=-b^{-1}[\sum_{n=0}^{\infty} \alpha_n \theta_{n+1,t-1}-\theta-\varepsilon_t]$ and $E_t[\tilde{P}_t \| F_{s,t-1}]=-b^{-1}[\sum_{n=0}^{\infty} \alpha_n m_{n+1,t-1}(i)-m_{0,t-1}(i)]$ for $i \in A_s$. Therefore, the output that maximizes conditional expected profit is $\frac{-a}{b}[\sum_{n=0}^{\infty} \alpha_n m_{n+1,t-1}(i)- m_{0,t-1}(i)]= \sum_{n=0}^{\infty} \alpha_n m_{n,t-1}(i)$.

REFERENCES

Allen, B., "Neighboring Information and Distribution of Agents' Characteristics under Uncertainty," *Journal of Mathematical Economics*, 12 (1983), 63–101.

———, "The Existence of Fully Rational Expectations Equilibria with Noisy Price Observations," *Journal of Economic Theory*, 37 (1985a), 213–253.

———, "The Existence of Fully Rational Expectations Equilibria with Noisy Price Observations," Caress Working Paper 82–07, Center for Analytic Research in Economics and the Social Sciences, University of Pennsylvania, (1982a).

———, "The Existence of Fully Rational Expectations Equilibria in a Large Economy with Noisy Price Observations," *Journal of Mathematical Economics*, 14 (1985b), 67–103.

———, "The Existence of Fully Rational Expectations Equilibria in a Large Economy with Noisy Price Observations," Caress Working Paper 82–06, Center for Analytic Research in Economics and the Social Sciences, University of Pennsylvania, (1982b).

ANDERSON, R. AND H. SONNENSCHEIN, "On the Existence of Rational Expectations Equilibrium," *Journal of Economic Theory*, 26 (1982), 261–278.

ARROW, K., *Essays in the Theory of Risk-Bearing* (Amsterdam: North-Holland, 1970 Chopter 2).

BILLINGSLEY, P., *Convergence of Probability Measures*, (New York: John Wiley & Sons, 1968).

———, *Probability and Measure*, (New York: John Wiley & Sons, 1979).

BLACKWELL, D. AND L. DUBINS, "Merging of Opinions with Increasing Information," *Annals of Mathematical Statistics*, 33 (1962), 882–886.

BLUME, L. E., M. M. BRAY, D. EASLEY, "Introduction to the Stability of Rational Expectations Equilibrium," *Journal of Economic Theory*, 26 (1982), 313–317.

BLUME, L. E. AND D. EASLEY, "Learning to be Rational, "Journal of Economic Theory, 26 (1982), 340–351.

———, "Rational Expectations Equilibrium: An Alternative Approach," *Journal of Economic Theory*, 34 (1984), 116–129.

BOYLAN, E., "Equiconvergence of Martingales," *Annals of Mathematical Statistics*, 42 (1971), 552–559.

BRAY, M., "Learning, Estimation and the Stability of Rational Expectations," *Journal of Economic Theory*, 26 (1982), 318–339.

——— AND D. KREPS, "Rational Learning and Rational Expectations," mimeo, Standord University, (1981).

CHUNG, K. L., *A Course in Probability Theory*, (New York: Academic Press, 1974).

COTTER, K., "Similarity of Information and Behavior with a Pointwise Convergence Topology," *Journal of Mathematical Economics*, 15 (1986), 25–38.

CYERT, R. AND M. DEGROOT, "Rational Expectations and Bayesian Analysis," *Journal of Political Economy*, 82 (1974), 521–536.

DECANIO, S., "Rational Expectations and Learning from Experience," *Quarterly Journal of Economics*, 92 (1979) 45–57.

DIACONIS, P. AND D. FREEDMAN, "On the Consistency of Bayes Estimates," *Annals of Statistics*, 14 (1986), 1–26.

FELDMAN, MARK, "Bayesian Learning and Convergence to Rational Expectations," (forthcoming) *Journal of Mathematical Economics.*

FRYDMAN, R., "Towards an Understanding of Market Processes: Individual Expectations, Learning and Convergence to Rational Expectations Equilibrium," *American Economic Review*, 72 (1982), 652–658.

HALL, P. AND C. C. HEYDE, *Martingale Limit Theory and Its Application* (New York: Academic Press, 1980).

HARSANYI, J., "Games with Incomplete Information Played by Bayesian Players," *Management Science*, 14 (1967–1968), 156–182, 320–334, 486–502.

JORDAN, JAMES, "Convergence of Rational Expectations in a Stationary Linear Game," mimeo, University of Minnesota, (1986).

———, "The Generic Existence of Rational Expectations Equilibrium in the Higher Dimensional Case," *Journal of Economic Theory*, 26 (1982), 224–243.

———, "Learning Rational Expectations: The Finite State Case," *Journal of Economic Theory*, 36 (1985), 257–276.

KADANE, J. AND D. CHUANG, "Stable Decision Problems," *Annals of Statistics*, 6 (1978), 1095–1110.

KELLEY, J., *General Topology*, (Germany: Springer-Verlag, 1955).

MARCET, A. AND T. SARGENT, "Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," mimeo, University of Minnesota, (1986).

NEVEU, J., *Mathematical Foundations of the Calculus of Probability* (New York: Holden-Day, 1965).

RADNER, R., "Equilibrium under Uncertainty," Chapter 20, in, K. Arrow and M. Intriligator eds., *Handbook of Mathematical Economics* (New York: North-Holland, 1982).

SMART, D. R., *Fixed Point Theorems*, (Cambridge, Cambridge University Press, 1974).

TOWNSEND, R., "Market Anticipations, Rational Expectations and Bayesian Analysis," *International Economic Review*, 19 (1978), 481–494.

WOODFORD, MICHAEL, "Learning to Believe in Sunspots", mimeo, Columbia University and New York University, (1986).